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# Theoretical Mechanics. Kinematics

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The textbook covers all the sections of kinematics of the discipline «Theoretical Mechanics» in the following sequence: first, the theoretical material is presented, and then the practical part is presented, where the methodology for solving problems from this section is provided, as well as a series of problems with demonstration solutions, accompanied by a full explanation, after which problems for independent solution are proposed. Then you are provided with a number of activities to check your knowledge: self-control questions, test questions, and test tasks. The textbook is recommended for for students of all specialties where Theoretical Mechanics is studied.

For full-time and part-time students.

УДК 531.1 (075.8)

**Content**

## Preface

Kinematics is a branch of theoretical mechanics that studies the motion of a point or solid body from a geometric point of view, regardless of the forces acting on them of the forces acting on them. It studies the dependencies between the space-time characteristics of mechanical motion.

When studying the motion of a solid body, you should know with respect to which reference system the motion is being considered. As you know, the movements of all the objects under study can occur in both moving and stationary reference frames.

The motion of a solid body relative to a chosen reference frame is considered known if its position in this frame can be determined at any time.

Dependence of parameters characterizing the position of a solid relative to the reference frame, is determined by the corresponding equations, which are called the laws of motion of a body.

Since the motion of a geometric image of a body is known when the law of motion of all its points is known, its motion should be considered after studying the motion of one of its points. This logic is the basis for the division of Kinematics into such sections such as “Kinematics of a point” and “Kinematics of a solid body” (depending on the type of the body) and the kinematics of the combination of a solid and a point. That is why the textbook first deals with the kinematics of a point, and then with the kinematics of a solid and synthesis of movements.

The textbook presents all sections of kinematics in the following sequence: first, the theoretical material based on the textbook [1] is presented, then the practical part is presented, which describes the methodology of solving problems from this section, as well as a series of problems with demonstration and a series of problems with demonstration solutions accompanied by a full explanation [2, 3]. After that problems for self-study are offered [4]. Next, there are a number of activities to test the acquisition of knowledge: questions for self-control, test questions and test tasks.

## Basic concepts of kinematics

In the definition of mechanical motion, the subject of study of kinematics - uses the concepts of space and time, which are basic in kinematics.

The units of length and time are the meter (m) and the second (s). Physical quantities whose units of measurement are expressed through the units of length and time are called kinematic.

The position of any material object in a chosen frame of reference is determined by a set of numbers, which are called coordinates of the material object in this reference frame.

The reference frame is usually referred to by the same letters that designate the axes associated with the reference frame.

Thus, for example, if the Earth with the coordinate axes Oxyz set on it is chosen as the reference frame *A*, then we speak about the reference frame Oxyz associated with the Earth. The reference frame in which the reference frame is the carriage of a moving train, let us call the system $O_1x_1y_1z_1$, if the axes attached to the car are designated by these letters.

When the positions of various bodies in the system *Oxy*z with reference frame *A* are determined, these bodies should be said to be in the space *Oxyz* associated with body *A*, or in short - in the space of body *A*.

From the set of reference frames we should distinguish heliocentric and geocentric. For the first one the Sun together with the fixed stars is the reference frame, and for the second one - the Earth.

Naturally, in engineering practice, the geocentric reference frame is most often used, since most engineering calculations are related to determining the positions of parts of structures and machine parts in relation to the Earth calculations are related to determining the positions of parts of structures and machine parts in relation to the Earth.

Mechanical motion has been defined as a change in the mutual arrangement of material bodies in space over time. Since the position of material bodies is determined only in relation to a certain body of reference, it means that before we talk about the motion of bodies, we must first introduce a frame of reference, in the space of which this motion is considered.

Let us consider the position of different bodies in *Oxyz* space (bodies of frames of reference $\gamma$). The words in the bracket can sometimes be omitted, because space is inconceivable without a reference frame, which is assigned before we speak about the motion of any material objects.

If the position of some body *B* (a point, a mechanical system) in the space *Oxyz* changes with the passage of time, we will say about the body *B* that it performs mechanical motion in this space. Thus, mechanical motion is a change in the position of material bodies in the space of a selected reference frame with time in the space of the selected reference frame with the passage of time.

If the position of the body B in the space *Oxyz* does not change with the passage of time then body B is called stationary, at rest or in equilibrium in this space.

The motion and rest of material bodies in the space of the chosen reference frame is usually called their mechanical state in this space.

It is senseless to speak about motion and rest of a body until the reference frame is established. Rest and motion are defined relative to the selected reference frame. Therefore, these concepts are relative.

For example, a passenger sitting in a car is stationary with respect to the car but is moving relative to the platform

A body stationary in one frame of reference can move in another frame of reference. It is clear that the motion of the same body in different spaces turns out to be different.

The motion of a material object (a material object; from the position it occupies at the moment of time $t$, for a finite $\Delta t$ or any small intervals $dt$ will be called finite or elementary, respectively. The elementary motion of the material object is also called instantaneous or the motion of the material object at time $t$. Thus, when one speaks of the instantaneous motion or motion of the material object at a certain moment of time, it means the motion of the material object in during time $dt$ from the position it occupied at that moment. The distances between the same points of the material object at times $t$ and $t + dt$ are small quantities of order $(dt)^2$ or higher: a small time interval also corresponds to a small change in the position of the moving body in space of the moving body in space. In this sense, the motion of material objects is said to be continuous.

Among the motions of a solid body it is customary to distinguish its two basic or the simplest motion: translational and rotational about a stationary axis.

These two motions of a solid are called basic or simplest, because any displacement of a body from one position to another can be can be accomplished by a sequence of translational and rotational motions.

## Lecture №1
## 1 Kinematics of a point. Basic concepts

Kinematics is a branch of mechanics that studies the geometric properties of the motion of bodies without taking into account their inertia (mass) and the forces acting on them. It is conventionally divided into two sections:

1) point kinematics, i.e., the kinematics of a body whose dimensions can be neglected and whose position can be described as that of a geometric point;

2) kinematics of a solid body.

The motion of a point is characterized by the law of motion, trajectory, velocity, and acceleration.

The law of motion of a point defines the relationship between an arbitrary position of a moving point in space and time. The motion of a point relative to a chosen reference system is considered to be given if it is known how the position of the point can be determined at any given time.

The line that a point follows as it moves is called a trajectory. If the trajectory of a point is a straight line, the motion of the point is called straight line, if the trajectory of a point is curved, it is called curved.

The main kinematic characteristics of a point's movement are its position, velocity, and acceleration.

The main task of point kinematics is to find ways to specify its position and methods for determining its velocity and acceleration. The motion of a point can be determined in three ways: vector, coordinate, and natural.

### 1.1 Three ways to define the motion of a point. Determining velocity and acceleration

1) The vector method.

The position of point M is determined by the radius vector $\bar{r}$ drawn from some fixed point *O* to this point *M* (Fig. 1). When moving, the radius vector $\bar{r}$ changes in size and direction. Each moment of time *t* corresponds to a certain value $\bar{r}$.

$$\bar{r} = \bar{r}(t) \tag{1.1}$$

Eq. (1.1) is called the kinematic equation of motion of a point in vector form, as well as the equation of the point's trajectory in vector form.

The velocity of a point $\bar{V}$ is equal to the vector derivative of the radius vector of the point ($\bar{r}$) in time and is always tangent to the point's trajectory:

$$\bar{V} = \frac{d\bar{r}}{dt} \tag{1.2}$$

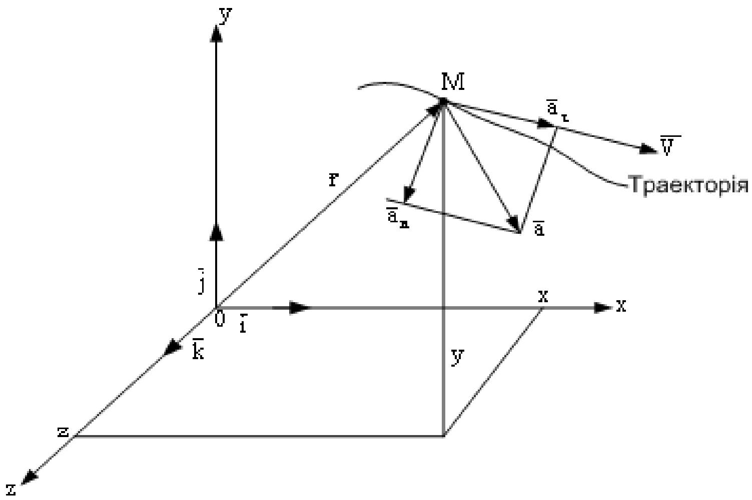



Figure 1

The acceleration of a point $\bar{a}$ is a measure of the change in the velocity of a point over time.

The acceleration of a point is equal to the second derivative of its velocity vector in time

$$\bar{a} = \frac{d\bar{V}}{dt} = \frac{d^2\bar{r}}{dt^2} = \ddot{\bar{r}}. \qquad (1.3)$$

The acceleration of a point $\bar{a}$ is always directed towards the concavity of its trajectory.

The law of motion of a point is considered to be known if the conditions are given that allow you to determine the position of the point at any given time (Any vector is completely determined in one of two ways: a) by projections of the vector onto the axis coordinates; b) by the vector's modulus and its direction in space).

2) The coordinate method

The radius vector of point $M$ is equal to (Fig. 1)

$$\bar{r} = \bar{\imath}x + \bar{\jmath}y + \bar{k}z,$$

where $\bar{\imath}, \bar{\jmath}, \bar{k}$– axis orthogonals $x, y, z$.

That is

$$r_x = x, r_y = y, z_z = z \text{ i } r = \sqrt{x^2 + y^2 + z^2}.$$

The projections of the radius vector on the axis $x, y, z$– are the Cartesian coordinates of this point.

Kinematic characteristics of a point in the coordinate method of motion.

When a material point $M$ moves in space, a coordinate system is introduced $Oxyz$. The motion of the point is given by three functions:

$$x = x(t),\ y = y(t), z = z(t) \quad (1.4)$$

Eq. (1.4) is the equation of motion of a point in coordinate form in the spatial case.

The velocity of a point $\bar{V}$ is determined by the projections of the velocity vector onto the coordinate axes, which are found using the formulas:

$$V_x = \dot{x}(t),\ V_y = \dot{y}(t),\ V_z = \dot{z}(t), \quad (1.5)$$

where $V_x, V_y, V_z$ – are the projections of the velocity vector onto the coordinate axes, respectively $x, y, z$.

According to the known values of the velocity projections (1.5) on the coordinate axes, the modulus can be calculated by directing the cosines of the velocity vector.

$$\bar{V} = \bar{V}_x + \bar{V}_y + \bar{V}_z = V_x \cdot \bar{\imath} + V_y \cdot \bar{\jmath} + V_z \cdot \bar{k}, \quad (1.6)$$

velocity modulus:

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2},$$

$$cos\,\alpha_V = \frac{V_x}{V};\ cos\,\beta_V = \frac{V_y}{V};\ cos\,\gamma_V = \frac{V_z}{V},$$

$\alpha_V, \beta_V, \gamma_V$ – are the corresponding angles between the velocity vector $\bar{V}$ and the corresponding axis.

The acceleration of a point $\bar{a}$ is determined by the projections of the acceleration vector onto the coordinate axes, which are found using the formulas:

$$a_x = \dot{V}_x = \ddot{x}(t), a_y = \dot{V}_y = \ddot{y}(t),\ a_z = \dot{V}_z = \ddot{z}(t). \quad (1.7)$$

$a_x, a_y, a_z -$ are the projections of the acceleration vector on the axes, respectively $x, y, z$.

According to the known values of the velocity projections (1.7) on the coordinate axes, the modulus can be calculated by directing the cosines of the acceleration vector:

$$\bar{a} = \bar{a}_x + \bar{a}_y + \bar{a}_z = a_x \cdot \bar{\imath} + a_y \cdot \bar{\jmath} + a_z \cdot \bar{k} \quad (1.8)$$

$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ – point acceleration module,

$$cos\,\alpha_a = \frac{a_x}{a}; cos\,\beta_a = \frac{a_y}{a};\ cos\,\gamma_a = \frac{a_z}{a},$$

$\alpha_a, \beta_a, \gamma_a$ – are the corresponding angles between the acceleration vector $\bar{a}$ and the corresponding axis.

In the case of a flat trajectory, one of the components is missing.

3) The natural way

If the trajectory is known in advance (for example, the trajectory of a train, tram, or trolleybus), then to determine the law of motion, it is enough to set the start and direction of motion. Therefore, one of the points $M_o$ on the trajectory (Fig. 2) is taken as the origin of the arc coordinates, since the position of the moving point $M$ is determined by its distance $S$ which is counted along the arc of the trajectory from the selected reference point.

$$S = S(t) \tag{1.9}$$

Eq. (1.9) is the law of motion of a point along a trajectory. The function (1.9) must be unambiguous, continuous, and differentiable.

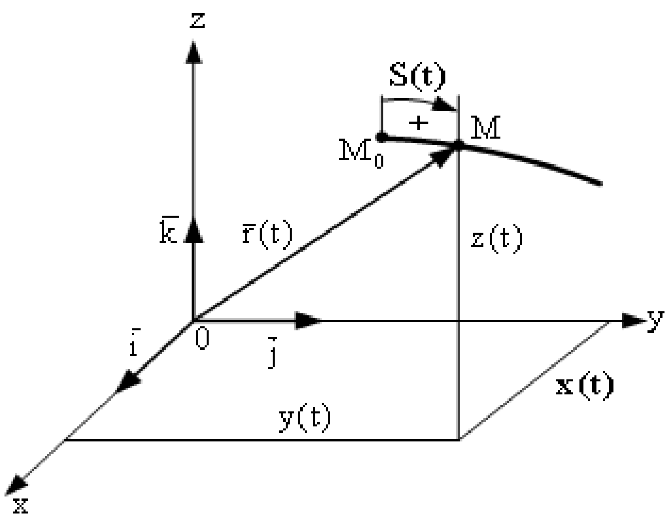



Figure 2

The velocity modulus is determined by the formula:

$$\bar{V} = \frac{d\bar{S}}{dt} \tag{1.10}$$

A vector $\bar{V}$ is directed tangentially $\bar{\tau}$ to the trajectory in the direction of the point movement (Fig. 3)

The acceleration is decomposed into two mutually perpendicular vectors $\bar{a}_n$ and $\bar{a}_\tau$ (Fig. 3) the modules of which are equal to

$$\bar{a} = \bar{a}_\tau + \bar{a}_n, \quad \bar{a}_\tau \perp \bar{a}_n \tag{1.11}$$

Tangential acceleration $\bar{a}_\tau$, characterizes the change in velocity by magnitude and is always directed along the tangent to the trajectory at a point.

$$\bar{a}_\tau = \frac{dV}{dt} = \frac{d^2S}{dt^2}.$$

$$\bar{a}_\tau = \frac{V_x a_x + V_y a_y + V_z a_z}{V}$$

If $\bar{a}_\tau$>0, then the vector $\bar{a}_\tau$ vector coincides with the direction of the velocity vector $\bar{V}$ or directed in the direction opposite to the velocity, if $\bar{a}_\tau$<0.

Normal acceleration $\bar{a}_n$ characterizes the change in speed in the direction. The vector $\bar{a}_n$ is always directed along the principal normal $\bar{n}$ to the trajectory of the point in the direction of curvature, i.e. along $\rho$ (Figure 3).

$$a_n = \sqrt{a^2 - a_\tau^2}\ ,$$

$$\bar{a}_n = \frac{V^2}{\rho},$$

where $\rho$ – is the radius of curvature of the trajectory at the point $\rho = \frac{V^2}{a_n}$.

The full acceleration is determined:

$$\bar{a} = \bar{a}_\tau + \bar{a}_n = a_\tau \cdot \bar{\tau} + a_n \cdot \bar{n} = \frac{V^2}{\rho} \cdot \bar{n} + \frac{dV}{dt} \cdot \tau, \quad \bar{a}_\tau \perp \bar{a}_n \qquad (1.12)$$

where $\bar{\tau}$ – is the tangent; $\bar{n}$ – is the normal.

Acceleration module

$$a = \sqrt{a_\tau^2 + a_n^2}.$$

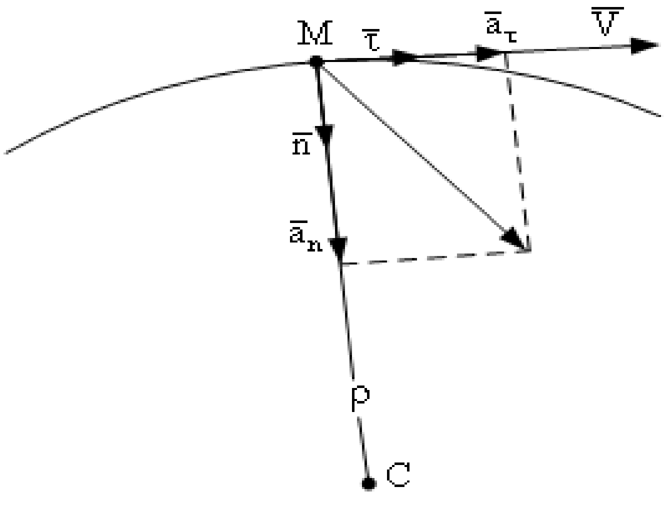



Figure 3

**1.2 Example. Kinematic study of the motion of a point according to the given equations of motion.**

Given equations of motion of a point, find the trajectory, velocity, acceleration with its components tangent and normal acceleration, and the radius of curvature of the trajectory.

Build a trajectory in the figure, indicate the position of the point at the initial moment of movement and at the time $t = t_1$. In the figure of the trajectory at time $t_1$, plot the velocity and acceleration vectors of a point at the appropriately selected scales.

Plot velocity dependencies on time $v(t)$ and acceleration from time $a(t)$.

The equations of motion of the point $x = x(t), y = y(t)$ and the values of the moment $t_1$ are in Tables 1.2, a and 1.2, b, and the values of the coefficients *A*, *B*, *C*, *D*, *K* and $\varphi$ are in Tables 1.1, a and 1.1, b for the respective variants. The dimension of coordinates is centimeters, time is seconds.

Table 1.1, a

| Варі-ант | *A* | *B* | *C* | *D* | *k* |
|---|---|---|---|---|---|
| 1 | 3 | 2 | 6 | 4 | $2\pi$ |
| 2 | 7 | 4 | 2 | 9 | $3\pi$ |
| 3 | 4 | 3 | 6 | 5 | $8\pi$ |
| 4 | 6 | 5 | 4 | 2 | $4\pi$ |
| 5 | 8 | 4 | 7 | 6 | $5\pi$ |
| 6 | 5 | 7 | 3 | 4 | $6\pi$ |
| 7 | 3 | 4 | 7 | 2 | $2\pi$ |
| 8 | 5 | 3 | 6 | 3 | $4\pi$ |
| 9 | 4 | 5 | 8 | 6 | $3\pi$ |
| 0 | 2 | 6 | 9 | 8 | $2\pi$ |

Table 1.1, b

| Варі-ант | *A* | *B* | *C* | *D* | $\varphi$ |
|---|---|---|---|---|---|
| | *см* | | | | *рад* |
| 1 | 70 | 20 | 35 | 25 | $\pi t$ |
| 2 | 40 | 10 | 10 | 15 | $t^2$ |
| 3 | 50 | 25 | 10 | 30 | $2\pi t$ |
| 4 | 64 | 16 | 30 | 18 | $\pi t^2$ |
| 5 | 80 | 40 | 36 | 50 | $2t$ |
| 6 | 55 | 11 | 20 | 25 | $2\pi t^2$ |
| 7 | 65 | 13 | 18 | 20 | $2t$ |
| 8 | 70 | 10 | 25 | 20 | $3t^2$ |
| 9 | 75 | 25 | 30 | 15 | $3\pi t$ |
| 0 | 72 | 24 | 16 | 18 | $t^3$ |

**Example of a task**

A point moves in the plane according to functions of time

$$x = 4 \cdot t - 1,$$

$$y = 2 \cdot t^2 + 3 \qquad (1.13)$$

Find the trajectory, velocity, acceleration, tangent, and normal. The acceleration of the point and the radius of curvature of the trajectory in time $t_1 = 1c$.

Draw a graph of the trajectory, indicate the intervals of movement on it, mark the initial position of the point at $t = 0$ and at $t = t_1$ display the velocity and acceleration vectors by their components at these moments.

Plot the velocity versus time $v(t)$ and acceleration versus time $a(t)$.
Table 1.2, a

| Варіант | $x(t)$, $y(t)$, см | $t_1$, с |
|---|---|---|
| **1** | $x = -At^2 + C, \quad y = -Dt$ | 1 |
| **2** | $x = A\cos^2(\frac{k}{3}t), \quad y = B\sin^2(\frac{k}{3}t)$ | 0,5 |
| **3** | $x = -C\cos(\frac{k}{3}t^2), \quad y = D\sin(\frac{k}{3}t^2)$ | 0,3 |
| **4** | $x = At + B, \quad y = -\frac{D}{t+1}$ | 2 |
| **5** | $x = A\sin(\frac{k}{3}t), \quad y = -B\cos(\frac{k}{3}t) + C$ | 1 |
| **6** | $x = At^2 + C, \quad y = -Bt$ | 2 |
| **7** | $x = 3t^2 - t + C, \quad y = Dt^2 - \frac{D}{3}t - B$ | 1,5 |
| **8** | $x = B\sin(\frac{k}{6}t^2), \quad y = A - C\cos(\frac{k}{6}t^2)$ | 4 |
| **9** | $x = -\frac{A}{t+B}, \quad y = At + 2B$ | 2 |
| **10** | $x = -D\cos(\frac{k}{3}t), \quad y = -C\sin(\frac{k}{3}t) - D$ | 1 |

Continuation of Table 1.2, a

| *Варіант* | $x(t)$, $y(t)$, *см* | $t_1$, *с* |
|---|---|---|
| **11** | $x = -At^2 + D, \quad y = -Bt$ | 1,5 |
| **12** | $x = B\sin^2(\frac{k}{6}t), \quad y = -B\cos^2(\frac{k}{6}t)$ | 1 |
| **13** | $x = D\cos(\frac{k}{3}t^2), \quad y = -D\sin(\frac{k}{3}t^2)$ | 1 |
| **14** | $x = -At - C, \quad y = -\frac{A}{t+1}$ | 1 |
| **15** | $x = D\cos(\frac{k}{3}t), \quad y = -D\sin(\frac{k}{3}t)$ | 0,5 |
| **16** | $x = At, \quad y = Bt^2 - D$ | 1 |
| **17** | $x = C\sin^2(\frac{k}{6}t) - D, \quad y = -A\cos^2(\frac{k}{6}t)$ | 1 |
| **18** | $x = -Dt^2 - C, \quad y = At$ | 0,5 |
| **19** | $x = A + B\cos(\frac{k}{3}t), \quad y = C\sin(\frac{k}{3}t) + D$ | 1 |
| **20** | $x = B - Ct - At^2, \quad y = C - Ct - At^2$ | 1 |

Continuation of Table 1.2, a

| *Варіант* | $x(t)$, $y(t)$, *см* | $t_1$, *с* |
|---|---|---|
| **21** | $x = A\sin(\frac{k}{6}t^2) - B$, $y = A\cos(\frac{k}{6}t^2) + C$ | 2 |
| **22** | $x = Bt^2 - D$, $y = Ct$ | 1 |
| **23** | $x = At^2 + t$, $y = D - Bt^2 + \frac{B}{C}t$ | 0,5 |
| **24** | $x = -D\sin(\frac{k}{6}t^2) - C$, $y = -A\cos(\frac{k}{6}t^2)$ | 1 |
| **25** | $x = -At$, $y = -Bt^2 - C$ | 1 |
| **26** | $x = D\sin^2(\frac{k}{4}t) + C$, $y = -D\cos^2(\frac{k}{4}t) - B$ | 1 |
| **27** | $x = -C - B\sin(\frac{k}{3}t^2)$, $y = -B\cos(\frac{k}{3}t^2) + A$ | 1 |
| **28** | $x = -Dt^2 + A$, $y = -Ct$ | 1,5 |
| **29** | $x = At^2 + \frac{A}{B}t - C$, $y = Bt^2 + t + C$ | 1 |
| **30** | $x = D\cos(\frac{k}{3}t^2) - A$, $y = -D\sin(\frac{k}{3}t^2) + B$ | 0,5 |

**Solution**

Let's find the trajectory of the point in the analytical form. To do this, we remove the parameter *t* from the given equations of motion (1.14).

$$t = \frac{(x+1)}{4}, \quad y = \frac{(x+1)^2}{8} + 3 \tag{1.14}$$

or after simplification:

$$y = \frac{(x^2+2x+25)}{8} \tag{1.15}$$

which is the trajectory of the point.

Find the speed of movement.

$$v_x = \frac{dx}{dt} = 4,$$

$$v_y = \frac{dy}{dt} = 4t, \tag{1.16}$$

$$v = \left(v_x^2 + v_y^2\right)^{0,5}.$$

Find the acceleration of the point.

$$a_x = \frac{dv_x}{dt} = 0,$$

$$a_y = \frac{dv_y}{dt} = 4, \tag{1.17}$$

$$a = \left(a_x^2 + a_y^2\right)^{0,5}.$$

Find the tangent, normal acceleration and radius of the curve The trajectory of the company.

$$a_\tau = \frac{dv}{dt} = \frac{\left(v_x \cdot a_x + v_y \cdot a_y\right)}{v},$$

$$a_n = (a^2 + a_\tau^2)^{0,5} \tag{1.18}$$

$$\rho = \frac{v^2}{a_n}.$$

According to formulas (1.14), (1.16) – (1.18), we calculate the values of coordinates $x$ and $y$, velocity $v$ with components $v_x$ and $v_y$ , acceleration a with components $a_x$, $a$ , $a_y$ τ , an and radius of curvature at time $t = 0$, $t = 0{,}5s$, etc. every half second up to $t = 7s$. The values of $v_x$, $a_x$, $a_y$ and $a$ do not depend on time and are equal at any moment: $v_x = 4$, $a_x = 0$, $a_y = 4$, $a = 4$. The rest of the results are listed in Table 1.3.

Table 1.3

| $t$ | $x$ | $y$ | $v_y$ | $v$ | $a_\tau$ | $a_n$ | $\rho$ |
|---|---|---|---|---|---|---|---|
| 0 | -1 | 3 | 0 | 4 | 0 | 4 | 4 |
| 0,5 | 1 | 3,5 | 2 | 4,47 | 1,79 | 3,58 | 5,59 |
| 1 | 3 | 5 | 4 | 5,66 | 2,83 | 2,83 | 11,3 |
| 1,5 | 5 | 7,5 | 6 | 7,21 | 3,33 | 2,22 | 23,4 |
| 2 | 7 | 11 | 8 | 8,84 | 3,58 | 1,79 | 44,7 |
| 2,5 | 9 | 15,5 | 10 | 10,8 | 3,71 | 1,49 | 78,1 |
| 3 | 11 | 21 | 12 | 12,8 | 3,79 | 1,26 | 126 |
| 3,5 | 13 | 27,5 | 14 | 14,6 | 3,85 | 1,1 | 193 |
| 4 | 15 | 35 | 16 | 16,5 | 3,88 | 0,97 | 280 |
| 4,5 | 17 | 43,5 | 18 | 18,4 | 3,91 | 0,87 | 352 |
| 5 | 19 | 53 | 20 | 20,4 | 3,92 | 0,78 | 530 |
| 5,5 | 21 | 63,5 | 22 | 22,4 | 3,94 | 0,72 | 699 |
| 6 | 23 | 75 | 24 | 24,3 | 3,95 | 0,68 | 900 |
| 6,5 | 27 | 101 | 28 | 28,3 | 3,96 | 0,57 | 1414 |
| 7 | 29 | 115,5 | 30 | 30,3 | 3,97 | 0,53 | 1733 |

Based on the results, we build the trajectory of the point in Fig. 4. The analytical form of the trajectory is given by formula (1.15), which in combination with the image gives a complete picture of the line along which the point is moving. This is a parabola that is symmetric to the vertical line passing through the point $x = -1$ with a minimum at the point $(-1, 3)$. But the point $M$ moves only along the right-hand side of this parabola, i.e., $x > -1$. In the table of results at $t = 0$, we have $x = -1$, $y = 3$. This is the initial position of the point $M$ on the trajectory, which is denoted by the point $M_O$. At $t = 1s$ we have: $x = 3$, $y = 5$. This corresponds to the position of the point *M* on the trajectory at time *1 s*, which is marked by $M_1$. The points $M_O$ and $M_1$ are shown in Fig. 4

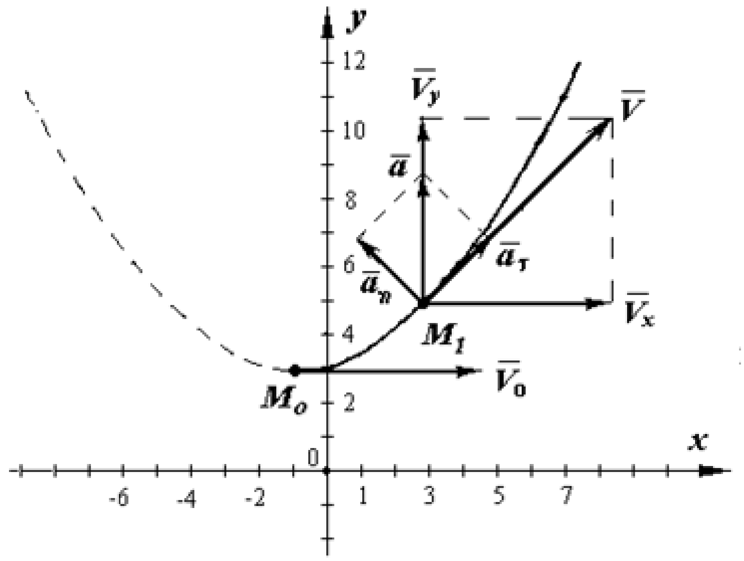



Figure 4

We plot the velocity versus time $v(t)$ and acceleration as a function of time $a(t)$ and show them in Figures 5 and 6.

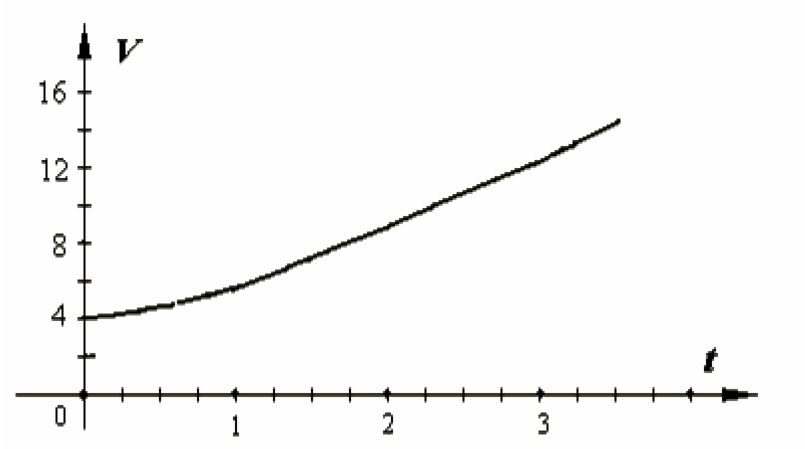



Figure 5

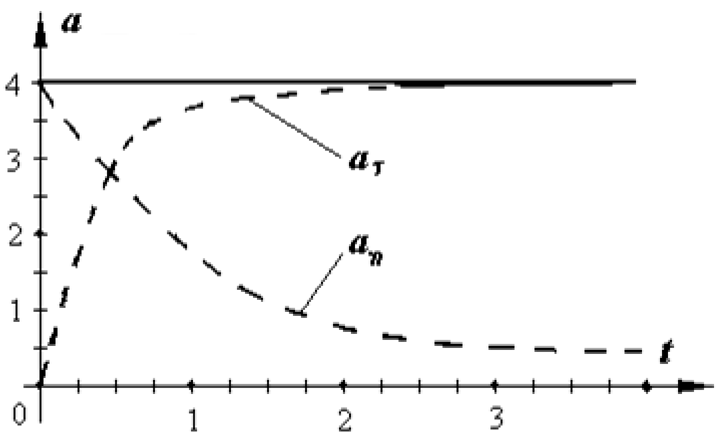



Figure 6

A more detailed view of the nature of the point's movement along the trajectory will be demo when we show the vectors of velocities and accelerations at points $M_0$ and $M_1$ in Fig. 4. At the initial moment of motion, at $t = 0$, $v_x = 4$, $v_y = 0$, $v = 4$. This means that at the point $M_0$ the speed of the moving point is directed parallel to the axis $O_x$. At time $t = 1s$ we have: $v_x = 4$, $v_y = 4$, $v = 5{,}66$.

Then at the point $M_1$ in Fig. 4, the velocity vector dice makes an angle of $45^{\circ}$ with the axis $O_x$. The velocity vectors at points $M_0$ and $M_1$ are built on the scale shown in Fig. 4. In the same way, only at the appropriate scale, we plot for point $M_1$ the vectors accelerations.

As can be seen from Fig. 5, the velocity of the point at the beginning of the motion grows nonlinearly. After a sufficiently long period of time, the function $v(t)$ becomes almost linear, i.e., the point moves equally accelerated. Fig. 6 shows that the total acceleration is a constant value and the tangential and normal accelerations asymptotically approach certain limits: $a_\tau$ tends to 4, and $a_n$ to zero. This means that somewhere in infinity, the point moves equally accelerated and straight.

Answer: according to the given equations, the point moves along the trajectory, which is the right branch of the parabola $y = (x^2 + 2x + 25)/8$, at the initial time it is at the point (-1,3), at the time 1s at the point (1, 5) it has a speed $v = 5{,}66\ sm/s$ and an acceleration $a = 4\ sm/s$.

Questions for self-control

1. What variables in the kinematics of a point are considered independent?

2. Does the type of trajectory of a point depend on the choice of coordinate system?

3. What is the essence of the main problem of point kinematics?

4. How to construct a natural trihedron (Fresnel trihedron)?

5. Define the velocity and acceleration of a point.

6. How is the velocity of a point in the Cartesian coordinate system determined?

7. How is the velocity of a point determined in the natural way of setting of motion?

8. What restrictions are imposed on the functions that describe the law of motion of a point?

9. What is the radius of curvature of the trajectory at the point of its inflection?

10. What are the ways to specify the position of a point in space?

11. How is the acceleration of a point determined in the natural way of setting the motion?

**Lecture № 2**
# 2 Kinematics of a solid body

The lecture discusses the kinematics of a solid body – translational and rotational motion

Kinematics of a rigid state is a branch in which the kinematics of an absolutely rigid body is studied. The main tasks of kinematics:

1. In finding ways to specify the motion of a solid body as a whole and its kinematic characteristics;

2. In determining the motion of each point of a solid individually.

A rigid body is the definition of motion and the determination of the kinematic characteristics of the motion of the body as a whole, as well as the determination of the kinematic characteristics of the motion of the points belonging to this body.

Depending on how the motion of a body is specified (on the type of equation that uniquely determines the position of the body in the chosen reference frame at any given time), five types of motion of a solid are distinguished: translational, rotational, complex (special case of plane), spherical, and general motion.

## 2.1 Translational motion of a solid body

A translational motion is a motion of a solid body in which any line *AB* drawn in the body remains parallel to its original position during the motion (Fig. 7)

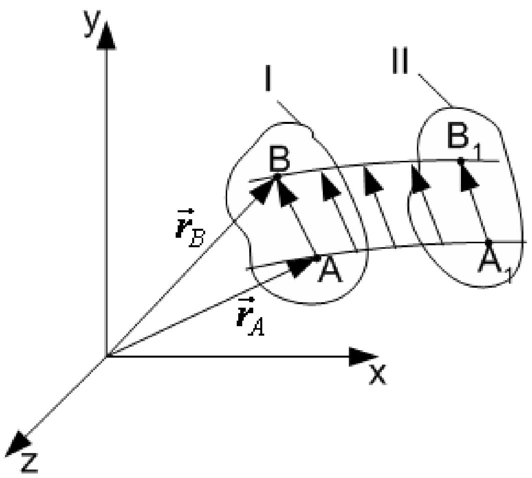



Figure 7

When moving the body from position $I$ to position $II$ $AB = A_1B_1 = const(solid)$ (Fig. 7):

$$\overline{v_A}=\overline{v_B},\ \overline{a_A}=\overline{a_B}.$$

The velocities and accelerations of all points of a body in translational motion are the same. The trajectories are also the same, but parallel to each other. The law of motion is given by the law of motion of any point. The velocity $\bar{v}$ and acceleration $\bar{a}$ are studied in the same way as for an isolated point. Thus, the study of translational motion of a solid body is reduced to the study of the motion of any of its points, i.e., to the problem of point kinematics.

### 2.2 Rotational motion of a solid body

The rotational motion of a solid is the motion of a body (Fig. 8) in which there is a line rigidly connected to the body (called the axis of rotation) that remains stationary at all times.

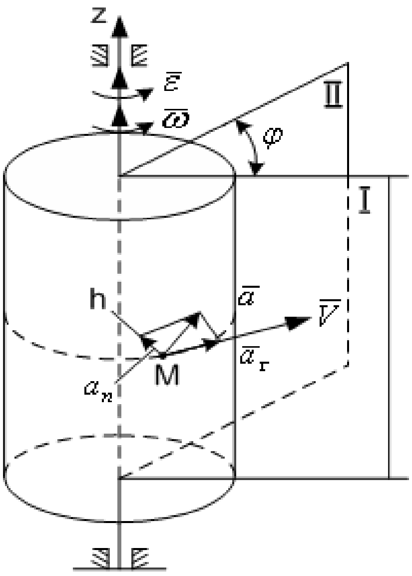


Figure 8

When a solid body rotates around the Z-axis (Fig. 8), its position is determined by the angle of rotation $\varphi$. The angle $\varphi$ is considered positive if the transition from plane I, which is fixed in space, to plane II, which is invariably connected to the body, occurs counterclockwise, and negative if it occurs clockwise.

The law of body movement:

$$\varphi = \varphi(t) \tag{2.1}$$

Law (2.1) is called the kinematic equation of rotational motion of a body about a fixed axis.

The main kinematic characteristics of the rotational motion of a solid body are: angle of rotation $\varphi$, angular velocity $\omega$, angular acceleration at a given time $\varepsilon$.

Angular velocity

$$\omega = \frac{d\phi}{dt} = \dot{\varphi} \quad [s^{-1}],$$

Thus, the angular velocity is equal to the first time derivative of the angle of rotation $\varphi$. In engineering, the angular velocity is often given by the number $n$ revolutions per minute. The relationship between $\omega$ i $n$ is determined by the formula

$$\omega = \frac{2\pi n}{60} = \frac{\pi n}{30}.$$

If the angular velocity of a body is constant ($\omega$ = c$onst$) then the rotation of the body is called uniform. In this case, the angle of rotation changes proportionally to time:

$$\varphi = \omega \cdot t + \varphi_0,$$

where $\varphi_0$ – is the initial angle of rotation.

This equation is called the equation of uniform rotation of a body about a fixed axis.

Angular velocity $\bar{\omega}$ is a vector quantity (Fig. 9).

The angular velocity vector $\bar{\omega}$ is a vector that is numerically equal to the absolute value of the time derivative of the angle of rotation and is directed along the axis of rotation in the direction from which the body's rotation is seen to be counterclockwise (Fig. 8, Fig. 9).

Angular acceleration

$$\bar{\varepsilon} = \frac{d\bar{\omega}}{dt} = \frac{d^2\varphi}{dt^2} = \ddot{\varphi} \quad [s^{-2}].$$

Angular acceleration of a body $\varepsilon$ characterizes the rate of change of the angular velocity $\omega$ in time.

Angular acceleration $\bar{\varepsilon}$ vector quantity equal to the first time derivative of the angular velocity or the second time derivative of the angle of rotation, the angular acceleration vector is directed along the axis of rotation in the direction from which the body rotation is seen to be counterclockwise.

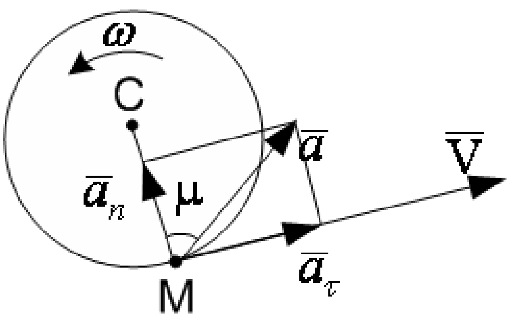



Figure 9

If the direction $\bar{\varepsilon}$ coincides with the direction of the vector $\overline{\omega}$ then the rotation is accelerated and the opposite $\overline{\omega}$ in case of slow motion.

### 2.3 The trajectory, velocity and acceleration of a point on a body performing rotational motion.

The trajectory of any point *M* of a rotating body is a circle with the center *C* on the axis of rotation (Fig. 8).

The velocity $\bar{v}$ of any point *M* of the rotating body is directed perpendicular to the line connecting it with the axis of rotation, in the direction of rotation of the body and is equal to

$$V = \omega \cdot R,$$

where *R* – is the distance from point *M* to the axis of rotation, or the radius of the *MC* circle that describes point *M*.

The acceleration of point *M* is decomposed into tangential (rotational) ($\bar{a}_\tau$) and normal (centripetal) ($\bar{a}_n$ ) components:

$$\bar{a} = \bar{a}_\tau + \bar{a}_n, \quad \bar{a}_\tau \perp \bar{a}_n$$
$$a_\tau = \varepsilon \cdot R;$$
$$a_n = \omega^2 \cdot R.$$

By module $a = R\sqrt{\varepsilon^2 + \omega^4}$.

The deviation of the total acceleration vector $\bar{a}$ from the radius described by the point of the circle is determined by the angle $\mu$:

$$tg\mu = \frac{a_\tau}{a_n} = \varepsilon/\omega^2.$$

Vector $a_\tau$ vector is directed tangentially in the direction (rotation) of motion when the body is accelerated and in the opposite direction when it is decelerated. The vector $\bar{a}_n$ - vector is directed along the radius of the *MS* to the axis of rotation, i.e., in the direction of the concavity of the trajectory of the point *M*.

Questions for self-control

1. What types of motion of a solid are called the simplest?
2. What is the task of motion of a solid body?
3. What kind of trajectory have the points of a solid body rotating around a fixed axis?
4. What kind of motion of a solid is called translational?
5. What kind of motion of a solid is called rotational about a fixed axis?
6. What are the kinematic characteristics of the body in translational motion?

7. What are the kinematic characteristics of the body in rotational motion around a about a fixed axis?

8. What is the formula for determining the speed of a body point rotating around a fixed axis?

9. What formula is used to determine the acceleration of a body point rotating about a fixed axis?

10. What acceleration of a point is called rotational? What is its direction?

11. What is the acceleration of a point is called pre-axial? What is its direction?

12. What formula can be used to convert angular velocity from revolutions per minute to radians per second?

13. What is the ratio between linear velocities in mechanisms with gears?

14. Can the kinematics of translational motion of a solid body be reduced to the kinematics of a point?

**Lecture № 3**
**3 Transformation of the simplest movements of a solid body**

Goals and objectives:

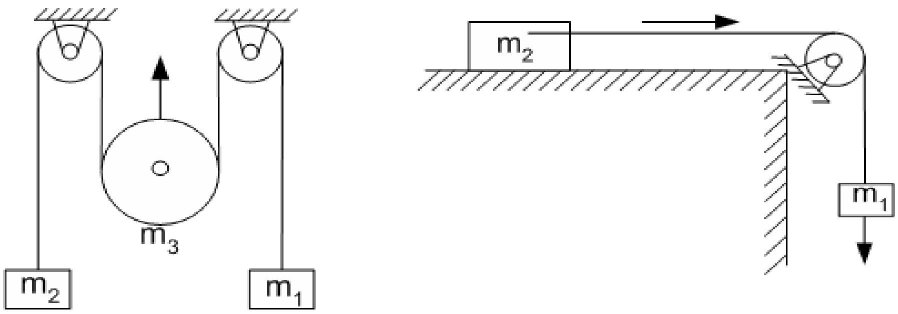
The lecture discusses the transformation of the simplest motions of a solid body.

**3.1 Converting of one translational motion into another translational motion**

The simple motions of a solid body include translational and rotational motion, which were discussed in the previous lecture.

In practice, there are four groups of problems involving the transformation of the simplest motions of a solid body:

1.1. Converting one translational motion into another translational motion. Examples of this transformation are provided by block mechanisms (Fig. 10):




a) б)

Figure 10

In Fig. 10, a) with the help of two loads of masses $m_1$and $m_2$ moving, for example, downward, a third load of mass $m_3$ moves progressively upward. Fig. 10, b) shows an example of converting the translational motion of a body of mass $m_1$ into translational motion of a body of mass $m_2$ in the direction perpendicular to the direction of motion of the body with mass $m_1$.

**3.2 Conversion of rotational motion about one axis into rotational motion about another axis**

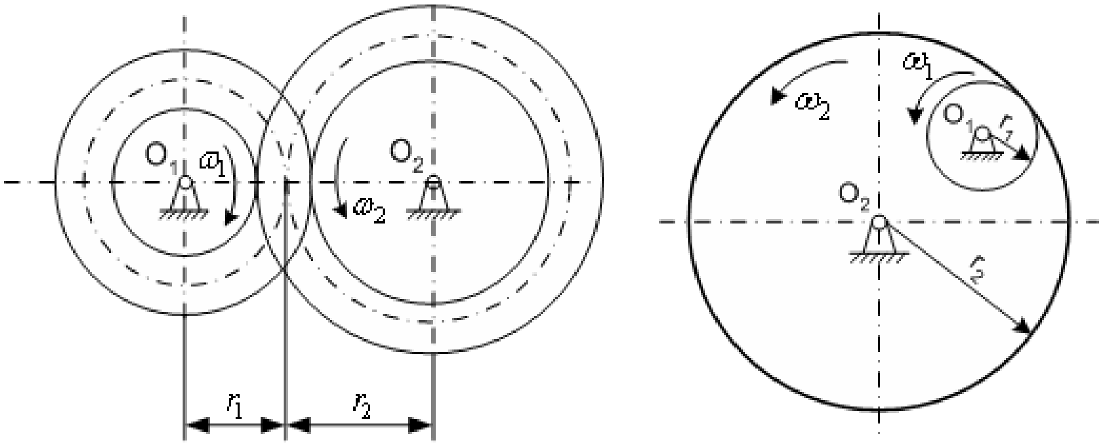
If the axes of rotation are parallel or intersecting, the rotation can be transmitted by means of gears or friction gears. In this case, the engagement can be either external (Fig. 11, a) or internal (Fig. 11, b).

The basis for the kinematic calculation of these gears is the assumption that there is no sliding or gap between the teeth in the system. This means that the speeds on the rim of the gears that are in engagement are the same. In other words, in these cases, the following relationship holds

$$v = \omega_1 \cdot r_1 = \omega_2 \cdot r_2.$$

The angular velocities of the gear wheels are therefore inversely proportional to the wheel radii:

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}.$$




a) b)

Figure 11

Ratio of the angular velocity of the drive wheel $\omega_1$ to the angular velocity of the driven wheel $\omega_2$ is called the gear ratio $U$:

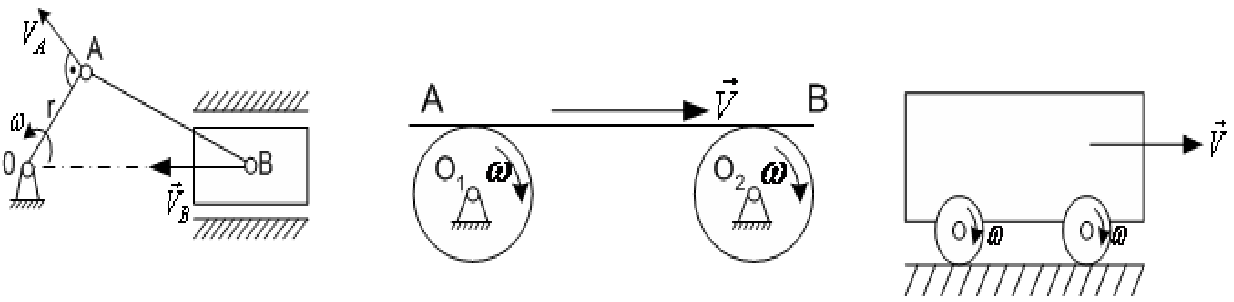
$$U = \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}.$$

If we take into account that the number of teeth $Z$ is proportional to the lengths of the circles, i.e. radii, the gear ratio is determined by the corresponding ratio of the number of teeth:

$$U = \frac{Z_2}{Z_1}.$$

### 3.3 Converting translational motion into rotational motion and vice versa

Examples of the transformation of translational motion into rotational motion are (Fig. 12): a) crank mechanisms; b) conveyors. A classic example of the transformation of translational motion into rotational motion and vice versa is wheeled vehicles of all kinds, trams, trolleybuses (Fig. 12, c).

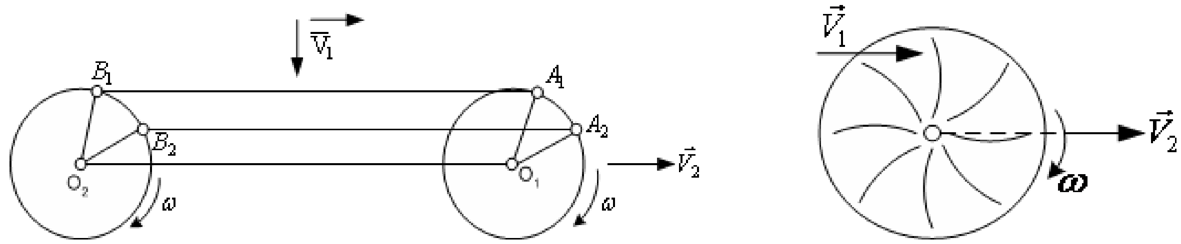



a) b) c)

Figure 12

### 3.4 Complex transformations of the simplest movements

Complex transformations of the simplest movements include those that have at least three simplest movements in any combination (Fig. 13):




a) b)

Figure 13

a) in a pair, the translational motion of the rod $A_1B_1$ with speed $\bar{v}_1$causes rotational motion of the wheels with angular velocity $\omega$ and translational motion of the mate with speed $\bar{v}_2$ in the horizontal direction;

b) in turbine engines, gas (air, steam) moving at a speed of $\bar{v}_1$ causes rotational motion of the turbine engine output shaft with an angular velocity $\omega$ which can then be converted into other angular velocities of the drive shafts and ultimately cause the translational motion of the object with a speed of $\bar{v}_2$.

**Example**

According to the given equation of motion of the load 1 $S = 5 + 60t^2 sm$, find the velocity, tangential and total acceleration of point $M$ of the mechanism (Fig. 14) at the moment when the load has traveled a distance $S = 10\ sm$, $R_2 = 40\ sm$, $r_2 = 20\ sm$, $R_3 = 40\ sm$, $R_3$=40 sm, $r_3 = 15\ sm$.

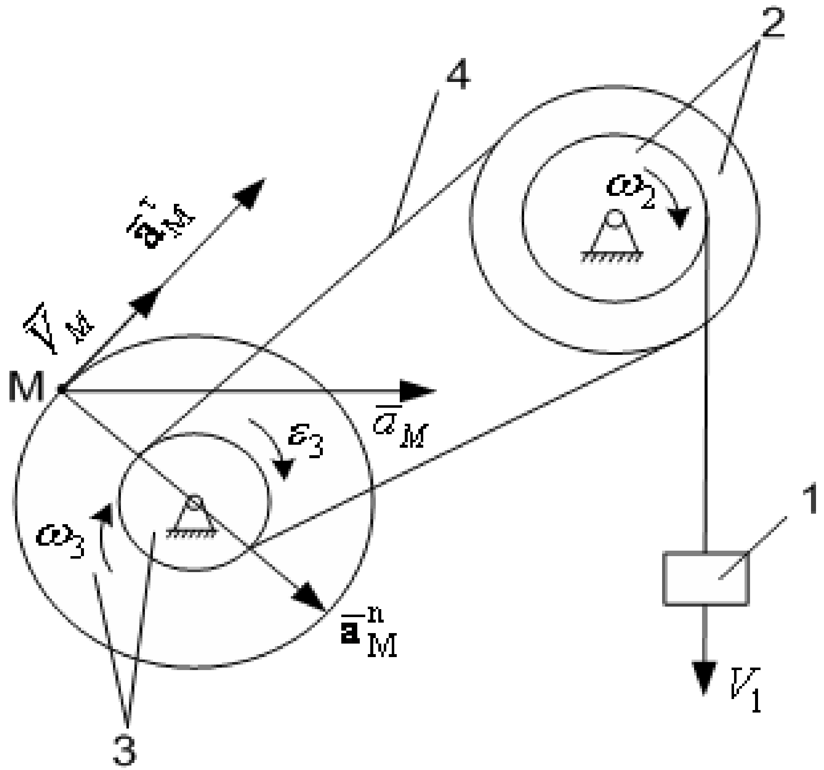



Figure 14

**Solution**

In this case, we have the following scheme for the complex transformation of the simplest movements:

$$v_1 \to \omega_2;\ \omega_2 \to v_4;\ v_4 \to \omega_3.$$

Speed of the load 1:

$$v_1 = \frac{dS}{dt} = 120t\ cm/s.$$

Wheel angular velocity 2:

$$\omega_2 = \frac{v_1}{r_2} = \frac{120t}{20} = 6\ t\quad 1/s.$$

The speed of the passing gear:

$$v_4 = \omega_2 \cdot R_2.$$

Wheel angular velocity 3:

$$\omega_3 = \frac{v_4}{r_3} = \frac{\omega_2 \cdot R_2}{r_3} = \frac{6t \cdot 40}{15} = 16t\,\frac{1}{s}.$$

Angular acceleration of the wheel 3:

$$\varepsilon_3 = \frac{d\omega_3}{dt} = \frac{d}{dt}(16t) = 16\,\frac{1}{s^2}.$$

Time of passage of 1 way by cargo $S = 10\ cm$

$$S = 5 + 60 \cdot t^2 = 10\ cm,$$
$$t = \sqrt{\frac{5}{60}} = 0{,}29\ s.$$

The velocity of the point *M:*

$$v_M = \omega_3 \cdot R_3 = 16t \cdot 40 = 16 \cdot 0.29 \cdot 40 = 185{,}6\ cm/s.$$

Tangential acceleration of point *M*:

$$a_M^\tau = \varepsilon_3 \cdot R_3 = 16 \cdot 40 = 640\ cm/s^2.$$

The normal acceleration of point *M*

$$a_M^n = \omega_3^2 \cdot R_3 = (16t)^2 \cdot 40 = (16 \cdot 0{,}29)^2 \cdot 40 = 860\ cm/s^2.$$

The total acceleration of point *M* at the moment when the load has traveled the path $S = 10cm$

$$a = \sqrt{(a_M^\tau)^2 + (a_M^n)^2} = \sqrt{640^2 + 860^2} = 1072\,{}^{cm}/_{s^2}.$$

Answer:

$$v_M = 185{,}6\,{}^{cm}/_{s},\ a_M = 1072\,{}^{cm}/_{s^2}.$$

### 3.5 Determination of velocities and accelerations of points of a solid in translational and rotational motion

The mechanism consists of three bodies: a stepped pulley 2, body 1, and pulley 3. Body 1 moves according to the law $x = f(t)$. Determine the velocity and acceleration of points $A$ and $M$ at $t_1 = 1s$.

The data for the calculation are in Table 3.1.

Table 3.1

| Option | $R_2$, m | $r_2$, m | $R_3$, m | $x = f(t)$, m |
|---|---|---|---|---|
| 1 | 0,1 | 0,05 | 0,3 | $0{,}4t^2 + 0{,}3$ |
| 2 | 0,3 | 0,2 | 0,15 | $0{,}5\,sin\left(\frac{\pi}{3}t^2\right)$ |
| 3 | 0,4 | 0,3 | 0,1 | $0{,}5t^3 + 0{,}33\left(\frac{\pi}{4}t^2\right)$ |
| 4 | 0,3 | 0,15 | 0,2 | $0{,}6\,cos\left(\frac{\pi}{3}t\right)$ |
| 5 | 0,2 | 0,1 | 0,15 | $0{,}6t^3$ |
| 6 | 0,3 | 0,15 | 0,2 | $0{,}3t^2 - 0{,}4$ |
| 7 | 0,2 | 0,1 | 0,15 | $0{,}2\,sin^2\left(\frac{\pi}{4}t\right)$ |
| 8 | 0,1 | 0,05 | 0,3 | $4\,cos\frac{\pi}{3}t^2$ |
| 9 | 0,2 | 0,1 | 0,3 | $5\,sin\,\pi\,t^2$ |
| 0 | 0,3 | 0,2 | 0,15 | $0{,}7t^3 + 0{,}1$ |

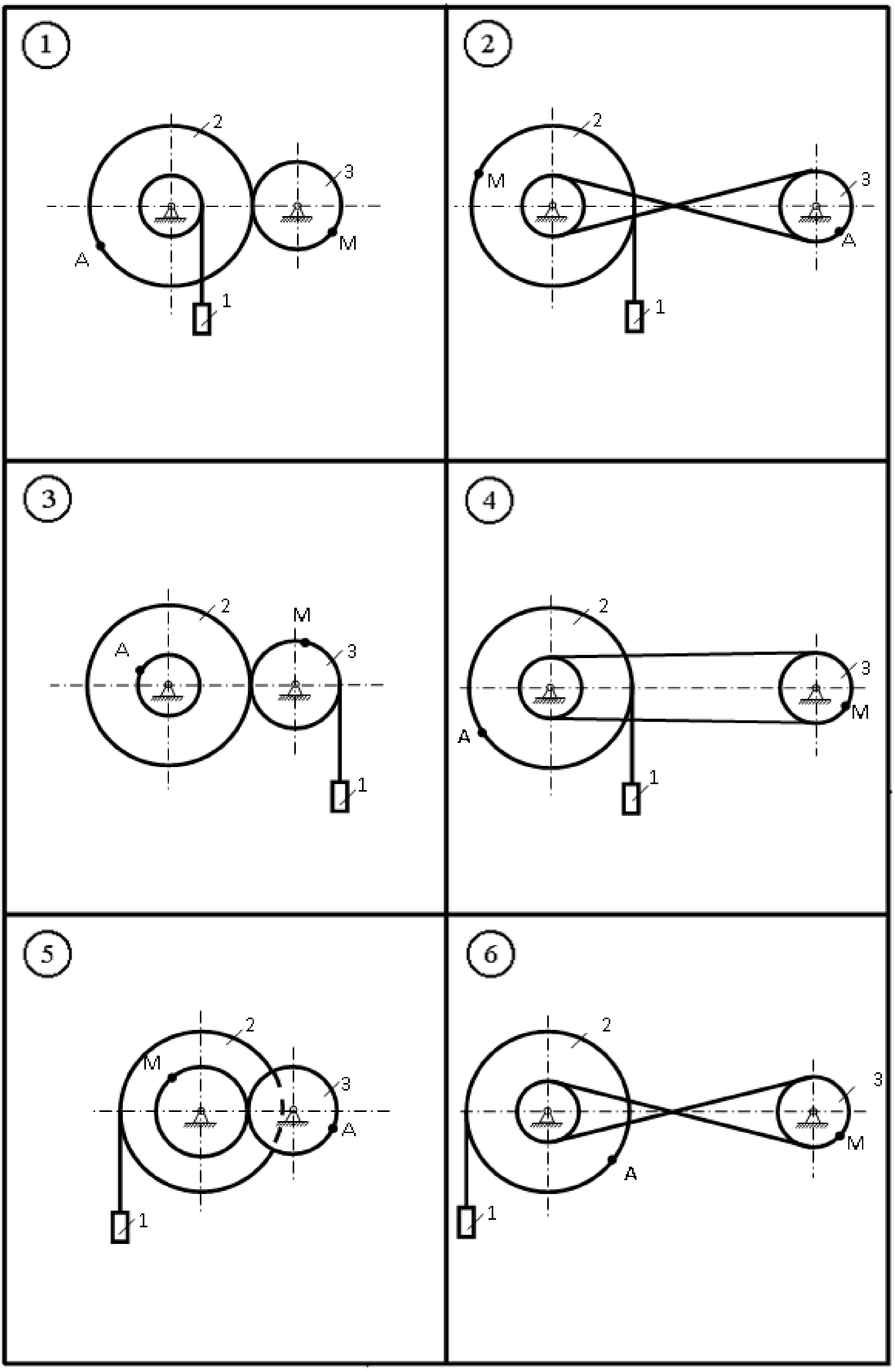



Figure 15

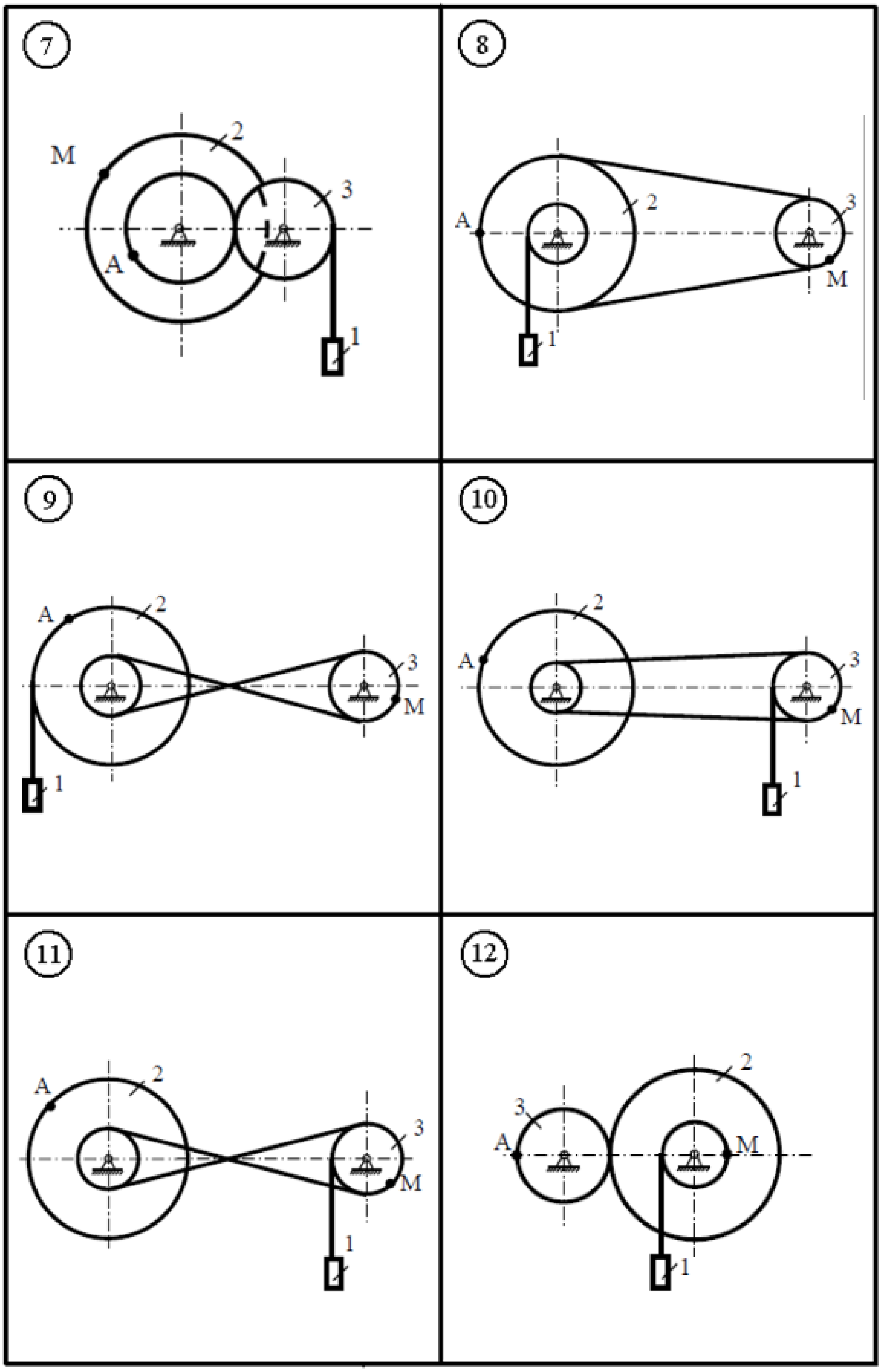


Figure 16

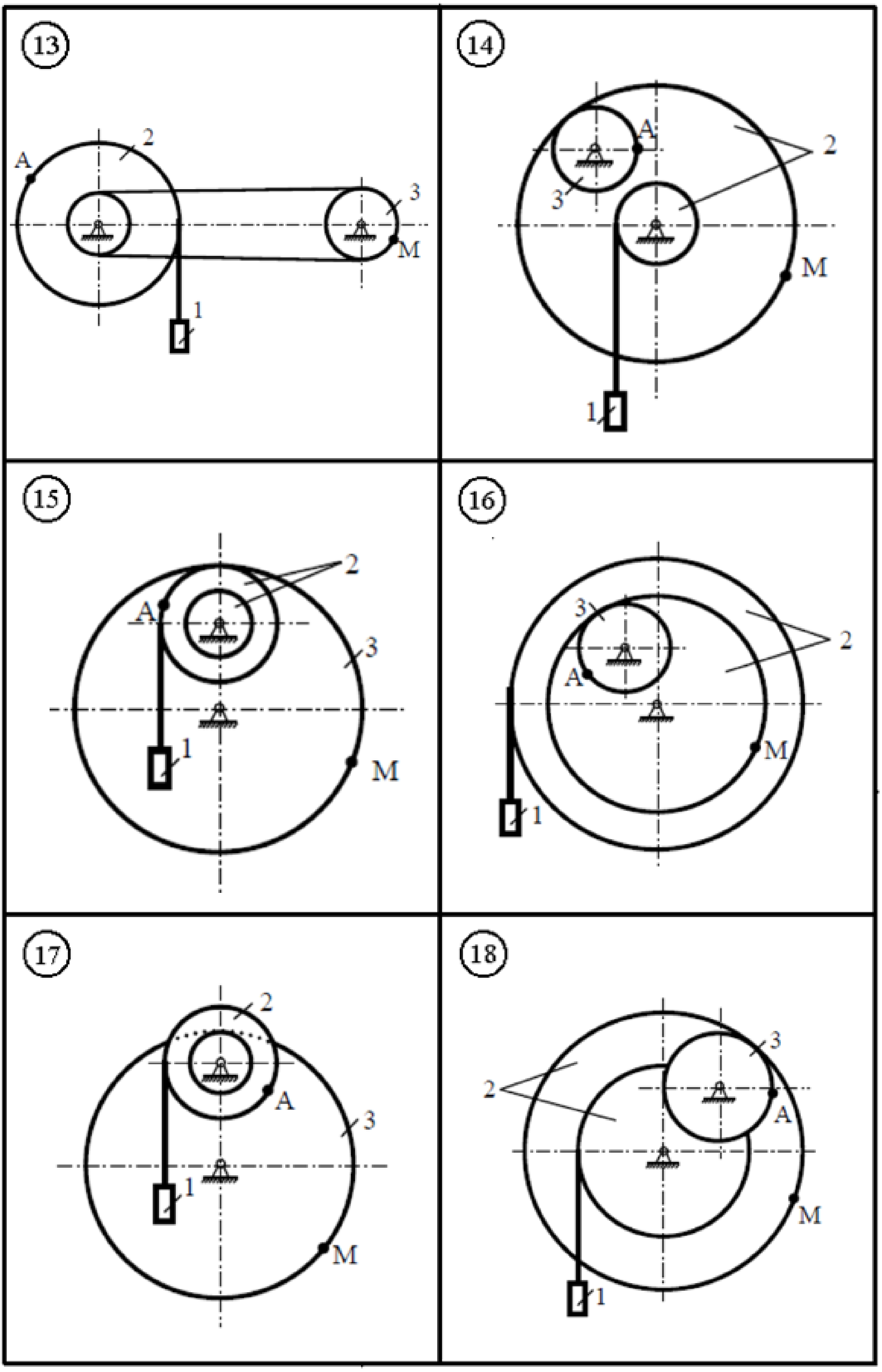



Figure 17

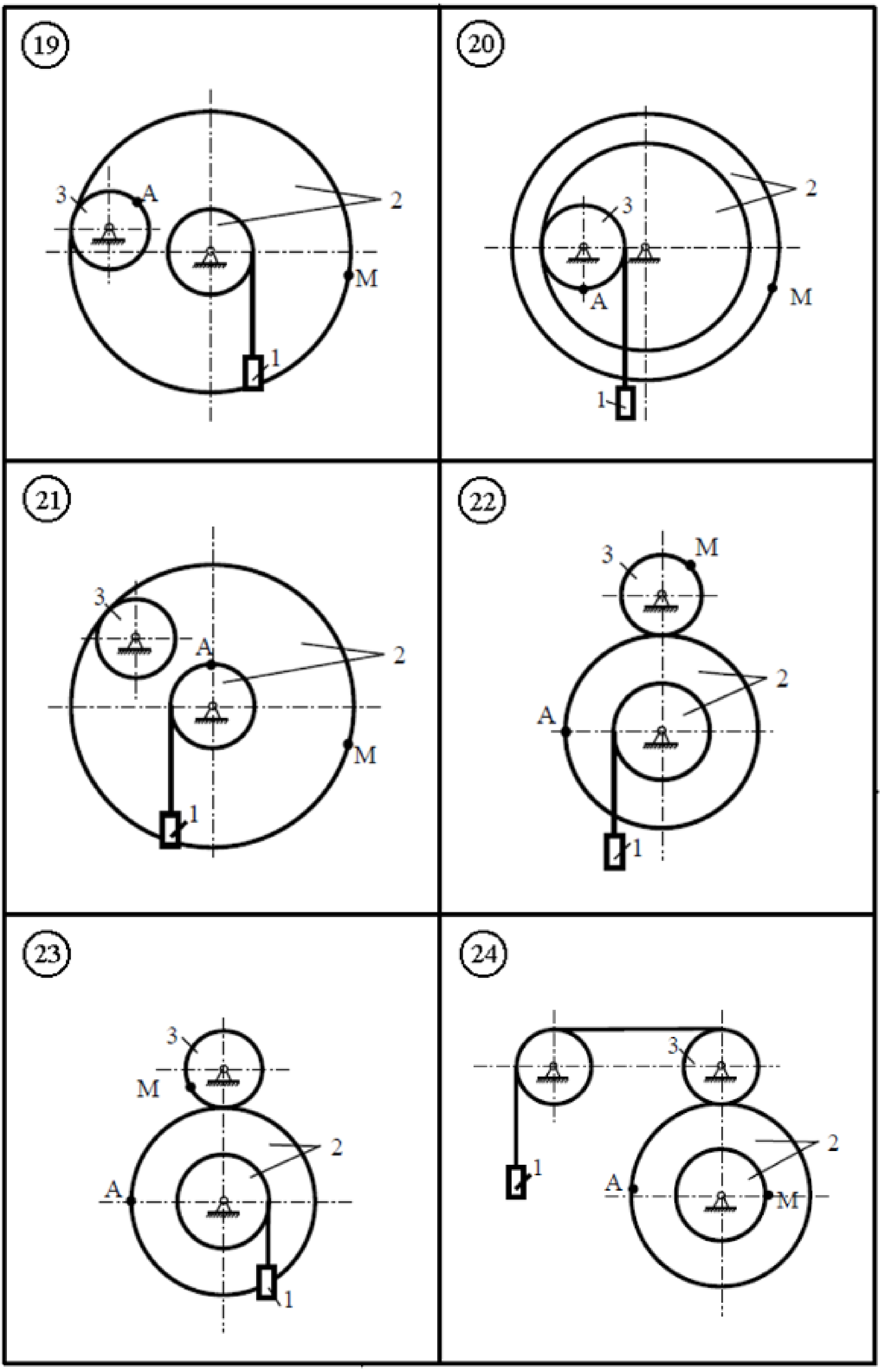



Figure 18

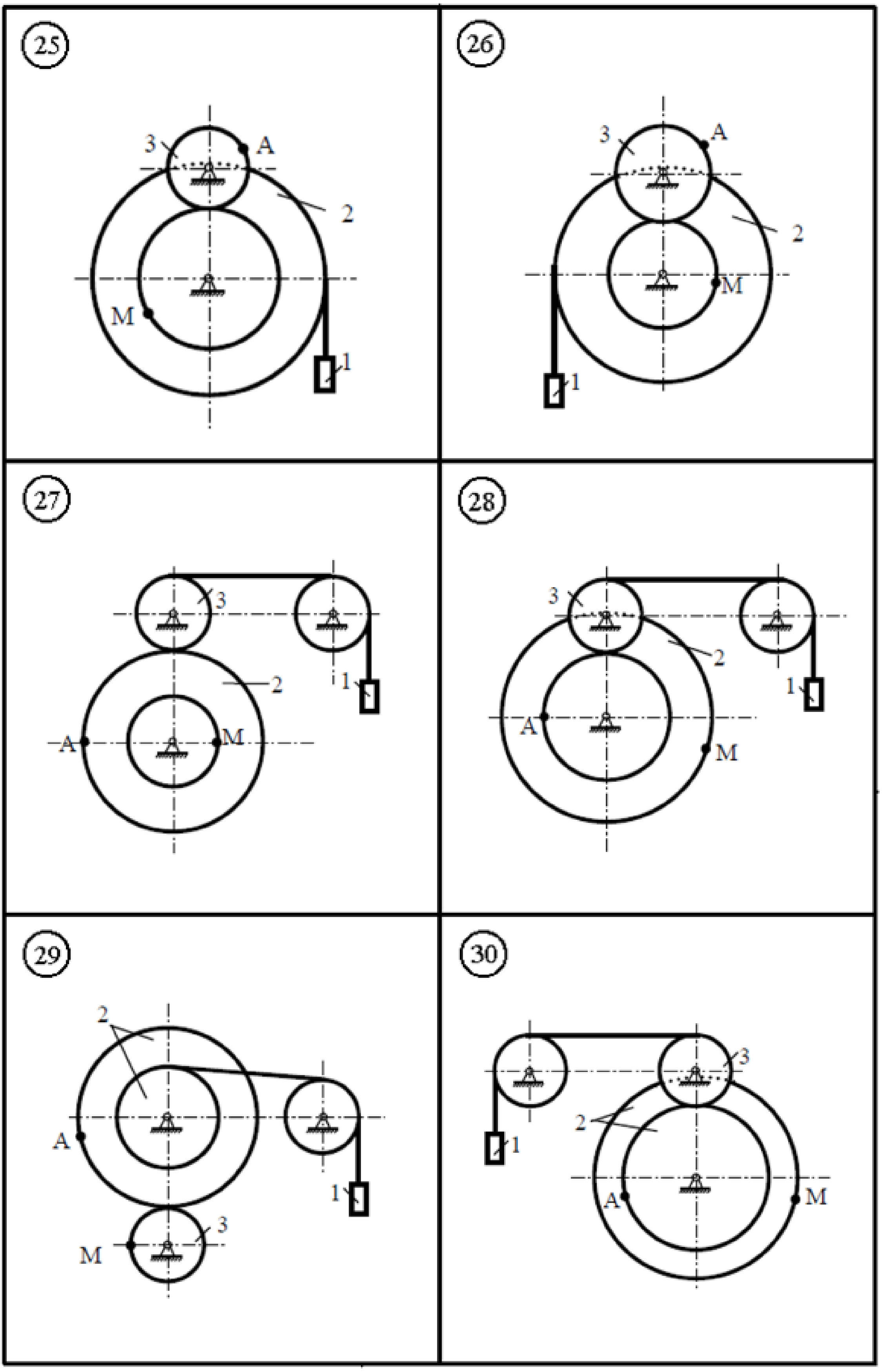



Figure 19

**Example**

For the mechanism (Fig. 20), find the velocity and acceleration of points *A* and *M* at $t_1 = 1s$ if body 2 moves according to the law $\varphi_2 = 8t^3 - 3t^2$ ($R_2 = 0{,}1m$; $R_1 = 0{,}4m, r_1 = 0{,}15m$).

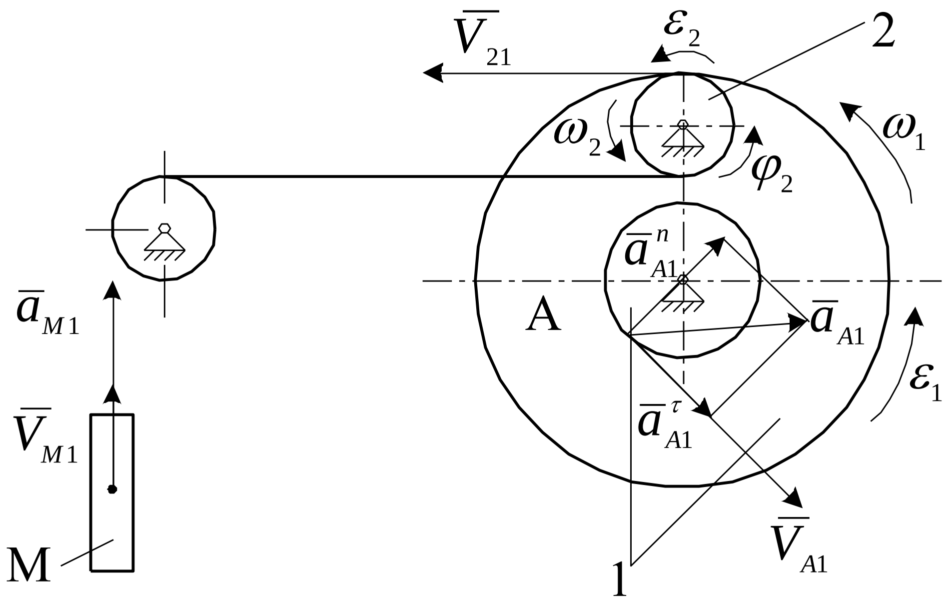



Figure 20

**Solution**

Find the angular velocity and angular acceleration of body 2

$$\omega_2 = \frac{d\phi_2}{dt} = (24t^2 - 6t)\frac{1}{c},$$
$$\varepsilon_2 = \frac{d^2\phi_2}{dt^2} = (48t - 6)\frac{1}{s^2}.$$

At $t_1 = 1s$:

$$\omega_{21} = 24 - 6 = 18\frac{1}{s},$$
$$\varepsilon_{21} = 48 - 6 = 42\frac{1}{s^2}.$$

Find the velocity $v_{21}$ of the points of contact of bodies 2 and 1

$$v_{21} = \omega_2 \cdot R_2 = \omega_1 \cdot R_1,$$
$$\omega_1 = \omega_2\frac{R_2}{R_1} = 0{,}25\omega_2.$$

Determine the velocity of point *A*

$$v_A = \omega_1 \cdot r_1 = 0{,}25 \cdot 0{,}15 \cdot \omega_2 = 0{,}0375 \cdot \omega_2.$$

Acceleration of point *A*

$$\overline{a_A} = \overline{a_A^n} + \overline{a_A^\tau},$$
$$a_A^n = \frac{v_A^2}{r_1} = \omega_1^2 r_1 = 9{,}375 \cdot \omega_2^2 \cdot 10^{-3}\,\frac{m}{s^2},$$
$$a_A^\tau = \frac{dV_A}{dt} = \varepsilon_1 \cdot r_1 = 0{,}0375 \cdot \varepsilon_2\,\frac{m}{s^2},$$

where

$$\varepsilon_1 = \frac{d\omega_1}{dt} = 0{,}25 \cdot \varepsilon_2.$$

The acceleration value of point A

$$a_A = \sqrt{(a_A^n)^2 + (a_A^\tau)^2}.$$

At $t_1 = 1s$:

$$v_{A1} = 0{,}0375 \cdot \omega_{21} = 0{,}0375 \cdot 18 = 0{,}675\,\frac{m}{s},$$
$$a_{A1}^n = 9{,}375 \cdot \omega_{21}^2 \cdot 10^{-3} = 9{,}375 \cdot 18^2 \cdot 10^{-3} = 3{,}038\,\frac{m}{s^2},$$
$$a_{A1}^\tau = 0{,}0375 \cdot \varepsilon_{21} = 0{,}0375 \cdot 42 = 1{,}575\,\frac{m}{s^2},$$
$$a_A = \sqrt{3{,}038^2 + 1{,}575^2} = 3{,}42\,\frac{м}{c^2}.$$

Determine the velocity and acceleration of a translationally moving body *M*

$$v_м = v_{21} = \omega_2 \cdot R_2 = 0{,}1 \cdot \omega_2,$$
$$a_м = \frac{dv_м}{dt} = 0{,}1 \cdot \varepsilon_2.$$

At $t_1 = 1s$:

$$v_{м1} = 0{,}1 \cdot \omega_{21} = 1{,}8\,\frac{m}{s},$$
$$a_{м1} = 0{,}1 \cdot \varepsilon_{21} = 4{,}2\,\frac{m}{s^2}.$$

We show in Fig. 15 the vectors $\overline{v_{A1}}, \overline{a_{A1}^n}, \overline{a_{A1}^\tau}, \overline{a_{A1}}, \overline{v_{м1}}, \overline{a_{м1}}$.

## Lecture № 4

The lecture deals with the issue of determining the velocity and acceleration of points on a body performing plane-parallel motion.

### 4.1 Three ways to represent flat motion

The motion of a solid body in space is called plane-parallel (plane) if all points of the body move in planes parallel to some fixed plane (Fig. 21).

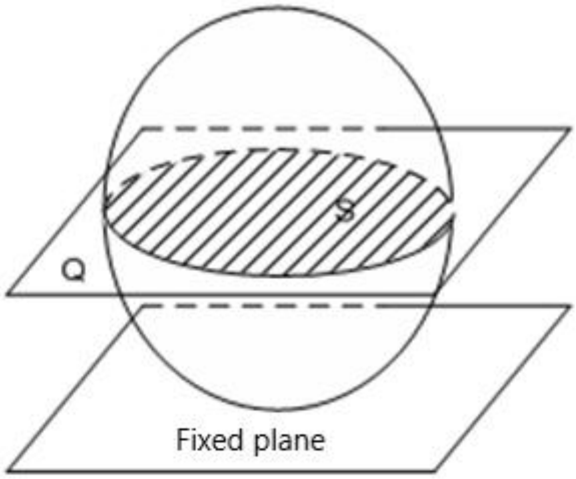



Figure 21

The plane motion of a solid body in space is determined by the motion of a plane figure $S$ created by the intersection of the body by any plane $Q$ parallel to a fixed plane (Fig. 21). The motion of a plane figure in its plane is determined by the motion of the constant line segment AB belonging to this plane (Fig. 22).

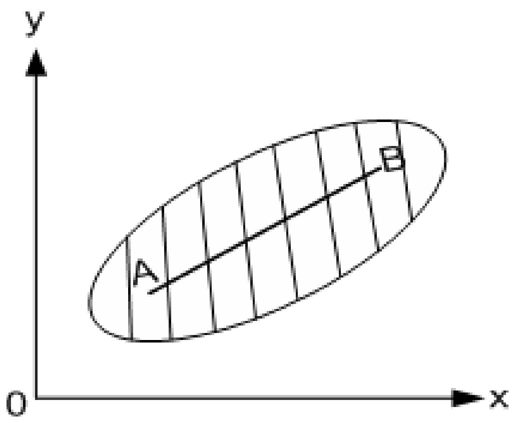



Figure 22

Since the distance between points $A$ and $B$ remains unchanged, only three of the four coordinates of points $A$ and $B$ remain independent. So, to describe the plane motion of a body, you need three independent coordinates as a function of time.

### 4.1.1 Schall's theorem

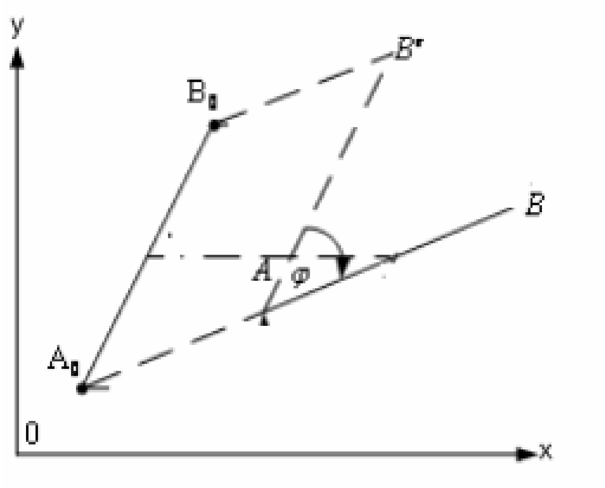



Figure 23

The motion of a flat figure in its plane is represented as a continuous sequence of instantaneous translational movements ($A_0B_0$ або AB) together with the pole and instantaneous rotations around the pole (point A) (Fig. 23). The position of the pole $A$ is described by two parameters $X_A, Y_A$, and the third parameter is the angle $\phi$ of rotation of the body in the plane around the pole point $A$.

Based on this, the law of motion of a plane figure is as follows

$$X_A = f_1(t)$$
$$Y_A = f_2(t)$$
$$\phi = f_3(t).$$

The first two equations describe the translational motion of the figure, in which all points move in the same way as the pole, and the third equation describes the rotational motion around the pole.

Rotational motion around the pole is characterized by angular velocity:

$$\omega = \frac{d\phi}{dt}$$

and angular acceleration:

$$\varepsilon = \frac{d\omega}{dt} = \ddot{\phi}(t).$$

The angular velocity and angular acceleration of the rotational motion of a shape do not depend on the choice of pole.

### 4.1.2 Euler-Schall theorem

The motion of a plane figure in its plane is represented as a continuous sequence of instantaneous rotations around the corresponding ICE (instantaneous centers of rotation).

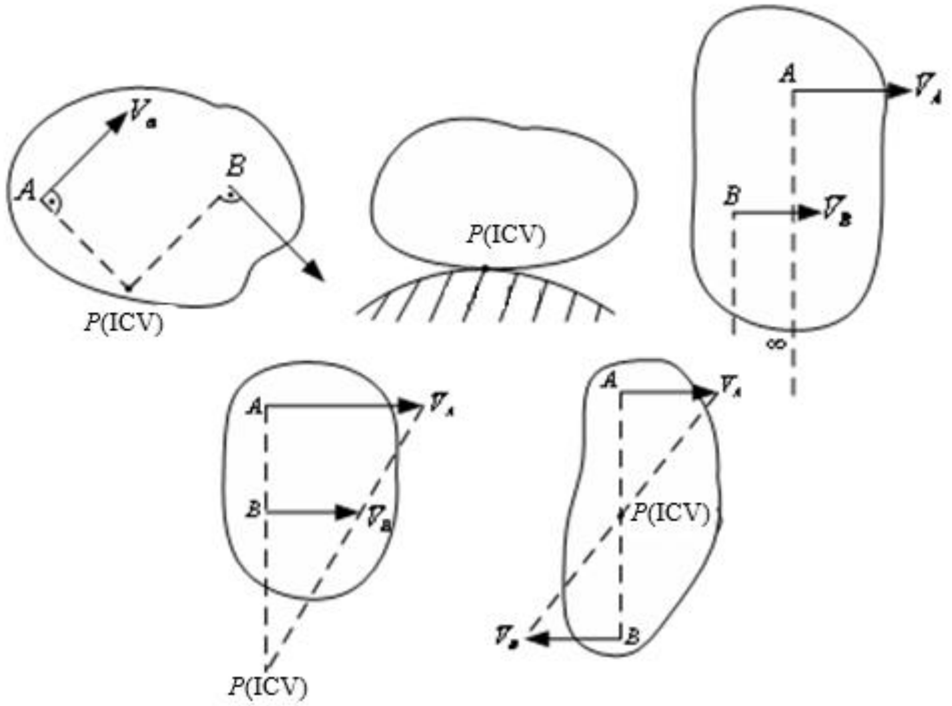



Figure 24

The instantaneous center of rotation is the point of the fixed plane with which the instantaneous center of velocity (ICV) coincides at a given time.

The instantaneous center of velocity (pole $P$) is a point of the moving plane rigidly attached to a figure whose velocity is zero at a given time (Fig. 24): $V_P = 0$.

The instantaneous center of velocity is the point of intersection of the instantaneous axis of rotation with the plane of motion.

### 4.1.3 Poinsot's theorem

Any continuous motion of a plane figure in its plane can be obtained by constructing a moving and a fixed centroid, rigidly connecting the former to the plane figure, and rolling the moving centroid without slipping on the fixed centroid. The centroid is the geometric location of the instantaneous centers of velocity (ICV) (Fig. 25).

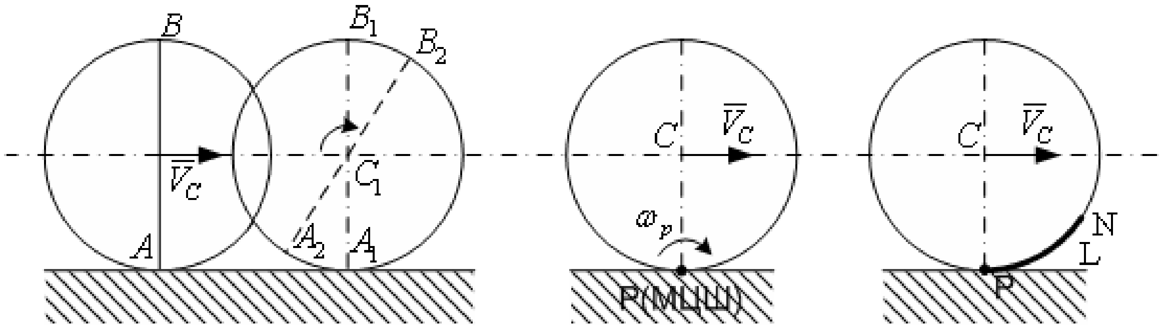



Figure 25

In plane-parallel motion, two centroids are formed, since the instantaneous center of velocity describes one curve in the fixed coordinate system and the other in the moving coordinate system.

A fixed centroid is the trajectory of the instantaneous center of velocity on a fixed plane, and a moving centroid is the trajectory of the instantaneous center of velocity on a moving plane.

The moving centroid *PN* rolls without slipping on the fixed *PL* (Fig. 25).

The concept of centroids is widely used in the theory of mechanisms and machines when profiling gears.

### 4.2 Velocities of points of a plane figure

There are two main methods for finding the velocity of any point of a plane figure at a given time, i.e. at a given position of the figure, based on the Schall and Euler-Schall theorems.

#### 4.2.1 The velocity theorem for points of a plane figure

The velocity of any point *B* of a plane figure consists of the velocity of the pole *A* and the velocity of the point *B* when the plane figure rotates around the pole *A*, which is perpendicular to *AB* (Fig. 26).

$$\overline{v_B} = \overline{v_A} + \overline{v_{BA}};\ \overline{v_{BA}}\ \perp BA.$$

The velocity $\overline{v_{BA}}$ is equal to

$$V_{BA} = |\omega| \cdot BA = \omega_{BA} \cdot h = \omega_{BA} \cdot l_{AB},$$

where $\omega$ – is the algebraic angular velocity of a plane figure, which does not depend on the choice of pole. The vector $\overline{v_{BA}}$ is directed perpendicular to *AB* counterclockwise around the pole *A* if $\omega > 0$ and vice versa, if $\omega < 0$.

### 4.2.2 Theorem on the projection of velocities of two points of a body

Theorem on the projection of velocities of two points of a body onto a line passing through these points.

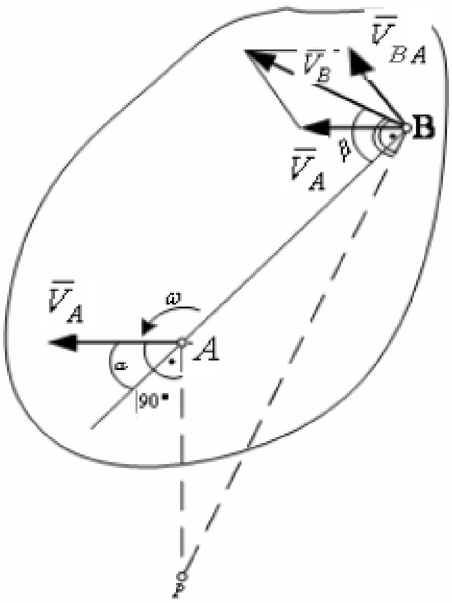



Figure 26

The projections of the velocities of two points of the figure on a line passing through these points are equal to each other and directed in the same direction (Fig. 26).

The projection $\overline{v_A}$ onto *AB* is equal to the projection $\overline{v_B}$ onto *AB*, i. e.

$$v_A \cdot cos\,\alpha = v_B \cdot cos\,\beta,$$

where $\alpha$ i $\beta$ – are the angles between $\overline{v_A}$ i $\overline{v_B}$ and the direction of the line *AB*, respectively.

### 4.2.3 Euler-Schall theorem

According to the Euler-Schall theorem, plane motion is represented as rotational motion around an instantaneous center of rotation, or center of velocity.

The pole is the instantaneous center of velocity (ICV), i.e., a point $P$ of a moving plane rigidly attached to a figure whose velocity at a certain moment of time is zero $\overline{v_P} = 0$.

Then

$$v_B = v_{BP} = |\omega| \cdot BP,$$

i.e., the velocity of any point of the figure is equal in magnitude to the product of the angular velocity modulus of the figure and the distance from this point to the CSC and is directed perpendicular to this segment ($\overline{v_B} \perp BP$) counterclockwise, if $\omega$>0 and vice versa.

The distribution of instantaneous velocities of the points of a plane figure is as if the figure were rotating around the CSC (point *P*) (Fig. 22). From this we have the following relations

$$\frac{v_B}{v_A} = \frac{BP}{AP}.$$

Thus, the ratio of the velocities of two points is equal to the ratio of their distances to the instantaneous center of velocity, or

$$\frac{v_A}{AP} = \frac{v_B}{BP} = \ldots = |\omega|.$$

**4.3 Cases of finding the instantaneous center of velocities and angular velocities of a plane figure**

1. In general, to find the instantaneous center of velocity, you only need to know the direction of the velocities of two points of the figure (Fig. 27). To do this, we draw perpendiculars from the beginning of the velocity vectors of these two points (for example, *A* and *B*). The point of intersection of these perpendiculars is the instantaneous center of velocity (point *P*).

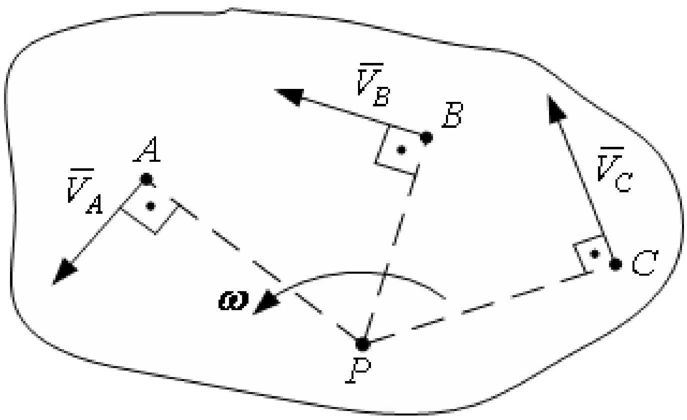



Figure 27

The angular velocity $\omega$ of a plane figure at each moment of time is equal to the ratio of the velocity of any point of the figure to its distance from the CMP:

$$\omega = \frac{v_A}{AP} = \frac{v_B}{BP} = \frac{v_C}{CP} = \ldots$$

2. Two points $A$ and $B$ are known by the velocity of two figures that are parallel to each other, directed in the same direction and perpendicular to the line $AB$ ($\overline{V_A}//\overline{V_B}$) (Fig. 28).

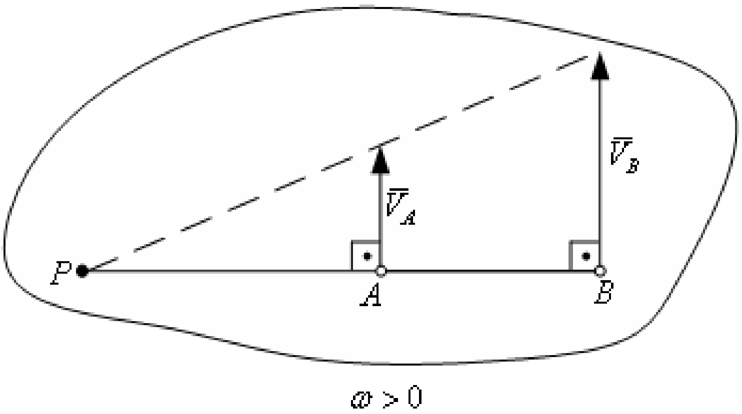



Figure 28

The MSP (point $P$) is located at the intersection of line $AB$ and the line connecting the ends of the velocity vectors of points $A$ and $B$:

$$\omega_{AB} = \frac{v_A}{AP} = \frac{v_B}{BP}; \ (v_B \perp AB, \ v_A \perp \mathrm{AB}).$$

3. If the velocities of two points of a plane figure are directed in different directions and perpendicular to the line segment connecting these points, then the instantaneous center of velocity lies at the intersection of the line connecting the ends of the velocity vectors with the above segment (Figure 29).

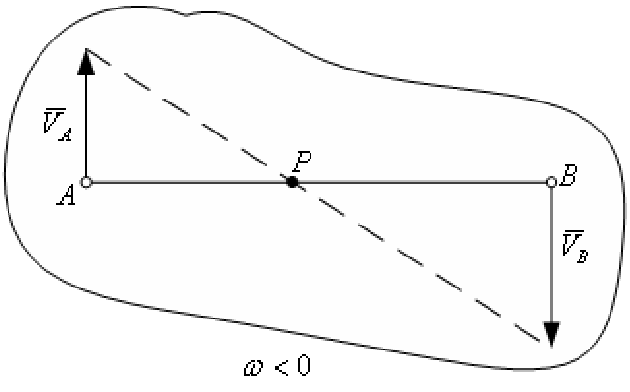



Figure 29

4. If the velocities of two points of a plane figure are parallel and equal to each other, directed in the same direction, then the instantaneous center of velocity is removed at an infinitely large distance, i.e., there is no ICV, $\omega$ = 0 and the instantaneous velocities of all points of the figure are geometrically equal to each other (Fig. 30).

$$v_A = v_B = v_C = \cdots$$

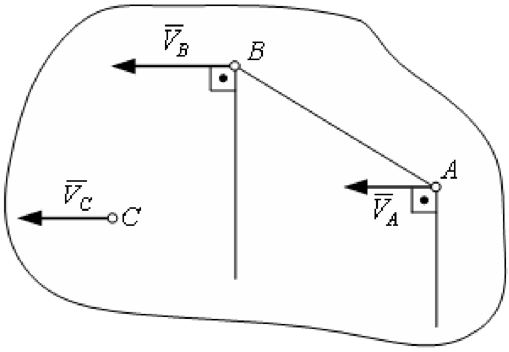


Figure 30

This is a case of instantaneous translational motion of the body:

$$ICP \to \infty;\ \ \omega = \frac{v_A}{\infty} = 0.$$

5. When a moving contour of a plane figure rolls without sliding on a fixed contour (Fig. 31), the instantaneous center of velocity lies at the point of contact of these contours. The angular velocity

$$\omega = \frac{v_A}{AP},\ v_B = \omega \cdot BP;\ v_C = \omega \cdot CP$$

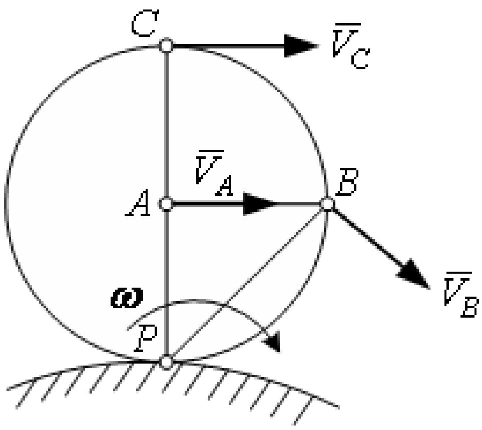


Figure 31

### 4.3.1 Solving the problem by the method of instantaneous centers of velocity

From the angular velocities of the driving links, we determine the velocity of point *A*, the axis of the gear pairing, and the velocity of point *B* of the contact of wheels 1 and 2 (Fig. 32):

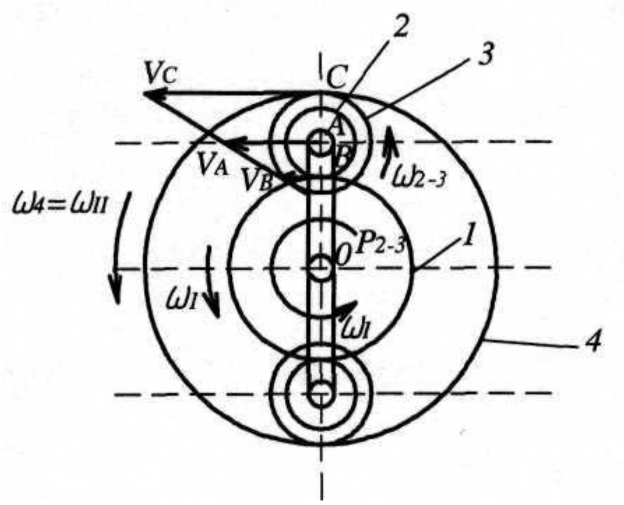



Figure 32

$$v_A = \omega_I \cdot (r_1 + r_2) = 80 \cdot 0{,}45 = 36\ m/s$$

$$v_B = \omega_1 \cdot r_1 = 20 \cdot 0{,}3 = 6\ m/s \qquad (4.1)$$

Let's set aside the vectors $\bar{v}_A$ and $\bar{v}_B$ (see Fig. 23), we find the instantaneous center of velocity of gears 2-3:

$$\frac{P_{2-3}A}{P_{2-3}B} = \frac{v_A}{v_B} = \frac{36}{6} = 6, \qquad P_{2-3}A = 6 \cdot P_{2-3}B$$

Since $P_{2-3} - P_{2-3}B = r_2 = 0{,}15\ m$, then:

$$P_{2-3}B = \frac{0{,}15}{5} = 0{,}03\ m;$$

$$P_{2-3}A = 0{,}18\ m;$$

$$P_{2-3}C = 0{,}48\ m.$$

Determine the velocity of point *C*:

$$\frac{v_C}{v_A} = \frac{P_{2-3}C}{P_{2-3}A} = \frac{0{,}48}{0{,}18}, \qquad v_C = \frac{0{,}48}{0{,}18} \cdot v_A = \frac{0{,}48}{0{,}18} \cdot 36 = 96\ m/s.$$

The speed of gear 2-3 is determined by the speed of point *A* (or *B* and *C*):

$$\omega_{2-3} = \frac{v_A}{P_{2-3}A} = \frac{36}{0{,}18} = 200\ \mathrm{s}^{-1}.$$

## Lecture № 5

Goals and objectives:

The lecture discusses the method of determining the acceleration in the plane motion of a solid body, and also provides an algorithm for solving problems of determining the velocities and accelerations of points of a body performing plane motion.

### 5.1 Accelerate points on a plane figure

The acceleration of any point of a plane figure geometrically consists of the acceleration of the pole and the acceleration of the point in the rotational motion of the body around the pole, consisting of centripetal (normal) and rotational (tangential) accelerations (Fig. 33).

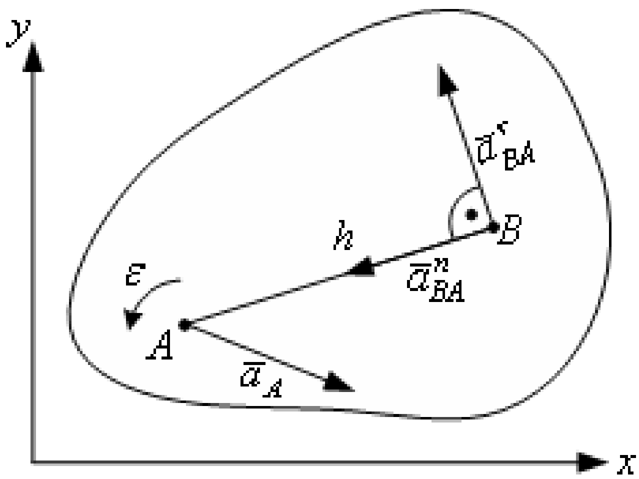



Figure 33

$$\bar{a}_B = \bar{a}_A + \bar{a}_{BA},$$

$$\bar{a}_{BA} = \bar{a}^n_{BA} + \bar{a}^\tau_{BA},$$

$$\bar{a}_B = \bar{a}_A + \bar{a}^n_{BA} + \bar{a}^\tau_{BA},$$

where $a^n_{BA} = \omega^2 BA$ – is directed along *AB* from point *B* to point *A*,

$a^\tau_{BA} = |\varepsilon| \cdot h = \varepsilon \cdot l_{BA}$ – is always directed perpendicular to the *AB* connecting point *B* with pole *A* (Fig. 34), in the direction of angular acceleration $\varepsilon$.

$$a_{BA} = BA \cdot \sqrt{\omega^4 + \varepsilon^2}$$

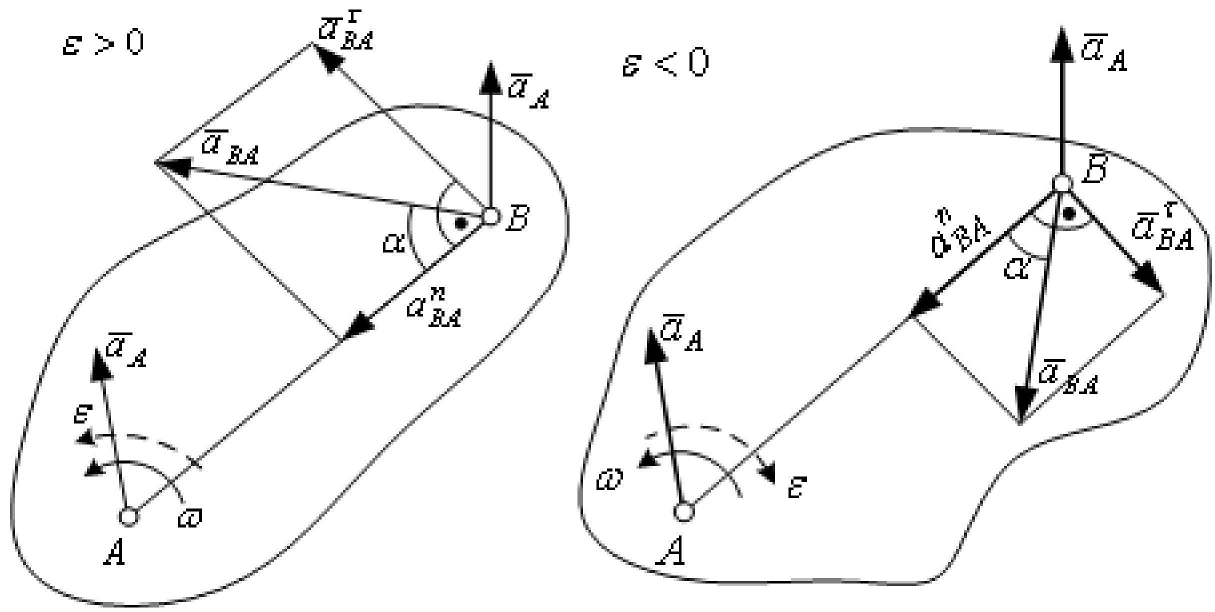



Figure 34

Thus, the acceleration $\bar{a}_B$ of any point *B* of a plane figure can be found as the geometric sum of three accelerations: the pole acceleration $\bar{a}_A$, centripetal acceleration $\bar{a}^n_{BA}$ and rotational acceleration $\bar{a}^{\tau}_{BA}$ of point *B* around pole *A*. The vector $\bar{a}_{BA}$ has a direction at an angle $\alpha$ to the line *BA*.

At the same time

$$tg\alpha = \frac{|\varepsilon|}{\omega^2}; \left(0 < \alpha < \frac{\pi}{2}\right).$$

**5.2 Guidelines for solving problems in kinematic analysis of a plane mechanism**

1. Consistently consider and determine the movement of individual links in the mechanism.
2. Start the calculation with the link whose motion is specified. If the link performs rotational motion (it has a fixed point), use the formulas of rotational motion to determine the velocities and accelerations. If the link performs plane-parallel motion (there is no fixed point), use the formulas of plane motion to determine the speed and acceleration.
3. Construct the MSC of all links and find the magnitudes and directions of the velocities of all specified points on the mechanism.
4. Determine the moduli and signs of angular velocities of all the links.
5. Determine the magnitudes and directions of the accelerations of all the given points, as well as the magnitudes and signs of the angular accelerations of all the links.

**Example**

The flat mechanism consists of a crank *OA*, a slider *B*, and a connecting rod *AB* (Fig. 35). *OA* = 10 sm, *AB* = 60 sm, *AC* = 20 sm. At a given time, the crank OA has an angular velocity $\omega_{\text{OA}} = 1{,}5\ s^{-1}$ and angular acceleration $\varepsilon_{OA} = 2\ s^{-2}$.

For a given position of the mechanism, determine the velocity *A*, *B*, *C* and the acceleration of points *A*, *B*, as well as the angular velocity and angular acceleration of the connecting rod *AB*: $\varepsilon_{AB}$; $\omega_{AB}$.

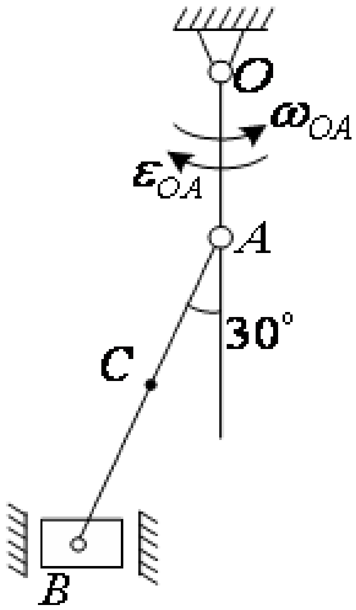



Figure 35

**Solution**

1. The link *OA* moves around the fixed point *O* and performs a rotational motion.

Calculate the velocity of point *A* of the crank *OA*.

$$V_A = \omega_{OA} \cdot OA = 1{,}5 \cdot 10 = 15\ sm/s$$

and is directed in the direction of rotation of the *OA* link, according to the "arrow" (direction) $\omega_{OA}$ (Figure 36). The velocity vector of point *A* is perpendicular to the crank *OA*:

$$\bar{V}_A \perp OA.$$

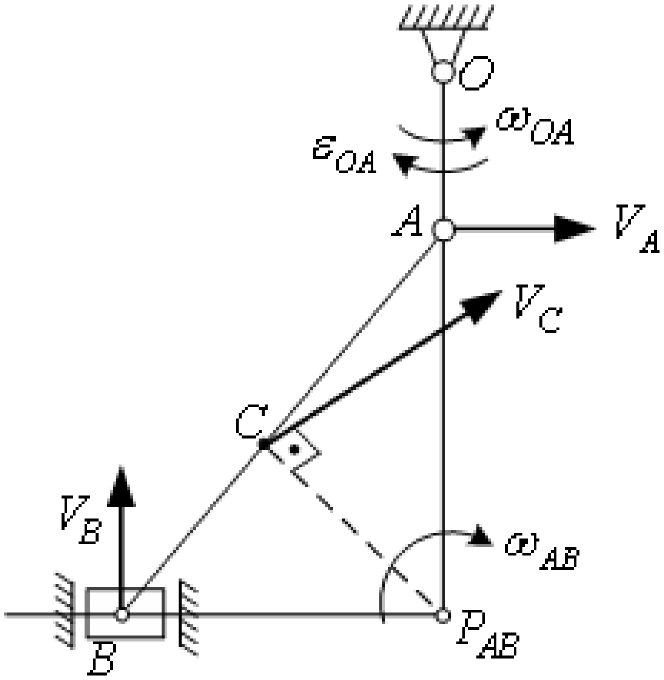



Figure 36

Slider *B* moves forward. Point *B* belongs to the slider, so the line of action of its velocity $\bar{V}_B$ is parallel to the direction of the slider.

The point *B* of the slider also belongs to the connecting rod *AB*, which moves in a plane-parallel direction.

The instantaneous center of velocity of the connecting rod *AB* is located at the intersection of the perpendiculars drawn from points *A* and *B* to the direction of their velocities (point $P_{AB}$).

The angular velocity of the connecting rod *AB* is equal to

$$\omega_{AB} = \frac{V_A}{AP_{AB}} = \frac{V_A}{AB \cdot cos\,30°} = \frac{15}{60 \cdot cos\,30°} = 0{,}29\ s^{-1}$$

and directed according to how the vector $\bar{V}_A$ vector rotates around the point $P_{AB}$.

The velocities of points *B* and *C* are respectively equal to

$$V_B = \omega_{AB} \cdot BP_{AB} = 0{,}29 \cdot 30 = 8{,}7\ sm/s,$$
$$V_C = \omega_{AB} \cdot CP_{AB} = 0{,}29 \cdot 36{,}1 = 10{,}5\ sm/s,$$

where

$$BP_{AB} = AB \cdot sin\,30° = 60 \cdot 0{,}5 = 30\ sm,$$
$$CP_{AB} = \sqrt{BC^2 + BP_{AB}^2 - 2 \cdot BC \cdot BP_{AB} \cdot cos\,60°} =$$
$$\sqrt{40^2 + 30^2 - 2 \cdot 40 \cdot 30 \cdot 0{,}5} = 36{,}1\ sm.$$

The vectors $\bar{V}_B$ and $\bar{V}_C$ are directed perpendicular to the segments $BP_{AB}$ and $CP_{AB}$ respectively and are directed according to the "arrow" $\omega_{AB}$.(Fig. 36)

2. Determine the acceleration of the points of the flat mechanism (Fig. 37).

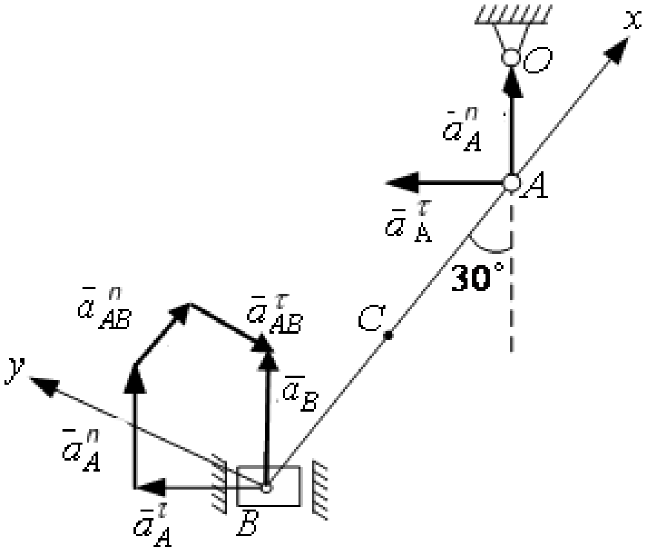



Figure 37

Acceleration of point *A* of the *OA* link.

The acceleration of point *A* consists of the geometric sum of the rotational $\bar{a}_A^\tau$ and centripetal $\bar{a}_A^n$ accelerations of the crank *OA*, which performs rotational motion:

$$\bar{a}_A = \bar{a}_A^\tau + \bar{a}_A^n;\ a_A = \sqrt{(a_A^\tau)^2 + (a_A^n)^2},$$

$$a_A^\tau = \varepsilon_{OA} \cdot OA = 2 \cdot 10 = 20\ sm/s^2,$$

$$a_A^n = \omega_{OA}^2 \cdot OA = 1{,}5 \cdot 10 = 22{,}5\ \ sm/s^2,$$

$$a_A = \sqrt{20^2 + 22,5^2} = 30{,}1\ sm/s^2.$$

The vector $a_A^n$ vector is directed along the *AA* from *A* to *O*.

The vector $\bar{a}_A^\tau \perp \bar{a}_A^n$ and directed according to the direction of angular acceleration $\varepsilon_{OA}$, according to the "arrow" (direction) $\varepsilon_{OA}$.

The acceleration of point *A*, the vector $\bar{a}_A$ is directed along the diagonal of the parallelogram constructed on the vectors $\bar{a}_A^\tau$ and $\bar{a}_n^n$.

The acceleration of the points of a line *AB* moving in plane parallel consists of the geometric sum of the pole acceleration and the rotational acceleration relative to the pole. The pole is taken to be the point of the link *AB*, the acceleration of which is already known - point *A*.

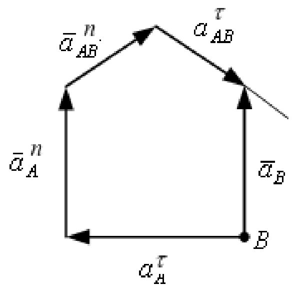



Figure 38

Acceleration of point *B* of the *AB* link (Fig. 38):

$$\overline{a_B} = \overline{a_A} + \overline{a_{BA}^{\tau}} + \overline{a_{BA}^{n}},$$

or

$$\bar{a}_B = \bar{a}_A^n + \bar{a}_A^{\tau} + \bar{a}_{BA}^{\tau} + \bar{a}_{BA}^n.$$

The module of the component $\bar{a}_{AB}^n$ is calculated by the formula

$$a_{BA}^n = \omega_{AB}^2 \cdot AB = 0{,}29^2 \cdot 60 = 5 \quad sm/s^2.$$

The vector $\bar{a}_{BA}^n$ is directed from *B* to *A*, and the rotational acceleration is perpendicular to it $\bar{a}_{BA}^{\tau} \perp a_{BA}^n$. The modulus of the component $\bar{a}_{BA}^{\tau}$ cannot be calculated yet, because it is unknown $\varepsilon_{AB}$, is the angular acceleration of the link *AB*.

The slider *B* moves gradually along a vertical line, along which the acceleration of point *B* is directed. $\bar{a}_B$, we plot the vectors $\overline{a}_A^{\tau}, \overline{a}_A^n, \overline{a}_{BA}^n$, which are defined by direction and module.

Through the end of the vector $a_{BA}^n$ we draw a straight line perpendicular to the vector $a_{BA}^n$, along which the vector $\bar{a}_{BA}^{\tau}$ should be directed, until it intersects with the line along which the acceleration *B* a is directed. The acceleration *B* a is defined as the closing vector of the acceleration polygon. Choose the coordinate axis XU. Find the acceleration of point *B* by projecting the vector equation onto the X-axis:

$$a_B \cdot cos\,30° = -a_A^{\tau} \cdot cos\,60° + a_A^n \cdot cos\,30° + a_{\mathrm{BA}}^n.$$

$$a_B = \frac{-20 \cdot 0{,}5 + 22{,}5 \cdot 0{,}866 + 5}{0{,}86} = 16{,}7 \ sm/s^2.$$

By projecting the vector equation onto the Y-axis, we determine the rotational acceleration:

$$a_{BA}^{\tau} = a_A^{\tau} \cdot cos\,30° + a_A^n \cdot cos\,60° - a_B \cdot cos\,60°$$
$$= 20 \cdot 0{,}866 + 22{,}5 \cdot 0{,}5 - 16{,}7 \cdot 0{,}5 = 20{,}2\ sm/s^2$$

The «+» sign means that the vector $\bar{a}_{BA}^{\tau} \perp BA$ vector coincides with the negative direction of the y-axis.

The angular acceleration of the link *AB*:

$$\varepsilon_{AB} = \frac{a_{AB}^{\tau}}{AB} = \frac{20{,}2}{60} = 0{,}34\ s^{-2}$$

and is directed as the vector $\bar{a}_{BA}^{\tau}$ rotates around the pole *A*.

Answer:

$$v_A = 15\,{}^{sm}/_{s};\ \ a_{\mathrm{A}}^{\tau} = 20\,{}^{sm}/_{s^2};\quad a_A^n = 22{,}5\,{}^{sm}/_{s^2};$$

$$v_B = 8{,}7\,{}^{sm}/_{s};\ a = 16{,}7\,{}^{sm}/_{s^2};\quad a_A = 30{,}1\,{}^{sm}/_{s^2};$$
$$v_C = 10{,}5\,{}^{sm}/_{s};\ \ \omega_{AB} = 0{,}29s^{-1};\ \varepsilon_{AB} = 0{,}34s^{-2}.$$

### 5.3 Kinematic analysis of a flat mechanism

Determine, for a given position of the mechanism, the velocity and acceleration of points *A* and *B*, the angular velocity and angular acceleration of link *AB*. Schemes of mechanisms are shown in Fig. 39-43, and the data for calculation are given in Table 5.1

Table 5.1

| Option | r, m | l, m | $\omega$, $s^{-1}$ | $\varepsilon$, $s^{-2}$ |
|---|---|---|---|---|
| 1 | 0,1 | 0,6 | 1 | 2 |
| 2 | 0,3 | 0,4 | 1,2 | 3 |
| 3 | 0,2 | 0,3 | 1,5 | 2 |
| 4 | 0,1 | 0,3 | 1,8 | 3 |
| 5 | 0,3 | 0,2 | 1,0 | 2 |
| 6 | 0,1 | 0,8 | 1,1 | 3 |
| 7 | 0,2 | 0,3 | 1,2 | 2 |
| 8 | 0,3 | 0,6 | 1,3 | 3 |
| 9 | 0,2 | 0,4 | 1,4 | 2 |
| 0 | 0,1 | 0,6 | 1,5 | 3 |

**An example of a task**

For the mechanism shown in Fig. 44, find the velocity and acceleration of points *A*, *B*, and the angular velocity and angular acceleration of links *AB* and *BC*. Answer: *OA* = 0,3 m; *OA*=*OB*; *BC* = 0,5 m; $\omega$ = 5 $s^{-1}$; $\varepsilon$ = 3 $s^{-2}$.

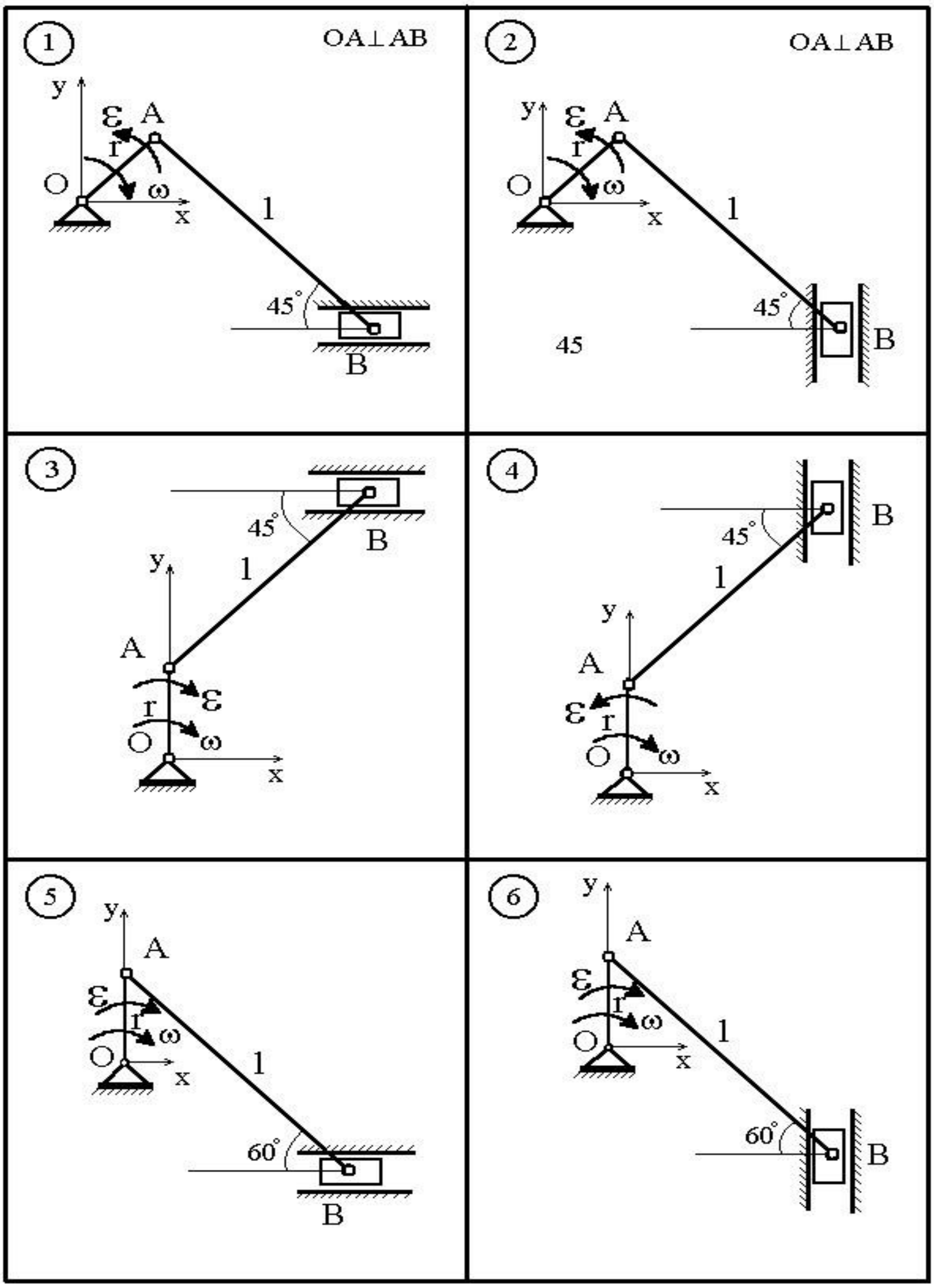
**Solution**

Let's determine the velocities of points *A*, *B* and the angular velocities of links *AB* and *CB*

The velocity of point *A* is directed perpendicular to the link *OA* in the direction of angular velocity ω.

$$v_A = \omega \cdot OA = 5 \cdot 0{,}3 = 1{,}5\,{}^{m}/_{s}$$

The velocity of point *B* will be directed along the line perpendicular to the link *BC* (line b-b, Fig. 44). Then the instantaneous center of velocity of link *AB* will be at point *O*.



1
OA⊥AB
y
ε
A
r
O
ω
x
1
45°
B
2
OA⊥AB
y
ε
A
r
O
ω
x
1
45°
45
B
3
45°
B
y
1
A
ε
r
O
ω
x
4
B
45°
1
y
A
ε
r
O
ω
x
5
y
A
ε
r
ω
O
x
1
60°
B
6
y
A
ε
r
ω
O
x
1
60°
B


Figure 39

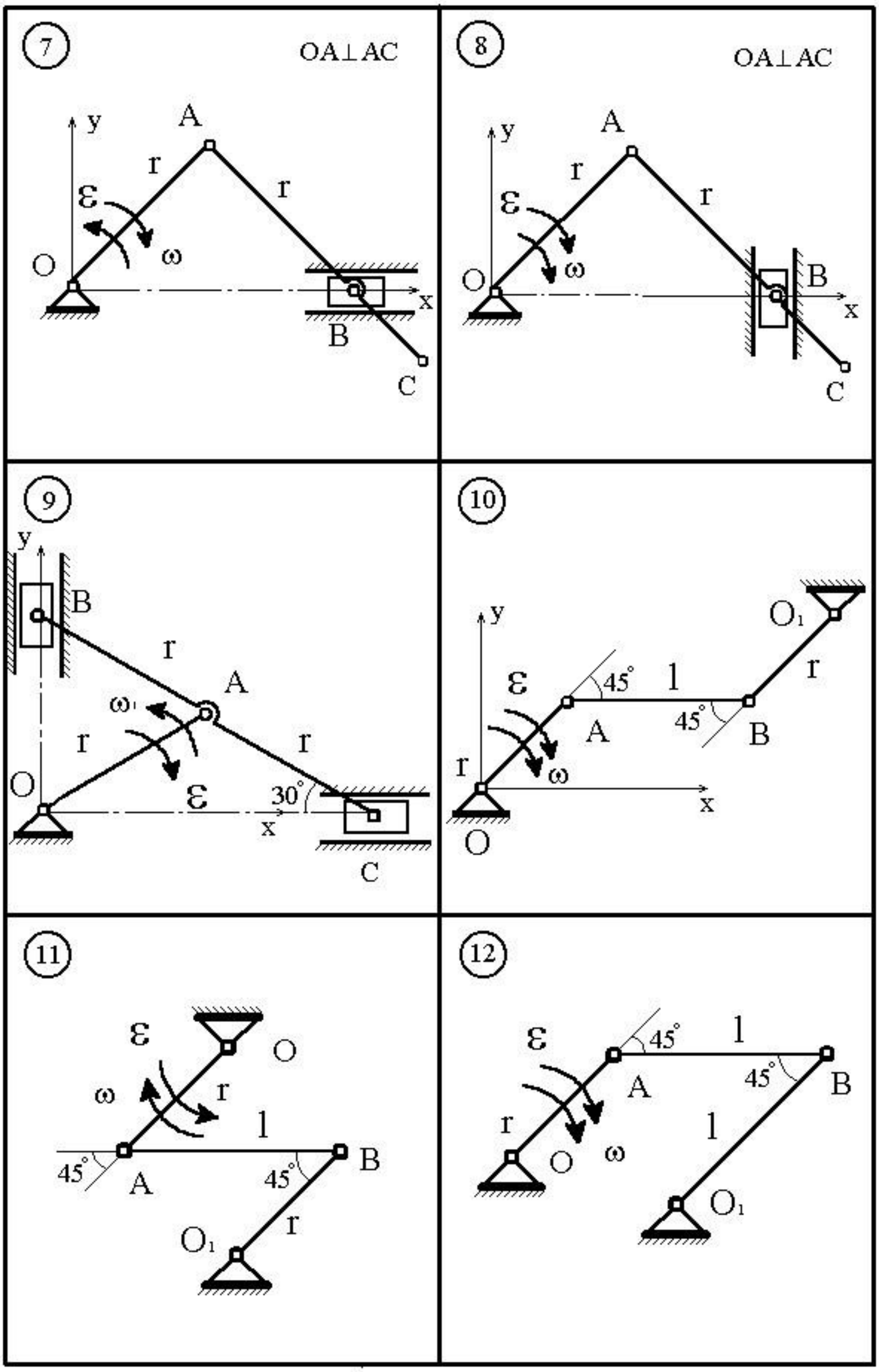

7
OA⊥AC
y
A
r
r
ε
ω
O
x
B
C
8
OA⊥AC
y
A
r
r
ε
ω
O
B
x
C
9
y
B
r
A
ω
r
r
O
ε
30°
x
C
10
y
O1
ε
45°
l
r
45°
A
B
r
ω
x
O
11
ε
O
ω
r
l
45°
A
45°
B
O1
r
12
ε
45°
l
A
45°
B
r
l
O
ω
O1

Figure 40

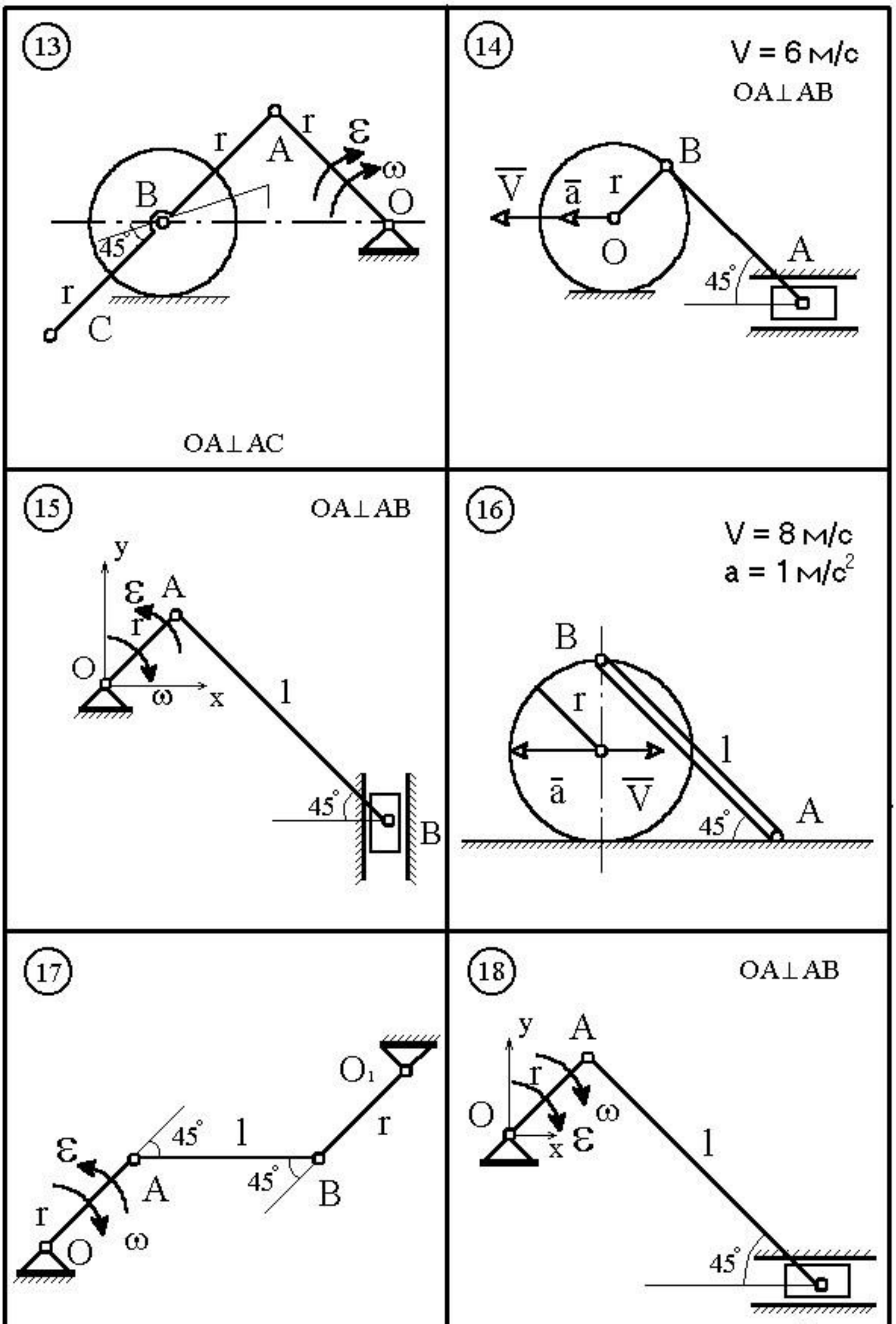

13
r
A
r
ε
ω
B
O
45°
r
C
OA⊥AC
14
V = 6 м/с
OA⊥AB
B
V
a
r
O
A
45°
15
OA⊥AB
y
ε
A
r
O
ω
x
l
45°
B
16
V = 8 м/с
a = 1 м/с²
B
r
l
a
V
45°
A
17
O₁
45°
l
r
ε
A
45°
B
r
O
ω
18
OA⊥AB
y
A
r
O
ω
x
ε
l
45°
B

Figure 41

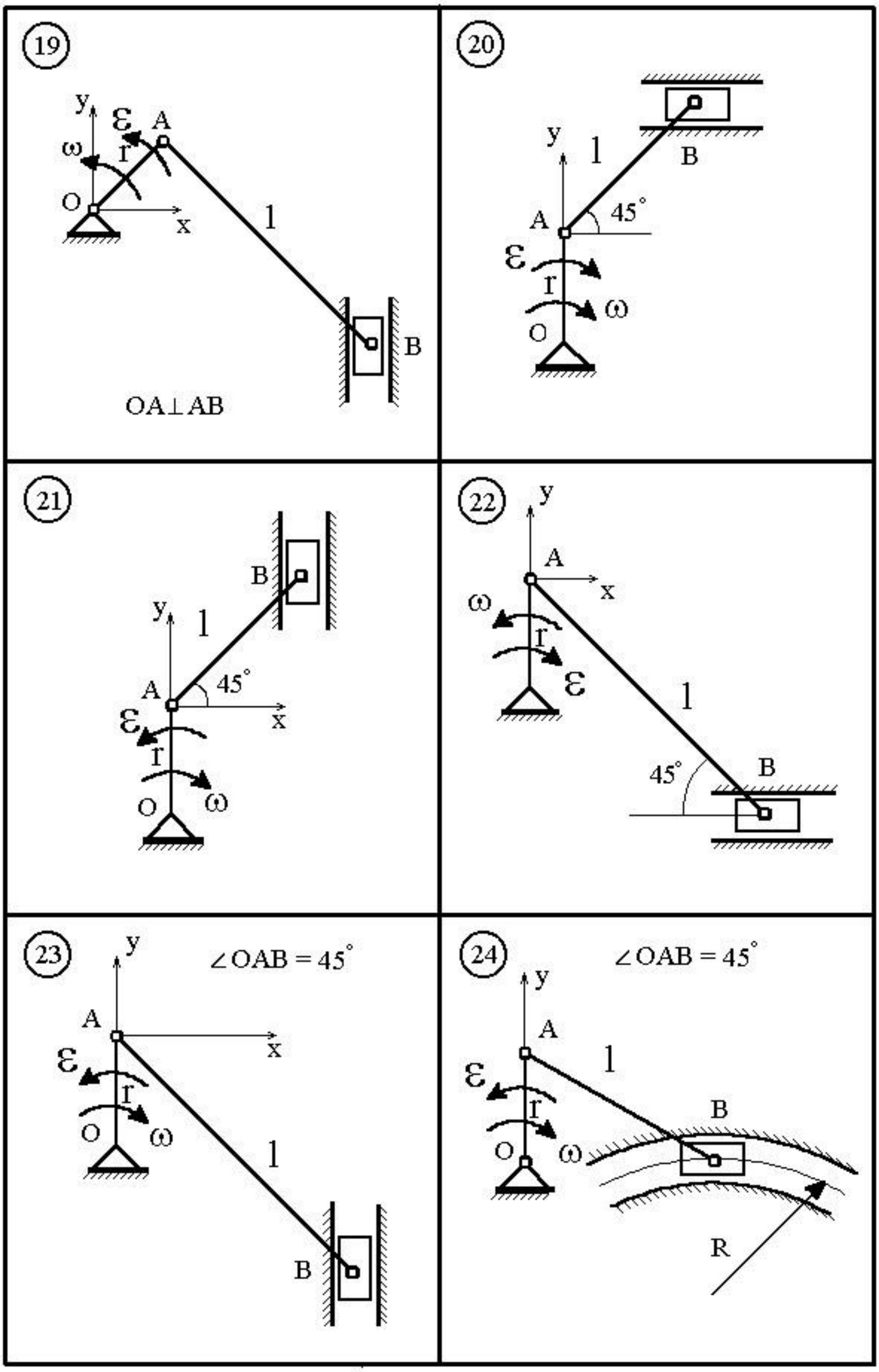
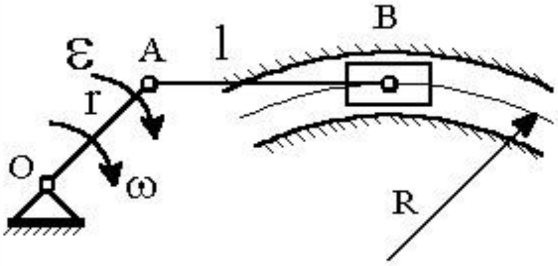

19
y
ε
A
ω
r
O
x
l
B
OA⊥AB
20
B
y
l
45°
A
ε
r
ω
O
21
B
y
l
A
45°
x
ε
r
O
ω
22
y
A
x
ω
r
ε
l
45°
B
23
y
∠OAB = 45°
A
x
ε
r
O
ω
l
B
24
∠OAB = 45°
y
A
l
ε
r
B
O
ω
R

Figure 42

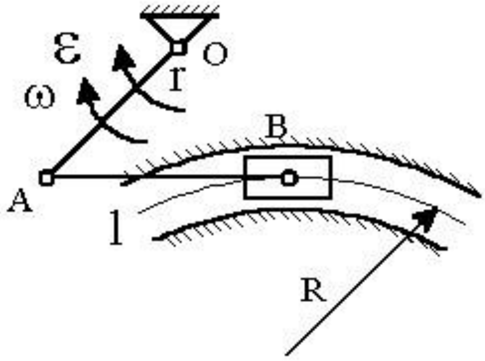
(25)

$\angle OAB = 135^\circ$



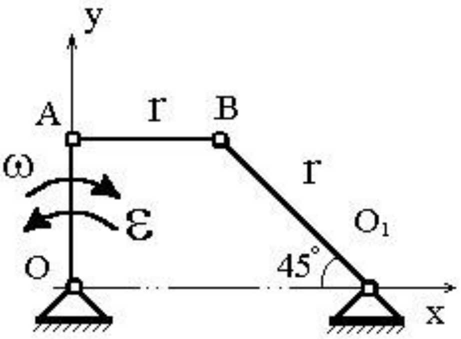

(26)

$\angle OAB = 45$



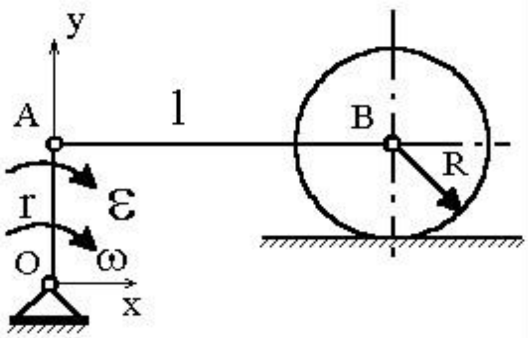

(27)

$OA \perp AB$



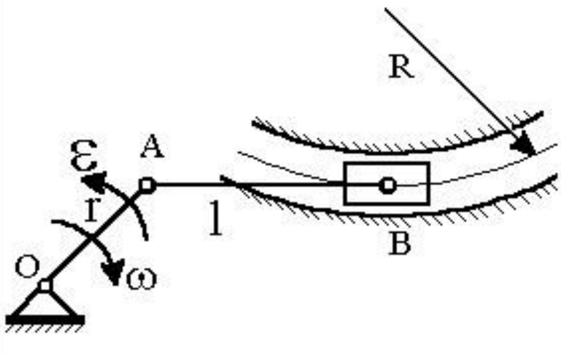

(28)

$OA \perp AB$




(29)

$\angle OAB = 135^\circ$




(30)

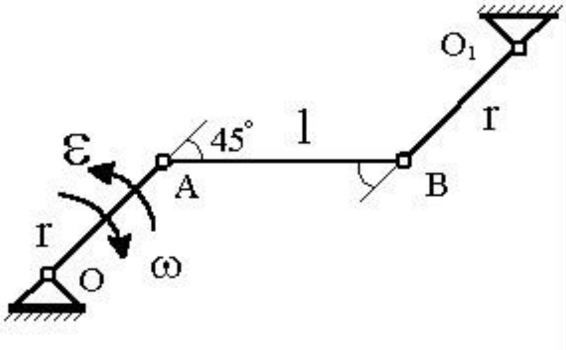


Figure 43

$$\omega_{AB} = \frac{v_A}{OA} = \frac{v_B}{BO}, \qquad \omega_{AB} = \omega = 5\ s^{-1}.$$

The direction of velocity of point *B* is determined by the direction of angular velocity $\omega_{CB}$ (Fig. 30).

$$v_B = v_A = 1{,}5\frac{м}{c}.$$

Angular velocity of the aircraft link *BC*

$$\omega_{CB} = \frac{v_B}{BC} = \frac{1{,}5}{0{,}5} = 3s^{-1}.$$

Now let's find the accelerations of points *A*, *B*, and the angular accelerations of links *AB* and *BC*.

Point *A* belongs to the body *OA*, which rotates around a fixed axis. Then:

$$\overline{a_A} = \overline{a_A^n} + \overline{a_A^\tau},$$
$$a_A^n = \omega^2 \cdot OA = 5^2 \cdot 0{,}3 = 7{,}5\frac{m}{s^2}$$
$$a_A^\tau = \varepsilon \cdot OA = 3 \cdot 0{,}3 = 0{,}9\frac{m}{s^2},$$
$$a_A = \sqrt{a_A^{n^2} + a_A^{\tau^2}} = \sqrt{7,5^2 + 0,9^2} = 7{,}55\frac{m}{s^2}.$$

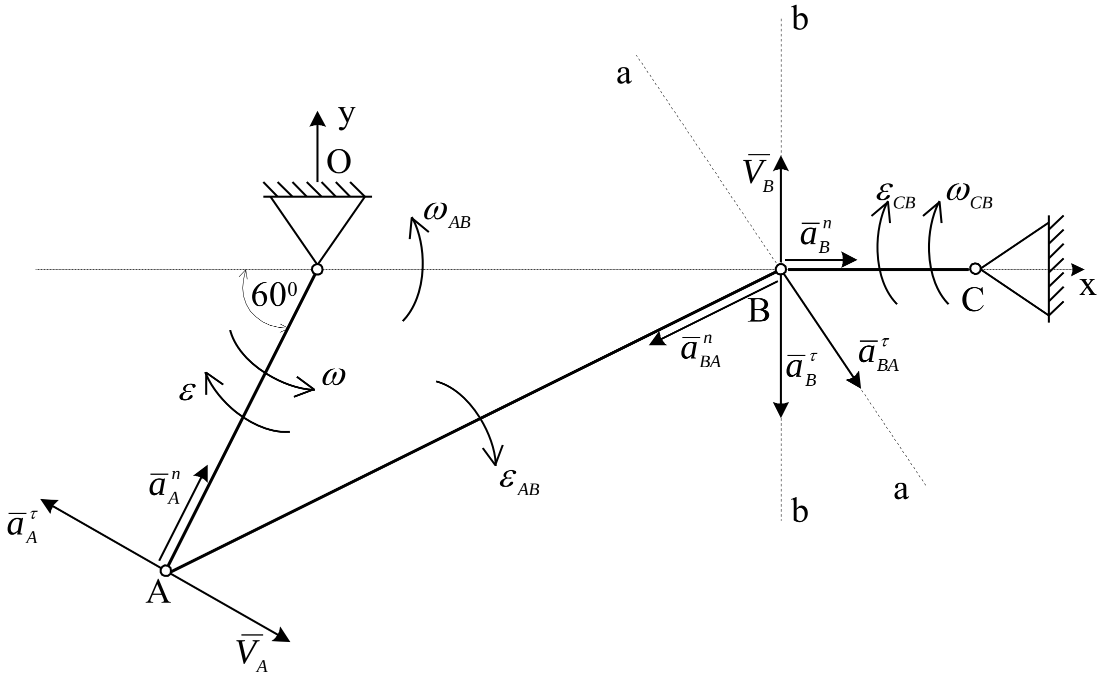



Figure 44

Let's take point *A* as the pole. Then, considering the movement of the link *AB*, we write:

$$\bar{a}_B = \bar{a}_A + \bar{a}_{BA}, \tag{5.1}$$

$$\bar{a}_{BA} = \bar{a}_{BA}^n + \bar{a}_{BA}, \tag{5.2}$$

$$a_{BA}^n = \omega_{AB}^2 \cdot AB = 5^2 \cdot 0{,}6 = 15 \ \frac{m}{s}.$$

where

$$AB = 2 \cdot OA \cdot cos30^{\circ} = 0{,}52\, m$$

The acceleration $\vec{a}_{BA}^{\tau}$ is directed along the line *AB* from point *B* to the pole, and the acceleration $\vec{a}_{BA}^{\tau}$ - is perpendicular to the line *AB* (line a-a, Fig. 30).

Since point *B* belongs to the *BC* link, then

$$\overline{a}_B = \overline{a}_B^n + \overline{a}_B^{\tau}, \tag{5.3}$$

$$a_B^n = \omega_{CB} \cdot BC = 3^2 \cdot 0{,}5 = 4{,}5 \ \text{m/s}^2.$$

The acceleration $\bar{a}_{BA}^{\tau}$ is directed along the line b-b.

Expression (5.1), given (5.2) and (5.3), is written:

$$\overline{a}_B^n + \overline{a_B^{\tau}} = \overline{a_A^n} + \overline{a_A^{\tau}} + \overline{a_{BA}^n} + \overline{a_{BA}^{\tau}}. \tag{5.4}$$

Let's project the vector equation (5.4) on the axes OX and OY. Axis X:

$$a_B^n = a_A^n \cdot cos\, 60^{\circ} - a_A^{\tau} \cdot sin\, 60^{\circ} - a_{BA}^n \cdot cos\, 30^{\circ} + \left(\overline{a_{BA}^{\tau}}\right)_x. \tag{5.5}$$

Axis OY:

$$(\bar{a}_{BA}^{\tau})_y = a_A^n \cdot sin\, 60^{\circ} + a_A^{\tau} \cdot cos\, 60^{\circ} - a_{BA}^n \cdot sin\, 30^{\circ} + (\bar{a}_{BA}^{\tau})_y. \tag{5.6}$$

In formulas (5.5) and (5.6), two unknown quantities $\bar{a}_{BA}^{\tau}$ and $\bar{a}_B^{\tau}$.

$$(\bar{a}_{BA}^{\tau})_x = a_B^n + a_B^{\tau} \cdot sin\, 60^{\circ} + a_{BA}^n \cdot cos\, 60^{\circ} - a_A^n \cdot cos\, 60^{\circ} = 4{,}5 + 0{,}9 \cdot \frac{\sqrt{3}}{2} + 15 \cdot \frac{\sqrt{3}}{2} - 7{,}5 \cdot 0{,}5 = 14{,}52,$$

$$(\bar{a}_{BA}^{\tau})_x = a_{BA}^{\tau} \cdot cos\, 60^{\circ} = 14{,}52,$$

$$a_{BA}^{\tau} = \frac{14{,}52}{cos\, 60^{\circ}} = 29{,}04 \frac{m}{s^2},$$

$$(\bar{a}_B^\tau)_y = a_A^n \cdot sin\,60^\circ + a_A^\tau \cdot cos\,60^\circ - a_{BA}^n \cdot sin\,60^\circ - a_{BA}^\tau \cdot cos\,60^\circ$$
$$= 7{,}5 \cdot \frac{\sqrt{3}}{2} + 0{,}9 \cdot \frac{1}{2} - 15 \cdot \frac{1}{2} - 29{,}04 \cdot \frac{\sqrt{3}}{2} = -25{,}70,$$
$$a_B^\tau = 25{,}7\,\frac{m}{s^2}.$$

Find the angular accelerations of links *AB* and *BC*

$$\varepsilon_{BA} = \frac{a_{BA}^\tau}{AB} = \frac{29{,}04}{0{,}52} = 55{,}85\,\frac{1}{s^2},$$
$$\varepsilon_{BC} = \frac{a_B^\tau}{BC} = \frac{25{,}7}{0{,}5} = 51{,}4\,\frac{1}{s^2}.$$

The direction of angular accelerations $\varepsilon_{BA}$ and $\varepsilon_{BC}$ are determined by the direction of the vectors $\bar{a}_{BA}^\tau$ and $\bar{a}_B^\tau$.

## Lecture № 6

Goals and objectives:

The lecture discusses the complex motion of a point, defines the concepts of relative, translational, and absolute motion, and finds the kinematic characteristics of complex motion,

### 6.1 Absolute, relative, and figurative motion

Complex motion of a point is said to occur when a point *M* moves relative to some reference frame, which in turn moves relative to a second reference frame (Fig. 45).

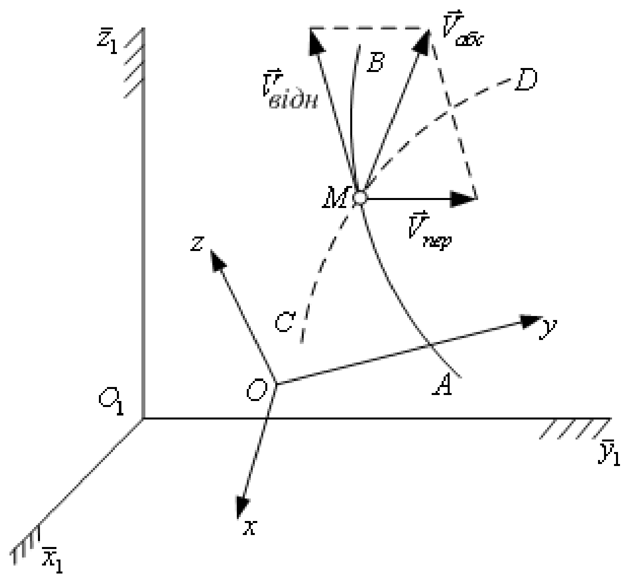



Figure 45

We consider the motion of a point *M* simultaneously with respect to two reference frames, one of which is assumed to be conditionally stationary $X_1,\ O_1, Y_1, Z_1$. It is associated with a solid body whose motion can be neglected under these conditions. The second reference system $X\ O\ Y\ Z$ is moving relative to the fixed $X_1,\ O_1, Y_1, Z_1$. The point *M* moves with the moving reference frame $X\ O\ Y\ Z$ and relative to it. This motion of the point is called complex.

The motion of point *M* in accordance with the moving reference frame $X\ O\ Y\ Z$ is called relative. The trajectory *AB* described by point *M* in relative motion is called a relative trajectory. The speed of the point M relative to the system $X\ O\ Y\ Z$ along *AB* is called the relative velocity $\bar{v}_r$ and the acceleration of the point in this motion is called the relative acceleration $\bar{a}_r$.

Motion of a moving reference frame $XOYZ$ relative to a stationary system $X_1, O_1, Y_1, Z_1$ is called translational. The moving system $XOYZ$ seems to "carry" a point *M* that moves in space. The velocity of the point m, which is invariably associated with the moving system $XOYZ$ point m, with which the moving point *M* coincides at a given time, is called the translational velocity of point *M* at a given time $\bar{v}_e$, and the acceleration of this point is called the relative acceleration of point $M$ $\bar{a}_e$.

The motion of a point $M$ relative to a fixed reference frame $X_1, O_1, Y_1, Z_1$ is called absolute or complex. The trajectory *CD* of this motion is called the absolute trajectory, the velocity is called the absolute velocity $\bar{v}_a$, and acceleration - absolute acceleration $\bar{a}_a$.

Thus, a complex motion of a point is a motion considered in two or more reference frames. One of these reference frames is conventionally considered to be stationary. The motion of a material point referred to the axes conditionally fixed in space is called absolute.

Relative motion is motion relative to moving coordinate axes. The kinematic characteristics of relative motion have the index $r$. Translational motion is the motion of a trajectory described by a point in relative motion.

The kinematic characteristics of translational motion have the index $e$.

The transferred motion of the same point is the motion of the point of the moving coordinate system with which the point whose motion is being considered coincides at a certain moment of time is considered. The main task of the kinematics of complex point motion is to establish a relationship between the kinematic characteristics of the absolute, translational, and relative motions of a point.

The solution of this problem involves dividing the absolute motion of a point into relative and translational motions with the analysis of each separately.

Relative motion can be defined as the imaginary motion of a point in which the moving coordinate system is conditionally stopped, and translational motion can be defined as the following movement of a point, in which the position of the point in the moving system of coordinates (i.e., there is no relative motion of the point).

Let's formulate the theorem of velocity addition in the case of complex motion of a point.

### 6.2 Determining the absolute velocity of a point in a complex motion (Velocity addition theorem.)

Theorem 1. The absolute velocity of a point in complex motion is equal to the vector sum of relative and translational velocities (Fig. 46):

$$\overline{v_a} = \overline{v_e} + \overline{v_r}. \qquad (6.1)$$

According to the cosine theory, the absolute velocity modulus

$$v_a = \sqrt{v_r^2 + v_e^2 + 2v_r v_e \cos(\overline{v_r}, \overline{v_e})}.$$

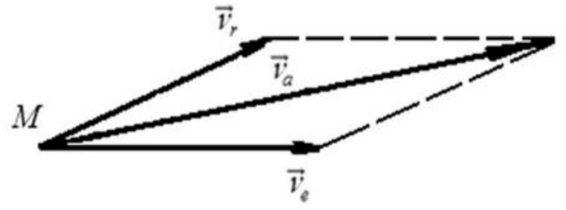



Figure 46

The absolute velocity of a point is defined as its speed relative to a fixed coordinate system. The relative velocity is the velocity of the point of the moving coordinate system with which the point whose motion is being studied coincides at a certain moment time coincides with the point whose motion is being studied. Absolute velocity is directed tangentially to the absolute trajectory, and relative velocity is tangent to the relative trajectory.

If in the problem it is necessary to determine the relative velocity, then, as can be seen from of formula (6.1), it is necessary to geometrically add the absolute velocity vector with vector of the oppositely directed translational velocity:

$$\overline{v_r} = \overline{v_a} + (-\overline{v_e}). \qquad (6.2)$$

It should be noted that regardless of the nature of the translational motion (translational or rotational), formula (6.1) is valid for determining the absolute velocity of a point.

Meanwhile, in the case of determining the absolute acceleration of a point, the nature of the translational motion plays an important role.

If the absolute and relative motions of a point are given in coordinate form, i. e:

$$x = x(t);$$
$$y = y(t);$$
$$z = z(t),$$

then these equations determine the absolute motion of the point and, at the same time, in the parametric form of the equation of the absolute trajectory of the point. The absolute velocity in this case and the absolute acceleration of the point and the absolute acceleration of the point are determined by their projections:

$$v_{ax} = \dot{x};$$
$$v_{ay} = \dot{y};$$

$$v_{az} = \dot{z};$$
$$a_{ax} = \dot{v}_{ax} = \ddot{x};$$
$$a_{ay} = \dot{v}_{ay} = \ddot{y};$$
$$a_{az} = \dot{v}_{az} = \ddot{z}.$$

The modules $\overline{v_a}$ and $\overline{a_a}$, respectively, are defined as follows:

$$v_a = \sqrt{x^2 + y^2 + z^2};$$
$$a_a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2},$$

and their directions are given by the directional cosines.

The equations of relative motion of a point are as follows:

$$\xi = \xi(t);$$
$$\eta = \eta(t); \tag{6.3}$$
$$\zeta = \zeta(t).$$

Simultaneously, in the parametric form of equation (6.3), the equation of the relative trajectory.

The relative velocity and relative acceleration of a point are determined by their projections on the moving coordinate axes:

$$v_{r\xi} = \dot{\xi};$$
$$v_{r\eta} = \dot{\eta};$$
$$v_{r\zeta} = \dot{\zeta};$$
$$a_{r\xi} = \dot{v}_{r\xi} = \ddot{\xi};$$
$$a_{r\eta} = \dot{v}_{r\eta} = \ddot{\eta};$$
$$a_{r\zeta} = \dot{v}_{r\zeta} = \ddot{\zeta}.$$

Accordingly, their modules:

$$v_r = \sqrt{\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2};$$
$$a_r = \sqrt{\ddot{\xi}^2 + \ddot{\eta}^2 + \ddot{\zeta}^2}.$$

The directions $\bar{v}_r$and $\overline{a_r}$ are determined by the directional cosines.

If the motion is translational, then the translational velocity and translational acceleration of the point are equal to the velocity and acceleration of the origin of the moving coordinate system $O_1$ (Fig. 47):

$$\bar{v}_{eM} = \bar{v}_{O1};$$

$$\bar{a}_{eM} = \bar{a}_{O1}.$$

In the case of rotational motion of a moving coordinate system around the origin $O_1$ (Fig. 47), the translational velocity of the point

$$\bar{v}_{eM} = \bar{v}_{O1} + \bar{\omega} \cdot \bar{\rho},$$

where $\bar{\omega} \cdot \bar{\rho}$, according to Euler's formula determines the velocity of point $M$ in rotational motion around point $O_1$.

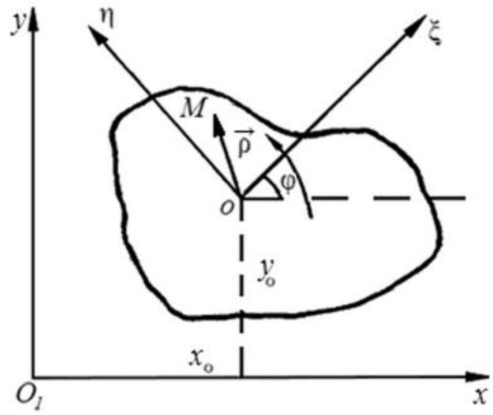



Figure 47

Accordingly, the transferred acceleration $\bar{a}_{eM}$ will be equal:

$$\bar{a}_{eM} = \bar{a}_{O1} + \bar{a}_{O1M},$$

where $\bar{a}_{O1M}$ is the acceleration of the point in its rotation around the point $O_1$.

### 6.3 Determining the absolute acceleration of a point in a complex motion

Coriolis' theorem on the addition of accelerations: The absolute acceleration of a point in a complex motion is equal to the vector sum of the relative, transfer, and Coriolis accelerations:

$$\overline{a_a} = \overline{a_e} + \overline{a_r} + \overline{a_{cor}}. \tag{6.4}$$

Absolute acceleration is defined as the vector sum of three accelerations: acceleration in translational motion $\bar{a}_e$ acceleration in relative motion $\bar{a}_r$ and the Coriolis acceleration $\bar{a}_{cor}$.

The Coriolis acceleration (rotational) characterizes the change in the relative velocity of a point in translational motion and the translational velocity in relative motion.

Coriolis' acceleration is defined as the double vector product of angular velocity in translational motion and linear velocity in relative motion.

$$\bar{a}_{кор} = 2\bar{\omega}_e \times \bar{v}_r,$$

where $\bar{\omega}_e$ – is the angular velocity vector of the moving system (transferred angular velocity);

$\bar{V}_r$ – is the relative velocity of the point.

The Coriolis acceleration modulus is equal to

$$a_{cor} = 2\omega_e \cdot v_r \cdot sin(\bar{\omega}_e, \bar{v}_r)$$

The Coriolis acceleration turns to zero in three cases:

1) when $\omega_e = 0$, the translational motion is translational;

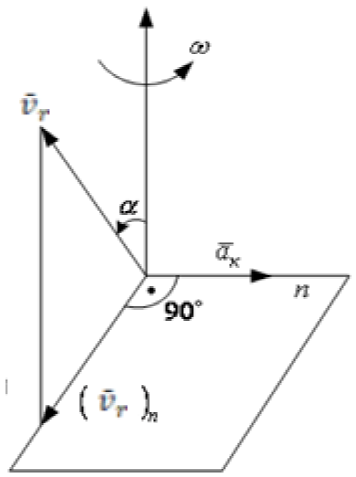
2) when $v_r = 0$, the relative velocity is zero;

3) when $\bar{\omega}_e // \bar{v}_r,\ \sin(\bar{\omega}_{e,}\ \bar{v}_r) = 0$.

The direction of Coriolis acceleration is determined either by the usual rule for finding the direction of a vector product or by the Zhukovsky rule.

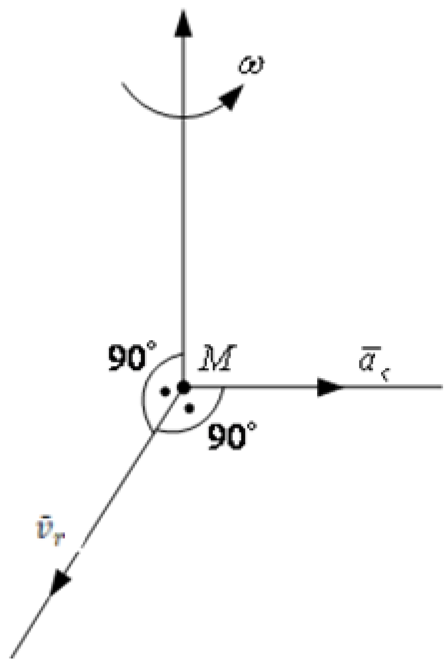
According to Zhukovsky's rule, the vector $\bar{v}_r$ is projected onto a plane perpendicular to the vector $\bar{\omega}_e$. In this plane, the projection vector $\bar{v}_r$ is rotated by $90^\circ$ in the direction of the translational rotation (Figure 30).

If the translational motion is rotational or flat, then it is always $\bar{\omega}_e \perp \bar{v}_r$ so there is no need to project it. The vector $\bar{v}_r$ must be rotated by $90^\circ$ in the direction of the translational rotation. This will be the direction of Coriolis acceleration (Figure 48).




$$a_{cor} = 2|\omega_e| \cdot |v_r| \cdot sin\,\alpha$$

Figure 48




$$a_{cor} = 2|\omega_e| \cdot |v_r|$$

Figure 49

In the case of various complex motions of the point, determining $\overline{v_a}$ and $\overline{a_a}$ we use formulas (1) and (6). Therefore, when considering the following sections of the kinematics of a solid, in particular in the cases of plane motion, the synthesis of motion i.e., when the motion of a solid consists of two rotational motions, the Coriolis theorem is used to determine the acceleration of points of the body.

When studying the complex motion of a point and solving problems, we may have the following special cases when it is necessary to determine:

1) by the equation of motion of a point - speed and acceleration;
2) given the known velocities - the equation of motion;
3) by known absolute and relative velocities - relative velocity;
4) by known velocities - the acceleration of the point;
5) according to the data of the problem - Coriolis acceleration.

**Example**

Rectangular plate *D* (Fig. 50) rotates about a fixed axis according to the law $\phi_e = 2t^2 - 0{,}5t$. The positive direction of the angle $\phi_e$ is shown by an arc arrow. The axis of rotation is perpendicular to the plane and passes through the point $O_1$ (the plate rotates in its plane). The point *M* moves along the plate along *AB*, the law of its relative motion $S_2 = OM = 25\,sin\left(\frac{\pi \cdot t}{3}\right)$ ($S$ in centimeters, $t$ seconds). Point *M* is shown in the position at which $S = OM > 0$ (when $S < 0$ point *M* is on the other side of point *O*).

Find the absolute velocity and absolute acceleration of point $M$ at time $t_1 = 4\ s$, if $a = 25\ sm$.

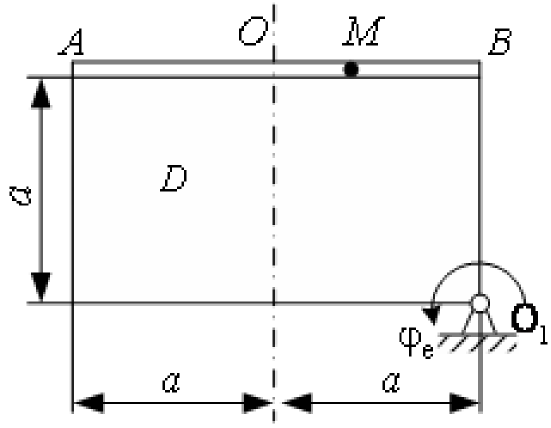



Figure 50

**Solution**

1. The point *M* moves with the plate *D* and relative to it along *AB*. The moving reference frame is associated with the plate *D*. Translational motion is the rotation of the plate *D* about the axis $O_1$, relative motion is the motion of point *M* along *AB* – a straight line motion

2. Determine the position of point *M* on the line *AB* when $t_1 = 4\ s$ (Fig. 50):

$$OM_1 = 25\,sin\left(\frac{4}{3}\pi\right) = -25 \cdot sin\,6\,0 = -25 \cdot 0{,}866 = -21{,}65\,sm.$$

Then the distance from the point $M_1$ to the axis of rotation in translational motion $t = 4\ s$.

$$O_1M_1 = \sqrt{(a + OM_1)^2 + a^2} = \sqrt{46{,}65^2 + 25^2} = 52{,}926\ sm.$$

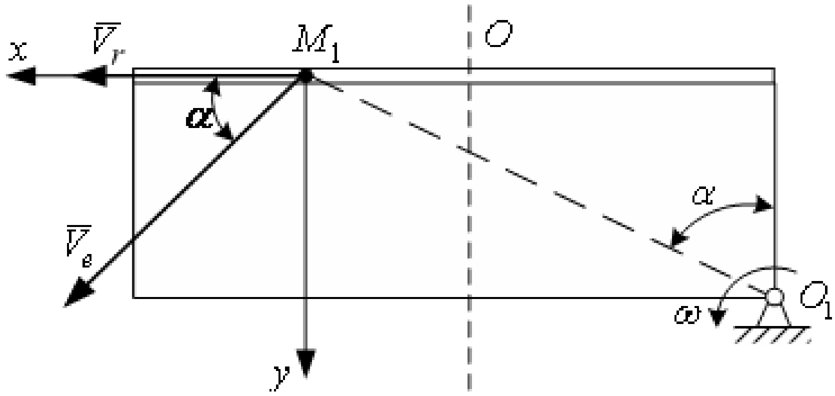



Figure 51

3. Determination of the absolute velocity of point $M_1$, when $t = 4\ s$.

$$\overline{v_a} = \overline{v_e} + \overline{v_r}$$

The translational velocity ($v_e$) of point $M$ is the velocity of the place on the plate where point $M$ is located at a given time:

$$v_e = \omega_e \cdot O_1 M_1 = 15{,}5 \cdot 52{,}926 = 820{,}36\ sm/s,$$

where

$$\omega_e = \frac{d}{dt}(\phi_e) = \frac{d}{dt}(2t^2 - 0{,}5t) = 4t - 0{,}5.$$

When $t = 4\ s$, $\omega_e = 4 \cdot 4 - 0{,}5 = 15{,}5\ s^{-1}$, pointing in the arc $\phi$ because it is the same sign.

The vector $\bar{\omega}$ vector is directed perpendicular to the plane of the plate towards the viewer and passes through the point $O_1$.

The vector $\bar{v}_e \perp O_1 M_1$ and directed in the direction of the arc arrow $\omega$.

The relative velocity of point $M_1$ is tangential to the circle of radius $O_1 M_1$. The motion of the point is given in a natural way. Therefore

$$v_r = \frac{d}{dt}(S_r) = \frac{d}{dt}\left(25\, sin\frac{\pi \cdot t}{3}\right) = 25 \cdot \frac{\pi}{3} cos\frac{\pi}{3} t.$$

At $t = 4\ s$, $v_r = 25 \cdot \frac{\pi}{3} cos\frac{4}{3}\pi = -\frac{25 \cdot \pi}{3} cos\, 6\, 0° = -13{,}08\ sm/s$. The directional vector of relative velocity $\bar{V}_r$ along $OM_1$ and coincides in direction with $OM_1$.

The absolute velocity is determined by the method of projections, for which we choose the coordinate system $XM_1Y$.

Projection of the absolute velocity of point $M$ on the $X$-axis

$$(v_a)_x = v_r + v_e \cdot cos\,\alpha = 13{,}08 + 820{,}36 \cdot 0{,}47 = 399{,}3\ sm/s,$$
$$(v_a)_y = v_e \cdot sin\,\alpha = 820{,}36 \cdot 0{,}88 = 721{,}91\ sm/s,$$

where

$$cos\,\alpha = \frac{a}{O_1 M_1} = \frac{25}{52{,}926} = 0{,}47,$$
$$sin\,\alpha = \frac{a + OM_1}{O_1 M_1} = \frac{25 + 21{,}65}{52{,}926} = 0{,}88,$$
$$v_a = \sqrt{v_{ax}^2 + v_{ay}^2} = \sqrt{399{,}3^2 + 721{,}9^2} = 824{,}9\ sm/s.$$

A vector $\bar{v}_a$ is directed along the diagonal of the parallelogram constructed on the vectors $\bar{v}_r$ i $\bar{v}_e$.

4. Determination of the absolute acceleration of point $M_1$, when $t_1 = 4\ s$ (Fig. 52):

$$\bar{a}_a = \bar{a}_e^\tau + \bar{a}_e^n + \bar{a}_r^n + \bar{a}_r^\tau + \bar{a}_{cor}.$$

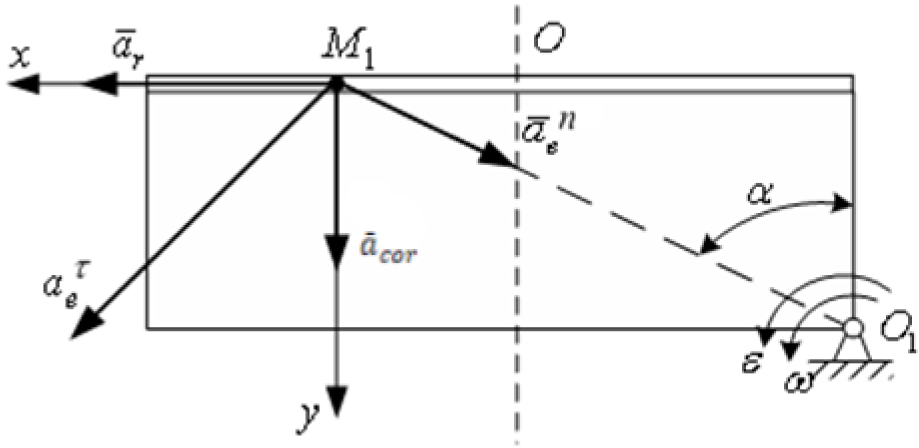



Figure 52

The translational acceleration of point *M₁* is the acceleration of the place on the plate *D* where point *M* is located at a given time, i.e., at point *M₁*:

$$a_e^n = \omega_e^2 \cdot O_1 M_1 = 15,5^2 \cdot 52{,}925 = 12715\ sm/s^2,$$
$$a_e^\tau = \varepsilon_e \cdot O_1 M = 4 \cdot 52{,}925 = 211{,}7\ sm/s^2,$$

where is the angular acceleration during the rotational motion of the point *M*:

$$\varepsilon_e = \frac{d}{dt}(\omega_e) = \frac{d}{dt}(4t - 0{,}5) = 4\ s^{-2}.$$

The relative motion of point *M* is given in the natural way. Therefore, the normal acceleration of the point $M_1$, which moves in a straight line ($\rho = \infty$),

$$a_r^n = \frac{v_r^2}{\rho} = 0.$$

Tangential acceleration of point $M_1$

$$a_r^\tau = \frac{d}{dt}(v_r) = \frac{d}{dt}\left(25 \cdot \frac{\pi}{3} cos \frac{\pi}{3} t\right) = -25 \cdot \frac{\pi^2}{3^2} sin \frac{\pi}{3} t.$$

When $t_1 = 4s$

$$a_r^\tau = -25 \cdot \frac{\pi^2}{3^2} sin \frac{4}{3}\pi = -\frac{\pi^2}{3^2} \cdot 25 \cdot (-sin\,6\,0°) = \frac{\pi^2}{3^2} \cdot 25 \cdot 0{,}866 = 25{,}68\ sm/s^2$$

vector $\bar{a}_r^\tau$ directed along the $OM_1$.

The direction $\bar{a}_{cor}$ is determined by the rule of M.E. Zhukovsky: the vector $\bar{v}_r$ rotated by an angle of $90^0$ in the direction of the arc arrow $\omega = \omega_e$, i.e., in the direction of the clock.

The modulus of Coriolis acceleration is found by the formula:

$$(\bar{\omega}_e \perp \bar{v}_r)$$
$$\bar{a}_{кор} = 2|\omega_e||v_r| \cdot sin\, 9\,0^{\circ} = 2 \cdot 13{,}08 \cdot 15{,}5 = 405\ sm/s^2.$$

The absolute acceleration modulus of point $M_1$ - $\bar{a}_a$ is determined by the method of projections:

$$(a_a)_x = a_r^{\tau} + a_e^{\tau} \cdot cos\,\alpha - a_e^n \cdot sin\,\alpha = 25{,}68 + 2{,}17 \cdot 0{,}47 - 127{,}1 \cdot 0{,}88 = -85{,}15\ sm/s^2;$$
$$(a_a)_y = a_e^{\tau} \cdot sin\,\alpha + a_e^n \cdot cos\,\alpha + a_{cor} = 2{,}17 \cdot 0{,}88 + 127 \cdot 0{,}47 + 405{,}5 = 467{,}1\ sm/s^2.$$

Then
$$a_a = \sqrt{(a_a)_x^2 + (a_a)_y^2} = \sqrt{85{,}12^2 + 467{,}1^2} = 474{,}82\ sm/s^2.$$

Answer:
$$v_a = 824{,}9\ sm/s = 8{,}25\ m/s,$$
$$a_a = 474{,}8\ sm/s^2 = 4{,}74\ m/s^2.$$

## Questions for self-control

1. What kind of point motion is called complex?
2. What motion of a point is called relative?
3. What is the main task of complex motion of a point?
4. How is the absolute velocity of a point in complex motion determined?
5. What is the Coriolis acceleration?
6. In what cases is the Coriolis acceleration equal to zero?
7. How to determine the direction of Coriolis acceleration?
8. How is the theorem of addition of velocities in complex motion formulated?
9. How to formulate the Coriolis theorem.
10. Why, when a point moves towards the north and south poles of the Earth Coriolis acceleration will be directed in different directions?
11. What motion of a point is called translational?
12. What motion of a point is called absolute?
13. How is the absolute velocity of a point determined in complex motion?
14. How is the translational velocity of a point in complex motion determined?
15. How is the relative motion of a point determined in complex motion?
16. How is the translational motion of a point determined in complex motion?

17. How is the absolute acceleration of a point determined in complex motion?
18. How is the axial acceleration of a point in complex motion determined?
19. How is the translational acceleration of a point in complex motion determined?
20. What is the Coriolis acceleration, if the translational motion is translational?
21. Does the type of translational motion affect the Coriolis acceleration?
22. What coordinate systems should be used when solving a problem of complex motion of a point and why?
23. Can a fixed coordinate system in a problem be associated with a moving body?
24. Can there be two components of the absolute acceleration of a point in a problem of complex motion acceleration of a point? In what case is this possible?
25. How is the modulus of absolute acceleration of a point determined in problems of complex motion acceleration of a point in the case of three, four, five components?

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**Тетяна Ігорівна Молодецька**

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