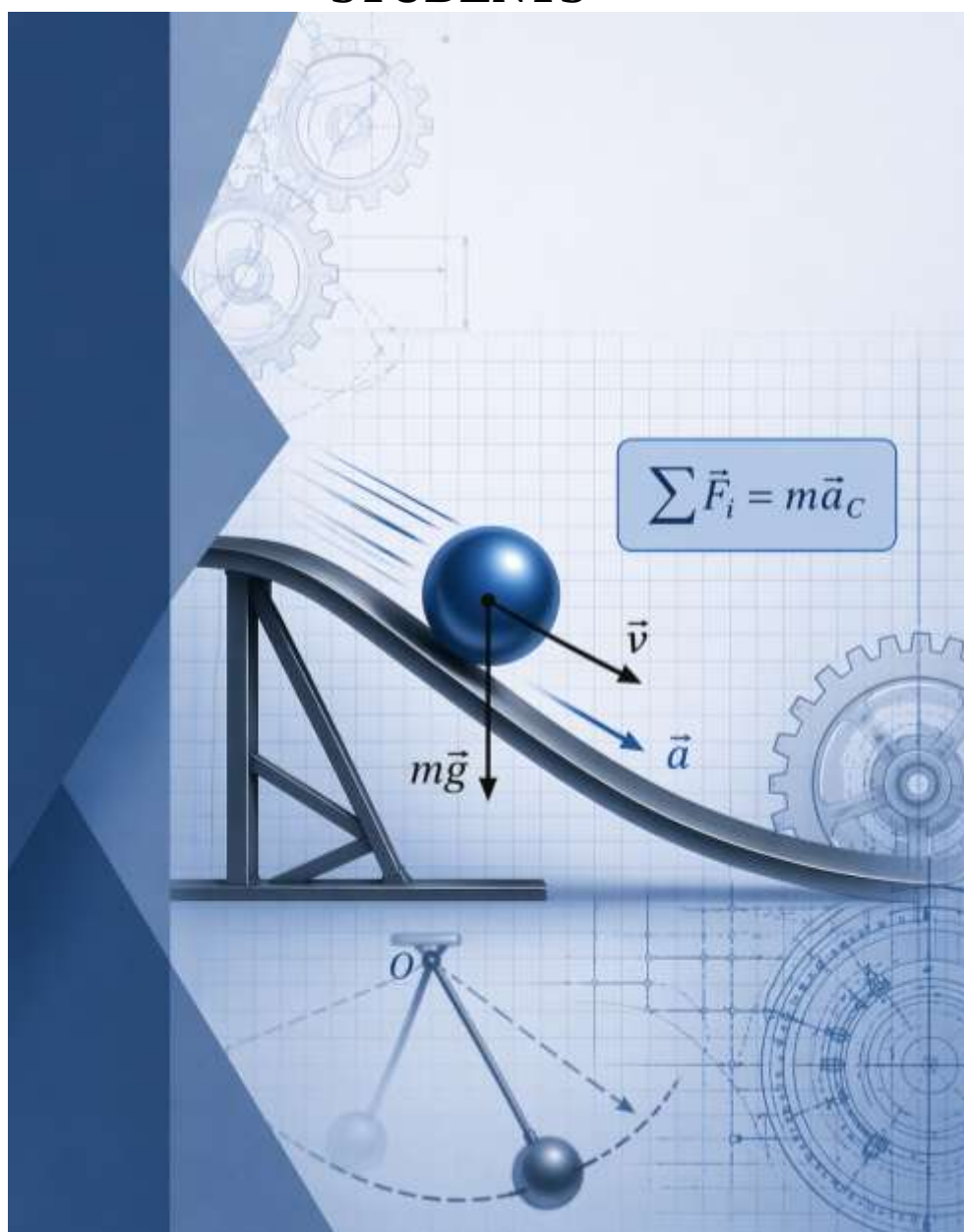


**THEORETICAL MECHANICS.
DYNAMICS
INDEPENDENT AND INDIVIDUAL WORK OF
STUDENTS**



Ministry of Education and Science of Ukraine
Vinnytsia National Technical University

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The textbook provides basic theoretical information on the discipline «Theoretical Mechanics», namely the section «Dynamics». Examples of using theoretical material to solve problems are considered. Each section includes tasks for independent work by students (sets of tasks). Each task is illustrated with a detailed calculation example with explanations when solving the task.

The manual is intended for both Ukrainian-speaking students and foreign applicants for the following specialties: G9 – «Applied Mechanics», G8 – «Materials Science», G11.03 – «Mechanical Engineering», J8 – «Motor Transport», G19 – «Construction and Civil Engineering», G4.02 – «Energy Production (Thermal Power Engineering)», G3 – «Electrical Engineering», G4.03 – «Energy Production (Renewable Energy Sources and Hydropower)» of higher education institutions studying theoretical mechanics.

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INTRODUCTION

The subject of dynamics.

A historical overview of the development of dynamics

Dynamics is the branch of mechanics that studies the motion of material bodies depending on the forces acting on them.

Dynamics is the most general section of mechanics, which is of particular importance for solving many practical problems in various fields of engineering.

The founder of dynamics is the great Italian scientist Galileo (1564 – 1642).

He was the first to introduce the concepts of velocity and acceleration of a moving point in non-uniform rectilinear motion into mechanics and established the laws of falling bodies in a vacuum. Galileo formulated the first law of dynamics - the law of inertia. He established that the motion of a body thrown at an angle to the horizon in a vacuum occurs along a parabola.

The Dutch scientist Huygens (1629 – 1695) introduced the concept of moment of inertia, created the theory of the pendulum, and invented the clock. Generalizing the concept of acceleration to the case of curvilinear motion of a point, Huygens established the concept of centrifugal force.

The work started by Galileo on the creation of dynamics was completed by the great English scientist Newton (1643–1727), who in his famous work *Philosophiae naturalis principia mathematica* (1687) formulated the basic laws of classical mechanics and, based on these laws, gave a systematic exposition of dynamics. Newton discovered the law of universal gravitation.

Of particular importance was the law of equality of action and reaction established by Newton, which allowed us to move from the dynamics of a material point to the dynamics of a mechanical system.

Developing the Descartes' (1596–1650) idea of the conservation of momentum, Newton established that the change in momentum of a mechanical system is determined only by external forces.

The application of the laws of classical mechanics, created by Galileo and Newton, as was shown by the discoveries made at the end of the 19th and the first quarter of the 20th centuries, is limited. These laws do not agree with experience in studying the motion of bodies whose speed is of the same order as the speed of light. The new relativistic mechanics (theory of relativity), created at the beginning of the 20th century. by the German physicist Albert Einstein (1879 – 1955), radically changed the idea of mechanics about space, time, mass and energy.

However, the results obtained based on the laws of classical and relativistic mechanics for bodies whose speed is less than the speed of light practically coincide..

In the light of the theory of relativity, the classical mechanics of Galileo and Newton acquired the character of its special case and retains its significance to the present day, being the scientific and theoretical basis of most branches of technology.

Based on the laws of Galileo and Newton, theorems were later proved and the principles of mechanics were established, which constitute the content of the modern course of theoretical mechanics.

The theorem on the change of kinetic energy, or, as it was previously called, the theorem of living forces, was formulated by I. Bernoulli (1667–1748) and D. Bernoulli (1700–1782). The theorem on the change of angular momentum was established almost simultaneously by Euler and D. Bernoulli in 1746.

In 1716, Ya. Herman (1678–1733) established a principle of mechanics that provides a general method by which the equations of dynamics are given the form of the equations of statics, which was called the method of kinetostatics.

In 1737, Euler generalized this principle and applied it to the study of vibrations of flexible bodies. In 1743, d'Alembert expressed a principle that became the basis for the construction of the mechanics of systems subject to constraints. D'Alembert's principle allowed to extend the application of the Hermann-Euler principle to the case of complex systems consisting of a significant number of interconnected bodies. Lagrange (1736–1813) linked the Hermann-Euler-d'Alembert principle with the general principle of statics, the principle of possible displacements, and gave it a form convenient for practical application. The principle of possible displacements was first established by Stevin (1545–1620). Galileo supplemented Stevin's research with considerations about the inclined plane and gave the famous formulation of the golden rule of mechanics: what is gained in force is lost in speed.

I. Bernoulli, Fourier, Poisson, Ampère and Lagrange worked on the scientific proof of the principle of possible displacements. Academician M. V. Ostrogradsky generalized the principle of possible displacements and applied it to the solution of many new problems of mechanics. The differential equations of motion of a mechanical system in generalized coordinates were obtained by Lagrange and bear his name. Lagrange's equations determine the motion of a mechanical system in the most general form. Lagrange applied these equations to the study of small oscillations of the system, which are of great practical importance.

In the 19th and 20th centuries, the work of scientists such as O. Lyapunov, M. Zhukovsky, S. Chaplygin, I. Meshchersky, K. Tsiolkovsky, O. Krylov, etc. was of great importance for the development of dynamics.

O. Lyapunov (1857–1918) is the developer of the modern theory of stability of motion. He also studies the stability of equilibrium forms of a rotating fluid.

M. Zhukovsky (1847–1921) is the founder of one of the most important areas of mechanics – aerodynamics. In addition, he wrote a large number of outstanding works on hydromechanics, hydraulics and solid body dynamics. M. Zhukovsky's work "On attached vortices" served as the theoretical basis for determining the lifting force of an airplane wing.

Academician S. Chaplygin (1869–1942) – a student of M. Zhukovsky also played a major role in the development of aviation. He introduced generalized equations of motion, in which boundary conditions are imposed not only on the positions of points, but also on their velocities. The theory of unsteady motion of an aircraft wing and high-speed aerodynamics created by S. Chaplygin are the foundation of aircraft calculations.

I. Meshchersky (1859–1935) is the author of a well-known collection of problems in theoretical mechanics. In his work “Dynamics of a Point of Variable Mass” (1897), he opened a new branch of mechanics – the mechanics of bodies of variable mass, one of the sections of which is the theory of motion of jet vehicles.

The creation of the foundations of the calculation of jet propulsion belongs to the outstanding Russian scientist and inventor K. Tsiolkovsky (1857–1935), who developed the design of the first space rocket.

The works of I. Meshchersky and K. Tsiolkovsky formed the basis of the theory of motion of modern multi-stage rockets, which allowed the launch of artificial satellites of the Earth and spacecraft-satellites.

The works of Academician O. Krylov (1863–1945) on ship theory, gyroscope theory, oscillation theory, equations of mathematical physics, external ballistics, and the theory of elasticity significantly influenced the development of mechanics. The above-mentioned scientists and inventors in the field of mechanics, who based their works on a close connection between theory and practice, made a great contribution to the treasury of world science.

1 BASIC CONCEPTS OF DYNAMICS. DYNAMICS OF A POINT

1.1 Introduction to the course

The two previous sections of the course of mechanics – statics and kinematics – are essentially little connected with each other. Each of them corresponds to its own separate range of concepts, problems and methods of their solution. Statics considers equilibrium problems, as well as problems of equivalent transformations of systems of forces; with such transformations, the question of what motion of the body is caused by the applied forces does not even arise. Kinematics studies motion “in itself” without connection with the forces under the action of which it occurs.

Dynamics is the main section of theoretical mechanics, which summarizes the provisions and conclusions obtained in statics and kinematics; that is, dynamics studies the mechanical motion of material objects that occurs under the action of forces applied to these objects.

It is in dynamics that two main problems of mechanics are posed and solved:

- according to the known law of motion of a material object, it is necessary to determine the forces that cause this motion (the first or direct problem);
- given the known forces acting on a material object, it is necessary to find the law of its motion (second or inverse problem).

Usually, dynamics, depending on the specific concept of a material object, is divided into three parts: the dynamics of a material point, the dynamics of a system of material points, and the dynamics of a rigid body.

1.2 Basic concepts and laws of dynamics

Dynamics is based on the laws of I. Newton, set out in the "Mathematical Principles of Natural Philosophy". When formulating the basic laws of dynamics, such concepts as **force, inertia, mass, material point** are used.

Force is a measure of the mechanical manifestation of a physical effect on material bodies. Force, as a vector quantity, is subject to all the laws of vector calculus.

The main types of forces: gravitational force ($\vec{P} = m \cdot \vec{g}$, also $m = P/g$); friction force ($\vec{F} = f \cdot \vec{N}$); elastic force ($\vec{F} = c \cdot x$, where c – stiffness coefficient [N/m]); viscous friction force ($\vec{R} = \mu \cdot \vec{V}$), aerodynamic (hydrodynamic) drag forces.

The unit of measurement for force is **N** (Newton) – it is the force that gives a mass of 1 kilogram an acceleration of 1m/s^2 ($1\text{H} = 1 \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$). An independent primary concept in theoretical mechanics is **mass**. Mass is understood as the inertia of a body. On the other hand, mass can be defined as the amount of matter in a body that is proportional to its weight ("gravitational mass"). In theoretical mechanics, it is assumed that mass does not change over time, its magnitude does not depend on either the velocity of a point or its position in space.

Material point (*particle*) – this is a material body (a body that has mass) which dimensions can be neglected when studying its motion.

Laws of dynamics

Newton's First Law (Law of Inertia)

An isolated material point maintains a state of rest or uniform and rectilinear motion until the influence of other forces removes it from this state.

Newton's Second Law (the fundamental law of dynamics)

The rate of change of momentum of a material point is equal to the force acting on this point. The acceleration of a material point is proportional to the force applied to it and has the same direction as it:

$$m \cdot \vec{a} = \vec{F}.$$

Newton's Third Law (law of equality of action and reaction)

The forces of interaction between two material points or two bodies (action and reaction) are equal in magnitude, directed in opposite directions, and have a common line of action.

Newton's Fourth Law (Superposition Principle)

The acceleration of a material point that occurs under the simultaneous action of several forces is equal to the vector sum of the accelerations that the individual forces exert on the point.

1.3 Differential equations of motion of a free material point

The differential equations of motion of a free material point are obtained by projecting the vector equation $m \cdot \bar{a} = \bar{F}$ on the coordinate axes:

a) in the Cartesian coordinate system (Fig. 1.1):

$$m \cdot a_x = \sum_{i=1}^n F_{ix} ; m \cdot a_y = \sum_{i=1}^n F_{iy} ; m \cdot a_z = \sum_{i=1}^n F_{iz} ,$$

або
$$m\ddot{x} = \sum_{i=1}^n F_{ix} ; m\ddot{y} = \sum_{i=1}^n F_{iy} ; m\ddot{z} = \sum_{i=1}^n F_{iz} ,$$

where m – mass of the point,

$\ddot{x} = a_x ; \ddot{y} = a_y ; \ddot{z} = a_z$ – projection of the acceleration of a point on the axis,

$\sum_{i=1}^n F_{ix}, \sum_{i=1}^n F_{iy}, \sum_{i=1}^n F_{iz}$ – algebraic sums of projections onto coordinate axes of forces

acting on a point.

These equations are called dynamic equations of motion of a material point in coordinate form.

If a point moves in a plane, then its motion is described by the first two equations, and if along a straight line, then only by one of them (in this case, the axis should be directed along the motion of the point).

б) in the natural coordinate system in natural form (tangent, normal and binormal) (Fig. 1.1):

$$\begin{aligned} \frac{m \cdot dV_\tau}{dt} &= \sum_{i=1}^n F_{i\tau} ; \\ m \frac{V^2}{\rho} &= \sum_{i=1}^n F_{in} ; \\ 0 &= \sum_{i=1}^n F_{i\theta} , \end{aligned}$$

where V_τ – projection of velocity onto tangent τ ;

ρ – radius of curvature of the trajectory at a given point;

$\sum_{i=1}^n F_{i\tau}, \sum_{i=1}^n F_{in}, \sum_{i=1}^n F_{i\theta}$ – algebraic sums of the projections of all forces acting on a

point onto the natural axes τ, n, θ .

These equations are called dynamic equations of motion of a point in natural form or in **Euler** form.

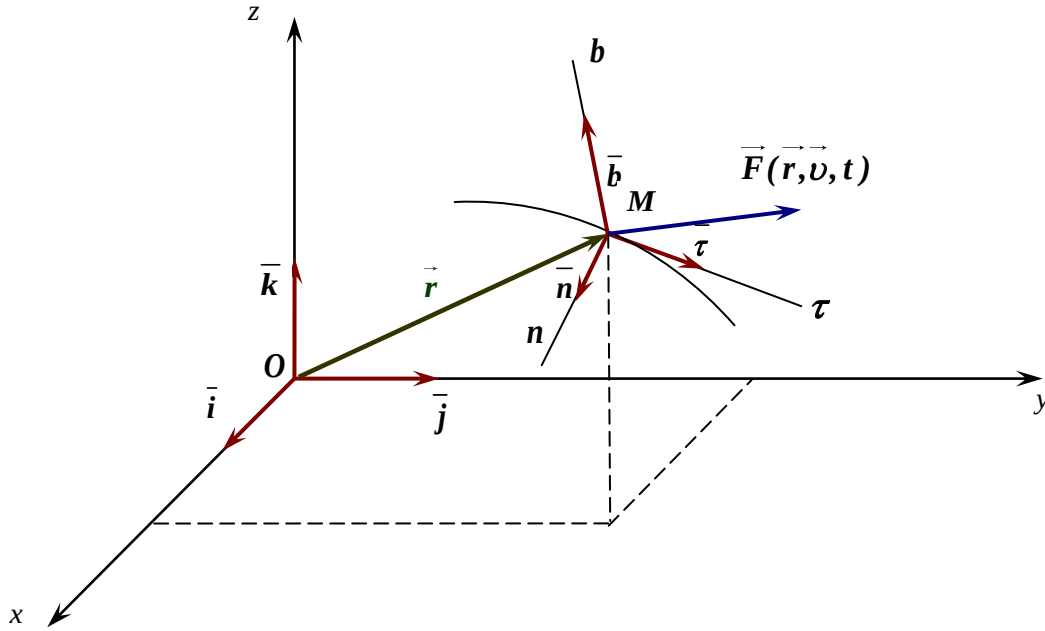


Figure 1.1

This system is used if the motion of a point is not free, when, due to the presence of constraints, the point moves along a known trajectory or surface.

The first equation is differential if the projections of forces onto the tangent to the trajectory of motion are known. The other two equations allow us to determine the reactions of the constraints.

The differential equations of motion of a material point allow us to solve two main problems of dynamics.

1.4 Two main tasks of dynamics

The first or direct main task of dynamics: knowing the law of motion of a point and its mass, determine the forces that cause this motion.

If the motion of a material point with mass m is given by the coordinate method

$x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$, then

1. Differentiating these relations twice with respect to time, we obtain the projections of accelerations onto the coordinate axes:

$$a_x = \ddot{x} = \frac{d^2x}{dt^2}; \quad a_y = \ddot{y} = \frac{d^2y}{dt^2}; \quad a_z = \ddot{z} = \frac{d^2z}{dt^2}.$$

2. Using the dynamic equations of motion of a material point in coordinate form, we determine the force projections:

$$F_x = m \frac{d^2x}{dt^2}; \quad F_y = m \frac{d^2y}{dt^2}; \quad F_z = m \frac{d^2z}{dt^2}.$$

3. We find the modulus of the resultant force $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$.

4. The direction of the resultant force is determined by the directed cosines

$$\cos(\bar{F}, OX) = \frac{F_x}{F}, \quad \cos(\bar{F}, OY) = \frac{F_y}{F}, \quad \cos(\bar{F}, OZ) = \frac{F_z}{F}.$$

Example 1. The body of a tram car together with a load $P_1 = 100 \text{ kH}$ performs vertical oscillations according to the law $x_1 = 0,02 \sin 10t \text{ m}$ on a trolley with wheels when moving on springs (Fig. 1.2). Determine the largest and smallest pressure of the car on the rails 3 of a horizontal straight section of the track, if the trolley with wheels weighs $P_2 = 10 \text{ kH}$.

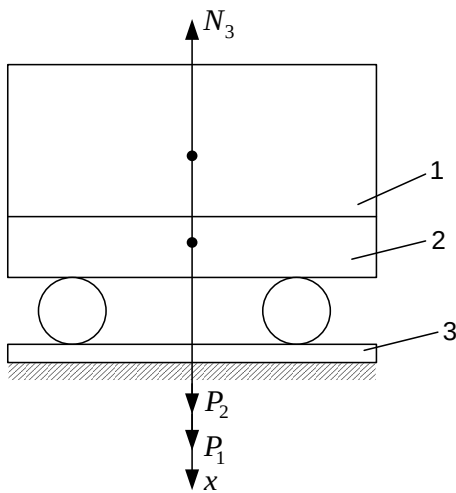
The solution:

Let us consider the motion of the body. This motion is rectilinear and passes along the x-axis. Therefore, from the three equations of motion, we will use one, which in this case takes the form

$$m_1 \ddot{x}_1 = \sum_{i=1}^n F_{ix} = P_1 + P_2 - N_3,$$

where P_1 i P_2 – known from the conditions of the example of the weight of the body and the trolley with wheels;

N_3 – reactive force from the connections, rails.



$$\text{From here } N_3 = P_1 + P_2 - m_1 \ddot{x}_1.$$

From the example condition we have:

$x_1 = 0,02 \sin 10t \text{ m}$ – the law of motion of a body on springs.

$$\text{Тоді } \dot{x}_1 = 0,2 \cos 10t,$$

$\ddot{x}_1 = -2 \sin 10t$ – projection of body acceleration in the direction of the axis x

$$(\ddot{x}_1)_{\max} = 2 \text{ m/s}^2;$$

$$(\ddot{x}_1)_{\min} = -2 \text{ m/s}^2.$$

Figure 1.2

$$\text{From here } (N_3)_{\max} = 100 + 10 + m_1 (\ddot{x}_1)_{\min} = 100 + 10 + \frac{100}{10} \cdot 2 = 130 \text{ kN};$$

$$(N_3)_{\min} = 100 + 10 - m_1 (\ddot{x}_1)_{\max} = 100 + 10 - \frac{100}{10} \cdot 2 = 90 \text{ kN}.$$

$$\text{Answer: } (N_3)_{\max} = 130 \text{ kN},$$

$$(N_3)_{\min} = 90 \text{ kN}.$$

The second or inverse problem of dynamics: determine the kinematic equations of motion of a point, if its mass m , the force $\bar{F}$ applied to it, and the initial conditions of motion are known.

The solution of the second dynamics problem is reduced to integrating the differential equations of motion of a material point.

1. Compilation of dynamic equations of motion of a material point according to the conditions of the problem (first, the projections of the forces F on the coordinate axis are found (F_x, F_y, F_z)).

2. Integrating the resulting system of differential equations.

3. Determining the values of the constants of integration. The constants of integration are determined from the initial conditions. The **initial** conditions are the quantities that determine the position of the point and the projection of the velocity vector at the initial moment of time. ($t = t_0$).

4. Finding the law of motion: $x = x(t)$; $y = y(t)$; $z = z(t)$.

Example 2. We have the motion of a material point thrown at an angle to the horizon.

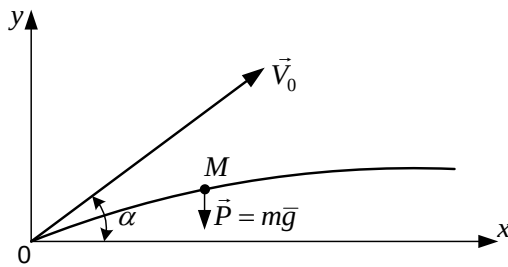


Figure 1.3

Determine the motion of a point M with mass m , which is thrown with an initial velocity V_0 at an angle to the horizon. Neglect air resistance (Fig. 1.3). Determine the trajectory of the point.

The solution

The origin of coordinates is shown at the initial position of the point. Let us depict a moving material point M at an arbitrary point of the trajectory and show the gravitational force acting on it $\vec{P} = m \cdot \vec{g}$.

At the initial moment of time ($t = 0$) The point was at the origin of coordinates, so when $t = 0$, $x_0 = 0$; $y_0 = 0$.

Projections of the initial velocity on the coordinate axis:

When $t = 0$, $\dot{x}_0 = V_0 \cdot \cos \alpha$, $\dot{y}_0 = V_0 \cdot \sin \alpha$.

To determine the dependence of the coordinates x, y of a point on time, we use the differential equations of motion of a point:

$$m\ddot{x} = \sum_{i=1}^n F_{ix} = 0,$$

$$m\ddot{y} = \sum_{i=1}^n F_{iy} = -mg, \text{ or after shortening to } m$$

$$\ddot{x} = 0,$$

$$\ddot{y} = -g.$$

Integrating these equations, we get

$$\dot{x} = C_1,$$

$$\dot{y} = -gt + C_2.$$

We find the integration constants from the initial conditions:

when $t=0$,

$$\dot{x} = V_0 \cos \alpha = C_1,$$

$$\dot{y} = V_0 \sin \alpha = C_2.$$

So

$$\dot{x} = V_0 \cos \alpha,$$

$$\dot{y} = -gt + V_0 \sin \alpha.$$

Hence, after integration, we obtain

$$x = V_0 \cos \alpha \cdot t + C_3,$$

$$y = V_0 \sin \alpha \cdot t + C_4 - \frac{gt^2}{2}.$$

Constants of integration C_3 i C_4 we find from the initial conditions of motion.

When

$t = 0$, $x_0 = 0$; $y_0 = 0$, therefore $C_3 = C_4 = 0$.

Then the law of motion of point M will be of the form $x = V_0 \cdot t \cdot \cos \alpha$,

$$y = V_0 \cdot t \cdot \sin \alpha - \frac{gt^2}{2}.$$

These expressions are the equations of the trajectory in parametric form. By removing time from these equations, we find the trajectory of the point M :

$$y = x \cdot \operatorname{tg} \alpha - \frac{gx^2}{2V_0^2 \cos^2 \alpha}.$$

The resulting trajectory is a parabola that belongs to the plane XOY .

The method of solving the second basic problem of dynamics is used when solving problems about rectilinear oscillations of a material point.

The axiom of liberation from connections allows the problem of the motion of a non-free material point to be considered as the motion of a free material point, if the action of the connections is replaced by the corresponding forces - the reactions of the connections.

Self-awareness questions

1. What does the section of theoretical mechanics – dynamics – study?
2. What laws underlie dynamics?
3. What is called a material point?
4. What is force?
5. How are the differential equations of motion of a free material point obtained?
6. What is the essence of the first and second problems of dynamics?

2 STRAIGHT LINEAR MOVEMENT OF A MATERIAL POINT

This section considers the rectilinear motion of a material point under the action of a force that depends on: time, position of the point, velocity of the point; and also presents the features of integration of differential equations.

In order for a material point to move in a straight line, it is necessary and sufficient that the force acting on it is always parallel to the initial velocity of the point. If the initial velocity is zero, then the movement will occur in a straight line, parallel to the direction of the force. If the force acting on the material point is not equal to the initial velocity of the point, then the movement occurs in the field of gravitational forces.

Let the material point move in a straight line (for example OX), then

$$y = z = 0, \quad \dot{z} = \ddot{z} = \dot{y} = \ddot{y} = 0.$$

$$\text{Accordingly, the equalities will hold } F_y = 0, \quad F_z = 0 \quad (2.1)$$

or

$$\left. \begin{aligned} m\ddot{y} &= F_y = 0 \\ m\ddot{z} &= F_z = 0 \end{aligned} \right\} - \text{necessary conditions.}$$

$$\text{So, the only equation is: } m\ddot{x} = F_x(t, x, \dot{x}) \quad (2.2)$$

Let us consider the features of integrating equation (2.2) if separate forces act on a material point $F_x(t)$, $F_x(x)$, $F_x(\dot{x})$.

2.1 Rectilinear motion of a material point under the action of a force that depends only on time. Features of integration of differential equations

The differential equation of motion in this case will have the form:

$$m\ddot{x} = F_x(t),$$

From which

$$\frac{d\dot{x}}{dt} = \frac{1}{m} F_x(t).$$

$$\text{Integrating, we get } \dot{x} = \frac{1}{m} \int F_x(t) dt + C_1,$$

In which $\int F_x(t) dt$ is understood as the original function.

Integrating further, we get

$$x = \frac{1}{m} \int \left[\int F_x(t) dt \right] dt + C_1 t + C_2.$$

Example. A material point with mass $m = 3\text{kg}$ moves in a straight line under the action of a force that coincides with the direction of motion, is measured in newtons and is given analytically by the dependence

$$F = -15e^{-3t}.$$

Find the kinematic equations of motion of a point if at the initial moment of motion the coordinate and velocity are respectively equal:

$$S_0 = 3 \quad ; \quad V_0 = 6.$$

Distances are given in meters, time in seconds..

The solution

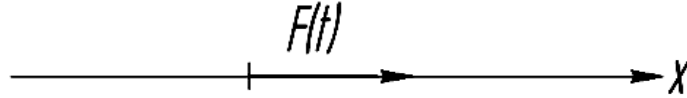


Figure 2.1

$$m\ddot{x} = F_x(t). \quad (2.3)$$

$$F = -15e^{-3t}. \quad (2.4)$$

$$m\ddot{x} = -15e^{-3t}. \quad (2.5)$$

$$\ddot{x} = \frac{-15}{m}e^{-3t}. \quad (2.6)$$

$$\ddot{x} = -5e^{-3t}. \quad (2.7)$$

We integrate the differential equation (2.7) twice:

$$\dot{x} = \frac{5}{3}e^{-3t} + C_1, \quad (2.8)$$

$$x = -\frac{5}{9}e^{-3t} + C_1t + C_2. \quad (2.9)$$

Initial conditions:

when $t = 0$; $x = S_0 = 3$; $\dot{x} = v_0 = 6$,

then $6 = \frac{5}{3} + C_1$; $C_1 = \frac{13}{3}$; $3 = -\frac{5}{9} + C_2$; $C_2 = \frac{32}{9}$,

so, $\dot{x} = \frac{5}{3}e^{-3t} + \frac{13}{3}$; $x = -\frac{5}{9}e^{-3t} + \frac{13}{3}t + \frac{32}{9}$.

2.2 Rectilinear motion of a material point under the action of a force that depends only on the position of the point. Features of integration of differential equations

In this case, the differential equation of motion will be:

$$m\ddot{x} = F_x(x).$$

Let us introduce $v = \dot{x}$, we obtain

$$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx},$$

accordingly, the differential equation will have the form

$$v dv = \frac{1}{m} F_x(x) dx$$

After integration, we find:

$$\frac{v^2}{2} = \frac{1}{m} \int F_x(x) dx + C_1.$$

From which

$$v = \sqrt{\frac{2}{m} \int F_x(x) dx + C_1} ,$$

or, moving on to the velocity projection

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m} \int F_x(x) dx + C_1} ,$$

$$dt = \frac{dx}{\sqrt{\frac{2}{m} \int F_x(x) dx + C_1}} .$$

Integrating this equation, we have

$$t = \pm \int \frac{dx}{\sqrt{\frac{2}{m} \int F_x(x) dx + C_1}} + C_2 .$$

Note. The sign before the root is chosen depending on the direction of movement of the point in the time interval under consideration. If $X_0 > 0$, then the sign "+" is taken, in the opposite case – « – ». The sign can change to the opposite at the moment when $\dot{x} = 0$.

Let us introduce the notation

$$\Phi(x, C_1, C_2) = \pm \int \frac{dx}{\sqrt{\frac{2}{m} \int F_x(x) dx + C_1}} + C_2 .$$

From which $t = \Phi(x, C_1, C_2)$.

Solving this equation with respect to x , we find the dependence of x on time and the constants of integration: $x = \varphi(t, C_1, C_2)$.

2.3 Rectilinear motion of a material point under the action of a force that depends only on the velocity of the point. Features of integration of differential equations

Differential equation of motion:

$$m\ddot{x} = F_x(\dot{x}) .$$

Let's introduce a replacement $v = \dot{x}$, then

$$m \frac{dv}{dt} = F_x(v) .$$

Let's convert to the form

$$\frac{dv}{F_x(v)} = \frac{1}{m} dt .$$

Integrating this equation, we get:

$$\int \frac{dv}{F_x(v)} + C_1 = \frac{1}{m} t .$$

$$\text{Let } \Phi(v, C_1) = m \left[\int \frac{dv}{F_x(v)} + C_1 \right] .$$

Then $t = \Phi(v, C_1)$ or $t = \Phi(\dot{x}, C_1)$.

Solving this equation as for $\dot{x}$, we will have

$$\frac{dx}{dt} = f(t, C_1),$$

From which $x = \int f(t, C_1) dt + C_2$.

If the equation $t = \Phi(\dot{x}, C_1)$ cannot be solved as for $\dot{x}$, then by introducing the transformation $\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx} v$, we rewrite the differential equation $m\ddot{x} = F_x(\dot{x})$ as follows $m \frac{dv}{dx} v = F_x(v)$ or $m \frac{v dv}{F_x(v)} = dx$, from which

$$x = m \int \frac{v dv}{F_x(v)} + C_1.$$

From this equation we find $v = \dot{x}$:

$$\frac{dx}{dt} = \varphi(x, C'_1).$$

Integrating a second time

$$t = \int \frac{dx}{\varphi(x, C'_1)} + C'_2.$$

since x is defined as a function of time t .

Example. A bullet that has penetrated a board of thickness h changes its speed from v_0 to v_1 . The resistance force is proportional to the speed ($R = k\dot{x}$).

We need to determine: 1) the mass of the bullet.

2) time of movement of the bullet in the board.

The solution

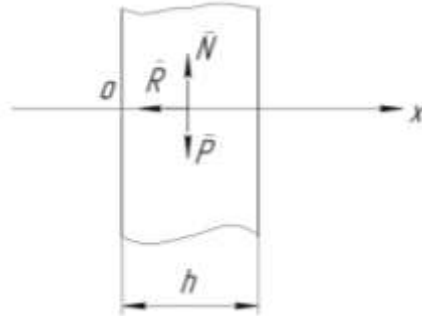


Figure 2.2

$$m\ddot{x} = \Sigma F_x(\dot{x}). \quad (2.10)$$

$$m\ddot{x} = -R. \quad (2.11)$$

$$m\ddot{x} = -k\dot{x}. \quad (2.12)$$

$$\ddot{x} = -\frac{k}{m} \dot{x}. \quad (2.13)$$

Since $\ddot{x} = \frac{d\dot{x}}{dt}$, then

$$\frac{d\dot{x}}{dt} = -\frac{k}{m} \dot{x}. \quad (2.14)$$

Separating variables:

$$\frac{d\dot{x}}{\dot{x}} = -\frac{k}{m} dt. \quad (2.15)$$

Integrating the equation (2.15):

$$\ln \dot{x} = -\frac{k}{m}t + C_1. \quad (2.16)$$

Initial conditions:

When $t = 0$; $\dot{x} = v_0$.

$\ln v_0 = C_1$, then ,

$$\ln \dot{x} = -\frac{k}{m}t + \ln v_0. \quad (2.17)$$

$$\ln \dot{x} - \ln v_0 = -\frac{k}{m}t. \quad (2.18)$$

$$\ln \frac{\dot{x}}{v_0} = -\frac{k}{m}t. \quad (2.19)$$

$$\dot{x} = v_0 e^{-\frac{k}{m}t}. \quad (2.20)$$

Integrating the equation (2.20):

$$x = -\frac{m}{k}v_0 e^{-\frac{k}{m}t} + C_2. \quad (2.21)$$

Initial conditions:

When $t = 0$; $x = 0$.

$0 = -\frac{m}{k}v_0 + C_2$; $C_2 = \frac{m}{k}v_0$, then

$$x = -\frac{m}{k}v_0 e^{-\frac{k}{m}t} + \frac{m}{k}v_0 = \frac{m}{k}v_0 \left(1 - e^{-\frac{k}{m}t}\right). \quad (2.22)$$

Initial conditions:

When $t = \tau$; $\dot{x}_\tau = v_1$; $x_\tau = h$.

$$v_1 = v_0 e^{-\frac{k}{m}\tau}. \quad (2.23)$$

$$h = \frac{m}{k}v_0 \left(1 - e^{-\frac{k}{m}\tau}\right). \quad (2.24)$$

From the equation (2.23):

$$e^{-\frac{k}{m}\tau} = \frac{v_1}{v_0}. \quad (2.25)$$

(2.25) $\rightarrow$ (2.24)

$$h = \frac{m}{k}v_0 \left(1 - \frac{v_1}{v_0}\right) = \frac{m}{k}(v_0 - v_1). \quad (2.26)$$

$$m = \frac{h \cdot k}{(v_0 - v_1)}. \quad (2.27)$$

From the equation (2.23):

$$\ln \frac{v_1}{v_0} = -\frac{k}{m}\tau. \quad (2.28)$$

$$\ln \frac{v_0}{v_1} = \frac{k}{m}\tau. \quad (2.29)$$

From the equation (2.29):

$$\tau = \frac{m}{k} \ln \frac{v_0}{v_1}. \quad (2.30)$$

(2.27) $\rightarrow$ (2.30)

$$\tau = \frac{h \cdot k}{(v_0 - v_1) \cdot k} \ln \frac{v_0}{v_1} = \frac{h}{(v_0 - v_1)} \ln \frac{v_0}{v_1}. \quad (2.31)$$

Therefore, the mass of the bullet is determined by the formula $m = \frac{h \cdot k}{(v_0 - v_1)}$, and the time of movement of the bullet in the board – $\tau = \frac{h}{(v_0 - v_1)} \ln \frac{v_0}{v_1}$.

Self-awareness questions

1. What conditions must be met for a material point to move in a straight line?
2. What is a material point?
3. What is a force?
4. What are the features of integrating differential equations in the rectilinear motion of a material point under the action of a force that depends:
 - a) on time; b) on the position of the point; c) on the speed?

2.4 Calculation-graphic and control tasks

D1 Motion of a material point under the action of constant forces

Variant 1

A body begins to move from rest downward along a smooth inclined plane that makes an angle $\alpha = 30^\circ$ with the horizon. Determine the time it will take for the body to travel a distance of 30 m.

Variant 2

A body with a mass $m = 4$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 10$ m/s. The point is acted upon by a resistance force $R = 0.8v$. Determine the velocity of the body at time $\tau = 6$ s.

Variant 3

A body with mass $m = 25$ kg starts moving from rest along a horizontal straight plane under the action of a force $F = 20t$, which is directed along the same straight line. Determine the path the body will travel in 4 s.

Variant 4

A body moves downward along a smooth inclined plane, which makes an angle $\alpha = 45^\circ$ with the horizon. At the initial moment of movement, the velocity of the body was equal to $v_0 = 10$ m/s. Determine the time it will take for the body to travel a distance of 50 m.

Variant 5

A body of mass $m = 200$ kg begins to move from rest downward along a smooth inclined plane that makes an angle $\alpha = 60^\circ$ with the horizon. Determine the time it takes for the body to travel a distance of 8 m.

Variant 6

A body with a mass $m = 3$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 2$ m/s. The point is acted upon by a resistance force $R = 0.3v$. Determine the velocity of the body at time $\tau = 3$ s.

Variant 7

A body starts to move from rest downwards along a rough inclined plane, which makes an angle $\alpha = 45^\circ$ with the horizon. Determine the path that the body will cover in 5 s if the coefficient of sliding friction $f = 0.3$.

Variant 8

A body with a mass $m = 5$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 3$ m/s. The point is acted upon by a resistance force $R = 0.5v$. Determine the path the body will travel in 3 s.

Variant 9

A body moves downward along a smooth inclined plane, which makes an angle $\alpha = 45^\circ$ with the horizon. At the initial moment of movement, the velocity of the body was equal to $v_0 = 5$ m/s. Determine in what time the body will cover a distance of 40 m.

Variant 10

A body with a mass $m = 6$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 4$ m/s. The point is acted upon by a resistance force $R = 0.6v$. Determine the velocity of the body at time $\tau = 4$ s.

Variant 11

A body with a mass $m = 15$ kg moves along a horizontal straight plane under the action of a force $F = 15t$, which is directed along the same straight line. At the initial moment of motion, the body's velocity was $v_0 = 10$ m/s. Determine the path the body will travel in 8 s.

Variant 12

A body with a mass $m = 7$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 6$ m/s. The point is subjected to a resistance force $R = 0.7v$. Determine the path the body will travel in 4 s.

Variant 13

A body with a mass $m = 100$ kg moves upwards along a smooth inclined plane, which makes an angle $\alpha = 30^\circ$ with the horizon. At the initial moment of movement the velocity of the body was equal to $v_0 = 15$ m/s. Determine the time in which the body will cover a distance of 10 m.

Variant 14

A body moves downward along a smooth inclined plane that makes an angle of $\alpha = 60^\circ$ with the horizon. At the initial moment of motion, the velocity of the body was $v_0 = 7$ m/s, and the coordinate $x_0 = 1$ m. Determine the time it will take for the body to travel a distance of 20 m.

Variant 15

A body moves downward along a rough inclined plane, which makes an angle $\alpha = 30^\circ$ with the horizon. At the initial moment of movement, the velocity of the body was $v_0 = 18$ m/s. Determine the path that the body will travel in 10 s if the coefficient of sliding friction $f = 0.2$.

Variant 16

A body of mass m starts to move from rest downwards along a smooth inclined plane making an angle $\alpha = 60^\circ$ with the horizon. Determine the time it takes for the body to travel a distance of 9 m.

Variant 17

A body with a mass $m = 8$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 7$ m/s. The point is acted upon by a resistance force $R = 0.8v$. Determine the velocity of the body at time $\tau = 5$ s.

Variant 18

A body starts to move from rest downwards along a rough inclined plane, which makes an angle $\alpha = 45^\circ$ with the horizon. Determine the path that the body will cover in 10 s, if the coefficient of sliding friction $f = 0.3$.

Variant 19

A body with a mass $m = 18$ kg moves along a horizontal straight plane under the action of a force $F = 9t$, which is directed along the same straight line. At the initial moment of motion, the body's velocity was $v_0 = 6$ m/s, and the coordinate $x_0 = 0.5$ m. Determine the path the body will travel in 15 s.

Variant 20

A body with a mass $m = 2$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 10$ m/s. The point is acted upon by a resistance force $R = 0.4v$. Determine the path the body will travel in 6 s.

Variant 21

A body with a mass $m = 80$ kg moves upwards along a smooth inclined plane, which makes an angle $\alpha = 45^\circ$ with the horizon. At the initial moment of movement, the velocity of the body was $v_0 = 4$ m/s, and the coordinate $x_0 = 0.7$ m. Determine the time in which the body will travel a distance of 5 m.

Variant 22

A body with a mass $m = 3$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 11$ m/s. The point is acted upon by a resistance force $R = 0.6v$. Determine the velocity of the body at time $\tau = 7$ s.

Variant 23

A body moves downward along a rough inclined plane, which makes an angle $\alpha = 45^\circ$ with the horizon. At the initial moment of movement, the velocity of the body was equal to $v_0 = 13$ m/s, and the coordinate $x_0 = 0.2$ m. Determine the path that the body will travel in 12 s, if the coefficient of sliding friction $f = 0.3$.

Variant 24

A body with a mass $m = 2$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 5$ m/s. The point is acted upon by a resistance force $R = 0.4v$. Determine the path the body will travel in 7 s.

Variant 25

A body begins to move from rest downward along a smooth inclined plane that makes an angle $\alpha = 30^\circ$ with the horizon. Determine the time it will take for the body to travel a distance of 20 m.

Variant 26

A body with a mass $m = 3$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 6$ m/s. The point is acted upon by a resistance force $R = 0.9v$. Determine the velocity of the body at time $\tau = 8$ s.

Variant 27

A body with a mass $m = 23$ kg starts moving from rest along a horizontal straight plane under the action of a force $F = 20t$, which is directed along the same straight line. Determine the path the body will travel in 3 s.

Variant 28

A body with a mass $m = 4$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 7$ m/s. The point is acted upon by a resistance force $R = 0.8v$. Determine the path the body will travel in 8 s.

Variant 29

A body of mass $m = 150$ kg starts to move from rest downwards along a smooth inclined plane making an angle $\alpha = 30^\circ$ with the horizon. Determine the time it takes for the body to travel a distance of 7 m.

Variant 30

A body with a mass $m = 5$ kg moves along a horizontal straight plane with an initial velocity $v_0 = 8$ m/s. The point is acted upon by a resistance force $R = 0.5v$. Determine the velocity of the body at time $\tau = 9$ s.

2.5 Example of completing the task for D1

Example 1. A load of mass m is lifted up a rough inclined plane (Fig. 2.3), which makes an angle $\alpha = 30^\circ$ with the horizon. At the initial moment, the speed of the load was equal to $v_0 = 15$ m/s. The coefficient of friction $f = 0.1$. What path will the load travel before stopping? In what time will the load travel this path?

The solution

1. To study the motion of the load, we write the differential equation of the free load in projections onto the x -axis x .

$$m\ddot{x} = \Sigma F_x \quad (2.32)$$

The load is moved under the action of the weight P , the normal component N and the friction force F_{mp} of the rough surface.

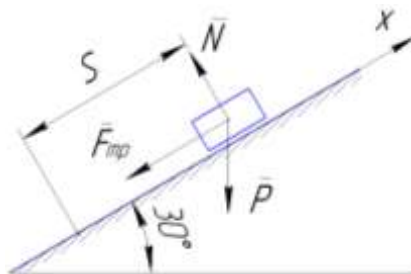


Figure 2.3

Taking into account the forces applied to the load (Fig. 2.3), equation (2.32) can be written as:

$$m\ddot{x} = -P \sin 30^0 - F_{mp} \quad (2.33)$$

Weight of cargo near the Earth's surface.

$$P = mg \quad (2.34)$$

According to the Amonton-Coulomb law, at critical equilibrium of the load, we obtain:

$$F_{mp} = fN \quad (2.35)$$

So, as the load moves along the x -axis, then:

$$N = P \cos 30^0 \quad (2.36)$$

Let's substitute the formulas (2.34), (2.35) and (2.36) into a differential equation (2.33)

$$m\ddot{x} = -mg \sin 30^0 - fmg \cos 30^0; m \neq 0 \quad (2.37)$$

$$\ddot{x} = -g(\sin 30^0 + f \cos 30^0) \quad (2.38)$$

$$\ddot{x} = -9,8 \left(\frac{1}{2} + 0,1 \frac{\sqrt{3}}{2} \right) = -5,75 \quad (2.39)$$

$$\ddot{x} = -5,75 \quad (2.40)$$

2. We integrate the differential equation (2.27) twice:

$$\dot{x} = -5,75t + C_1 \quad (2.41)$$

$$x = -5,75 \frac{t^2}{2} + C_1 t + C_2 \quad (2.42)$$

3. To determine the constants of integration in equations (2.41) and (2.42), we use the initial conditions:

Initial conditions: When $t = 0$; $\dot{x}_0 = 15\text{m/s}$; $x_0 = 0$.

So, $C_1 = \dot{x}_0$; $C_2 = x_0$. Correspondingly: $C_1 = 15$; $C_2 = 0$, then

$$\dot{x} = -5,75t + 15. \quad (2.43)$$

$$x = -5,75 \frac{t^2}{2} + 15t + 0. \quad (2.44)$$

4. For the moment τ , when the cargo stops.

When $t = \tau$; $\dot{x} = 0$; $x = S$.

Then

$$0 = -5,75\tau + 15. \quad (2.45)$$

$$S = -5,75 \frac{\tau^2}{2} + 15\tau. \quad (2.46)$$

From formula (2.45) we find τ :

$$\tau = 15 / 5,75 = 2,61 \text{ s}. \quad (2.47)$$

Let's substitute the value τ (2.47) into equation (2.46) and we get the path that the cargo will take to stop

$$S = -5,75 \frac{2,61^2}{2} + 15 \cdot 2,61 = 19,57 \text{ m}.$$

Answer: $\tau = 2,61 \text{ s}$; $S = 19,57 \text{ m}$.

Example 2. A body with a mass $m = 2 \text{ kg}$ moves along a horizontal straight plane with an initial velocity $v_0 = 8 \text{ m/s}$. The body is subjected to a resistance force $R = 0.8v$ (Fig. 2.4). Determine:

- body velocity at time instant $\tau = 5 \text{ s}$;
- the path the body will travel in 5 s.

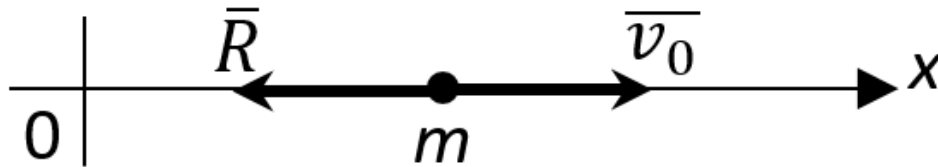


Figure 2.4

The solution

The differential equation of motion in this case will have the form:

$$m\ddot{x} = F_x(\dot{x}). \quad (2.48)$$

Let's project all the acting forces onto the axis X:

$$m\ddot{x} = -R, \quad (2.49)$$

$$R = 0,8\dot{x}, \quad (2.50)$$

(2.50) $\rightarrow$ (2.49)

$$m\ddot{x} = -0,8\dot{x}. \quad (2.51)$$

Since $m \neq 0$, then : m .

$$\ddot{x} = -\frac{0,8}{2}\dot{x} \quad (2.52)$$

$$\ddot{x} = -0,4\dot{x} \quad (2.53)$$

Considering that $\ddot{x} = \frac{d\dot{x}}{dt}$,

then

$$\frac{d\dot{x}}{dt} = -0,4\dot{x}. \quad (2.54)$$

Separating variables

$$\frac{d\dot{x}}{\dot{x}} = -0,4dt. \quad (2.55)$$

Integrate the equation (2.55)

$$\ln \dot{x} = -0,4t + C_1. \quad (2.56)$$

We use the initial conditions to determine the constant of integration C_1 .

Initial conditions:

When $t = 0$; $\dot{x} = v_0 = 8m/s$

$$\ln 8 = -0,4 \cdot 0 + C_1; \quad (2.57)$$

$$C_1 = \ln 8.$$

$$\ln \dot{x} = -0,4t + \ln 8 \quad (2.58)$$

$$\ln \dot{x} - \ln 8 = -0,4t \quad (2.59)$$

By the property of logarithms

$$\ln \frac{\dot{x}}{8} = -0,4t \quad (2.60)$$

$$\dot{x} = 8e^{-0,4t} \quad (2.61)$$

Integrate the equation (2.61)

$$x = -\frac{10}{4}8e^{-0,4t} + C_2. \quad (2.62)$$

We use the initial conditions to determine the constant of integration C_2 .

Initial conditions:

When $t = 0$; $x = 0$

$$0 = -\frac{10}{4}8e^{-0,4 \cdot 0} + C_2 \quad (2.63)$$

$$C_2 = 20.$$

Then,

$$x = -\frac{10}{4}8e^{-0,4t} + 20 \quad (2.64)$$

or

$$x = -20e^{-0,4t} + 20 = 20(1 - e^{-0,4t}). \quad (2.65)$$

Boundary conditions:

By $t = \tau = 5s$; $\dot{x} = v_\tau$; $x = S_\tau$.

Substitute the boundary conditions into the equation (2.61) and (2.65):

$$v_\tau = 8e^{-0,4\tau} = 8e^{-0,4 \cdot 5} = 1,1 \text{ (m/s)}$$

$$S_\tau = 20(1 - e^{-0,4\tau}) = 20(1 - e^{-0,4 \cdot 5}) = 17,3 \text{ (m)}$$

Answer: $v_\tau = 1,1 \text{ (m/s)}$; $S_\tau = 17,3 \text{ (m)}$.

2.6 Calculation, graphic and control tasks

D2. Motion of a material point in a plane under the action of time-dependent forces

A material point M with a mass m moves in a vertical plane xOy under the action of the weight force P and the time-dependent force F . The direction of the force F is determined by the angle α . The figures (pages 28 – 32) for options 1 to 30 depict the point M with the acting forces, where the analytical form of the dependence $F(t)$ is indicated, as well as the initial position of the point M in the coordinate system with the parameters h and l . The point M begins to move with a speed determined by the components along the axes V_{Ox} and V_{Oy} .

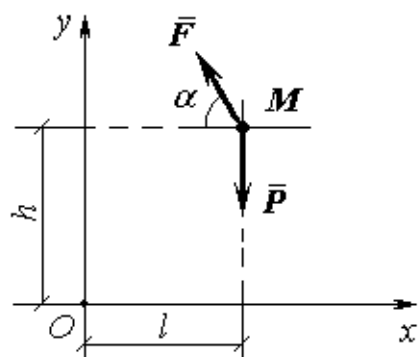
Find the kinematic equations of motion of the point M : $x(t)$, $y(t)$, $V(t)$, $a(t)$. Plot the graphs of the dependencies $V(t)$, $a(t)$ and the trajectory $y(x)$. Mark the position of the point M on the trajectory at time t_1 , construct the velocity and acceleration vectors for this moment, and depict these vectors on the trajectory drawing at the selected scale.

The values of the coefficients A , B , C , D and k are constants, they, together with other data, are given in Table D2.1 for the variants.

Table D2.1

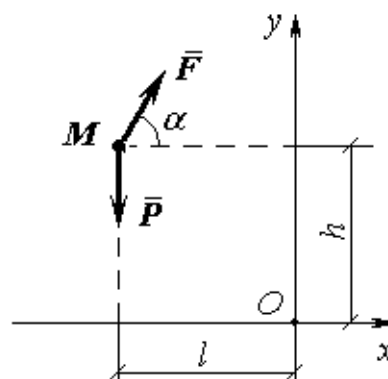
Variant	m <i>kg</i>	α <i>deg</i>	A	B	C	D	h	l	k <i>s⁻¹</i>	V_{Ox} <i>m/s</i>	V_{Oy} <i>m/s</i>	t_1 <i>s</i>
			m									
1	20	30	100	120	60	40	4	3	0.5π	5	10	1
2	15	45	60	0	25	15	0.5	2	2π	8	14	1.5
3	6	0	98	24	0	36	6	0.4	3π	6	4	1.8
4	12	60	80	36	72	0	1	5	π	0.8	2	2
5	14	0	120	90	60	42	0.6	3	0.5π	12	5	0.5
6	24	45	102	0	96	45	10	6	0.4π	3	2	0.8
7	4	0	32	8	0	16	8	2	2π	6	10	1
8	9	30	54	42	36	0	12	8	π	5	4	0.4
9	10	45	80	98	42	60	0.2	1	$\pi/3$	9	3	0.6
0	8	60	50	60	80	24	3	5	$\pi/4$	7	0	1.5

1



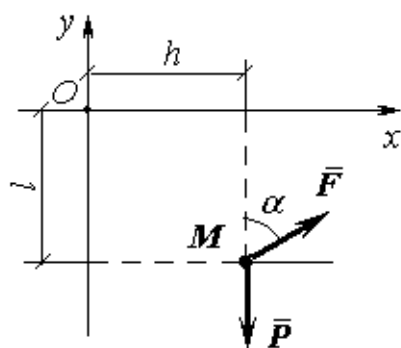
$$F = A - B \cdot t - C \cdot t^2$$

2



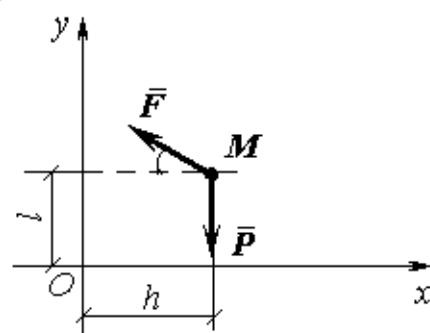
$$F = B \cdot \sin(2kt) + D$$

3



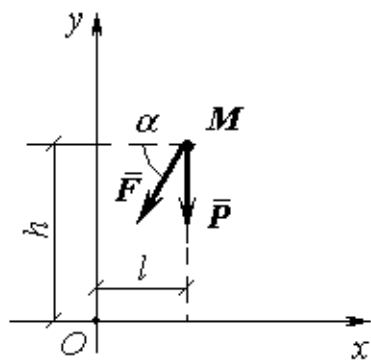
$$F = D - A \cos(kt)$$

4



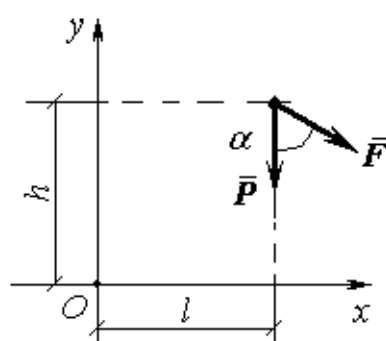
$$F = D \sqrt{3+t} + B$$

5



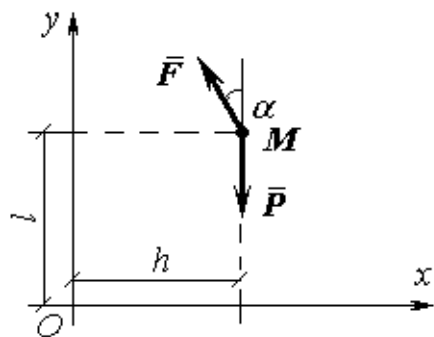
$$F = D(1 - e^{0.5t}) + B$$

6



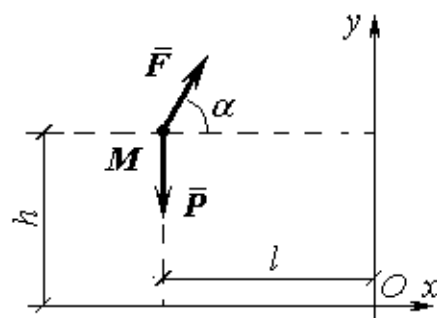
$$F = B \cdot \cos(2kt) + A$$

7



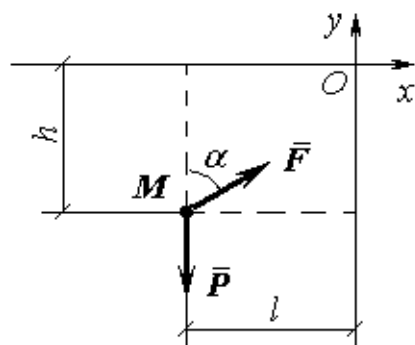
$$F = B + A \cdot t + C \cdot t^2$$

8



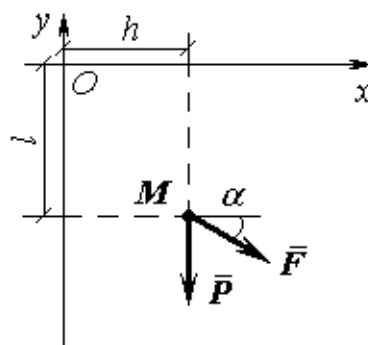
$$F = B \sin(kt) + A$$

9



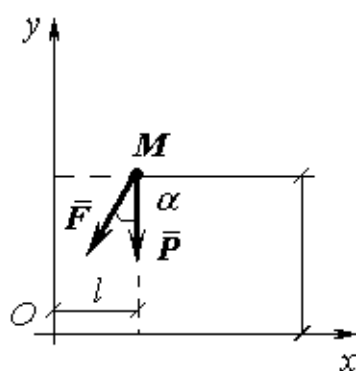
$$F = A(1 - e^{-0.5t}) + 0.5B$$

10



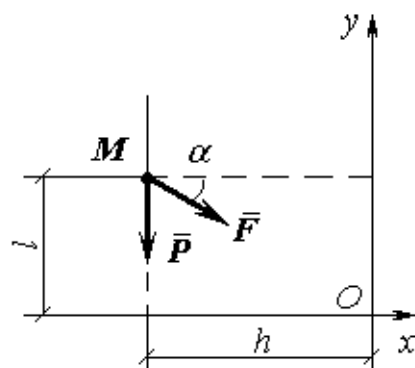
$$F = A\sqrt{5+t} + B \cdot t$$

11



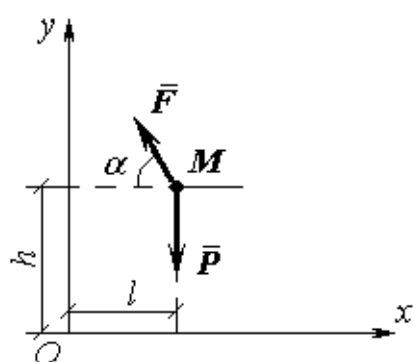
$$F = D \cos(2kt) + A$$

12



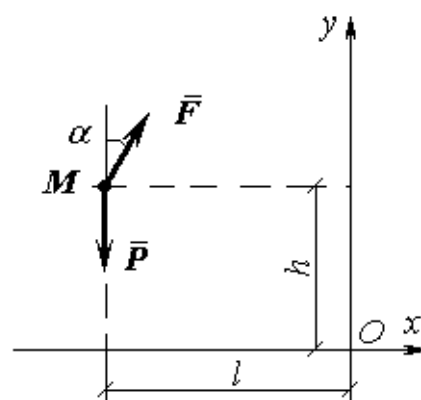
$$F = B + A(1 - e^{0.5t})$$

13



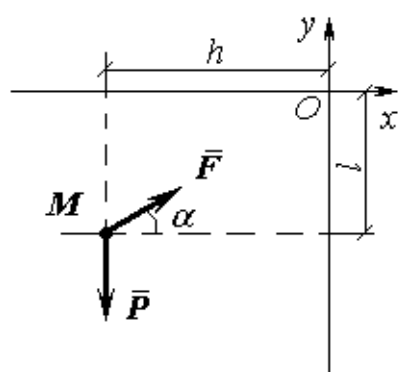
$$F = -C + A \cdot t - B \cdot t^2$$

14



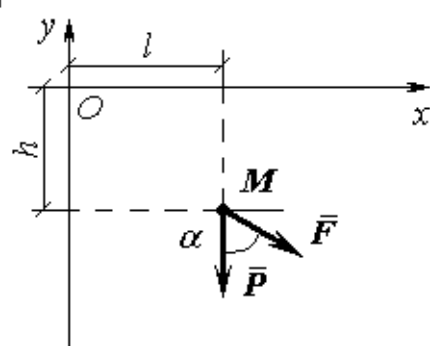
$$F = C(1 - e^{-0.5t}) + D$$

15



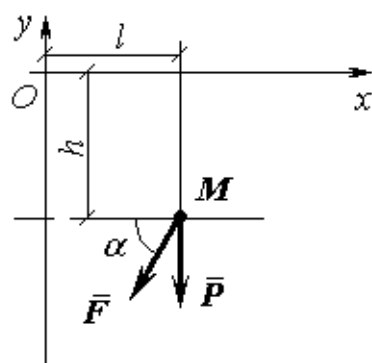
$$F = C \sin(kt) + B$$

16



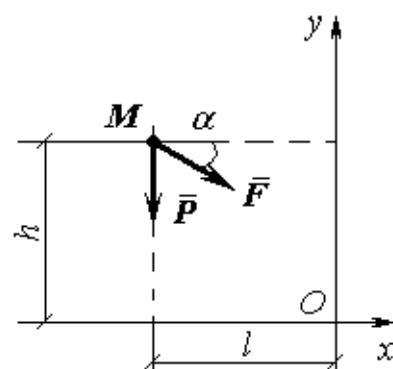
$$F = D\sqrt{2+t} + B \cdot t$$

17



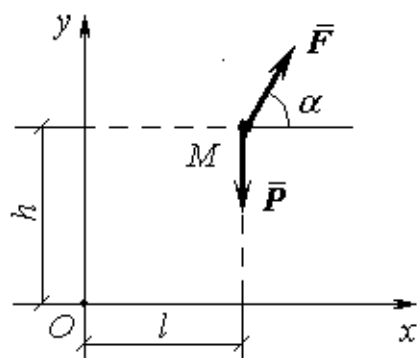
$$F = A(4 - e^{0.5t}) + D$$

18



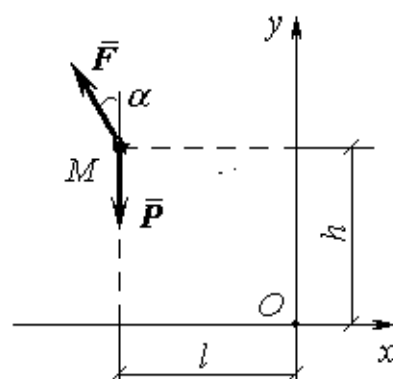
$$F = 2B \cos(2kt) + A$$

19



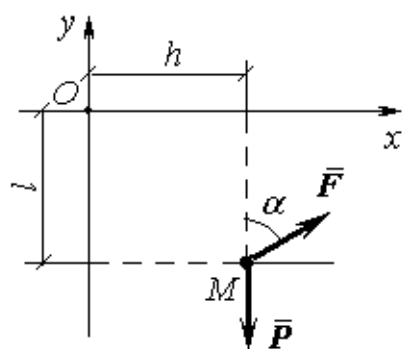
$$F = B \cdot \sin\left(\frac{k}{2}t\right) - D$$

20



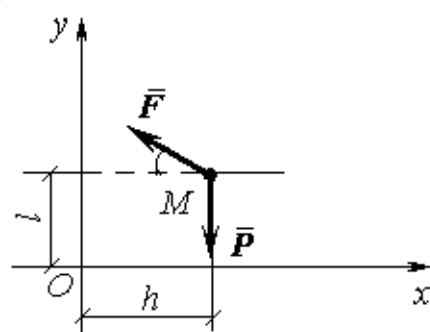
$$F = B + 2A(l - e^{0.5t})$$

21



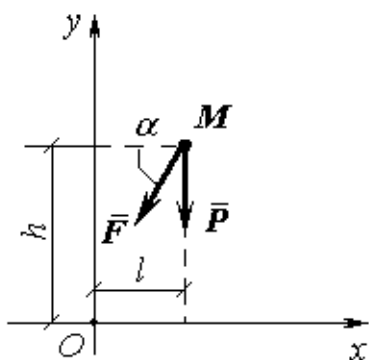
$$F = A\sqrt{10+t} + B \cdot t$$

22



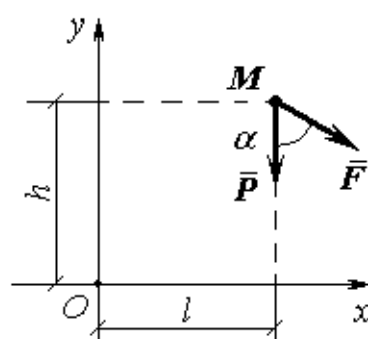
$$F = B \sin(kt) + A \cdot t$$

23



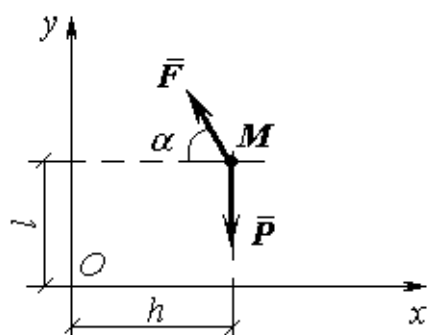
$$F = A(2 - e^{0.5t}) + B \cdot t$$

24



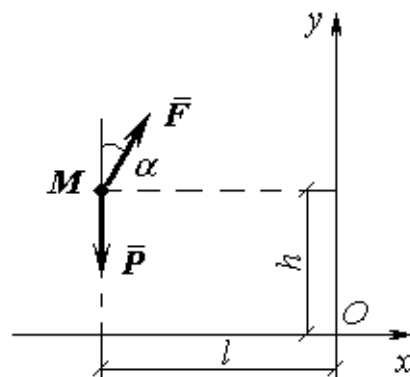
$$F = A \cdot (2 + t) + C \cdot t^2$$

(25)



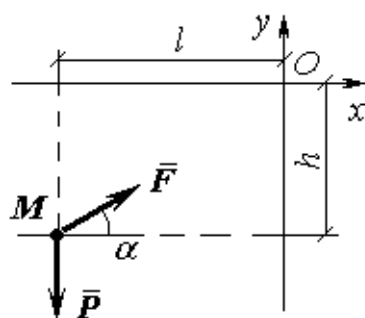
$$F = A \sin(kt) + B$$

(26)



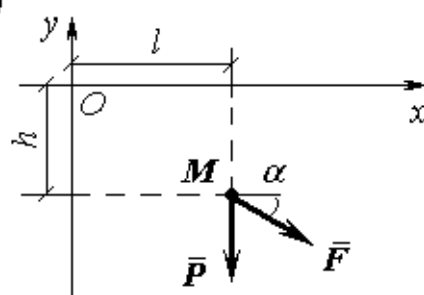
$$F = 0,5A + B \cdot t - 2A \cdot t^2$$

(27)



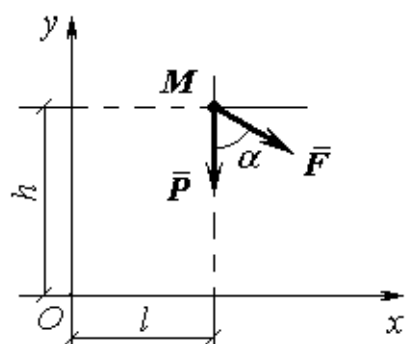
$$F = B \cos(2kt) + A$$

(28)



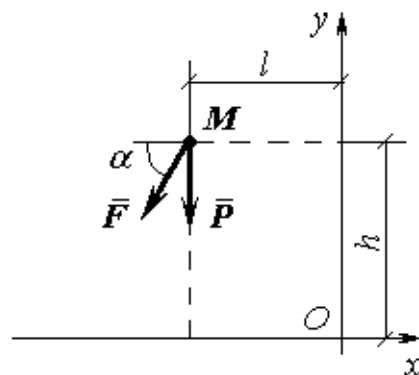
$$F = A(1 - e^{-0,5t}) + D$$

(29)



$$F = A(1 - e^{0,5t}) + B$$

(30)



$$F = A\sqrt{2+t} + B \cdot t$$

2.7 An example of a task to D2

Condition of the problem. A material point M with a mass $m = 0.5 \text{ kg}$ is under the action of the gravitational force P and the force $F = 50\sin(\pi t) + 4$, directed at an angle $\alpha = 15^\circ$ to the horizon. The position of the point M at the beginning of the motion in the vertical coordinate system xOy , fig. 2.5, is indicated by the parameters $a = 10 \text{ m}$, $b = 2 \text{ m}$.

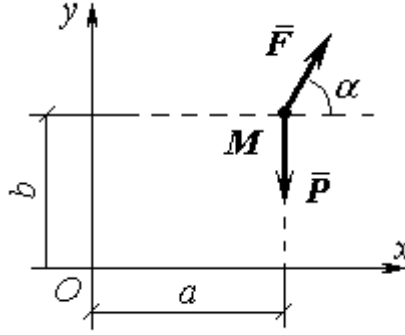


Figure 2.5

The components of the initial velocity are given by the quantities:

$$V_{ox} = 10 \text{ m/s}, V_{oy} = 2 \text{ m/s}.$$

Find the kinematic characteristics of the motion of point M : coordinates $x(t)$, $y(t)$, velocity $V(t)$, acceleration $a(t)$. Build the graphs of the dependences $V(t)$, $a(t)$ and the trajectory $y(x)$. Indicate on the trajectory the position of point M at $t = 3 \text{ s}$ with the velocity and acceleration vectors.

The solution

We compose the differential equations of motion of the point M in projections on the axis Ox and Oy .

$$m\ddot{x} = \sum X_k, \quad m\ddot{y} = \sum Y_k,$$

$$\sum X_k = F \cos \alpha = [50\sin(\pi t) + 4]\cos 15^\circ = 48,3\sin(\pi t) + 3,86,$$

$$\sum Y_k = F \sin \alpha - P = [50\sin(\pi t) + 4]\sin 15^\circ - mg = 12,9\sin(\pi t) - 38,16.$$

After substituting the found sums and numerical reductions, the differential equations of motion of point M have the form

$$\ddot{x} = 24,15\sin(\pi t) + 1,93, \quad (2.66)$$

$$\ddot{y} = 6,45\sin(\pi t) - 19,08. \quad (2.67)$$

Integrate the differential equation (2.66). Since $\ddot{x} = \frac{dV_x}{dt}$, then

$$\begin{aligned}\int dV_x &= \int 24,15 \sin(\pi t) dt + \int 1,93 dt, \\ V_x &= -\frac{24,15}{\pi} \cos(\pi t) + 1,93t + C_1.\end{aligned}\quad (2.68)$$

Since $V_x = \frac{dx}{dt}$, then

$$\int dx = \int -\frac{24,15}{\pi} \cos(\pi t) dt + \int 1,93t dt + \int C_1 dt.$$

After integration we get

$$x = -\frac{24,15}{\pi^2} \sin(\pi t) + 1,93 \frac{t^2}{2} + C_1 t + C_2. \quad (2.69)$$

For the integrals (2.68) and (2.69), the initial conditions will be:

$$x = a = 10 \text{ m}, V_x = V_{ox} = 10 \text{ m/s} \quad \text{when } t = 0.$$

Then the constants of the integration C_1 and C_2 levels

$$C_1 = 10 + \frac{24,15}{\pi} = 17,69, \quad C_2 = 10.$$

Integrate the differential equation (2.67). Since $\ddot{y} = \frac{dV_y}{dt}$, then

$$\int dV_y = \int 6,45 \sin(\pi t) dt - \int 19,08 dt.$$

and

$$V_y = -\frac{6,45}{\pi} \cos(\pi t) - 19,08t + C_3. \quad (2.70)$$

Since $V_y = \frac{dy}{dt}$, to

$$\int dy = \int -\frac{6,45}{\pi} \cos(\pi t) dt - \int 19,08t dt + \int C_3 dt,$$

$$y = -\frac{6,45}{\pi^2} \sin(\pi t) - 19,08 \frac{t^2}{2} + C_3 t + C_4. \quad (2.71)$$

Initial conditions for integrals (2.70) i (2.71) there will be

$$y = b = 2 \text{ m}, \quad V_y = V_{oy} = 2 \text{ m/s},$$

Then the constants of the integration C_3 i C_4 levels:

$$C_3 = 2 + \frac{6,45}{\pi} = 5,23, \quad C_4 = 2.$$

Taking into account the constants of the integration C_1 , C_2 , C_3 i C_4 form the first integrals (2.68) and (2.70) we find the components of the velocity vector

$$V_x = 17,69 + 1,93t - 7,69 \cos(\pi t), \quad (2.72)$$

$$V_y = 5,23 - 19,08t - 2,05 \cos(\pi t).$$

And from the integrals (2.69) i (2.71) we receive the coordinates

$$x = 10 + 17,69y + 0,97t^2 - 2,45 \sin(\pi t), \quad (2.73)$$

$$y = 2 + 5,23t - 9,54t^2 - 6,65 \sin(\pi t).$$

The components (2.72) allow us to find the velocity modulus

$$V = \sqrt{V_x^2 + V_y^2} \quad (2.74)$$

The components of the acceleration vector are written by the formulas (2.66) and (2.67) $a_x = \ddot{x}$, $a_y = \ddot{y}$, therefore the acceleration modulus is equal to

$$a = \sqrt{a_x^2 + a_y^2}. \quad (2.75)$$

Using the dependences (2.66), (2.67), (2.72), (2.73) and formulas (2.74), (2.75) we perform calculations on a PC in the EXCEL program with a step of 0.25 for time t . The results are given in the table A2.2.

Numerical results allow us to visually analyze the motion of a point in the field of gravity and the force specified in the problem condition. Fig. 2.6 shows the trajectory $y(x)$, and Fig. 2.7 and 2.8 show the graphs of the velocity $V(t)$ and acceleration $a(t)$.

Table D2.2

t	x	y	V_x	V_y	V	a_x	a_y	a
0	10	2	10	3,18	10,493	1,93	-19,08	19,18
0,5	16,636	-4,424	18,649	-4,3116	19,141	26,08	-12,63	28,98
1	28,651	-2,3206	27,31	-11,8	29,75	1,968	-19,07	19,17
1,5	41,156	-4,966	20,603	-23,385	31,167	-22,22	-25,53	33,85
2	49,248	-25,679	13,86	-34,98	37,626	1,853	-19,1	19,19
2,5	57,806	-51,204	22,484	-42,478	48,062	26,08	-12,63	28,98
3	71,743	-68,202	31,17	-49,96	58,886	2,045	-19,05	19,16
3,5	86,186	-89,906	24,488	-61,539	66,232	-22,22	-25,53	33,85
4	96,216	-129,68	17,72	-73,14	75,256	1,776	-19,12	19,2
4,5	106,7	-174,3	26,32	-80,645	84,831	26,08	-12,63	28,98
5	122,56	-210,4	35,03	-88,12	94,827	2,122	-19,03	19,15
5,5	138,94	-251,17	28,372	-99,692	103,65	-22,22	-25,53	33,84
6	150,9	-310	21,58	-111,3	113,4	1,7	-19,1	19,2
6,5	163,31	-373,72	30,155	-118,81	122,58	26,08	-12,63	28,98
7	181,09	-428,92	38,89	-126,28	132,13	2,199	-19,01	19,13
7,5	199,41	-488,75	32,257	-137,85	141,57	-22,22	-25,53	33,84
8	213,31	-566,64	25,441	-149,46	151,61	1,622	-19,16	19,23

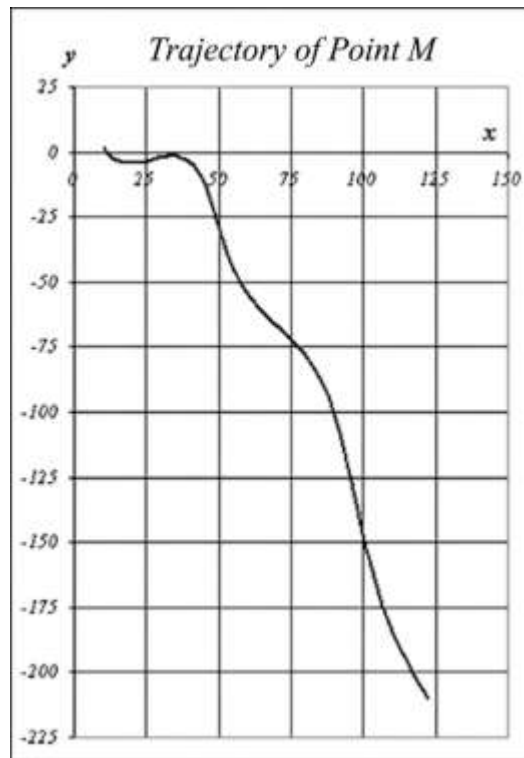


Figure 2.6

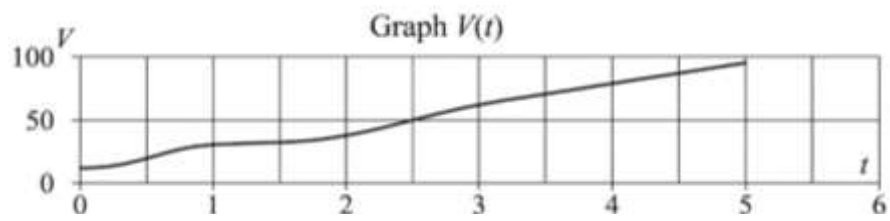


Figure 2.7

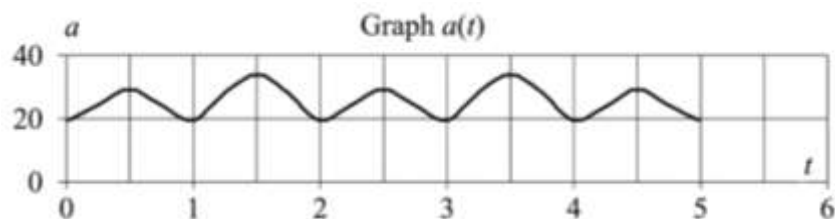


Figure 2.8

3 RECTILINEAR OSCILLATIONS OF A MATERIAL POINT

Mechanical movements that are periodically repeated are called **mechanical vibrations**. A material point can be affected by: **recuperative force**, which tries to return the point to the equilibrium position; a **force of resistance**, which depends on the speed of the point; a **revolving force**, given as a function of time.

Depending on the combination of these forces, we will consider three types of oscillatory motion of a material point: **free vibrations** under the action of only a linear restoring force, **fading vibrations** under the action of a restoring force and a resistance force that depends on the speed, and **forced vibrations** under the action of a restoring force and a disturbing force that varies according to a harmonic law.

3.1 Free (natural) vibrations of a material point

Free (natural) oscillations of a material point are called its oscillations under the action of forces determined by initial conditions: deviation of the point from the equilibrium position or giving it initial velocities.

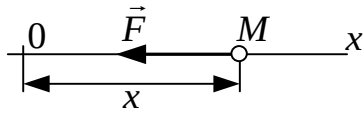


Figure 3.1

Let us consider the rectilinear motion of a material point M with mass m under the action of only a force $\vec{F}$ directed towards the fixed center O and proportional to the distance of point M from the center O : $F = -cx$ (linear dependence) (fig. 3.1).

Force $\vec{F}$ tries to return point M to the equilibrium position O , so the force $\vec{F}$ is called a restoring force. An example of such a force is the force of elasticity of a spring.

Determine the law of motion of point M , that is, the law of change of coordinate x with time.

Let us write the differential equation of motion of a point with respect to time in the projection onto the axis x :

$$m\ddot{x} = F_x, m\ddot{x} = -Cx \quad \text{or} \quad \ddot{x} = -\frac{c}{m}x,$$

$$\ddot{x} + \frac{c}{m} = 0, \quad \ddot{x} + k^2x = 0,$$

where $k^2 = \frac{c}{m}$, $k = \sqrt{\frac{c}{m}}$ – **circular (natural) frequency** of oscillations of a material point with mass m , measured in rad/s;

c – the coefficient of stiffness (elasticity) of a spring, which is numerically equal to the force that must be applied to the spring to change its length by one unit (measured in N/m).

The law of oscillations of point M has the form (solution of the differential equation of oscillations):

$$x = A \cos kt + B \sin kt ,$$

$$\dot{x} = -A \cdot k \sin kt + B \cdot k \cdot \cos kt ,$$

where A i B – integration constants determined from initial conditions. Initial conditions: at $t = 0$; $x = x_0$; $\dot{x} = V_0$.

Given the initial conditions, we have: $x_0 = A$; $\frac{\dot{x}_0}{k} = B$.

Finally, the law of oscillations of point M has the form

$$x = x_0 \cos kt + \frac{\dot{x}_0}{k} \sin kt .$$

If instead of the constants of integration A and B , we introduce the constants a and α and , we obtain the oscillation law in the form $x = a \sin(kt + \alpha)$, where a and α are also determined from the initial conditions.

$a = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{k^2}}$ – the amplitude of oscillations, which is equal to the largest deviation of point M from the center O and depends on the initial conditions,

$$\operatorname{tg} \alpha = \frac{x_0 \cdot k}{\dot{x}_0} ,$$

where $\alpha = \operatorname{arctg} \frac{x_0 \cdot k}{\dot{x}_0}$ – the initial phase of oscillations, which also depends on the initial conditions.

The rectilinear motion of a material point M is considered with the mass m under the influence of force $\overline{F}$, whose magnitude is proportional to the deviation of the point from the static equilibrium position, is a **harmonic oscillatory motion with a natural**

circular frequency $K = \sqrt{\frac{c}{m}}$ **and period** $T = \frac{2\pi}{k} = 2\pi \sqrt{\frac{m}{C}}$.

Oscillation period T – the time interval between two successive passages of a point through the position of static equilibrium in a certain fixed direction.

The circular frequency of natural oscillations K and their period T do not depend on the initial conditions and the amplitude of oscillations a . When the initial conditions are equal to zero at the same time ($x_0 = 0, \dot{x} = 0$), to $x = 0$, i.e. free (natural) vibrations do not arise.

The effect of a constant force on free vibrations

Let us consider the example of the oscillation of a load of mass m , suspended by a spring of stiffness C to a fixed plane, along the axis OX (fig. 3.2).

At point M (load weighing $P = mg$) acts the force of the spring elasticity and the force of gravity $\bar{P} = m\bar{g}$ (constant force).

Point O – is the beginning of the axis OX , directed towards the action of the force $\bar{P}$, l_0 – length of the spring in the undeformed state, x – current coordinate of the cargo during movement, f_{st} – static deformation of a spring under the action of a load.

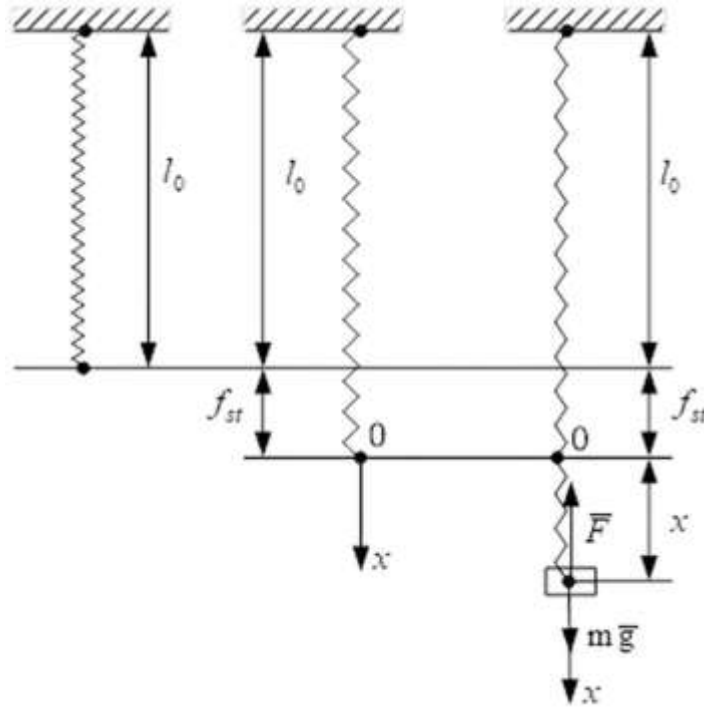


Figure 3.2

Point O is the point of static equilibrium, where the force P is balanced by the spring force. $\bar{P} = \bar{F}$

$$mg = c \cdot f_{st}; f_{st} = \frac{mg}{c} = \frac{P}{c}.$$

When moving, the elastic force of the spring in the projection onto the axis OX

$$F_x = -c(x + f_{st}), P_x = P = mg = cf_{st}.$$

The differential equation of motion of point M in the projection onto the OX axis has the form

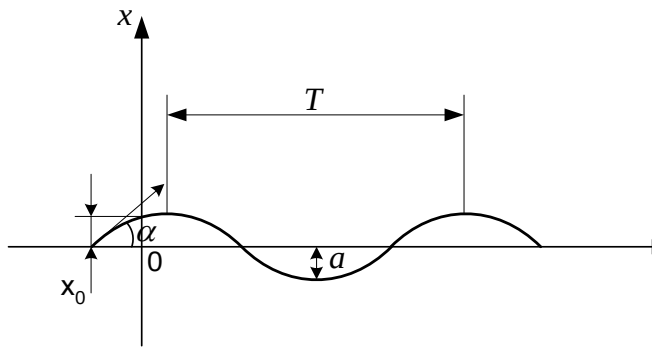
$$m\ddot{x} = P_x - F_x = cf_{st} - c(x + f_{st}) = -cx$$

or

$$\ddot{x} + k^2x = 0,$$

$$k^2 = \frac{c}{m} = \frac{g}{f_{st}}.$$

The law of oscillation of the load (solution of the differential equation of motion) under the action of a constant force and a linear restoring force in this case has the form $x = \alpha \sin(kt + \alpha)$, as in the case of the action of only the restoring force (Fig. 3.3).

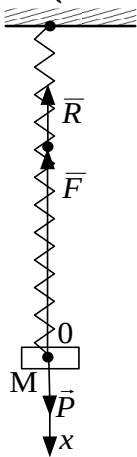


A constant force does not change the nature of the oscillations (according to the law of sine or cosine), shifting the center of oscillations in the direction of the constant force along the axis OX by the amount of static deformation f_{cm} .

Figure 3.3

$$T = \frac{2\pi}{k} = 2\pi \sqrt{m \cdot f_{st} / p}, \quad P = mg, \quad T = 2\pi \sqrt{f_{st} / g}.$$

3.2 Free oscillations of a material point in the presence of linear viscous resistance (friction)



A material point M (a load of mass m) moves under the action of gravity in a viscous medium (liquid or gas), during which a resistance force arises, proportional to the first power of the velocity. This resistance force is called the viscous friction force: $\vec{R} = -\mu \vec{V}$ (the sign "-" shows that the force $\vec{R}$ directed oppositely $\vec{V}$). Thus, the restoring force acts on point M during motion $\vec{F}$ and the force of resistance $\vec{R}$ (рис 3.4).

Figure 3.4

The differential equation of motion of a point (load) has the form:

$$m\ddot{x} = P - F - R$$

or $m\ddot{x} = mg - c(f_{st} + x) - \mu\dot{x}$

or $m\ddot{x} + \mu\dot{x} + cx = 0$

or $\ddot{x} + 2n\dot{x} + k^2x = 0,$

where $k = \sqrt{\frac{c}{m}}$ – natural frequency of oscillations (circle);

$n = \frac{\mu}{2m}$ – relative damping coefficient, which has the dimension of the circular frequency k of oscillations ($[n] = \text{rad/s}$).

The law of oscillation of the load (solution of the differential equation) depends on the ratio of the parameters k and n .

1. Law of oscillation of the load in the case of small resistance $n < k$ looks like

$$x = e^{-nt} (A \cos k_1 t + B \sin k_1 t)$$

or

$$x = a e^{-nt} \cdot \sin(k_1 t + \alpha),$$

where the integration constants are determined, as before, from the initial conditions;

$k_1 = \sqrt{k^2 - n^2}$ – frequency of damped oscillations;

$a = \sqrt{x_0^2 + \frac{(x_0 \cdot n + V_0)^2}{k_1^2}}$ – maximum deviation of a point from the equilibrium

position, $\alpha = \text{arctg} \frac{x_0 \cdot k_1}{x_0 \cdot n + V_0}$ – initial phase.

Oscillations that occur according to this law are called decaying, because the magnitude x decreases over time, due to the multiplier e^{-nt} , towards zero (Fig. 3.5).

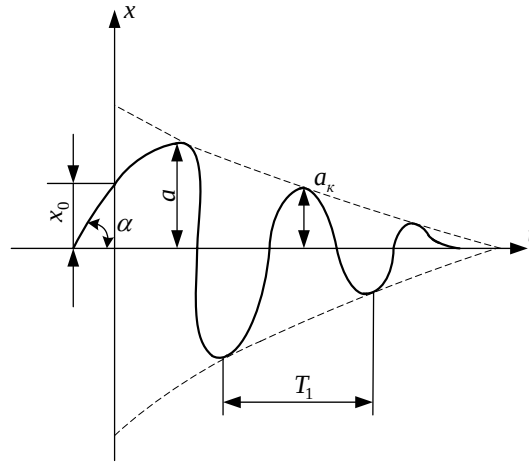


Figure 3.5

The period of damping oscillations T_1 is the time interval between two successive passages of a point through the position of static equilibrium in a certain fixed direction:

$$T_1 = \frac{2\pi}{k_1} = \frac{2\pi}{\sqrt{k^2 - n^2}}.$$

From this it can be seen that the period of damped oscillations is greater than the period of natural oscillations $T_1 > T$. The resistance of the medium, which is proportional to the velocity in the first degree, increases the period of oscillations.

Correlation $\frac{a_K}{a} = e^{-nT_1}$ called the **attenuation decrement** or attenuation factor.

The amplitude of the decaying oscillations decreases in geometric progression.

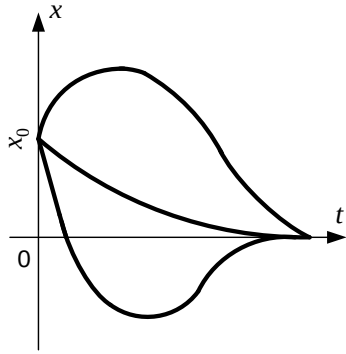


Figure 3.6

2. At $n = k$ (limiting case) the solution of the differential equation of motion: $x = e^{-nt}(A \cdot t + B)$.

The motion of the point will be aperiodic, decaying, will not be oscillatory, and will tend to zero over time. ($x \rightarrow 0$ at $t \rightarrow \infty$). The motion of a point depends on the initial conditions (fig. 3.6).

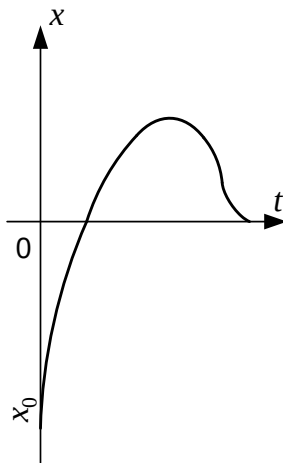


Figure 3.7

3. At $n > k$, when the resistance is large compared to the restoring force, the solution to the differential equation of motion has the form:

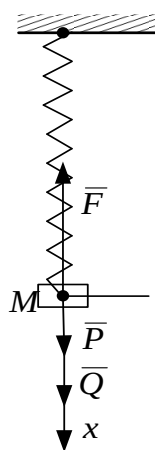
$$x = A \cdot e^{-(n-k_2)t} + B \cdot e^{-(n+k_2)t},$$

$$\text{where } k_2 = \sqrt{n^2 - k^2}.$$

The motion of the point in this case is also not oscillatory and under the action of the restoring force it will gradually (asymptotically) move towards the equilibrium state $x = 0$. The motion of the point also depends on the initial conditions and is aperiodic (fig. 3.7).

3.3 Forced oscillations of a material point

Forced oscillations occur under the condition that, in addition to the restoring force $\bar{F}$, a disturbing force $\bar{Q}$ acts on the material point, which varies according to a harmonic law, i.e. $Q = H \cdot \sin pt$, where H – the maximum value of the disturbing force, p is the circular frequency of the disturbing force (Fig. 3.8).



The differential equation of motion of a material point M (load) has the form $m\ddot{x} = \bar{P} + \bar{F} + \bar{Q}$

or in projection onto the axis x :

$$m\ddot{x} = mg - c(f_{st} + x) + H \sin pt,$$

or after dividing by m :

$$\ddot{x} + k^2 x = h \sin pt,$$

where $k^2 = \frac{c}{m}; h = \frac{H}{m}.$

Figure 3.8

The general solution of a non-homogeneous differential equation consists of two solutions (Fig. 3.9): $x = x_1 + x_2$

where $x_1 = A \cdot \cos kt + B \sin kt$ – **general solution of a homogeneous equation;**

x_2 – **partial solution of a non-homogeneous equation.**

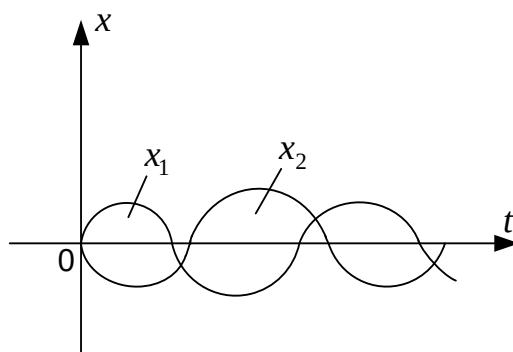


Figure 3.9

When determining a partial solution, we will consider three cases:

$(k \neq p, \quad k \approx p, \quad k = p).$

1. The frequency of free oscillations is not equal to the frequency of the disturbing force

$(k \neq p).$

In this case $x_2 = \frac{h}{k^2 - p^2} \cdot \sin pt.$

Then the law of oscillation of the load

$$x = x_1 + x_2 = A \cos kt + B \sin kt + \frac{h}{k^2 - p^2} \cdot \sin pt$$

or

$$x = a \cdot \sin(kt + \alpha) + \frac{h}{k^2 - p^2} \cdot \sin pt,$$

where A, B, a, α – integration constants determined from initial conditions.

The law of oscillation states that the oscillations consist in this case of oscillations with amplitude a (depending on the initial conditions) and natural frequency k , and oscillations with amplitude $\frac{h}{k^2 - p^2}$ (independent of initial conditions) and frequency p , which are called forced oscillations.

In contrast to free oscillations described by homogeneous differential equations, forced oscillations in the presence of a disturbing periodic force are also excited under zero initial conditions ($x_0 = 0; \dot{x}_0 = 0$).

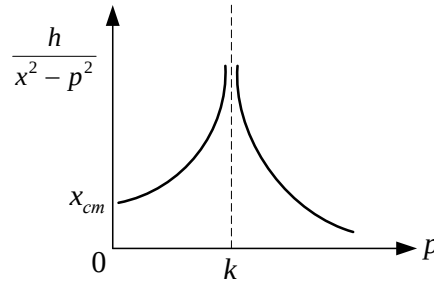


Figure 3.10

At $p \rightarrow k$ amplitude of forced oscillations $\frac{h}{k^2 - p^2}$ grows indefinitely.

At $p \approx 0$ (also $p \ll k$) the amplitude is equal to the static deflection of the point under the action of force H (the maximum value of the disturbing force) (Fig. 3.10).

At $p \gg k$ the amplitude $\frac{h}{k^2 - p^2} \rightarrow 0$.

Forced oscillations, the frequency p of which is less than the frequency k of free oscillations, are called low-frequency forced oscillations. ($p < k$) i $x_2 = \frac{h}{k^2 - p^2} \cdot \sin pt$

.The phase of the oscillations pt coincides with the phase of the disturbing force $\bar{Q}$.

Forced oscillations, the p frequency of which is greater than the frequency k of free oscillations, are called high-frequency forced oscillations. $p > k$ i

$$x_2 = \frac{h}{p^2 - k^2} \cdot \sin(pt - \pi).$$

In this case, the amplitude of forced oscillations is equal to $\frac{h}{p^2 - k^2}$, and a phase $(pt - \pi)$ differs from the phase of the disturbing force pt by π , i.e., the phases of the forced oscillations and the disturbing force are opposite.

2. The frequency of free oscillations is approximately equal to the frequency of the disturbing force ($k \approx p$), if at $t=0$, $x_0 = 0$, $\dot{x}_0 = 0$, that is, the initial conditions are

zero (fig. 3.11). In such a case $x_2 = \frac{2h}{k^2 - p^2} \sin\left(\frac{p-k}{2}\right)t \cdot \cos pt$.

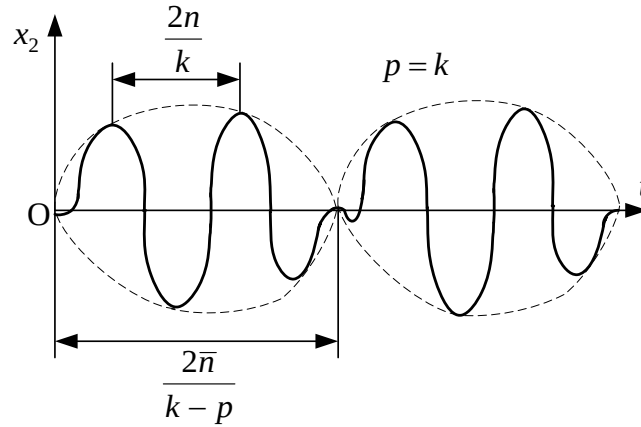


Figure 3.11

Oscillatory motion occurs at the frequency of the disturbing force, the amplitude of which slowly changes with a frequency corresponding to half the difference between the frequencies p and k . Such a motion is called **beating**.

3. The frequency of free oscillations is equal to the frequency of the disturbing force ($p = k$), the nature of the oscillations changes qualitatively and resonance occurs.

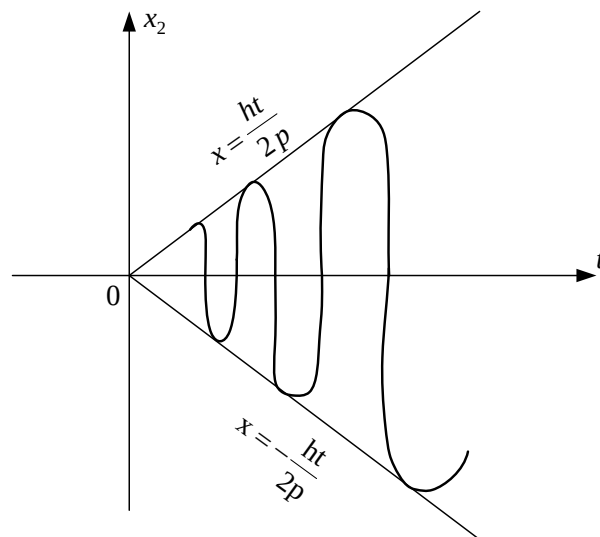


Figure 3.12

The partial solution of inhomogeneous differentiation has the form

$$x_2 = -\frac{h \cdot t}{2 \cdot p} \cdot \cos pt.$$

Amplitude of forced oscillations x_2 , which is defined as $-\frac{h \cdot t}{2p}$, increases directly proportional to time and at $t \rightarrow \infty$ grows indefinitely (fig. 3.12).

In the case of non-zero initial conditions, the general solution of the inhomogeneous differential equation has the form:

$$x = x_1 + x_2 = A \cdot \cos kt + B \sin kt - \frac{h \cdot t}{2p} \cos pt.$$

Methodology for solving problems in the study of oscillatory motion of a material point

1. In the given schemes (Table 3.1), the springs connecting the load to the base can be connected in parallel, mixed and series. Therefore, these schemes must be reduced to a design scheme with one spring, which has an equivalent stiffness C .

2. Choose a reference system, taking the point of static equilibrium of the load as the origin of coordinates ($c \cdot f_{cm} = m \cdot g$).

3. Direct the x -axis in the direction of movement of the cargo.

4. Draw the load in an intermediate position and the forces acting on it.

5. Determine the initial conditions of the load motion.

6. Write the differential equation of the load motion in the projection onto the axis.

7. Write the solution of the differential equation of the load motion.

8. Using the initial conditions, determine the integration constants of the law of motion of the load.

9. Determine the oscillation parameters.

Table 3.1

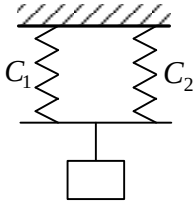
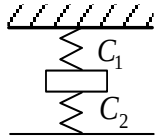
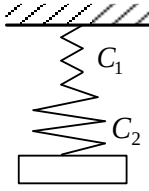
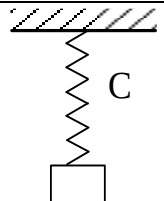
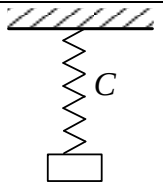
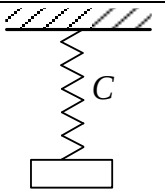
Given schemes	a)	б)	в)
			
Equivalent schemes	 $C = C_1 + C_2$	 $C = C_1 + C_2$	 $C = \frac{C_1 \cdot C_2}{C_1 + C_2}$

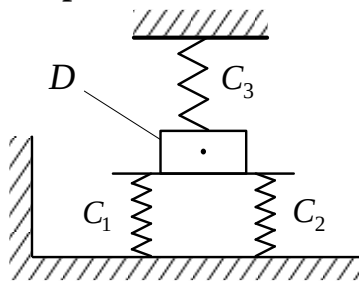
Table 3.2 – Rectilinear harmonic oscillations of a material point

Type of oscillation	Free	Fading		
Acting forces	$F = -c\chi$	$F = -c\chi; R = -m\dot{\chi}$		
Differential equation of motion	$m\ddot{\chi} = -c\chi$ $m\ddot{\chi} + c\chi = 0$ $\ddot{\chi} + k^2\chi = 0$ $k^2 = \frac{c}{m}$	$m\ddot{\chi} = -c\chi - \mu\dot{\chi}$ $m\ddot{\chi} + \mu\dot{\chi} + c\chi = 0$ $\ddot{\chi} + 2n\dot{\chi} + k^2\chi = 0$ $k^2 = \frac{c}{m}; 2n = \frac{\mu}{m}$		
Solution of the differential equation, law of motion	$\chi = A\cos kt + B\sin kt$ $\chi = a\sin(kt + \alpha)$	$n < k$ $\chi = e^{-nt}(A\cos k_1 t + B\sin k_1 t)$ $\chi = a \cdot e^{-nt}(\sin k_1 t + \alpha)$	$n = k$ $\chi = e^{-nt}(At + B)$	$n > k$ $\chi = A \cdot e^{-(n-k_2)t} + B \cdot e^{-(n+k_2)t}$
Motion parameters	$\kappa = \sqrt{\frac{c}{m}}$ – natural frequency $T = \frac{2\pi}{k}$ – Period $a = \sqrt{A^2 + B^2} = \sqrt{x_0^2 + \frac{V_0^2}{k^2}}$ – amplitude $\alpha = \arctg \frac{x_0 \cdot k}{V_0}$ – initial phase	$k_1 = \sqrt{k^2 - n^2}$ – frequency of damped oscillations $T_1 = \frac{2 \cdot \pi}{k_1}$ – period of decaying oscillations $a = \sqrt{x_0^2 + \frac{(x_0 \cdot n + V_0)^2}{k_1^2}}$ – amplitude of damped oscillations $\alpha = \arctg \frac{x_0 \cdot k_1}{x_0 \cdot n + V_0}$ – initial phase	Damping aperiodic motion	$\kappa_2 = \sqrt{n^2 - k^2}$ Damping aperiodic motion

Continuation of the table 3.2

Type of oscillation	Forced	
Acting forces	$F = -cx; Q = c \cdot \epsilon \cdot \sin pt$	
Differential equation of motion	$m \ddot{x} = -cx + c \cdot \epsilon \cdot \sin p \cdot t$ $m \ddot{x} + cx = c \cdot \epsilon \cdot \sin p \cdot t$ $\ddot{x} + k^2 x = h \cdot \sin p \cdot t$ $k^2 = \frac{c}{m}; h = \frac{c \cdot \epsilon}{m}$	
Solution of differential equation, law of motion	$k \neq p$ $x = x_1 + x_2$ $x_1 = A \cdot \cos k \cdot t + B \cdot \sin kt$ $p < k$ $x_2 = \frac{h}{k^2 - p^2} \cdot \sin p \cdot t$ $p > k$ $x_2 = \frac{h}{p^2 - k^2} \cdot \sin(pt - \pi)$	$p = k$, resonance $x_1 = A \cdot \cos kt + B \cdot \sin kt$ $x_2 = -\frac{h \cdot t}{2p} \cdot \cos pt$
Motion parameters	$k = \sqrt{\frac{c}{m}}$ – natural frequency of forced oscillations, p – circular frequency of forced oscillations, $T = \frac{2\pi}{\kappa}$ – period of natural oscillations, $T_B = \frac{2\pi}{p}$ – period of forced oscillations.	

An example of solving problems from the study of the oscillatory motion of a material point



The plate (Fig. 3.13) rests on two identical parallel springs, the stiffness coefficients of which are $c_1 = c_2 = 400 \text{ N/sm}$. At some point in time, the load D ($m=200\text{kg}$) is placed in the middle of the plate and simultaneously fixed to a third undeformed spring located above the load. With a stiffness coefficient $c_3 = 200 \text{ N/sm}$. At the same time (with undeformed springs) the load is given a speed $V_0 = 0,6 \text{ m/s}$, directed downwards.

Figure 3.13

Neglecting the mass of the plate and assuming that the load does not separate from the plate during further movement, find the equation of the oscillatory motion of the load D . Refer the motion of the load to the x -axis, taking the position of static equilibrium of the load as the origin (with static deformation of the springs).

Solution

We find the equivalent stiffness. Two parallel springs with the same stiffnesses c_1 and c_2 have equivalent stiffness $c_{1,2} = c_1 + c_2 = 400 \cdot 2 = 800 \text{ N/sm} = 80000 \text{ N/m}$.

Static deformations of springs with equivalent stiffness $c_{1,2}$ and c_3 are the same. At the same time, strength $\bar{P} = m\bar{g}$ balanced by elastic forces $c_{1,2} \cdot f_{ST}$ and $c_3 \cdot f_{ST}$ of the springs, that is $P = (c_{1,2} + c_3) f_{ST}$.

Hence $c_{EKB} = c = c_{1,2} + c_3 = 80000 + 20000 = 100000 \text{ N/m}$.

$$mg = c \cdot f_{ST},$$

$$f_{ST} = \frac{mg}{c} = \frac{200 \cdot 10}{100000} = 0,02 \text{ m}.$$

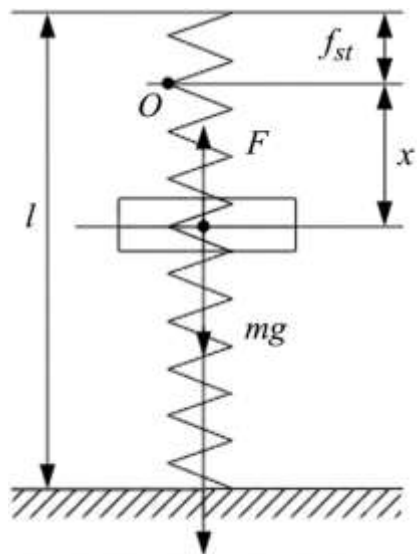


Figure 3.14

1. We choose the origin of coordinates. Point O determines the position of static equilibrium of the load. The x -axis is directed downwards.

2. We depict the load in an intermediate position and the forces acting on it:

$\bar{P} = m\bar{g}$ – gravity, $\bar{F}$ – elastic force.

3. We determine the initial conditions for cargo movement

At $t = 0$, $x_0 = -f_{ST} = -0,02m$, $v_0 = 0,6 \frac{m}{s}$.

4. The load moves under the force of the load's gravity. $P = m \cdot g$, spring elastic forces $F = -c(x + f_{ST})$ and performs free oscillations.

5. Differential equation of motion of a load

$$m\ddot{x} = mg - c(x + f_{ST}) = -cx,$$

$m\ddot{x} + cx = 0$ – differential equation of free vibrations,

$$\ddot{x} + \kappa^2 x = 0,$$

where $k^2 = \frac{c}{m} = \frac{100000}{200} = 500$;

$$k = \sqrt{500} = 22,4 s^{-1}.$$

6. Law of cargo movement $x = A \cdot \cos 22,4t + B \cdot \sin 22,4t$.

7. We use the initial conditions and determine the constants of integration of A and B .

At $t = 0$: $x_0 = -0,02 m$; $V_0 = 0,6 \frac{m}{s}$,

$$x_0 = -0,02 = A \rightarrow A = -0,02m,$$

$$\dot{x} = -22,4 \cdot A \cdot \sin 22,4t + 22,4 \cdot \cos 22,4t,$$

$$\dot{x}_0 = V_0 = 0,6 = 22,4B \rightarrow B = \frac{0,6}{22,4} = 0,027m.$$

Hence the law of motion of cargo $x = f(t)$ looks as

$$x = -0,02 \cdot \cos 22,4t + 0,027 \cdot \sin 22,4t(m).$$

8. We determine the parameters of cargo oscillation:

$k = 22,4 s^{-1}$ – natural frequency of oscillations,

$T = \frac{2 \cdot \pi}{22,4} = 0,28 s$ – period of oscillation,

$a = \sqrt{A^2 + B^2} = \sqrt{0,02^2 + 0,027^2} = 0,034 m = 3,4 cm$ – amplitude of oscillations (the largest deviation of the load from the equilibrium position),

$\alpha = \arctg \frac{A}{B} = \arctg \frac{-0,02}{0,027} = \arctg(-0,74) = 324^\circ$ – initial phase of oscillations,

$$\sin \alpha = \frac{x_0}{a} = -0,58, \quad \alpha = 324^\circ,$$

$$\cos \alpha = \frac{V_0}{ka} = \frac{0,6}{22,4 \cdot 0,034} = 0,78.$$

We build a graph of cargo fluctuations (fig. 3.15).

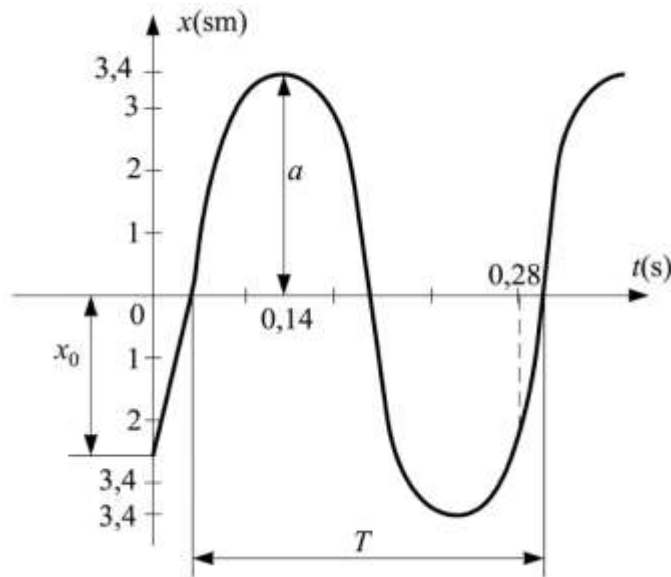


Figure 3.15

Answer: the load oscillates freely according to the law
 $x = -0,02 \cdot \cos 22,4t + 0,027 \cdot \sin 22,4t$ m .

Self-awareness questions

1. What forces are called restoring?
2. What are the main types of oscillatory motion of a material point?
3. What oscillations are called free?
4. What is the period of free oscillations?
5. How are the circular frequency and period of free oscillations determined?
6. What is the formula for determining the decrement of damping?
7. What is a disturbing force?
8. What is a beat?
9. What is a resonance?

3.4 Calculation-graphic and control tasks

D3 Study of the motion of a point under the action of elastic forces and resistance forces of the environment

Pages 55 – 84 present the texts of the tasks D3.1 – D3. 30 with the corresponding figures and data tables. The quantities marked in the tables with a question mark “?” must be found as a result of calculations. The quantities against which a dash “–” is marked are those that do not need to be determined or those that have no practical meaning for this problem.

Theoretical information.

In order to study the motion of a point under the action of forces, a differential equation of motion is used, which in the projection onto the Ox axis has the form

$$m\ddot{x} = \sum_{k=1}^n X_k, \quad (3.1)$$

where m – point mass,

x – its coordinate in the selected coordinate system;

X – projection of the acting force onto the axis x ;

k – force serial number, $k = 1, 2, 3, \dots$

The right-hand side of the differential equation (3.1) is fundamental because, on the one hand, it characterizes the process of motion as a mechanical phenomenon, and on the other hand, it affects the analytical form of the differential equation as a mathematical model. In this task, it is proposed to investigate the phenomenon of motion under the action of elastic forces that depend on coordinates, forces that depend on speed, which occurs when moving with the resistance of the environment, and disturbing forces that depend on time according to the harmonic law.

The differential equations obtained from its general form (3.1) depending on the listed forces have the form:

$$\ddot{x} + k^2 x = 0, \quad (3.2)$$

$$\ddot{x} + 2n\dot{x} + k^2 x = 0, \quad (3.3)$$

$$\ddot{x} + k^2 x = hf(\sin(pt), \cos(pt)). \quad (3.4)$$

where $x = x(t)$ – function to be found,

k, n, h, p – constant values.

The solution of the homogeneous differential equation (3.2) is the function

$$x = C_1 \sin(kt) + C_2 \cos(kt), \quad (3.5)$$

which corresponds to free harmonic oscillations.

For the differential equation (3.3), the solution depends on the ratio of the numbers n and k :

$$\text{when } n > k \quad x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \quad (3.6)$$

$$\text{when } k > n \quad x = e^{-nt} (C_1 \sin(k_1 t) + C_2 \cos(k_1 t)), \quad (3.7)$$

$$\text{when } k = n \quad x = e^{-nt} (C_1 + C_2 t). \quad (3.8)$$

when $\lambda_1, \lambda_2, k_1$ – constant values that depend on k and n ,

t – independent time parameter,

C_1 i C_2 – integration constants that depend on the initial conditions.

Function (3.6) corresponds to aperiodic motion, (3.7) to damped oscillations, and (3.8) to motion that is the limit between oscillatory and aperiodic.

The solution of the inhomogeneous differential equation (3.4) is the sum

$$x = x^* + x^{**} \quad (3.9)$$

where x^* – the solution of the corresponding homogeneous differential equation, which has the form of the function (3.5),

x^{**} – is a partial solution of the differential equation (3.4) and has the form:

$$x^{**} = f(B \sin(pt), D \cos(ht)). \quad (3.10)$$

Therefore, the final solution of the differential equation (3.4) is a function of the form

$$x = A \sin(kt + \varphi) + x^{**}, \quad (3.11)$$

where A, B, D and φ constant values obtained as results of calculations.

According to function (3.11), the point performs forced oscillations.

Task D3.1

A load P_1 is attached to a rigid weightless rod D , which is suspended from three vertical springs with stiffnesses c_1 and c_2 , as shown in Fig. D3.1, and a load P_2 is also attached to it. At the initial moment of time, the load P_2 is instantly disconnected from the load P_1 and the load P_1 is given an initial velocity V_0 , directed vertically downwards.

Assuming that the rod D moves translationally, find the equation of motion of the load P_1 , as well as the quantities indicated in Table D3.1, where the following are indicated: A is the amplitude of oscillations, k is the cyclic frequency, T is the period of oscillations. Construct a graph of the coordinate dependence on time.

The specified parameters and values that need to be found are in Table D3.1 by variant.

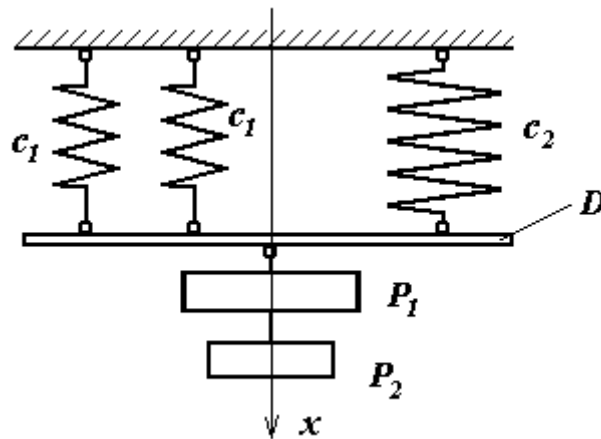


Figure D 3.1

Table D 3.1

Variant	m_1	m_2	c_1	c_2	V_0	k	T	A
	kg		N/m		m/s	s ⁻¹	s	m
1	5	4	20	?	0,4	15	?	?
2	20	15	110	500	-0,5	?	—	?
3	?	0,5	50	300	0,2	20	?	?
4	4	2,5	?	100	0,6	10	?	?
5	16	6	25	350	0,5	?	—	?
6	?	3	100	100	-0,3	5	?	?
7	10	5	300	?	0,2	8	?	?
8	0,5	0,2	200	50	0,8	?	—	?
9	3	4	?	58	-0,6	6	?	?
0	6	2	30	36	0,7	?	—	?

Task D 3.2

On an inclined plane that forms an angle α with the horizon, two loads P_1 and P_2 are placed, which are attached to two springs connected in series with stiffnesses c_1 and c_2 , Fig. D3.2. At some point in time, the load P_2 is instantly disconnected from the load P_1 and given an initial velocity V_0 , after which it moves along a plane with a friction coefficient equal to f . Find the equation of motion of the load P_1 , and construct a graph of the coordinate dependence on time.

The data for the final calculations are in Table D3.2 for the variants, where m_1 , m_2 are the masses of the cargoes P_1 and P_2 , respectively, A is the amplitude of oscillations, k is the cyclic frequency, and T is the period of oscillations.

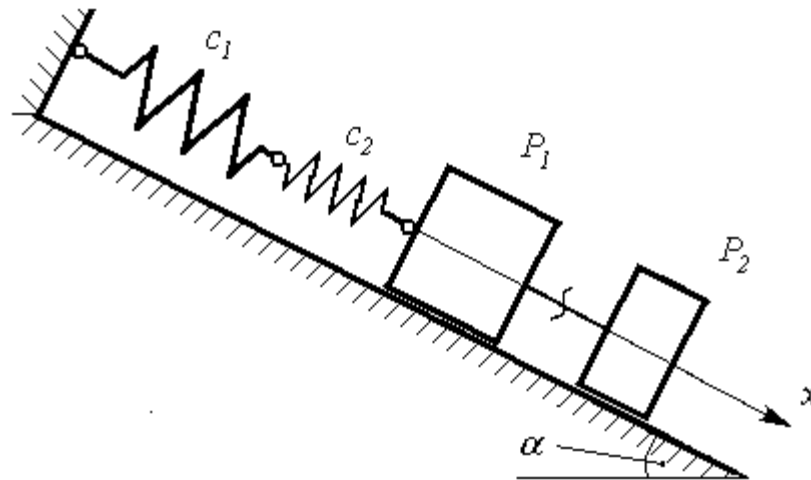


Figure D 3.2

Table D 3.2

Variant	m_1 kg	m_2 kg	c_1 N/m	c_2 N/m	f	α deg	V_0 m/s	k s^{-1}	T s	A m
1	4	0,8	100	?	0,1	20	0,1	20	?	?
2	?	1	180	90	0,05	15	0,4	5	–	?
3	5	0,2	30	60	0,1	30	0,3	?	?	?
4	0,4	0,1	?	45	0,2	45	0,2	5	?	?
5	2	0,5	40	20	0,1	30	0,3	?	–	?
6	?	4	120	30	0,3	60	0,5	8	?	?
7	10	2	?	200	0,15	45	0,2	12	?	?
8	8	5	180	360	0,1	20	0,1	?	–	?
9	?	10	60	120	0,2	75	0,4	4	?	?
0	12	3	?	96	0,1	45	0,3	6	–	?

Task D 3.3

Loads P_1 and P_2 , Fig. D3.3, with masses m_1 , m_2 , respectively, are suspended from a weightless rod B , which is attached to two springs with stiffnesses c_1 , c_2 and a damper D . At some point in time, the load P_2 is instantly disconnected from the load P_1 , and given an initial velocity V_0 . Determine the equation of motion of the load P_1 , and construct a graph of the coordinate versus time, if the rod B moves forward, and the damper D creates a resistance force proportional to the velocity $R = \mu V$.

The data for the final calculations are in Table D3.3 for the variants, where A is the amplitude of oscillations, k is the cyclic frequency, T is the oscillation period, δ is the decrement of damped oscillations.

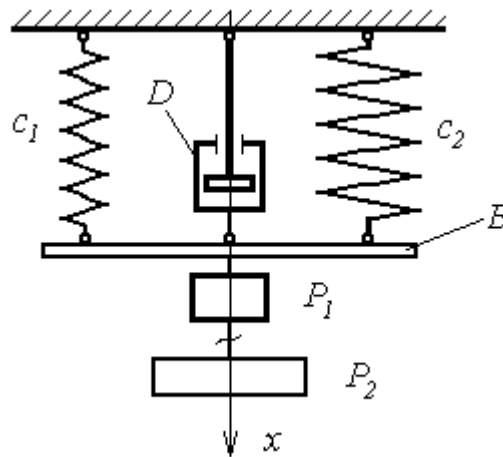


Figure D 3.3

Table D 3.3

Variant	m_1	m_2	c_1	c_2	μ	δ	V_0	k	T	A
	kg		N / m		N · s / m		m / s	s ⁻¹	s	m
1	5	8	120	80	4	?	-0,3	?	?	?
2	3	2	100	200	0,6	?	0,4	?	—	?
3	6	4	50	20	60	—	0,7	—	—	—
4	4,8	3	80	40	48	—	0,5	—	—	—
5	4	2	76	120	32	—	0,6	—	—	—
6	2	0,5	2	6	10	—	-0,6	—	—	—
7	8	2,5	10	22	24	?	0,3	—	?	?
8	12	5	2000	2800	480	—	0,5	—	—	—
9	5	6	25	55	50	—	-0,4	—	—	—
0	9	3	300	429	90	—	0,8	—	—	—

Task D 3.4

Two springs with stiffnesses c_1 and c_2 are attached to the vertical slider B in series, and loads P_1 and P_2 with masses m_1 and m_2 are suspended from them, respectively, Fig. D3.4. At some point in time, load P_2 is instantly disconnected from load P_1 , while load P_1 acquires a velocity V_0 , and slider B begins to perform vertical motion according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \sin(p_2 t)$$

Determine the equation of motion of the load P_1 and plot the coordinate versus time.

Data for final calculations in table D3.4 for options.

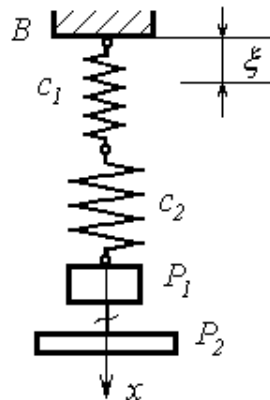


Figure D 3.4

Table D 3.4

Variant	m_1	m_2	c_1	c_2	V_0	H_1	H_2	p_1	p_2
	kg		N / m		m / s	m		s ⁻¹	
1	4	2	100	300	0,4	0	0,21	–	2
2	0,2	0,1	60	180	-0,6	0,3	0	15	–
3	0,5	0,2	160	640	0,32	0	0,15	–	4
4	10	3	800	500	0,1	0,5	0	3	–
5	2	0,5	150	450	2	0	-0,2	–	7,5
6	0,3	0,2	600	150	0,15	-0,3	0	5	–
7	3	0,6	200	100	0,5	0,2	0,6	2	3
8	6	2	120	480	0,3	0	-0,3	–	4
9	0,5	0,4	300	600	-0,2	0,15	0	20	–
0	2,34	0,8	70	350	0,5	0	0,12	–	4

Task D 3.5

A load P of mass m is placed on a horizontal platform D_1 , which is connected to a system of springs with stiffnesses c_1 , c_2 and c_3 , and a slider B acts on the spring with stiffness c_3 in the horizontal direction, as shown in Fig. D3.5. At some point in time, the load P is given an initial velocity V_0 , and the slider B begins to move according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \sin(p_2 t)$$

Determine the equation of motion of the load P and plot the coordinate versus time graph if bodies D_1 and D_2 are considered weightless, move translationally, and there is no friction.

Data for final calculations in table D3.5 for options.

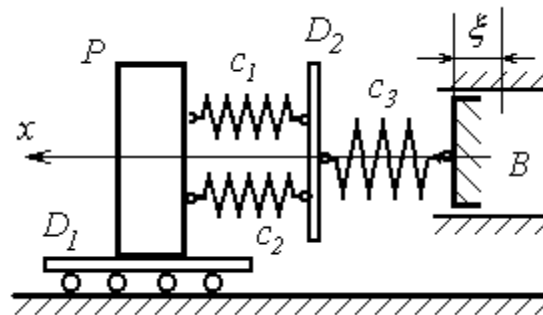


Figure D 3.5

Table D 3.5

Variant	m	c_1	c_2	c_3	V_0	H_1	H_2	p_1	p_2
	kg	N / m			m / s	m		s^{-1}	
1	7,5	100	200	500	0.2	0.05	0	5	—
2	3	700	600	300	5.46	0	0.2	—	12
3	0,43	200	150	16	2.8	0.4	0	10	—
4	2.41	400	300	500	3.2	0	0.5	—	11
5	8	800	200	250	-0.8	0.6	-0.2	8	3
6	5.27	160	360	100	0.9	-0.1	0	4	—
7	4.39	600	150	200	4	0	0.25	—	7
8	0.5	80	120	150	-0.3	0.28	0.4	6	10
9	1.2	300	60	160	2.4	0.1	0	14	—
0	2	720	300	500	-6	0	-0.3	9	—

Task D 3.6

A body P with a mass m is attached to horizontal springs with stiffnesses c_1 , c_2 and a damper D using a weightless rod B , as shown in Fig. D3.6. When the body P moves, the damper creates a resistance force R , which has a direction opposite to the velocity V and is described by the dependence $R = \mu V$, where μ is the viscosity coefficient of the fluid contained in the damper. At the initial moment of time, the body P is given an initial velocity V_0 and a displacement x_0 , after which it moves along the horizontal axis x .

Determine the equation of motion of the load P and construct a graph of the coordinate dependence on time, if the rod B moves forward, the deviations from the equilibrium position are small and horizontal.

The data for the final calculations are in table D3.6 for the options, where T is indicated as the period of damped oscillations

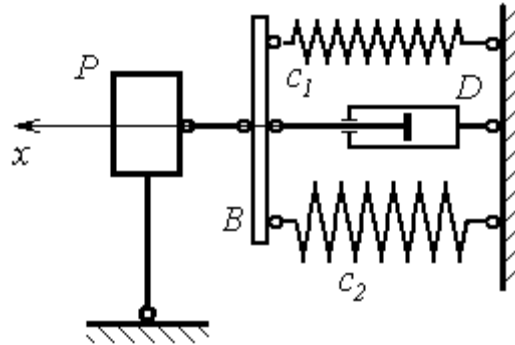


Figure D 3.6

Table D 3.6

Variant	m	c_1	c_2	μ	x_0	V_0	T
	kg	N/m	N/m	N s/m	m	m/s	s
1	5	2200	925	160	0,3	2	?
2	100	1000	600	2400	0,6	3	–
3	10	1000	3000	400	-0,2	1,5	–
4	6	5000	4600	360	0,15	-4	?
5	20	8000	10000	800	-0,1	5	?
6	4	80	20	96	-0,25	6	–
7	15	1000	1160	360	0,25	-2	–
8	12	20	88	1,5	0,2	-10	?
9	8	800	768	192	-0,3	5	?
0	3	17	10	60	0,2	4	–

Task D 3.7

A body P with a mass m falls from a height h onto a horizontal weightless platform B , which is supported by springs of stiffnesses c_1 , c_2 and a damper D , as shown in Fig. D3.7. At a height h , the body P is given an initial velocity U in the vertical direction and it falls until it meets the platform B . When the body P moves together with the platform, the damper creates a resistance force R , which has a direction opposite to the velocity V and is described by the dependence $R = \mu V$, where μ is the viscosity coefficient of the fluid contained in the damper.

Determine the equation of motion of the body P and construct a graph of the coordinate versus time if the platform B moves forward and the impact of the body P on the platform B is completely inelastic.

Data for final calculations are in Table D3.7 for variants, where T is the period of damped oscillations.

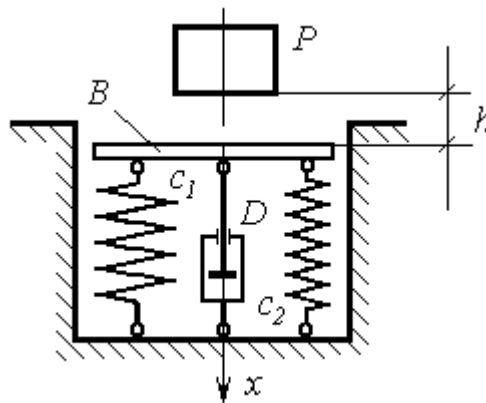


Figure D 3.7

Table D 3.7

Variant	m	h	c_1	c_2	μ	U	T
	kg	m	N/m	N/m	N s/m	m/s	s
1	20	0,5	800	1000	1000	1	?
2	16	1	150	250	128	0	–
3	50	0,4	3000	2000	1000	-2	–
4	5	0,75	2000	6000	300	0	?
5	10	2	1000	1250	400	3	–
6	18	1,5	4200	3000	720	0,5	–
7	40	0,8	1000	3000	320	-4	?
8	100	0,3	12000	28000	3200	0,7	–
9	25	0,6	100	300	200	-1	–
0	6	0,25	650	700	144	0,5	?

Task D 3.8

On an inclined plane with an angle α to the horizon, there is a weight P with a mass m , which is attached to the lower end of the springs connected in series c_1 i c_2 , and a slider B is attached to the upper end of the springs, as shown in Fig. D3.8. The system is in a position of static equilibrium. At some point in time, the slider B begins to move along the inclined plane according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \cos(p_2 t)$$

Determine the equation of motion of the load P and plot the coordinate versus time if the coefficient of friction to the inclined plane is f .

The data for the final calculations are in Table D3.8 by variant.

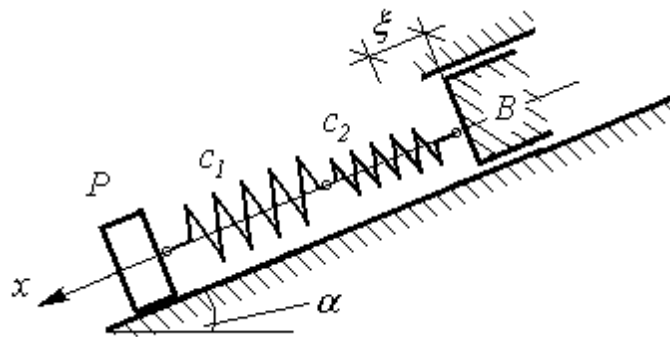


Figure D 3.8

Table D 3.8

Variant	m	f	α	c_1	c_2	H_1	H_2	P_1	P_2
	kg		deg	N/m	N/m	m	m	s^{-1}	s^{-1}
1	6	0,3	30	36	72	0	0,3	–	2
2	24	0,05	15	200	600	0,42	0	0,5	–
3	6	0,25	20	120	480	0	0,5	–	8
4	3	0,2	45	100	300	0,3	0	5	–
5	0,5	0,1	75	150	450	0	0,02	–	10
6	27,5	0,2	60	250	550	0,6	0	2,5	–
7	1,2	0,15	30	50	75	0	0,01	–	6
8	6	0,3	20	40	600	0,05	0	10	–
9	1,8	0,09	15	180	320	0	0,017	–	12
0	2,7	0,4	45	90	270	0,2	0	5	–

Task D 3.9

On an inclined plane with an angle α to the horizon, there is a load P with a mass m , which is attached to the upper end of the springs connected in series c_1 and c_2 and to the damper D , as shown in Fig. D3.9. The system is in a position of static equilibrium. When the body moves, the damper D creates a resistance force depending on the speed: $R = \mu V$. At the initial moment of time, the load P is given an initial displacement x_0 and an initial speed V_0 , after which it moves along the x axis.

Determine the equation of motion of the load P and construct a graph of the coordinate dependence on time, if the friction coefficient on the inclined plane is equal to f

Data for final calculations in Table D3.9 for variants

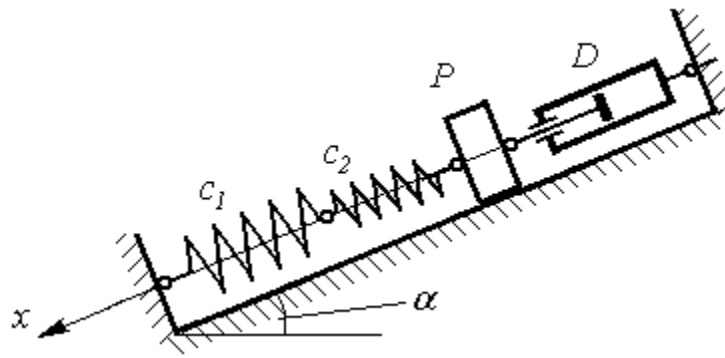


Figure D 3.9

Table D 3.9

Variant	m	f	α	c_1	c_2	μ	x_0	V_0
	kg		grad	N/m	N/m	N s/m	m	m/s
1	2,8	0,4	75	2400	2100	84	0,35	7
2	9,1	0,05	30	520	280	91	0,3	9
3	9,6	0,2	60	720	360	96	-0,1	5
4	2	0,15	45	2400	1200	64	0,4	6
5	10	0,1	15	9000	3000	400	-0,25	3
6	2	0,2	30	600	300	40	0,2	4
7	50	0,25	60	1800	600	200	-0,6	12
8	4,95	0,08	45	3300	2700	172,8	0,15	6
9	1,5	0,15	30	1800	900	60	0,5	10
0	75	0,3	60	5000	3000	600	-0,2	2

Task D 3.10

Two loads P_1 and P_2 with masses m_1 and m_2 , respectively, are located on a weightless platform B , which is supported by a system of springs with stiffnesses c_1 , c_2 and c_3 as shown in Fig. D3.10. The masses of loads P_1 and P_2 are in the ratio: $m_1/m_2 = n$, and during free oscillations of the specified loads, the cyclic frequency is equal to k . In the position of static equilibrium, platform B is displaced by λ_{cm} .

At the initial moment of time, the load P_2 is removed, while the load P_1 acquires the initial velocity V_0 in the vertical direction. Find the equation of the oscillatory motion of the load P_1 , construct a graph of the coordinate dependence on time, determine the amplitude (A), period (T) and initial phase (φ_0), if the platform B moves forward, and the resistance of the medium is neglected.

Data for final calculations – in table D3.10 for variants

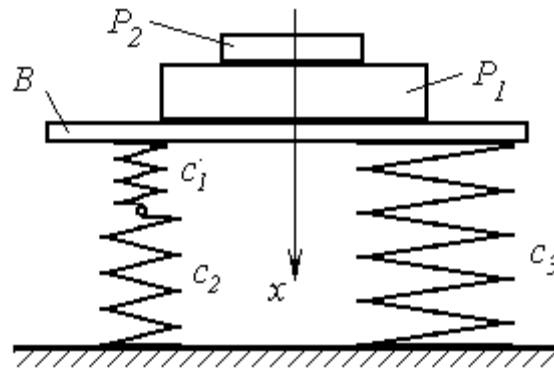


Figure D 3.10

Table D 3.10

Variant	n	λ_{cm}	k	c_1	c_2	c_3	V_0
		sm	s^{-1}	N/sm			m/s
1	1,5	9,8	10	6	18	5	8,3
2	3	9,8	10	20	80	24	5,2
3	2	29,4	20	27	54	30	8
4	3	39,2	5	32	48	10,8	4,5
5	4	3,02	18	22	28	20,08	2,8
6	2	6,8	12	30	50	11,25	9
7	0,6	4,4	15	12	24	10	2
8	3	1,6	25	24	60	30	4
9	2	2,45	20	8	12	13,2	6
0	1,5	12,25	20	10	40	12	10

Task D 3.11

A load P_1 is attached to a rigid weightless rod D , which is suspended from three vertical springs with stiffnesses c_1 and c_2 , as shown in Fig. D3.11. At the initial moment of time, a load P_2 is instantly attached to the load P_1 and the system of loads P_1 and P_2 is given an initial velocity V_0 , directed vertically downward. Assuming that rod D moves translationally, find the equation of motion of loads P_1 and P_2 as one point, as well as the quantities indicated in table D3.11, where the following are indicated: A – amplitude of oscillations, k – cyclic frequency, T – period of oscillations. Construct a graph of the dependence of the coordinate on time.

The specified parameters and values that need to be found are in Table D3.11 by variant.

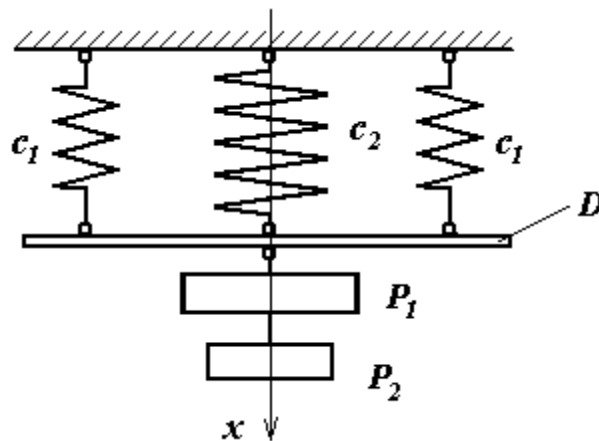


Figure D 3.11

Table D 3.11

Variant	m_1	m_2	c_1	c_2	V_0	k	T	A
	kg		N/m		m/s	s ⁻¹	s	m
1	6	3	100	?	0,4	8	?	?
2	16	12	200	300	-0,6	?	–	?
3	?	1	50	300	0,3	20	?	?
4	4	2	?	100	0,8	10	?	?
5	10	6	25	350	0,7	?	–	?
6	?	3	100	100	-0,4	5	?	?
7	10	5	300	?	0,2	8	?	?
8	8	2	200	90	0,8	?	–	?
9	1,5	2	?	58	-0,6	6	?	?
0	20	4	30	36	0,9	?	–	?

Task D 3.12

On an inclined plane that forms an angle α with the horizon, there is a load P_1 , attached to two springs connected in series with stiffnesses c_1 i c_2 , Fig. D3.12. At some moment in time, a load P_2 is instantly attached to the load P_1 and both are given an initial velocity V_0 , after which they move along the plane so that a friction force with a coefficient f acts on the load P_1 , and the load P_2 moves without friction.

Find the equations of motion of the loads P_1 and P_2 , and plot a graph of the coordinate dependence on time.

The data for the final calculations are in table D3.12 for the options, where m_1, m_2 are the masses of the loads P_1 and P_2 , respectively, A is the amplitude of oscillations, k is the cyclic frequency, and T is the period of oscillations.

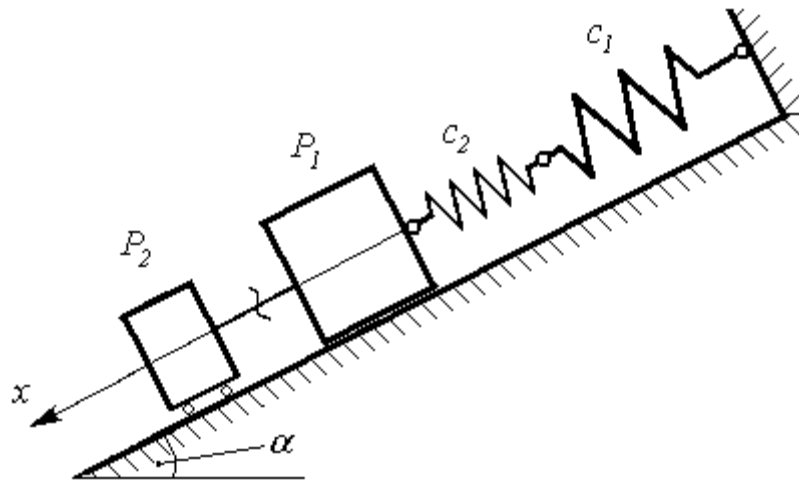


Figure D 3.12

Table D 3.12

Variant	m_1	m_2	c_1	c_2	f	α	V_0	k	T	A
	kg		N/m			grad	m/s	s ⁻¹	s	m
1	2	?	200	150	0,3	60	0,4	5	?	?
2	?	0,1	400	50	0,1	30	-0,2	12	—	?
3	5	3	300	600	0,2	45	0,6	?	?	?
4	0,57	?	100	75	0,2	20	0,3	7	?	?
5	?	1,5	360	180	0,15	15	0,2	4	—	?
6	6	2	250	200	0,25	75	-0,6	?	?	?
7	12	?	100	60	0,2	45	0,5	0,9	?	?
8	?	0,5	40	25	0,1	20	0,1	2	—	?
9	10	2	800	300	0,3	75	0,4	?	?	?
0	40	?	200	75	0,2	60	0,3	0,6	—	?

Task D 3.13

A load P P_1 with a mass m_1 is suspended from a weightless rod B , which is attached to two springs with stiffnesses c_1 , c_2 and a damper D , Fig. D3.13. At some point in time, a load P_2 with a mass m_2 is instantly attached to the load P_1 so that the loads P_1 , P_2 acquire an initial velocity V_0 . Determine the equations of motion of the loads P_1 , P_2 , and construct a graph of the coordinate versus time if the rod B moves forward and the damper D creates a resistance force proportional to the velocity $R = \mu V$.

The data for the final calculations are in Table D3.13 for the options, where A is the amplitude of oscillations, k is the cyclic frequency, T is the period of oscillations, δ is the decrement of damped oscillations.

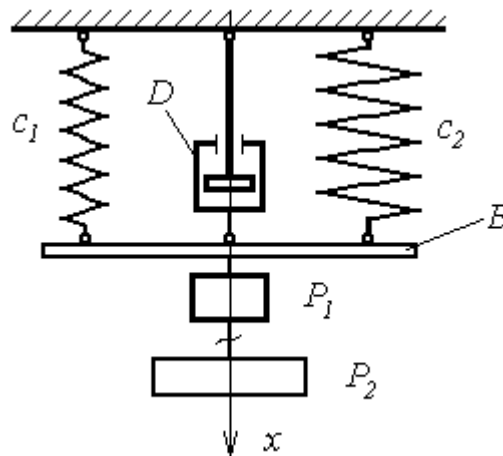


Figure D 3.13

Table D 3.13

Variant	m_1	m_2	c_1	c_2	μ	δ	V_0	k	T	A
	kg		N/m		N s/m		m/s	s ⁻¹	s	m
1	10	4	20	36	28	?	-0,4	?	—	?
2	16	12	10	18	112	—	0,5	—	—	—
3	8	2	300	700	200	—	0,7	—	—	—
4	5	3	200	600	96	?	0,2	?	?	—
5	0,8	1	30,4	34,4	21,6	—	0,4	—	—	—
6	50	20	1200	2230	980	—	-0,2	—	—	—
7	0,5	0,3	100	400	6,4	?	-0,3	?	?	—
8	12	4	700	900	384	—	0,5	—	—	—
9	15	6	200	325	210	—	0,3	—	—	—
0	25	5	920	1000	360	?	0,25	?	—	?

Task D 3.14

Springs with stiffnesses c_1 and c_2 are attached to the vertical slider B , and a load P_1 with mass m_1 is suspended from them, as shown in Fig. D3.14. At some point in time, a load P_2 with mass m_2 is attached to the load P_1 and the slider B begins to perform vertical motion according to the law

$$\xi = H_1 \sin(p_1 t) - H_2 \sin(p_2 t)$$

Determine the equations of motion of the loads P_1 , P_2 as one body and plot the graph of the coordinate versus time if their initial velocity is V_0 . The data for the final calculations are in Table D3.14 for the variants

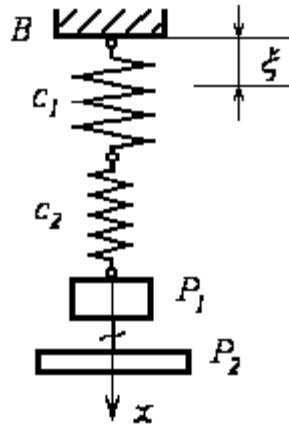


Figure D 3.14

Table D 3.14

Variant	m_1	m_2	c_1	c_2	V_0	H_1	H_2	p_1	p_2
	kg		N/m		m/s	m		s^{-1}	
1	5	1	480	120	2	0	0,12	—	4
2	1,93	1	250	750	1,5	0,37	0	5	—
3	12	3	960	320	4	-0,1	0,2	2	8
4	10	2	300	100	1,2	0,3	0	2,5	—
5	0,4	0,1	200	600	1	0	-0,3	—	12
6	1,7	0,3	800	450	-3	0,25	0	10	—
7	0,4	0,2	160	480	-2	0	0,15	—	3
8	0,15	0,05	180	60	0,5	-0,2	0	15	—
9	0,6	0,4	640	860	5	0	0,2	—	6
0	1,5	0,5	300	600	-0,6	0,15	0	6	—

Task D 3.15

A load P with a mass m is placed on a weightless platform D , which is supported by a system of springs with stiffnesses c_1 , c_2 and c_3 , and the spring with stiffness c_3 is supported by a slider B , as shown in Fig. D3.15. At some point in time, the load P is given an initial velocity V_0 , and the slider B begins to perform vertical motion according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \sin(p_2 t)$$

Determine the equation of motion of the load P and plot the coordinate dependence on time, if at the initial moment of time the system was in a position of static equilibrium, the platform D moves translationally, and there is no friction. The data for the final calculations are in table D3.15 by variants.

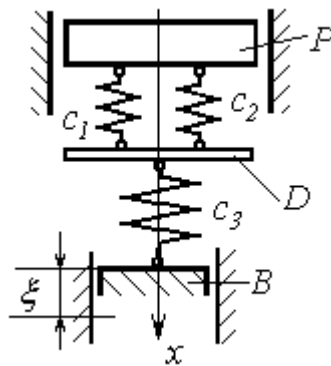


Figure D 3.15

Table D 3.15

Variant	m	c_1	c_2	c_3	V_0	H_1	H_2	p_1	p_2
	kg	N/m	N/m	N/m	m / s	m	m	s ⁻¹	s ⁻¹
1	5	1300	1200	600	2	0	0,3	–	10
2	1	800	600	1000	0,08	0,1	0	15	–
3	5	80	100	60	1	0	0,4	–	9
4	0,25	360	240	120	- 1	0,1	0	20	–
5	11,3	2000	2400	6000	0,5	0,12	0,07	4	6
6	8	120	180	600	- 1,2	0	0,12	–	5
7	3	540	520	720	3	- 0,6	0	12	–
8	0,79	90	150	300	4	0,2	0,1	9	16
9	3,9	180	640	250	5	0,14	0	14	–
0	5,9	320	700	900	- 0,5	0	0,25	–	4

Task D 3.16

A body P with a mass m is attached to horizontal springs with rigid bodies c_1 , c_2 , c_3 and a damper D by means of a weightless rod B , as shown in Fig. D3.16. When the body P moves, the damper creates a resistance force R , which has a direction opposite to the velocity V and is described by the dependence $R = \mu V$, where μ is the viscosity coefficient of the fluid contained in the damper. At the initial moment of time, the body P is given an initial velocity V_0 and a displacement x_0 , after which it moves along the horizontal axis x .

Determine the equation of motion of the load P and plot the coordinate versus time graph if the rod B moves forward, the deviations from the equilibrium position are small and horizontal.

The data for the final calculations are in table D3.16 for the variants, where T is the period of damped oscillations.

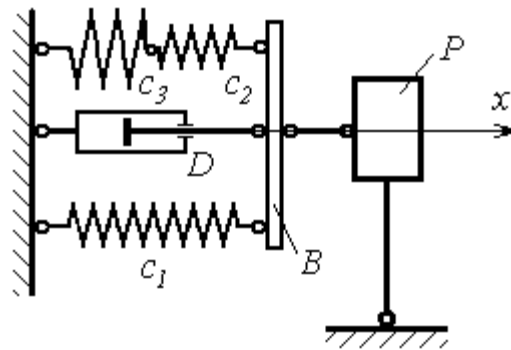


Figure D 3.16

Table D 3.16

Variant	m	c_1	c_2	c_3	μ	x_0	V_0	T
	kg	N/m			N s/m	m	m/s	s
1	14	2950	300	600	560	-0,1	5	—
2	10	1610	100	400	240	0,2	2	?
3	20	961	360	240	320	0,4	-3	—
4	8	38	120	360	240	-0,2	4	—
5	5	1440	450	300	120	0,16	-8	?
6	12	408	36	72	144	0,4	2,4	—
7	6	2200	300	600	144	0,25	3	?
8	4,5	0	120	180	108	-0,3	0,6	—
9	1,5	1110	360	720	15	0,15	8	?
0	18	795	1340	4020	360	0,4	-6	—

Task D 3.17

A body P of mass m falls from a height h onto a horizontal weightless platform B , which is supported by springs of stiffnesses c_1 , c_2 , c_3 and a damper D , as shown in Fig. D3.17. At a height h , the body P is given an initial velocity U in the vertical direction and it falls until it meets the platform B . When the body P moves, the damper together with the platform creates a resistance force R , which has a direction opposite to the velocity V and is described by the dependence $R = \mu V$, where μ is the viscosity coefficient of the fluid contained in the damper.

Determine the equation of motion of the load P and construct a graph of the coordinate dependence on time, if the platform B moves translationally, and the impact of the body P on the platform B is completely inelastic.

The data for the final calculations are contained in table D3.17 for the variants, where T is the period of damped oscillations.

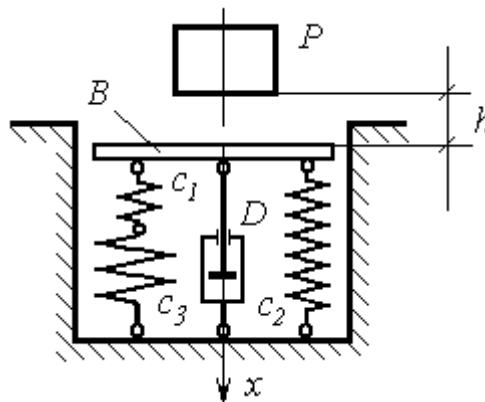


Figure D 3.17

Table D 3.17

Variant	m	h	c_1	c_2	c_3	μ	U	T
	kg	m	N/m	N/m	N/m	N·s/m	m/s	s
1	1.5	0,5	300	1300	600	450	2	—
2	5	1	30	1880	60	160	-0.2	?
3	1.2	1,5	720	240	360	38,4	0.3	?
4	50	0,2	100	720	400	400	-1	—
5	20	1	500	125	1500	160	0.5	?
6	30	0.3	120	400	240	300	-0.4	—
7	16	1,5	2000	8800	3000	128	0.3	?
8	18	2	150	315	350	180	0	—
9	40	0.4	600	480	400	240	1.5	—
0	25	0.8	200	280	300	200	0.2	—

Task D 3.18

On an inclined plane with an angle α to the horizon, there is a load P with a mass m , which is attached to the upper end of the springs connected in series c_1 and c_2 , and a slider B is attached to the lower end of the springs, as shown in Fig. D3.18. The system is in a position of static equilibrium. At some point in time, the slider B begins to move along the inclined plane according to the law

$$\xi = H_1 \cos(p_1 t) + H_2 \sin(p_2 t)$$

Determine the equation of motion of the load P and construct a graph of the coordinate versus time if the coefficient of friction for the load P to the inclined plane is equal to f .

Data for the final calculations are contained in Table D3.18 for the options.

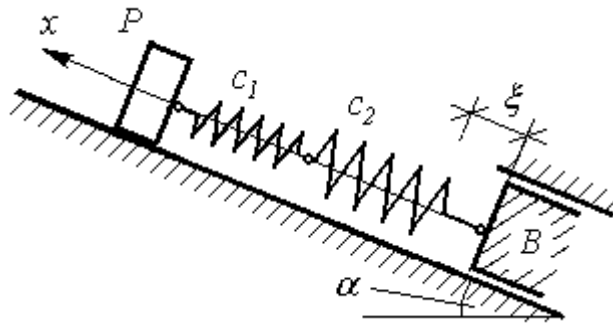


Figure D 3.18

Table D 3.18

Variant	m	f	α	c_1	c_2	H_1	H_2	p_1	p_2
	kg		grad	N/m		m		s^{-1}	
1	4	0,15	30	2000	8000	0	0,051	–	15
2	12	0,2	45	320	480	0,04	0	4	–
3	2	0,1	60	420	780	0	0.02	–	10
4	20,4	0,1	30	150	850	0,2	0	2,5	–
5	2	0,05	15	180	720	0	0,15	–	9
6	2,4	0,2	30	360	720	0,15	0	10	–
7	1,2	0,3	60	160	480	0	0,03	–	6
8	1,5	0,2	75	1000	1500	0,35	0	18	–
9	6,4	0,15	45	1600	3400	0	0,02	–	7
0	0,8	0,3	30	220	280	0,1	0	6	–

Task D 3.19

On an inclined plane with an angle α to the horizon, there is a load P of mass m , which is attached to two springs with stiffnesses c_1 and c_2 and to a damper D , as shown in Fig. D3.19. The system is in a position of static equilibrium. When the body moves, the damper D creates a resistance force depending on the speed: $R = \mu V$. At the initial moment of time, the body P is given an initial displacement x_0 and an initial velocity V_0 , after which it moves along the axis x .

Determine the equation of motion of the load P and plot the coordinate versus time graph if the friction coefficient for the load P to the plane is equal to f .

Data for final calculations are contained in Table D3.19 for the options.

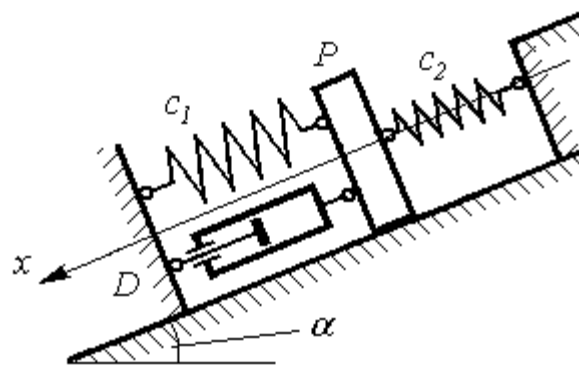


Figure D 3.19

Table D 3.19

Variant N_0	m kg	f	α grad	c_1 N/m	c_2 N/m	μ N·s/m	x_0 m	V_0 m/s
1	9	0,05	15	5400	2700	96	0,09	2
2	4	0	60	600	1000	160	0,1	5
3	36	0,2	75	688	212	936	0,2	8
4	10,5	0,1	30	3200	1000	672	-0,15	6
5	2	0	45	250	200	80	0,4	10
6	14	0,15	30	800	300	280	-0,6	1,2
7	7	0,05	20	1800	2400	700	0,8	2
8	10	0,3	60	3000	2000	250	0,3	0,8
9	36	0	75	8000	1200	624	0,4	-5
0	5	0,25	45	7200	2400	50	-0,2	3

Task D 3.20

The load P_1 is located on a weightless platform B , which is supported by a system of springs with stiffnesses c_1 , c_2 and c_3 as shown in Fig. D3.20. During free oscillations of the load P_1 , the cyclic frequency is k , and in the position of static equilibrium the platform B is displaced by λ_{st} .

At the initial moment of time, the load P_2 is placed on the load P_1 , and they acquire an initial velocity V_0 in the vertical direction. The masses of the loads P_1 and P_2 are in the ratio: $m_2/m_1 = n$. Find the equation of the oscillatory motion of the loads P_1 and P_2 as one body, construct a graph of the coordinate dependence on time, determine the amplitude (A), period (T) and initial phase (φ_0), if the platform B moves forward and the resistance of the medium is neglected.

Data for final calculations are contained in Table D3.20 by variant.

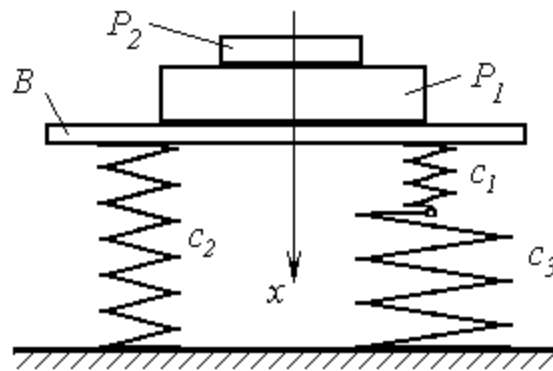


Figure D 3.20

Table D 3.20

Variant	n	λ_{st}	k	c_1	c_2	c_3	V_0
		sm	s^{-1}	N/sm			m/s
1	0,5	0,392	50	10	5	30	2
2	1,5	1,09	30	6	14	12	5
3	0,2	2,45	20	4	16,8	16	4
4	0,6	2,45	20	20	25	60	3
5	2	0,512	25	12	11	36	1
6	1	3,02	18	40	36	60	2
7	0,5	9,81	10	21	16	42	2,5
8	1,2	0,613	40	48	168	96	3,2
9	0,8	5	14	32	10,8	48	7
0	0,4	3,83	16	30	48	20	3,5

Task D 3.21

A rigid weightless rod D , suspended from four vertical springs with stiffnesses c_1 , c_2 and c_3 , as shown in Fig. D3.21, is attached to a load P_1 , and a load P_2 to it. At the initial moment of time, the load P_2 is instantly disconnected from the load P_1 and the initial velocity P_1 is given to the load P_1 , directed vertically.

Assuming that the rod D moves translationally, find the equation of motion of the load P_1 , as well as those quantities that are indicated in table D3.21, where it is indicated: A is the amplitude of oscillations, k is the cyclic frequency, T is the period of oscillations. Construct a graph of the dependence of the coordinate on time.

The given parameters and quantities that need to be found are in table D3.21 by variants.

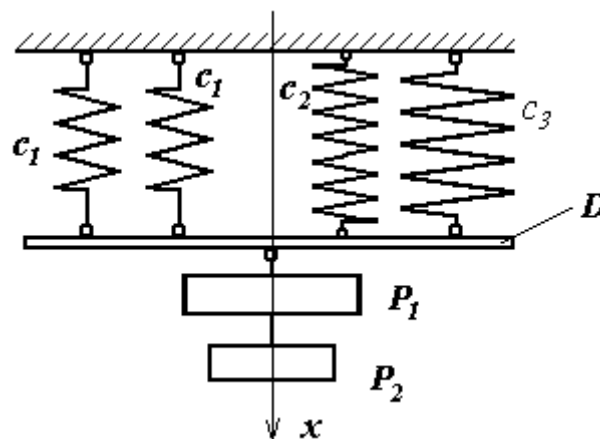


Figure D 3.21

Table D 3.21

Variant	m_1	m_2	C_1	C_2	C_3	V_0	k	T	A
	kg		N/sm			m/s	s^{-1}	s	m
1	?	0,5	80	60	100	4	40	?	?
2	5	2	20	10	14,8	8	?	?	?
3	15	6	6	18	30	-1	?	?	?
4	?	40	42	16	20	?	10	?	0,5
5	?	4,4	25	80	30	2	25	?	?
6	3	1	72	26	130	-5	?	?	?
7	25	10	55	60	80	3	?	?	?
8	?	6	88	14	220	-2	50	?	?
9	16	3	32	16	20	66	?	?	?
0	?	60	27	26	80	-4	?	0,63	?

Task D 3.22

On an inclined plane that forms an angle α with the horizon, two loads P_1 and P_2 are placed, which are attached to a system of springs with stiffnesses c_1 , c_2 and c_3 as shown in Fig. D3.22. At some point in time, the load P_2 is instantly disconnected from the load P_1 and given an initial velocity V_0 , after which it moves along a plane where there is no friction. Find the equation of motion of the load P_1 , and construct a graph of the coordinate dependence on time.

The data for the final calculations are contained in Table D3.22 for the variants, where m_1 , m_2 are the masses of the loads P_1 and P_2 , respectively, A is the amplitude of oscillations, k is the cyclic frequency, T is the period of oscillations.

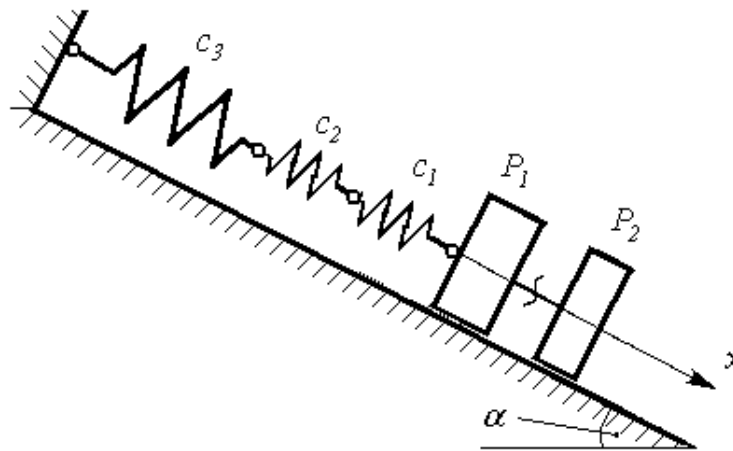


Figure D 3.22

Table D 3.22

Variant	m_1	m_2	c_1	c_2	c_3	α	V_0	k	T	A
	kg		N/sm			deg	m/s	s^{-1}	s	m
1	1,21	0,5	10	15	25	30	2	?	?	?
2	1,7	0,4	36	48	60	45	5	?	?	?
3	?	8	80	120	160	15	-3	25	?	?
4	2,1	1,5	42	60	80	75	-4	?	?	?
5	?	4	35	40	90	60	2,5	15	?	?
6	?	10	48	90	100	30	-2	?	0,63	?
7	2,09	?	12	42	80	45	0	?	?	0,01
8	?	4	45	60	120	15	3	12	?	?
9	?	2	28	42	60	75	4	?	0,42	?
0	1,2	0,7	35	40	80	60	8	?	?	?

Task D 3.23

To a weightless rod B , attached to three springs with stiffnesses c_1 , c_2 , c_3 and a damper D , are suspended loads P_1 and P_2 , Fig. D3.23, with masses m_1 , m_2 , respectively. At some moment in time, the load P_2 is instantly disconnected from the load P_1 and given an initial velocity V_0 . Determine the equation of motion of the load P , construct a graph of the coordinate versus time, if the rod B moves forward, and the damper D creates a resistance force proportional to the velocity $R = \mu V$.

The data for the final calculations are in Table D3.23 for the variants, where A is the amplitude of oscillations, k is the cyclic frequency, T is the period of damped oscillations.

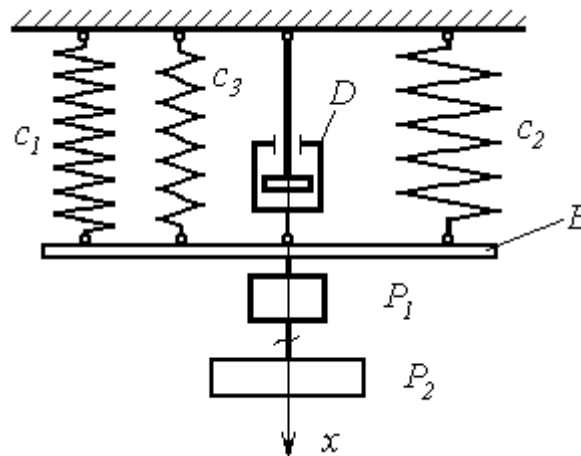


Figure D 3.23

Table D 3.23

Variant	m_1	m_2	c_1	c_2	c_3	μ	V_0	k	T	A
	kg		N/sm			N s/m	m/s	s ⁻¹	s	m
1	2,15	0,7	45	80	90	35	2	?	?	?
2	100	30	30	35	60	4000	-5	—	—	—
3	130	50	25	75	30	2600	6	—	—	—
4	32	12	40	95	65	640	2,5	?	?	?
5	8	4	30	60	40	800	-3	—	—	—
6	32	6	28	72	100	3200	8	—	—	—
7	46	20	100	60	25	920	-4	?	?	?
8	25	10	18	32	50	1500	3,5	—	—	—
9	40	15	35	45	80	1600	7	—	—	—
0	7,6	4	48	52	90	456	1,5	?	?	?

Task D 3.24

A system of three springs with stiffnesses c_1 and c_2 is attached to the vertical slider B , and loads P_1 and P_2 with masses m_1 and m_2 , are suspended from it, respectively, Fig. D3.24. At some point in time, load P_2 is instantly disconnected from load P_1 , while load P_1 acquires a velocity V_0 , and slider B begins to perform vertical motion according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \sin(p_2 t)$$

Determine the equation of motion of the load P_1 and plot the coordinate dependence on time. We neglect the mass of the connecting rod D .

Data for final calculations – in table D3.24 for variants

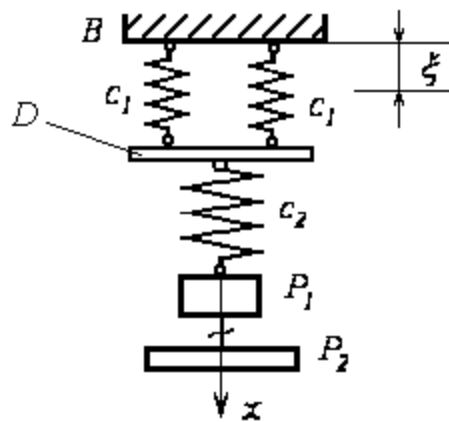


Figure D 3.24

Table D 3.24

Variant	m_1	m_2	c_1	c_2	V_0	H_1	H_2	p_1	p_2
	kg		N/sm		m/s	sm		s^{-1}	
1	6	2	20	60	-2	15	0	20	–
2	3	1	30	90	5	8	2	10	15
3	2,88	0,6	15	45	0	0,25	0	12	–
4	40	10	40	80	3,5	0	0,8	–	9
5	9,88	5	25	40	4	0,4	0	15	–
6	120	20	16	48	6	0	0,05	–	10
7	38,5	15	28	44	0	2,12	0	4	–
8	4,2	3	35	30	-4	0	0,15	–	20
9	9	2,5	32	36	2,5	0,35	0	16	–
0	3	1,5	80	45	8	0,2	-0,4	5	8

Task D 3.25

A body P with mass m is placed on a horizontal platform D_1 , which is connected to a system of springs with stiffnesses c_1 and c_2 , and a slider B acts on the springs with stiffness c_1 in the horizontal direction, as shown in Fig. D3.25. At some point in time, the body P is given an initial velocity V_0 , and the slider B begins to move according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \sin(p_2 t).$$

Determine the equation of motion of body P and plot the coordinate dependence on time if bodies D_1 and D_2 are considered weightless, move translationally, and there is no friction.

Data for final calculations are contained in table D3.25 for variants.

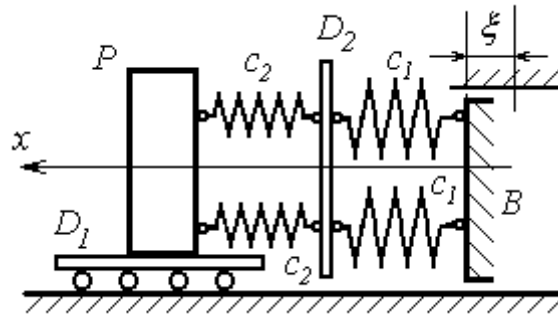


Figure D 3.25

Table D 3.25

Variant	m	c_1	c_2	V_0	H_1	H_2	p_1	p_2
	kg	N / sm		m / s	sm		s^{-1}	
1	6,4	30	60	2	0,8	0	25	—
2	6	25	75	5	0	-0,73	—	20
3	4,24	16	18	0	0,45	0,3	8	12
4	12	40	60	-2,5	0	0,5	—	15
5	12,18	42	58	4	0,5	0	20	—
6	11,2	50	90	3	0	0,24	—	14
7	8,75	14	56	-1,5	0,6	-0,2	10	6
8	6	20	30	4,2	0	0,4	—	20
9	19	60	80	-3	0,3	-0,25	6	18
0	20	24	36	1,8	2,4	0	12	—

Task D 3.26

A body P with a mass m is connected to horizontal springs with stiffnesses c_1 , c_2 , c_3 and a damper D using a weightless rod B , as shown in Fig. D3.26. When the body P moves, the damper creates a resistance force R , which has a direction opposite to the velocity V and is described by the dependence $R = \mu V$, where μ is the viscosity coefficient of the fluid contained in the damper. At the initial moment of time, the body P is given an initial velocity V_0 and a displacement x_0 , after which it moves along the horizontal axis x .

Determine the equation of motion of the body P and construct a graph of the coordinate versus time, if the rod B moves forward, the deviations from the equilibrium position are small and horizontal.

Data for final calculations are in table D 3.26 by options.

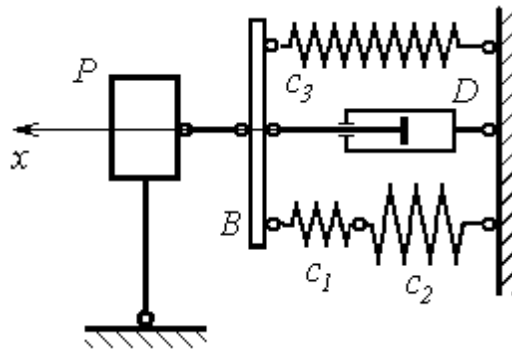


Figure D 3.26

Table D 3.26

Variant	m	c_1	c_2	c_3	μ	x_0	V_0
	kg	N/sm			N · s/m	m	m/s
1	10	12	36	5,4	140	0,3	-5
2	60	15	45	3,75	1200	0,01	2
3	45	30	60	25	900	0,02	4
4	6,67	18	32	3,48	54	-0,2	3
5	100	48	92	4,46	1600	0,15	1,8
6	30	25	50	13,33	600	-0,25	3,2
7	7,5	52	48	5,64	150	0,04	-2
8	61	62	28	10,73	1500	-0,03	2,5
9	50	22	66	15,5	800	0,2	-4
0	6,4	20	60	25	154	0,4	0,8

Task D 3.27

A body P with a mass m slides along an inclined smooth plane with an angle α to the horizon, after which it falls onto a weightless platform B , which is supported by springs of stiffnesses c_1 , c_2 , c_3 and a damper D , as shown in Fig. D3.27. At the beginning of the movement, the body P is given an initial velocity U in the direction of the plane and it travels a distance l before touching the platform B . When the body P moves, together with the platform, the damper D creates a resistance force R , which has a direction opposite to the velocity V and is described by the dependence $R = \mu V$, where μ is the viscosity coefficient of the fluid contained in the damper.

Determine the equation of motion of body P and plot the coordinate dependence on time if platform B moves translationally and the impact of body P on platform B is completely inelastic.

Data for final calculations in table D3.27 for variants.

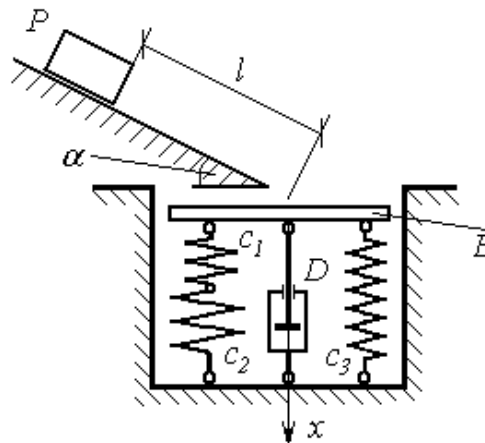


Figure D 3.27

Table D 3.27

Variant	m	l	α	c_1	c_2	c_3	μ	U
	kg	m	deg	N/sm	N/sm	N/sm	N s/m	m/s
1	13,33	2	30	25	75	11,25	320	0,5
2	12	1,5	45	10	50	16,7	440	1
3	60	1	60	40	60	36	1200	0
4	6,4	3	15	36	48	19,43	205	0,03
5	62,5	0,5	75	30	80	18,2	1750	0
7	16	0,6	20	28	56	17,3	480	0,2
7	15,6	2,5	60	45	80	11,2	440	0,6
8	40	10	15	8	32	3,6	800	0
9	11,1	1,6	75	12	48	6,4	267	2
0	17,8	0,75	45	17	56	27	285	4

Task D 3.28

On an inclined plane with an angle of 30° to the horizon, there is a load P with a mass m , which is attached to the lower end of the system of springs c_1, c_2 and c_3 , and a slider B is attached to the upper end of the springs, as shown in Fig. D3.28. The system is in a position of static equilibrium. At some point in time, the slider B begins to move along the inclined plane according to the law

$$\xi = H_1 \sin(p_1 t) + H_2 \cos(p_2 t).$$

Determine the equation of motion of the load P and plot the coordinate versus time graph if there is no friction and the mass of the plate D , which moves translationally, is neglected.

Data for final calculations are contained in table D3.28 for variants.

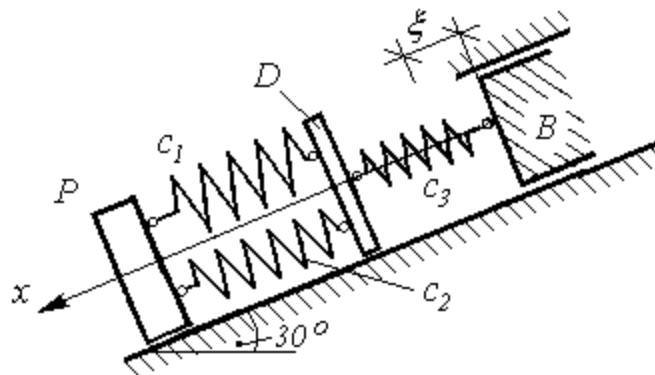


Figure D 3.28

Table D 3.28

Variant	m	c_1	c_2	c_3	H_1	H_2	p_1	p_2
	kg	N/sm			sm		s^{-1}	
1	6	15	45	40	0,4	0	20	—
2	12	20	40	15	0	0,8	—	6
3	24	32	64	54	2,5	4	6	5
4	7	48	50	80	3	0	25	—
5	16	24	36	90	0	0,5	—	10
6	8,2	20	70	120	5	-2	8	12
7	72	25	30	100	-0,36	0	14	—
8	5,3	60	40	90	0	0,8	—	30
9	8	30	50	110	3,5	0	10	—
0	10	40	90	120	0	0,4	—	15

Task D 3.29

On an inclined plane with an angle α to the horizon, there is a load P with a mass m , which is attached to the upper end of the system of springs c_1 , c_2 , c_3 and to the damper D , which is shown in Fig. D3.29. The system is in a position of static equilibrium. When the body moves, the damper D creates a resistance force depending on the speed: $R = \mu V$. At the initial moment of time, the body P is given an initial displacement x_0 and an initial velocity V_0 , after which it moves along the axis x .

Determine the equation of motion of the load P and construct a graph of the coordinate versus time if there is no friction and the mass of the body B , which moves translationally, is neglected.

The data for the final calculations are contained in Table D3.29 for the variants.

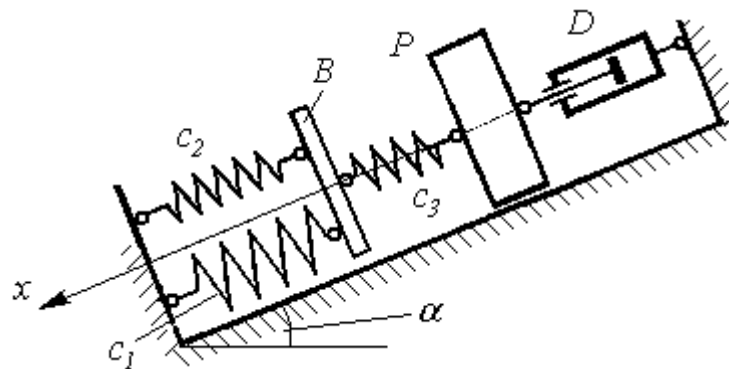


Figure D 3.29

Table D 3.29

Variant	m	α	c_1	c_2	c_3	μ	x_0	V_0
	kg	deg	N/sm			N s/m	sm	m/s
1	21	60	64	22	80	588	5	4
2	14	30	36	40	90	280	8	3
3	9,4	60	25	90	120	470	-2	6
4	20	45	84	110	140	800	7	-5
5	5	20	46	50	85	80	12	0
6	16	30	32	46	70	576	0	-4
7	31	75	85	105	110	930	10	0
8	9	60	60	85	90	270	0	2,5
9	17	15	35	90	70	680	7,5	0
0	15	45	85	120	140	600	-3	8

Task D 3.30

Two loads P_1 and P_2 with masses m_1 and m_2 , respectively, are located on a weightless platform B , which is supported by a system of springs with stiffnesses c_1 , c_2 , c_3 and c_4 as shown in Fig. D3.30. The masses of loads P_1 and P_2 are in the ratio $m_1/m_2 = n$, and during free oscillations of the specified loads, the cyclic frequency is k . In the position of static equilibrium, platform B is displaced by λ_{cm} .

At the initial moment of time, the load P_2 is removed, while the load P_1 acquires the initial velocity V_0 in the vertical direction. Find the equation of the oscillatory motion of the load P_1 , plot a graph of the coordinate versus time, determine the amplitude (A), period (T) and initial phase (φ_0), if the platform B moves forward, and the resistance of the medium is neglected.

Data for the final calculations are contained in table D3.30 for variants.

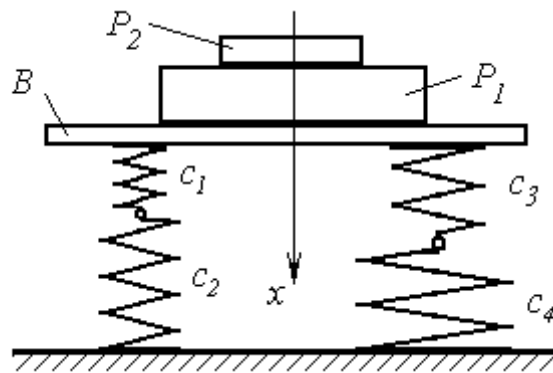


Figure D 3.30

Table D 3.30

Variant	n	λ_{st}	k	c_1	c_2	c_3	c_4	V_0
		sm	s^{-1}	N/sm				m/s
1	1,5	4,35	15	40	30	70	120	2
2	0,5	1,08	30	42	100	80	90	-4
3	0,7	5	14	18	32	40	65	1,5
4	2	11,9	25	25	60	45	90	5
5	1	39,2	5	68	80	45	120	-3
6	0,4	6,8	12	15	45	42	84	2,5
7	1,2	2,45	20	32	42	70	140	-2,4
8	0,75	1,7	24	75	90	85	130	7
9	0,6	9,8	10	24	65	72	160	9
0	0,8	15,3	8	30	80	90	110	-1

3.5 Example of completing the task for D3

A load of mass m_1 is suspended from two vertical springs connected in series with stiffnesses c_1 and c_2 and a damper D parallel to them, as shown in Fig. D3.31. At the initial moment of time, a load of mass m_2 is instantly attached to the load of mass m_1 , which is in a state of static equilibrium, so that both loads as a whole acquire a vertical velocity V_0 directed downwards. During the movement of the loads, the damper D creates a resistance force proportional to the velocity: $R = \mu V$.

Determine the dependence of the cargo coordinate on time $x = x(t)$, and plot this dependence. In the final calculations, accept

$m_1 = 20 \text{ kg}$, $m_2 = 12 \text{ kg}$, $c_1 = 10 \text{ N/sm}$, $c_2 = 40 \text{ N/sm}$, $\mu = 192 \text{ N s/m}$, $V_0 = 2 \text{ m/s}$.

The solution

Let us find the stiffness of the spring, which will be equivalent to springs with stiffness c_1 and c_2 . In series connection, the value of c is equal to

$$c = \frac{c_1 c_2}{c_1 + c_2} = \frac{10 \cdot 40}{10 + 40} = 8 \frac{\text{N}}{\text{sm}} = 800 \frac{\text{N}}{\text{m}}. \quad (3.12)$$

Let us show in Fig. D3.32, along with Fig. D3.31, the calculated diagram of the forces acting on the system of loads m_1 and m_2 at an arbitrary moment of motion.

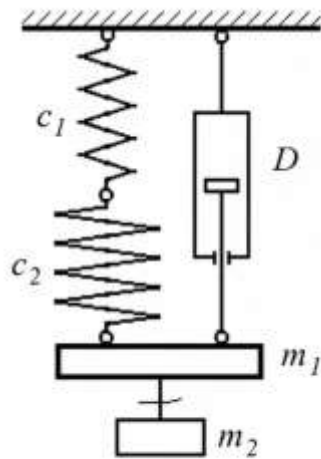


Figure D 3.31

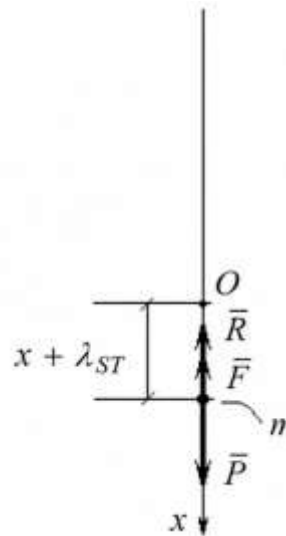


Figure D 3.32

The forces shown in Fig. D3.32 mean: $P = mg$ – total weight of two loads, $m = m_1 + m_2$, $R = \mu V$ – damper resistance force,

$F = c(x + \lambda_{ST})$ – сила реакції пружини.

We compose the differential equation of motion of a point of total mass m in the projection onto the axis Ox .

$$\sum_{k=1}^n X_k = P - F - R. \quad (3.13)$$

Then the differential equation (3.1) reduces to the form

$$m\ddot{x} = mg - c(x + \lambda_{cm}) - \mu V. \quad (3.14)$$

In the static equilibrium position, $mg = c\lambda_{cm}$ (3.15)

After simplification from (3.14) we obtain

$$\ddot{x} + \frac{\mu}{m}\dot{x} + \frac{c}{m}x = 0. \quad (3.17)$$

where $\dot{x} = V$, and c is determined by the formula (3.12).

Let's find the numerical values of the coefficients at $\dot{x}$ i x .

$$\frac{\mu}{m} = \frac{\mu}{m_1 + m_2} = \frac{192}{20 + 12} = 6s^{-1}, \quad \frac{c}{m} = \frac{c}{m_1 + m_2} = \frac{800}{20 + 12} = 25s^{-2}.$$

Differential equation with numerical coefficients (3.17) shall be written as

$$\ddot{x} + 6\dot{x} + 25x = 0. \quad (3.18)$$

Comparing (3.18) i (3.3) we determine that $n = 3$, $k = 5$, that is $k > n$. Following the example of (3.7), we write down the solution for (3.18), having previously established that

$$k_1 = \sqrt{k^2 - n^2} = \sqrt{5^2 - 3^2} = 4s^{-1},$$

$$x = e^{-3t} [C_1 \sin(4t) + C_2 \cos(4t)]. \quad (3.20)$$

Let us find the derivative of (3.20) and use the initial conditions to find the constants of integration C_1 and C_2 :

$$x = \lambda_{ST2} = -\frac{m_2 g}{c} = -\frac{12 \cdot 9,8}{800} = -0,147m, \quad \dot{x} = V_0 = 2 \frac{m}{s} \quad \text{when } t = 0.$$

$$\dot{x} = -3e^{-3t} [C_1 \sin(4t) + C_2 \cos(4t)] + e^{-3t} [C_1 4 \cos(4t) - C_2 4 \sin(4t)].$$

As for the C_1 and C_2 , we obtain a system of equations

$$\begin{aligned} -0,147 &= 0 + C_2 \\ 2 &= -3C_2 + 4C_1, \end{aligned}$$

Form which $C_1 = 0,389 m = 38,9 sm$, $C_2 = -0,147m = -14,7sm$.

Substituting C_1 and C_2 in (3.20) we obtain the desired equation of motion

$$x = e^{-3t} [38,9 \sin(4t) - 14,7 \cos(4t)]. \quad (3.21)$$

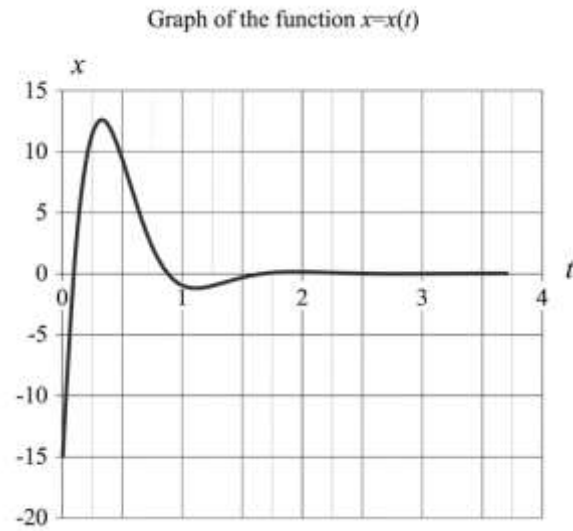


Figure Д3.33

Fig. D3.33 shows the graph of dependence (3.21), constructed using the EXEL program.

Function (3.21) and its graph in Fig. D3.33 indicate that the loads perform damped oscillations.

4 DYNAMICS OF A SYSTEM OF MATERIAL POINTS

4.1 General information about the mechanical system

A system of material points is a set of material points whose positions and motions are interconnected. There are free and non-free systems.

A system of material points is a set of material points whose positions and motions are interconnected. There are **free** and **non-free** systems.

If the motion of the points of the system is not subject to any predetermined restrictions that do not depend on the law of motion, then the system is called free. A system of material points whose motion is subject to constraints is called non-free.

Constraints are divided into **geometric** and **kinematic**. Geometric constraints impose restrictions on the coordinates of the points of the system. The equations of a geometric constraint have the form $f(t, x_i, y_i, z_i) = 0$, $(i = 1, 2, 3 \dots n)$.

Kinematic constraints impose restrictions on the velocities of points in the system. The kinematic constraint equations are written in the form $f(t, x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i) = 0$, $(i = 1, 2, 3 \dots n)$.

According to the classification of the German physicist Hertz, bonds are divided into **holonomic** and **non-holonomic**.

Holonomic are the equations whose equations can be integrated. Nonholonomic are the equations whose differential equations explicitly include velocities so that there is no integrating factor for these equations and the equations cannot be integrated.

Forces acting on a mechanical system are divided into:

1) external and internal;

2) active and connection reactions.

External forces $\bar{F}^e$ – these are the forces of interaction between material points of a certain system and other physical bodies that are not part of the system.

Internal forces $\bar{F}^i$ – forces of interaction between material points of the same system.

Properties of internal forces

1. 1. The main vector of the internal forces of the system is zero:

$$\bar{F}^i = \sum_{i=1}^n \bar{F}_i^i = 0, \quad i = 1, 2, 3 \dots n.$$

2. 2. The principal moment of the internal forces of the system relative to some point, for example O , is zero.:

$$\bar{M}_0^i = \sum_{i=1}^n \bar{M}_0(\bar{F}_i^i) = 0, \quad i = 1, 2, \dots n.$$

Active forces $-\bar{F}$, connection reaction $-\bar{R}$. This classification is used in the case of a non-free system of material points. The reactions of the joints are the forces with which the joints act on the system of material points.

The reactions of the connections, unlike the active forces, are unknown quantities and in the general case depend on the law of motion of the mechanical system..

Differential equation of motion of a system of material points

The differential equation of motion of a free system of material points has the form:

$$m_i \bar{a}_i = \bar{F}_i^e + \bar{F}_i^i, \quad i = 1, 2, 3 \dots n$$

or in coordinate form:

$$m_i \ddot{x}_i = X_i^e + X_i^i,$$

$$m_i \ddot{y}_i = Y_i^e + Y_i^i,$$

$$m_i \ddot{z}_i = Z_i^e + Z_i^i, \quad i = 1, 2, 3 \dots n.$$

The differential equation of motion of a non-free system of material points has the form

$$m_i \cdot \bar{a}_i = \bar{F}_i + \bar{R}_i, \quad i = 1, 2, 3 \dots n,$$

where $\bar{F}_i$ – resultant of active forces applied to the i -point of the system,

$\bar{R}_i$ – resultant reaction of the connections applied to the i -point of the system.

This equation is in coordinate form:

$$m_i \frac{d^2 x_i}{dt^2} = X_i + R_{ix},$$

$$m_i \frac{d^2 y}{dt^2} = Y_i + R_{iy},$$

$$m_i \frac{d^2 z}{dt^2} = Z_i + R_{iz},$$

where X, Y, Z – projection of the resultant of active or given forces on the axis of Cartesian coordinates;

R_{ix}, R_{iy}, R_{iz} – projections of the resultant reaction $\bar{R}_i$ onto the same coordinate axes.

4.2 The center of mass of the system

The mass of the system, consisting of n material points M , is called the sum of the masses of the points of the systems

$$m = \sum_{i=1}^n m_i.$$

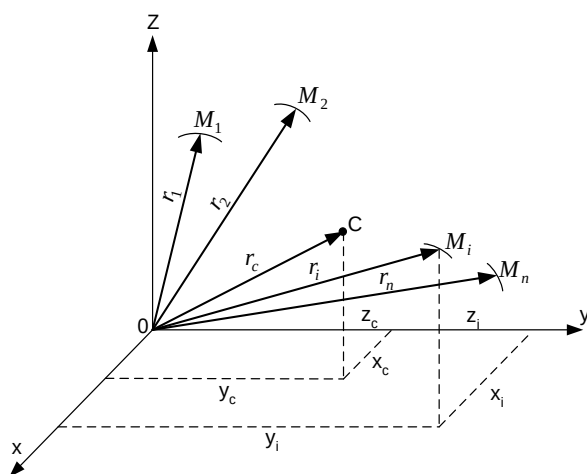


Figure 4.1

The center of mass of a mechanical system (рис. 4.1) is called a geometric point C, the position of which is determined by the radius vector

$$\bar{r}_c = \frac{\sum_{i=1}^n m_i \cdot r_i}{\sum_{i=1}^n m_i},$$

where $\sum_{i=1}^n m_i = m$ – mass of the system.

The coordinates of the center of mass of the system are determined by the formulas

$$x_C = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}; \quad y_C = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}; \quad z_C = \frac{\sum_{i=1}^n m_i z_i}{\sum_{i=1}^n m_i}.$$

The center of gravity of a body geometrically always coincides with the center of mass.

Self-awareness questions

1. What is a system of material points called?
2. What classification of bonds was proposed by the German physicist G. Hertz?
3. What classifications of forces are used in mechanics? What is their convention?
4. What do the differential equations of free and non-free systems of material points have in common and how do they differ?
5. What is the center of mass of the system called and by what formulas are its coordinates calculated?
6. What properties do internal forces have? What is the principal vector and principal moment of internal forces for a rigid body?

4.3 Elements of mass geometry

Moments of inertia of a mechanical system. Radius of inertia

When studying the motions of a mechanical system, it is not enough to know its mass and the position of the center of mass. To characterize the distribution of masses in systems whose elements participate in rotational motion, the concept of moment of inertia is introduced. There are moments of inertia relative to an arbitrary center (*polar moment of inertia*), relative to an axis (*axial moment of inertia*), relative to a plane (*planar moment of inertia*), and also *centrifugal* moments of inertia.

The moment of inertia of a material point relative to the axis (axial moment of inertia), is called the product of the mass of this point m by the square of its distance h from this axis, for example Oz :

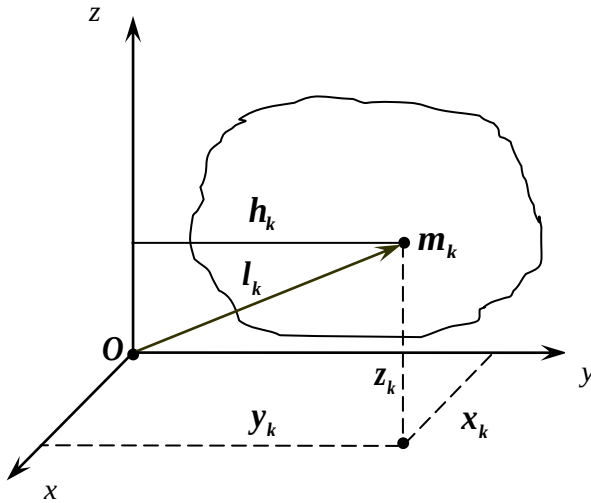
$$I_z = m \cdot h^2.$$

The moment of inertia of the system, consisting of n material points, is called the sum of the products of the masses of the points of the system by the squares of the distances relative to the axis h_k from points to axis $I_z = \sum_{k=1}^n m_k h_k^2$.

In the case of a continuous mass distribution, instead of a sum, there will be an integral that is extended over the entire mass.

The moment of inertia of a solid body about an axis, for example Oz , is called the integral that is extended over the entire mass and has the form

$$I_z = \int_{(m)} h_z^2 dm \text{ or}$$



$$I_x = \int_{(m)} (y^2 + z^2) \cdot dm,$$

$$I_y = \int_{(m)} (x^2 + z^2) \cdot dm,$$

$$I_z = \int_{(m)} (y^2 + x^2) \cdot dm,$$

where h_z – distance from the axis of the body part by mass dm ;

x, y, z – body part coordinates.

Figure 4.2

The polar moment of inertia of a mechanical system consisting of n material points, relative to some center O , is the sum of the products of the masses of the points of the system by the squares of their distances from this center (Fig. 4.2):

$$I_0 = \sum_{k=1}^n m_k l_k^2.$$

Polar moment of inertia of a body about the pole O:

$$I_0 = \int_{(m)} (x^2 + y^2 + z^2) \cdot dm.$$

Note. Polar and axial moments of inertia are always positive ($I_0, I_x, I_y, I_z > 0$).

The planar moment of inertia is a scalar quantity equal to the sum of the products of the mass of each point of the system by the square of the distance of this point to a certain plane.

Planar moments of inertia of a body relative to coordinate planes:

$$I_{xoy} = \int_{(m)} z^2 dm; \quad I_{xoz} = \int_{(m)} y^2 dm; \quad I_{yoz} = \int_{(m)} x^2 dm.$$

In addition to axial and polar moments of inertia, **centrifugal moments of inertia (CMI)** are used when solving some dynamics problems (for example, in the theory of rotor balancing).

The centrifugal moment of inertia of a mechanical system is called the quantity equal to the sum of the products of the mass of each point of the system by its two coordinates, i.e. $I_{yz} = \sum_1^n m_k y_k z_k$; $I_{xz} = \sum_1^n m_k x_k z_k$; $I_{xy} = \sum_1^n m_k x_k y_k$ or in the case of a continuous mass distribution, instead of the sum there will be an integral that is extended over the entire mass: $I_{xy} = \int_{(m)} xy dm$; $I_{xz} = \int_{(m)} xz dm$; $I_{yz} = \int_{(m)} yz dm$.

Centrifugal moments of inertia depend not only on the direction of the coordinate axes, but also on the choice of the origin of coordinates. Therefore, when we talk about the centrifugal moment of inertia at a certain point, we mean that the origin of coordinates coincides with this point.

Unlike axial moments, centrifugal moments of inertia can be positive, negative, or turn into zero.

The principal axes of inertia are called axes about which the centrifugal moments of inertia are zero.

The central axial moment of inertia is the moment of inertia of a body about an axis passing through the center of mass.

The relationship between polar, axial and planar moments of inertia is determined by the formulas

$$I_x + I_y + I_z = 2I_0;$$

$$I_{xoy} + I_{xoz} + I_{yoz} = I_0.$$

That is, the sum of the axial moments of inertia is equal to twice the polar moment of inertia, and the sum of the planar moments of inertia is equal to twice the polar moment of inertia.

Properties of the principal axes of inertia

1. The principal axis of inertia is always perpendicular to the plane of the material system of the body.

2. If the body has an axis of material symmetry, then this axis will be the principal central axis.

In some cases, the concept of the radius of inertia or the arm of inertia is used to calculate the moments of inertia.

If the body has an analytically indeterminate shape, then the moment of inertia is found by the general formula $I_z = m \cdot \rho^2$, где ρ – radius of inertia.

Radius of inertia of the body ρ is the distance from the axis of rotation to the point where the mass of the body m must be concentrated to obtain the equivalent moment of inertia $\rho = \sqrt{I_z / m}$.

The Huygens-Steiner theorem

The moments of inertia of a given body about different axes will have different values. The relationship between the moments of inertia of a body about two parallel axes is determined by the Huygens-Steiner theorem.

The moment of inertia I_z of a body about an arbitrary axis is equal to the sum of the moment of inertia of this body about the axis Z_C that passes parallel to it through the center of mass of the system, and the product of the mass of the body and the square of the distance between the axes:

$$I_z = I_{zC} + mh^2,$$

where h – the distance between the axes.

Calculation of axial moments of inertia of some homogeneous bodies

1. A thin uniform rod of length and mass M .

Let us calculate the moment of inertia of the rod relating the axis Az which passes through its end A perpendicular to the axis of the rod. The coordinate axis Ax is directed along the rod (Fig. 4.3).

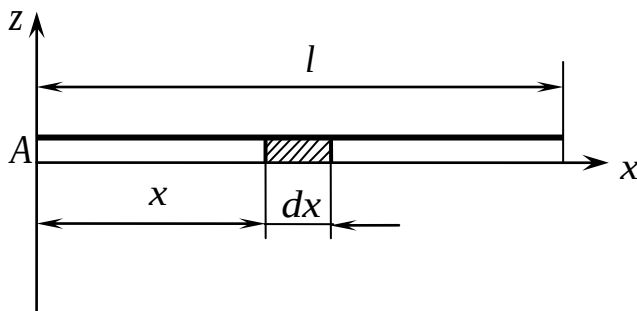


Figure 4.3

Since the rod is homogeneous, the mass of an elementary segment with length dx , which is located at a distance x from the axis Az ,

$$dm = \frac{M}{l} dx.$$

Then:

$$I_{Az} = \int_0^l x^2 dm = \frac{M}{l} \int_0^l x^2 dx = \frac{1}{3} Ml^2.$$

2. A thin uniform ring of mass M and radius R .

Let us determine the moment of inertia of the ring relative to the axis Cz passing through its center of mass C perpendicular to the plane of the ring (Fig. 4.4). Since all elementary masses of the ring are at the same distance R from the axis Cz , then:

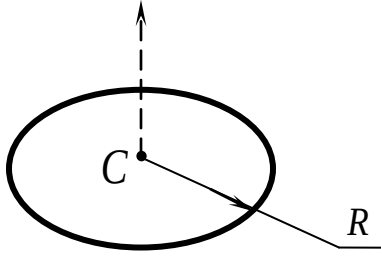


Figure 4.4

$$I_{Cz} = \sum_1^n m_k \cdot R^2 = \left(\sum_1^n m_k \right) \cdot R^2 = M \cdot R^2.$$

3. A circular uniform plate or cylinder with radius R and mass M .

Let us calculate the moment of inertia of a circular plate about an axis perpendicular to the plate (Fig. 4.5). The area of an elementary ring with radius r and width dr is $2\pi r \cdot dr$. Mass per unit area $\frac{M}{\pi R^2}$. Then the mass of the elementary ring is

$$dm = 2 \frac{M}{R^2} r dr, \text{ and the moment of inertia } dI_{Cz} = r^2 \cdot dm = 2 \frac{M}{R^2} r^3 dr.$$

$$\text{For the entire plate: } I_{Cz} = 2 \frac{M}{R^2} \int_0^R r^3 dr = \frac{1}{2} MR^2.$$

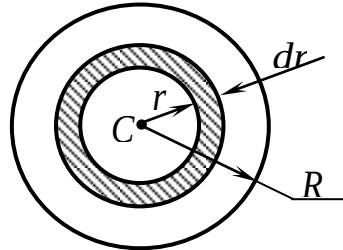


Figure 4.5

The moment of inertia I_{Cz} for a uniform circular cylinder of mass M and radius R relative to its longitudinal central axis (Fig. 4.6) will be determined by the same formula.

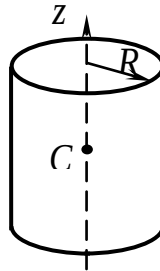


Figure 4.6

Self-awareness questions

1. What is the moment of inertia of a material point relative to an axis?
2. What is the moment of inertia of a system?
3. What is the polar moment of inertia of a mechanical system?
4. What is the planar moment of inertia?
5. What is the centrifugal moment of inertia of a mechanical system?
6. What are the axes of inertia of a body called the principal axes of inertia?
7. What is the central axial moment of inertia?
8. What properties do the principal axes of inertia have?
9. What is the radius of inertia of a body?
10. How is the Huygens-Steiner theorem formulated?

4.4 The Fundamental Theorems of Dynamics

The lecture presents the basic (general) theorems of dynamics, which allow solving problems that often arise in engineering.

Theorem on the motion of the center of mass of a mechanical system

The center of mass of a mechanical system moves as a free material point that has the mass of this system and to which all external forces acting on the system are applied:

$$m\bar{a}_C = \sum_{i=1}^n \bar{F}_i^e = \bar{F}^e, \quad i=1,2,3\dots n,$$

where $\sum_{i=1}^n \bar{F}_i^e$ – forces acting on the body and points of the system from bodies not included in the system.

Differential equations of motion of the center of mass:

$$m\ddot{x}_c = \sum_{i=1}^n F_{ix}^e = F_x^e;$$

$$m\ddot{y}_c = \sum_{i=1}^n F_{iy}^e = F_y^e;$$

$$m\ddot{z}_c = \sum_{i=1}^n F_{iz}^e = F_z^e.$$

Consequences of the theorem

1. Internal forces do not directly affect the motion of the center of mass of a material system.

2. If the main vector $\vec{F}^e$ of external forces acting on the system is zero ($\vec{F}^e = 0$), then the center of mass is at rest or moves uniformly and rectilinearly depending on the initial conditions.

3. If one of the projections of the main vector of external forces onto the axis of a fixed coordinate system is zero (for example $F_{ix}^e = 0$), then the projection of the velocity of the center of mass onto this axis is (V_{cx}) does not change, i.e. is a constant value $V_{cx} = \dot{x}_C = \text{const}$.

Example

On an inclined plane (Fig. 4.7) of a prism of mass 1 $m_1 = 10$ kg cargo is lowered 2 ($m_2 = 6$ kg), which pulls a load of mass 3 using a weightless string $m_3 = 4$ kg.

Find the displacement of prism 1 along a smooth horizontal plane if body 2 is lowered along an inclined plane by $S = 0,5$ m.

The solution

Let us show the external forces applied to the material system consisting of prism 1 and bodies 2, 3. These are force of the weight of the prism: $P_1 = m_1g$ – force of the weight of the prism, $P_2 = m_2g$ i $P_3 = m_3g$ – weight of the second and third loads, respectively, N – reaction of a smooth horizontal surface.

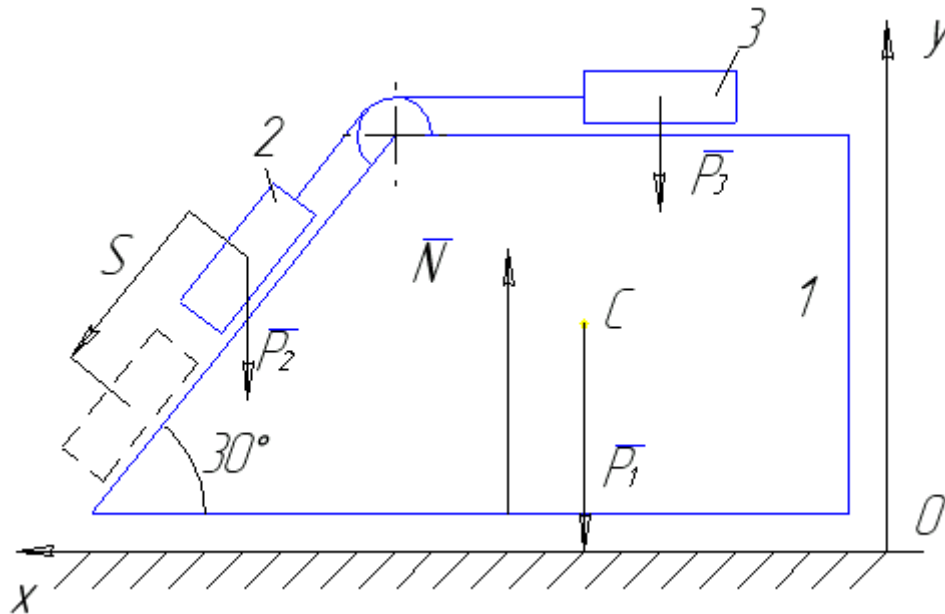


Figure 4.7

Let us write down the theorem on the motion of the center of mass of a material system in projections onto the axis X:

$$M\ddot{X}_C = F_X^e, \quad (4.1)$$

where $M = m_1 + m_2 + m_3$;

F_X^e – projection of the main vector of external forces onto the axis X.

Since $F_X^e = 0$, то $M\ddot{X}_C = 0$. Тоді $M\dot{X}_C = C_1$.

At the initial moment of time the system was at rest and therefore $C_1 = 0$. From formula (4.1) we have: $M\dot{X}_C = C_2$.

Thus, the coordinate X_C of the center of mass of the material system remains constant regardless of the movements of the bodies included in the system.

Let us determine the position of the center of mass of the system at the initial moment of time:

$$X_C = \frac{\sum_{k=1}^n m_k \cdot x_k}{M} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M}. \quad (4.2)$$

If load 2 moves by an amount ΔX_2 , then the body 3 – by ΔX_3 , and a prism 1 – ΔX_1 and the position of the center of mass X_C is found by the formula:

$$X_C = \frac{m_1(x_1 + \Delta x_1) + m_2(x_2 + \Delta x_2) + m_3(x_3 + \Delta x_3)}{M}. \quad (4.3)$$

Considering (4.2) from formula (4.3) we get:

$$m_1 \cdot \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 = 0. \quad (4.4)$$

Displacing ΔX_2 and ΔX_3 consists of what is relative to the prism and what is portable along with the prism. $\Delta x_2 = S \cdot \cos 30^\circ + \Delta x_1$,
 $\Delta x_3 = S + \Delta x_1$.

From formula (4.4) we find the displacement of the prism.

$$10 \cdot \Delta x_1 + 6(S \cdot \cos 30^\circ + \Delta x_1) + 4(S + \Delta x_1) = 0,$$

$$\Delta x_1 = -\frac{(6 \cdot \cos 30^\circ + 4)S}{10 + 6 + 4} = -0,229M.$$

The minus sign indicates that prism 1 has moved in the direction opposite to the positive direction of the axis. X.

Theorems on the change in momentum of a material point and momentum of a mechanical system

The quantity of motion (measure of mechanical motion) of a material point with the mass m_i is called the mass vector $\bar{q}_i$, which is equal to the product of the mass of the point by the vector of its velocity $\bar{q}_i = m_i \bar{V}_i$.

The quantity of motion of a system of material points is called the principal vector (vector sum) of the quantities of motion of all points of the system:

$$\bar{Q} = \sum_{i=1}^n \bar{q}_i = \sum_{i=1}^n m_i \cdot \bar{V}_i.$$

Or the amount of momentum of the entire system is equal to the amount of momentum of one material point, the mass of which is equal to the mass of the system, and the velocity – speed of the center of mass: $\bar{Q} = m \bar{V}_C$.

The vector form of this equation corresponds to three equalities in coordinate form::

$$Q_x = mV_{Cx}; \quad Q_y = mV_{Cy}; \quad Q_z = mV_{Cz}.$$

The impulse of a constant force is a vector quantity equal to the product of the force and the time of its action.

$$\bar{S} = \bar{F} \cdot \Delta t.$$

A full pulse of power $\bar{S} = \int_{t_0}^t \bar{F} \cdot dt.$

Projections of the momentum of the force onto the coordinate axes

$$S_x = \int_{t_0}^t F_x \cdot dt; \quad S_y = \int_{t_0}^t F_y \cdot dt; \quad S_z = \int_{t_0}^t F_z \cdot dt.$$

Theorem on the change in momentum of a material point in differential form:

The time derivative of the momentum of a material point is geometrically equal to the resultant of the forces applied to the point. $\frac{d\bar{q}}{dt} = \frac{d(m\bar{V})}{dt} = \bar{R}$:

$$\text{or } \frac{d(mV_x)}{dt} = R_x; \quad \frac{d(mV_y)}{dt} = R_y; \quad \frac{d(mV_z)}{dt} = R_z.$$

Theorem on the change in momentum of a material point in integral form: the change in the momentum of a material point over a certain period of time is equal to the impulse of the resultant force over the same period of time: $m\bar{V}_i - m\bar{V}_0 = \bar{S}$, or in projections on the coordinate axis:

$$mV_{1x} - mV_{0x} = S_x; \quad mV_{1y} - mV_{0y} = S_y; \quad mV_{1z} - mV_{0z} = S_z.$$

Theorem on the change in momentum of a system of material points in differential form: the time derivative of the momentum of a system of material points is equal to the principal vector of external forces acting on the system:

$$\frac{d\bar{Q}}{dt} = \bar{F}^e$$

or in projections on the coordinate axis:

$$\frac{dQ_x}{dt} = \sum_{i=1}^n F_{ix}^e; \quad \frac{dQ_y}{dt} = \sum_{i=1}^n F_{iy}^e; \quad \frac{dQ_z}{dt} = \sum_{i=1}^n F_{iz}^e.$$

Theorem on the change in momentum of a system of material points in integral form: The change in the momentum of a system over a period of time is equal to the sum of the impulses acting on the system by external forces over the same period of time:

$$\bar{Q}_1 - \bar{Q}_0 = \bar{S}^e,$$

in projections on the axis:

$$Q_{1x} - Q_{0x} = \sum_{i=1}^n S_{ix}^e;$$

$$Q_{1y} - Q_{0y} = \sum_{i=1}^n S_{iy}^e;$$

$$Q_{1z} - Q_{0z} = \sum_{i=1}^n S_{iz}^e.$$

Consequences of the theorem

1. The internal forces of the system do not directly affect the change in the momentum vector of the system.

2. If the sum of all external forces acting on a system is zero $\sum_{i=1}^n \bar{F}_i^e = 0$, then the momentum vector of the system will be constant in magnitude and direction $\bar{Q} = const$. $\bar{Q} = const$ – law of conservation of momentum of a system.

3. If the sum of the projections of all the acting forces onto any axis F_{ix}^e is equal to zero $\sum_{i=1}^n \bar{F}_{ix}^e = 0$, then the projection of the momentum of the system onto this axis is a constant quantity, $Q_x = const$.

Example

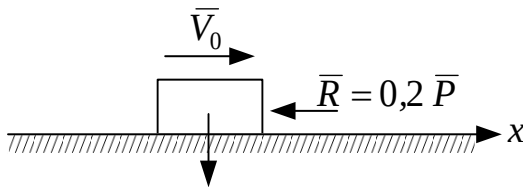


Figure 4.8

A body with mass m moves along a horizontal section of the path m with speed 10 m/s, and meets a resistance force $\bar{R}$, which is equal to 0.2 of the body weight (Fig. 4.8).

At some point in time, the driving force – the force of traction – is released, and the body continues to move. Determine after what time the body will stop?

The solution

By the momentum theorem

$$mV_x - mV_{ox} = \int_0^t -0,2P dt = -0,2Pt,$$

$$V_x = 0, \quad \bar{P} = m\bar{g}, \quad \bar{R} = 0,2m\bar{g},$$

$$-mV_{ox} = -0,2mgt.$$

Then

$$t = \frac{V_{ox}}{0,2g} = \frac{10}{0,2 \cdot 10} = 5 \text{ s}.$$

Answer: the body will stop 5 seconds after the traction is turned off.

Theorems on the change in the momentum of a material point and a mechanical system

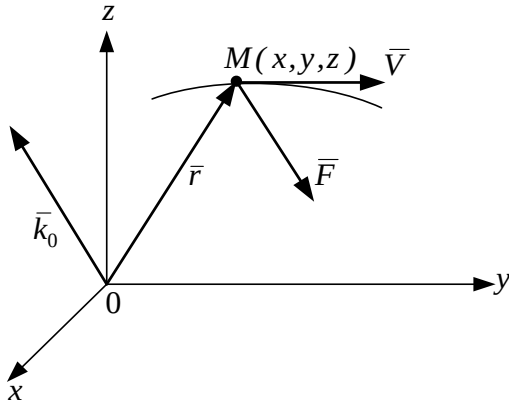


Figure 4.9

The moment of the amount of motion of a point M relative to the center O is called the quantity $\bar{k}_0$, which is equal to the vector product of the radius-vector of the material point, drawn from this center, by the amount of its motion (Fig. 4.9)

$$\bar{k}_0 = \bar{r} \times \bar{q} = \bar{r} \times m\bar{v}.$$

Theorem on the change in the momentum of a material point: the time derivative of the momentum of a material point relative to a fixed center O (or axis) is equal to the moment of the resultant forces applied to the point relative to the same center (or axis):

$$\frac{d\bar{k}_0}{dt} = \bar{M}_0(\bar{F})$$

or

$$\frac{dk_x}{dt} = \sum_{i=1}^n M_x(\bar{F}_i); \quad \frac{dk_y}{dt} = \sum_{i=1}^n M_y(\bar{F}_i); \quad \frac{dk_z}{dt} = \sum_{i=1}^n M_z(\bar{F}_i).$$

The kinetic moment of a system of material points or the principal moment of the quantity of motion of the system ($\bar{K}_0$) relative to the center O is the vector sum of the moments of the quantities of motion of the points of the system relative to the same center:

$$\bar{K}_0 = \sum_{i=1}^n \bar{k}_{0i} = \sum \bar{r}_i \times m_i \bar{V}_i,$$

where $\bar{k}_{0i}$ – angular momentum of the i - th point of the system;

$m_i \bar{V}_i$ – amount of movement of the i - th point;

$\bar{r}_i$ – radius vector connecting the fixed center O with the i - th point of the system.

Kinetic moment of the system relative to the coordinate axes.

$$K_x = \sum_{i=1}^n M_x(m_i \bar{V}_i); \quad K_y = \sum_{i=1}^n M_y(m_i \bar{V}_i); \quad K_z = \sum_{i=1}^n M_z(m_i \bar{V}_i).$$

Kinetic moment of a rigid body about the axis of rotation

Lets consider a rigid body rotating around the axis Oz with angular velocity ω (Fig. 4.10).

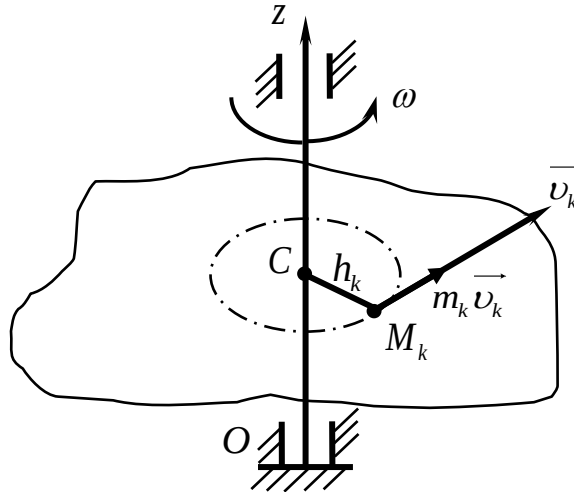


Figure 4.10

For an arbitrary point of a body M_k with mass m_k , which is located at a distance h_k from the axis of rotation and has a velocity $\vec{v}_k$, the angular momentum:

$$K_{kz} = M_z(m_k \vec{v}_k).$$

But $v_k = h_k \cdot \omega$, and the vector of the amount of motion of the point is perpendicular to h_k , that is, it lies in the plane perpendicular to the axis Oz . Therefore, it turns out that:

$$K_{kz} = h_k \cdot m_k \cdot v_k = m_k h_k^2 \omega.$$

Kinetic moment of the whole body: $K_z = \sum_1^n m_k h_k^2 \omega$.

As previously defined, $\sum_1^n m_k h_k^2$ is the moment of inertia of the body relative to the axis Oz , that is why, finally, $K_z = J_z \cdot \omega$

Thus, **the angular momentum of a rigid body about a fixed axis of rotation** is equal to the product of the moment of inertia of the body about this axis and the angular velocity of the body.

The sign of the angular momentum coincides with the sign of the angular velocity.

Theorem on the change in the momentum of a mechanical system: the time derivative of the kinetic momentum of a mechanical system relative to any fixed center $\bar{K}_0$ is equal to the sum of the moments of all external forces of the system relative to the same center:

$$\frac{d\bar{K}_0}{dt} = \sum_{i=1}^n \bar{M}_0(\bar{F}_i^e)$$

or in projections on the axis:

$$\frac{dK_x}{dt} = \sum_{i=1}^n M_x(\bar{F}_i^e); \quad \frac{dK_y}{dt} = \sum_{i=1}^n M_y(\bar{F}_i^e); \quad \frac{dK_z}{dt} = \sum_{i=1}^n M_z(\bar{F}_i^e)$$

or the time derivative of the kinetic momentum of the system about any fixed axis is equal to the principal moment of the external forces acting on the system about the same axis.

For axes moving gradually: $\frac{d\bar{K}_C}{dt} = \sum_{i=1}^n M_C(\bar{F}_i^e)$.

Consequences of the theorem

1. Internal forces do not directly affect the change in angular momentum of the system.

2. If the sum of the moments relative to a given center of all external forces applied to the system is zero $\bar{M}_0^e = 0$, then $\bar{K}_0 = const$. $\bar{K}_0 = const$ – the law of conservation of the principal momentum (the law of conservation of kinetic momentum relative to the center O). All navigation systems are created using this law.

3. If $\sum_{i=1}^n M_z(\bar{F}_i^e) = 0$, then the principal moment of the system's momentum relative to this axis will be a constant $K_z = const$.

Differential equation of rotation of a rigid body around a fixed axis

Suppose that a certain rigid body rotates around a fixed axis Oz under the action of a system of external forces, $\{\bar{F}_k^e\}$. Then:

$$\frac{dK_z}{dt} = \sum_1^n M_z(\vec{F}_k^e),$$

where K_z – angular momentum of a body about an axis Oz .

At the same time, as is known, $K_z = J_z \cdot \omega$, and the previous relation takes the form:

$$J_z \frac{d\omega}{dt} = \sum_1^n M_z(\vec{F}_k^e).$$

The obtained formula is the differential equation of rotation of a rigid body around a fixed axis.

When calculating the sum of the moments of external forces relative to the axis of rotation of a rigid body, it must be borne in mind that the reactions of ideal joints (reactions of the axis supports) are not included in the equation, because their lines of action intersect the axis.

From the differential equation of rotation of the body, we get:

1. If $\sum M_z(\vec{F}_k^e) = \text{const}$, then $\varepsilon = \frac{d\omega}{dt}$ is also constant, i.e. the rotation of the body is uniform.

2. If $\sum M_z(\vec{F}_k^e) = 0$, $\varepsilon = \frac{d\omega}{dt} = 0$ i $\omega = \text{const}$ – we have the case of uniform rotation of the body around the axis.

Note. The kinetic moment of a system of material points (the principal moment of the momentum of the system) is denoted by $\bar{K}_0$ аџ $\bar{L}_0$.

Example (Using the theorems on the motion of the center of mass and the kinetic momentum to study the motion of a material system)

The material system (Fig. 4.11) starts to move from rest under the action of the moment M applied to body 1. The axes of bodies 1 and 2 are horizontal. The coefficient of sliding friction is f . There is no sliding at the points of contact of the bodies. Neglect the mass of the belt. Body 1 is a uniform cylinder.

Determine the acceleration of the body 3, tension S_5 in the driven 5 and leading 4 (S_4) part of the pass (accept $S_4 = 2S_5$), 3 forces at the point of contact of bodies 1 and 2, reactions of the links of the cylindrical (fixed) hinges of bodies 1, 2 and 3.

Accept: $R_1 = 0,25$ m; $R_2 = 0,45$ m; $r_2 = 0,15$ m; $i_2 = 0,4$ m; $L = 0,7$ m; $m_1 = 0,5$ kg; $m_2 = 5$ kg; $m_3 = 4$ kg; $M = 3t^3$ N·m; $t_1 = 2$ s; $f = 0,4$.

The solution

Let us consider separately the motion of each body of the material system (Fig. 4.11).

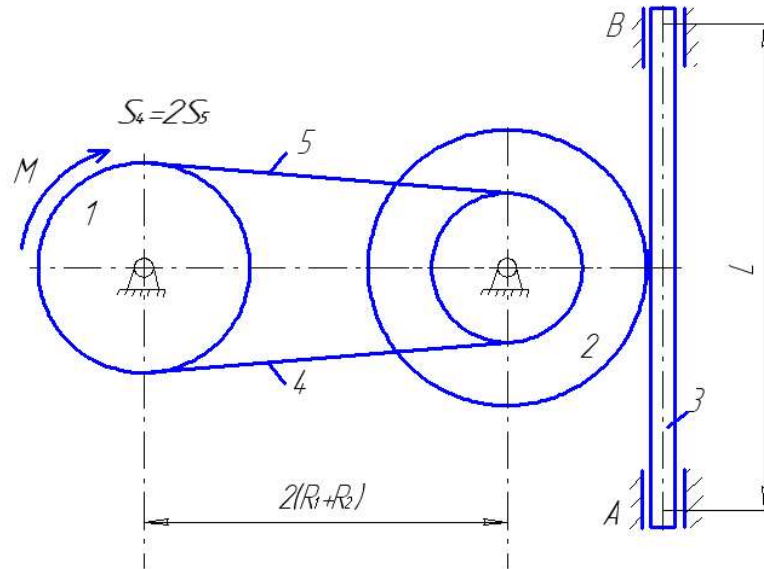


Figure 4.11

Determining the acceleration of a body 3

A homogeneous body 1 rotates around a horizontal axis under the action of a moment M (Fig. 4.12) and external forces are applied to it: gravitational force

$P_1 = m_1 g$; reactions of the cylindrical hinge X_1 and Y_1 ; forces in the belt links S_5 and S_4 .

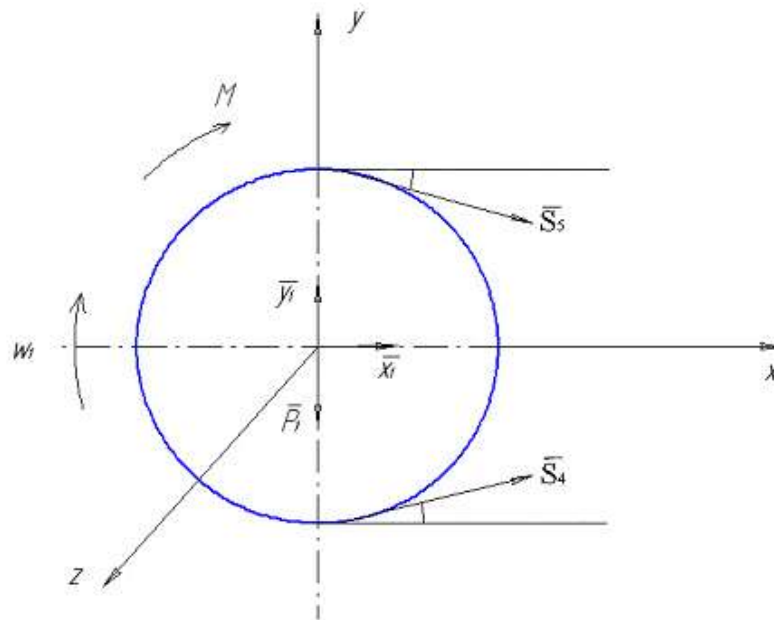


Figure 4.12

Let us write the differential equation of rotation of body 1 around a fixed axis Z :

$$I_{Z1}\ddot{\phi}_1 = M + S_5 R_1 - S_4 \cdot R_1. \quad (4.5)$$

It is necessary to take into account that the moment of a force or a pair of forces will be positive if it acts in the direction of the body's motion. Thus, in equation (4.5), the moment of force S_5 and the pair of forces with the moment M are taken with a positive sign, and the moment of force S_4 is taken with a negative sign.

Moment of inertia of body 1 relative to the axis Z .

$$I_{Z1} = \frac{1}{2} m_1 R_1^2. \quad (4.6)$$

External forces act on body 2 (Fig. 4.13) during motion: gravitational force $P_2 = m_2 g$, reactions of a fixed hinge X_2 and Y_2 , effort in the belt links S'_5 and S'_4 , body reactions 3 – S_2 and N_2 .

In the differential equation of motion of body 2 (Fig. 4.13) around the horizontal axis Z

$$I_{Z2} \cdot \ddot{\phi}_2 = -S'_5 r_2 + S'_4 r_2 - S_2 R_2 \quad (4.7)$$

Forces $S'_5 = S_5$ and $S'_4 = S_4$, and the moment of inertia of body 2 relative to the Z axis is found through the radius of inertia i_2

$$I_{Z2} = m_2 i_2^2. \quad (4.8)$$

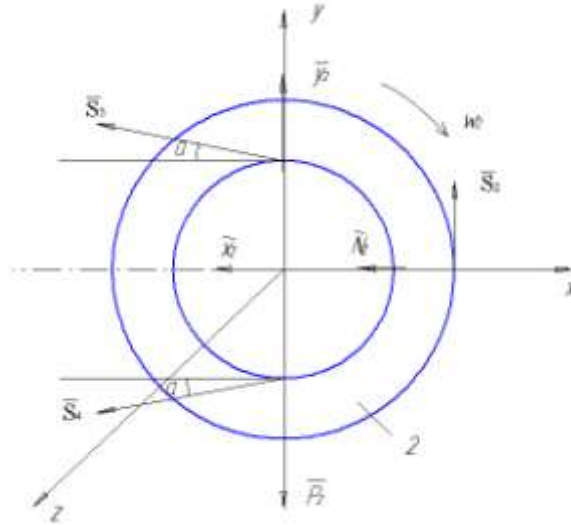


Figure 4.13

For body 3 (Fig. 4.14), moving under the action of gravity $P_3 = m_3 g$, reactions of fixed joints N_A and N_B , reactions S'_2 and N'_2 of the body 2, we write down the theorem about the motion of the center of mass in projections onto the axis Y (the Y axis is directed in the direction of body movement 3)

$$m_3 \ddot{y}_3 = P_3 + S'_2. \quad (4.9)$$

Of course $S'_2 = S_2$.

If the differential equations (4.5), (4.7) and (4.9) are considered with kinematic relations, then

$$\ddot{y}_3 = \ddot{\phi}_2 R_2, \quad \ddot{\phi}_2 r_2 = \ddot{\phi}_1 R_1. \quad (4.10)$$

Then we get five equations with unknowns $\ddot{y}_3, \ddot{\phi}_2, \ddot{\phi}_1, S_5, S_2, (S_4 = 2S_5)$.

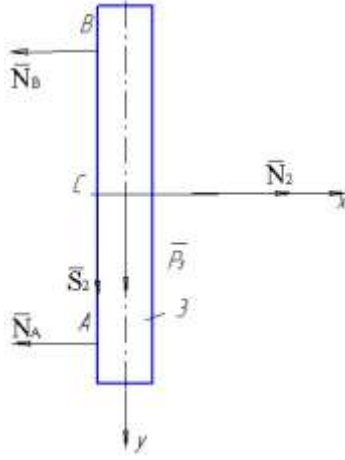


Figure 4.14

Solving the system of equations (4.5), (4.7), (4.9), (4.10), and taking into account (4.6) and (4.8), we have

$$\begin{aligned} S_5 &= \frac{M}{R_1} - \frac{1}{2} m_1 \frac{r_2}{R_2} \ddot{y}_3, \\ S_2 &= m_3 (\ddot{y}_3 - g), \\ \ddot{y}_3 &= \frac{M \cdot r_2 + m_3 R_1 R_2 \cdot g}{R_1 (m_2 \frac{r_2^2}{R_2} + \frac{1}{2} m_1 \frac{r_2^2}{R_2} + m_3 R_2)}. \end{aligned}$$

Or, substituting these conditions into the problem, we get

$$\begin{aligned} \ddot{y}_3 &= 4,92 + 0,5t^3, \\ S_2 &= 4y_3^3 - 39,24\ddot{y}_3, \\ S_5 &= 12t^3 - 0,083\ddot{y}_3. \end{aligned}$$

When $t_1 = 2$ s, $\ddot{y}_3 = 8,92 \frac{m}{s^2}, S_5 = 95,26 N, S_2 = -3,6 N$.

Theorems on the change in kinetic energy of a material point and a mechanical system

Kinetic energy

The kinetic energy of a material point is a scalar measure of mechanical motion in a fixed coordinate system, equal to half the product of the mass of the point and the square of its velocity $T = \frac{mV^2}{2}$.

The kinetic energy of a system of material points is the sum of the kinetic energies of all points included in the system:

$$T = \sum_{i=1}^n T_i = \frac{1}{2} \sum_{i=1}^n m_i V_i^2,$$

where m_i – mass of the i -th point of the system i ;

V_i – speed of the i -th point.

If a system consists of several solid bodies, then its kinetic energy is equal to the sum of the kinetic energies of all bodies included in the system..

1. During translational motion of a rigid body:

$$T = \frac{M \cdot V_C^2}{2}.$$

2. During rotational motion of a rigid body around a fixed axis z :

$$T = \frac{1}{2} I_z \cdot \omega^2.$$

3. In plane-parallel motion:

$$T = \frac{MV_C^2}{2} + \frac{1}{2} I_{cz} \cdot \omega^2,$$

where M – body weight;

V_C – speed of the center of mass of the body;

I_z – moment of inertia of a body about the axis of rotation z ;

I_{zc} – moment of inertia of a body about an axis Z_c , passing through the center of mass of the body perpendicular to the plane of motion;

ω – instantaneous angular velocity of the body.

For example, for a wheel of mass M that rolls without slipping on a straight rail (Fig. 4.15).

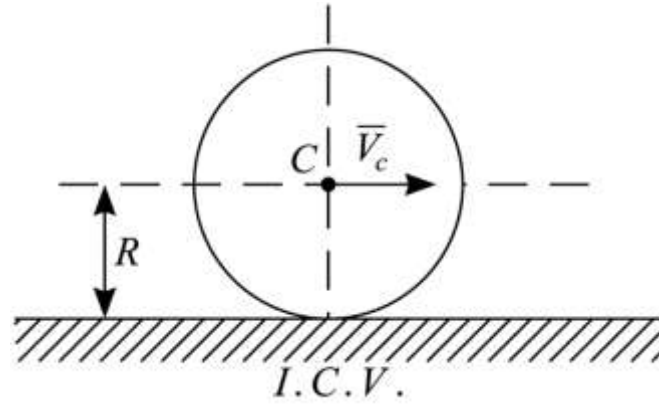


Figure 4.15

I. C. V. – instantaneous center of velocity.

$$T = \frac{1}{2} M V_c^2 + \frac{1}{2} \cdot \frac{M R^2}{2} \cdot \left(\frac{V_c}{R} \right)^2 = \frac{3}{4} M V_c^2.$$

Work of the force

The elementary work δA of a force F is a scalar quantity equal to the scalar product $\bar{F}$ of the vector of the elementary displacement of the point of its application (Fig. 4.16):

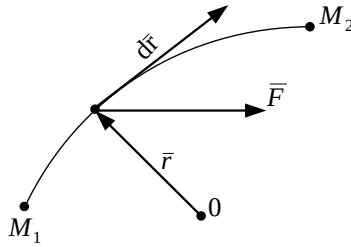


Figure 4.16

$$\delta A = \bar{F} \cdot d\bar{r} = F \cdot dr \cdot \cos(\bar{F}, d\bar{r}).$$

If, under the action of a force $\bar{F}$, a material point moves from position M_1 to position M_2 , then the work done by the force $\bar{F}$ on the path $M_1 M_2$ is equal to

$$A = \int_{M_1 M_2} \delta A = \int_{M_1 M_2} \bar{F} \cdot d\bar{r} = \int_{M_1 M_2} F \cdot dr \cdot \cos(\bar{F}, dr).$$

Taking into account the properties of the scalar product, we have

$$dA = F_x dx + F_y dy + F_z dz.$$

Work on finite displacement

$$A_{M_1 M_2} = \int_{M_1 M_2} F_x dx + F_y dy + F_z dz.$$

$$\text{Unit of measurement of work } [A] = H \cdot M = 1 \text{ Дж} = 1 \frac{\text{кг} \cdot \text{м}^2}{\text{с}^2}.$$

1. Work of a constant force $\vec{F}$ on rectilinear motion S (рис. 4.17):

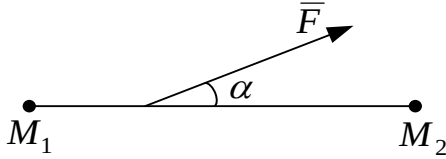


Figure 4.17

$$A = F \cdot S \cdot \cos \alpha.$$

2. The work of gravity (Fig.. 4.18)

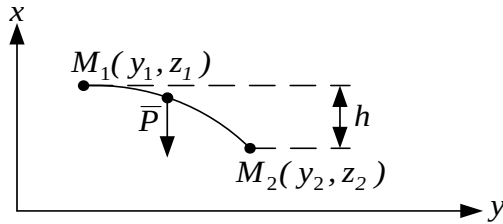


Figure 4.18

$$A = \pm F \cdot h$$

(„+” when the point drops,

„-” when the point rises),

where h – difference between the final and initial positions.

3. Work of elastic force:

$$A = \frac{c}{2} [(\Delta l_{init})^2 - (\Delta l_{ultim})^2],$$

where c – elasticity coefficient;

Δl_{init} – initial spring deformation;

Δl_{ultim} – ultimate spring deformation.

4. Work of friction force:

$$A = -F_{fr} \cdot S.$$

5. Work of a force applied to a rotating rigid body:

$$A = M_{OZ} \cdot \varphi,$$

where $M_{OZ} = const$ – moment of force about the axis of rotation;

φ – turning angle.

Power – is the work done per unit of time.

$$N = \frac{dA}{dt} = \vec{F} \cdot \vec{V} = F_{\tau} \cdot V,$$

where F_{τ} – projection of force onto the tangent axis;

V – speed of the point of application of force.

The power of the force applied to a rotating body

$$N = M_Z(\vec{F}) \cdot \omega,$$

where $M_Z(\vec{F})$ – moment of force about the axis of rotation z ;

ω – angular velocity of the body.

Theorems about the change in kinetic energy

Theorem on the change in kinetic energy of a material point: the increase in the kinetic energy of a point on a certain segment of the arc of its trajectory is equal to the work of the resultant of all forces applied to the point on this same segment of the arc of the trajectory:

$$\frac{mV_2^2}{2} - \frac{mV_1^2}{2} = A,$$

where V_2 – speed of the point at the end of the path traveled;

V_1 – speed of the point at the beginning of the path;

$A = \sum_{i=1}^n A_i$ – algebraic sum of the work of the forces applied to a point along the path traveled.

Theorem on the change in kinetic energy of a mechanical system in integral form: the increase in kinetic energy of a system of material points over a certain period of time is equal to the sum of the work of external forces acting on the points of the system during the considered period of time.

$$T - T_0 = \sum_{i=1}^n A_i,$$

where T_0, T – kinetic energy of the system in its initial and final positions, respectively;

$\sum_{i=1}^n A_i$ – the sum of the work of external and internal forces acting on the system.

In the case of a perfectly rigid body (an immutable system), the internal forces are balanced and their work is zero. The work of ideal bonds is also zero.

Theorem on the change in kinetic energy of a mechanical system in differential form:

$$\frac{dT}{dt} = \sum_{i=1}^n N_i,$$

that is, the derivative of kinetic energy with respect to time is equal to the sum of the powers of all external and internal forces applied to the system.

4.5 Calculation, graphic and control tasks

Д4 Determination of kinematic characteristics of bodies using the theorem on the change in kinetic energy of a material system

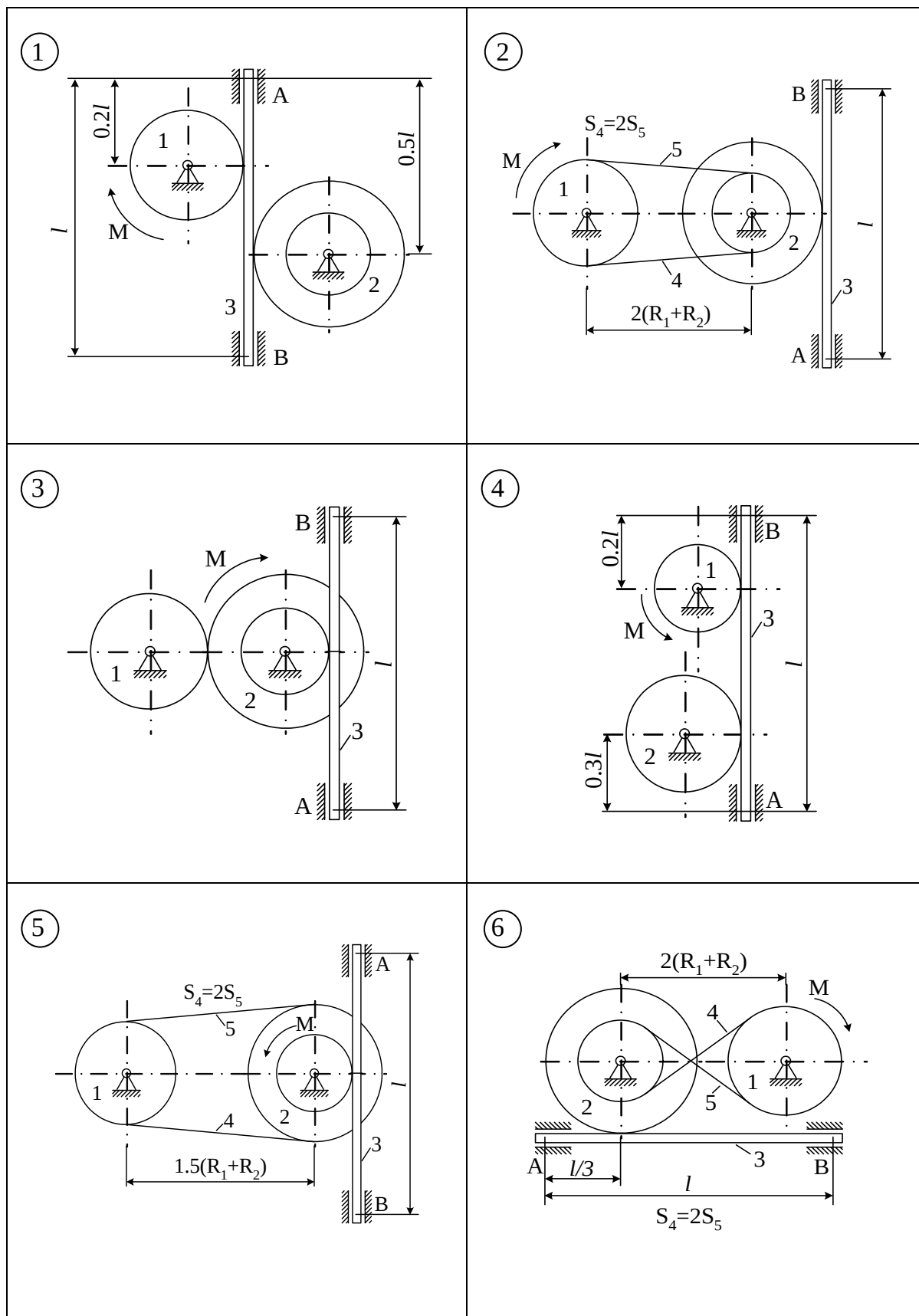
The material system (p. 112 – 116) is driven into motion by a moment M . Find the acceleration of body 3 and the angular accelerations of the bodies at time $t = t_1$.

Neglect the masses of the belts and their sliding on the pulleys. There is no sliding at the points of contact of the bodies. A homogeneous disk 1 and a stepped pulley 2 rotate around horizontal axes.

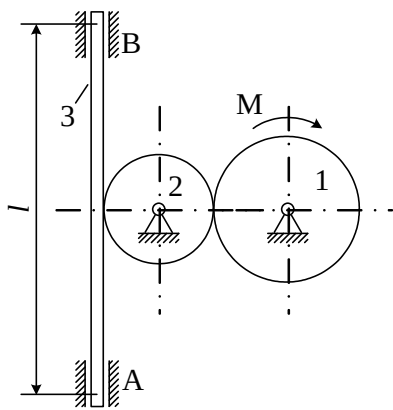
The data for the calculation are given in Table 4.1 where m_1, m_2, m_3 – masses of the bodies 1, 2 and 3; R_1, R_2, r_2 – sizes of the bodies 1 and 2; i_2 – radius and inertia of body 2 relative to the horizontal axis.

Table 4.1

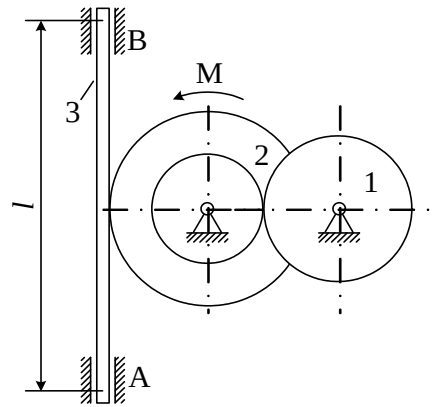
Variant	R_1, m	R_2, m	r_2, m	i_2, m	m_1, kg	m_2, kg	m_3, kg	$M, N \cdot m$	t_1, s
1	0,2	0,3	0,1	0,2	0,2	2	1	$1+3t$	2
2	0,3	0,4	0,2	0,3	0,3	3	2	$2+t^2$	1
3	0,4	0,5	0,3	0,4	0,4	4	3	$3+2t^2$	3
4	0,5	0,6	0,4	0,5	0,5	5	4	$4+t^2$	4
5	0,6	0,55	0,25	0,4	0,6	6	3	$6+5t$	5
6	0,1	0,15	0,1	0,1	0,1	1	1	$2+3t^2$	6
7	0,15	0,2	0,15	0,2	0,15	1,5	1	$4+5t$	2
8	0,25	0,3	0,2	0,25	0,25	2,5	2	$5+3t^2$	1
9	0,35	0,4	0,2	0,3	0,35	3,5	3	$7+2t$	3
0	0,45	0,5	0,4	0,45	0,45	4,5	4	$9+t^2$	2



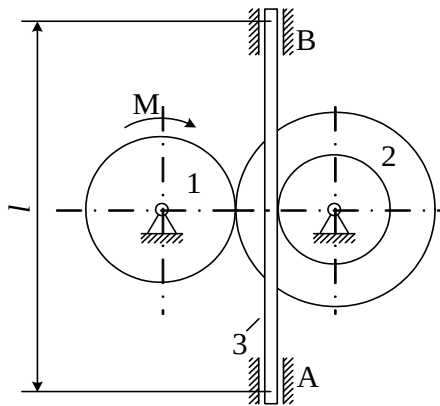
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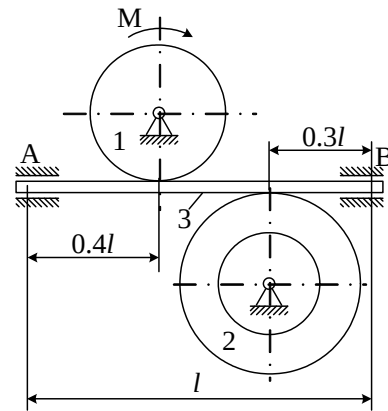
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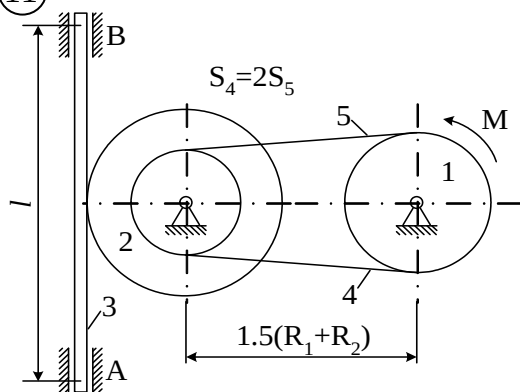
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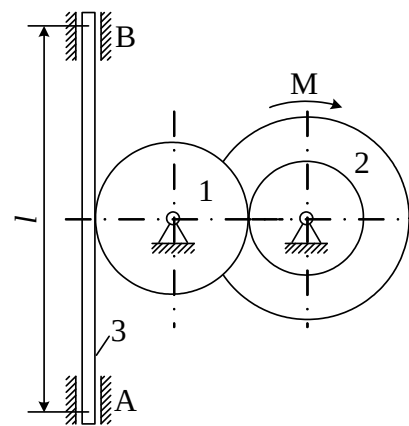
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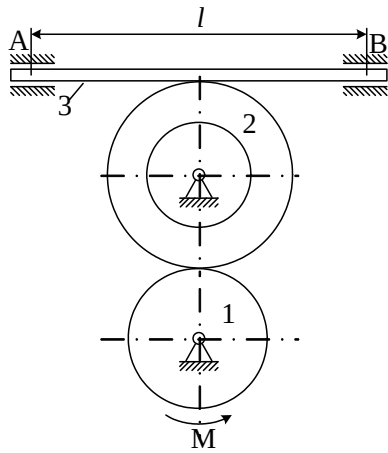
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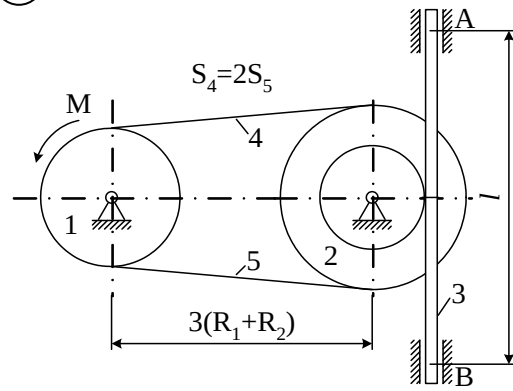
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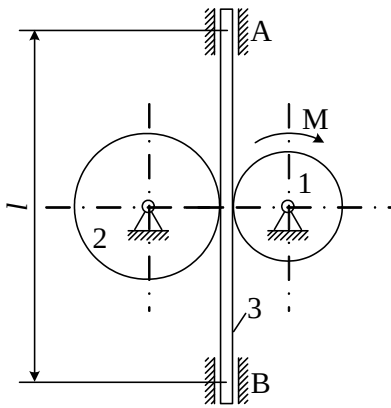
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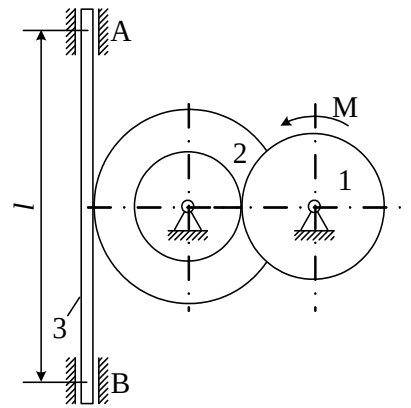
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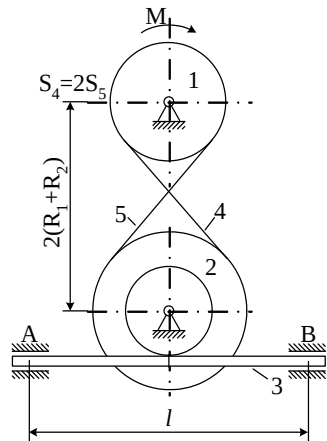
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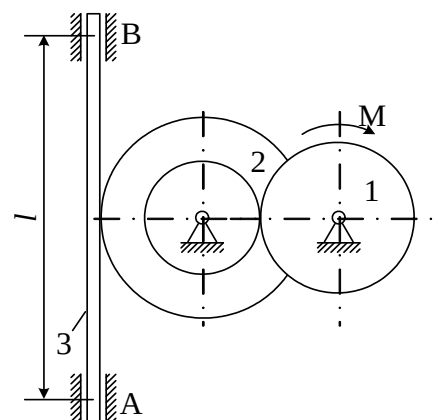
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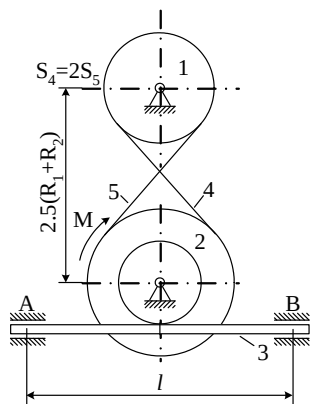
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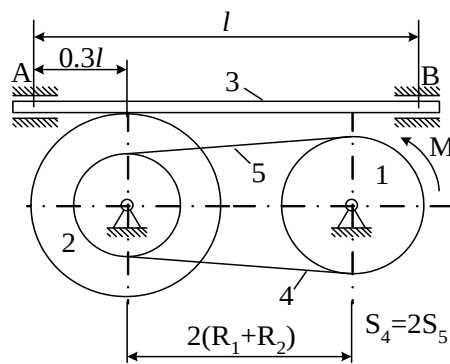
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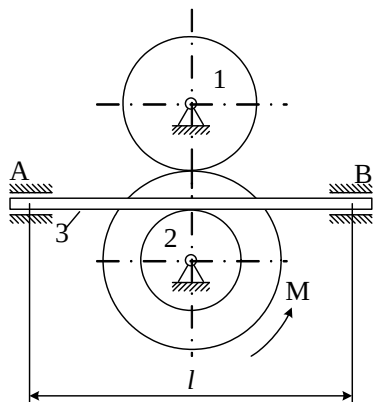
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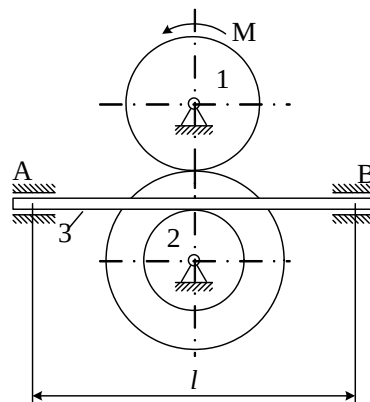
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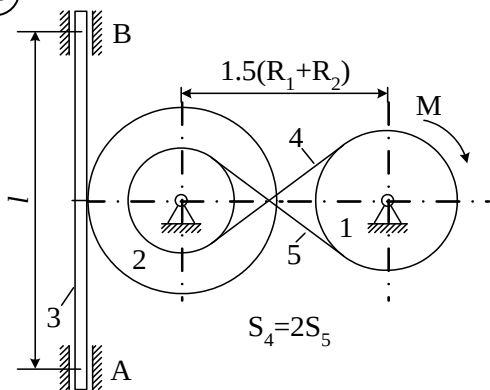
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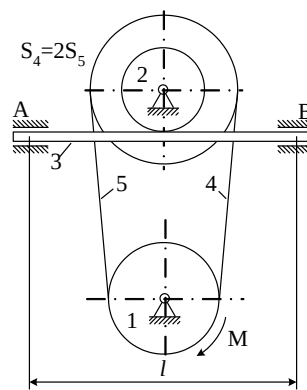
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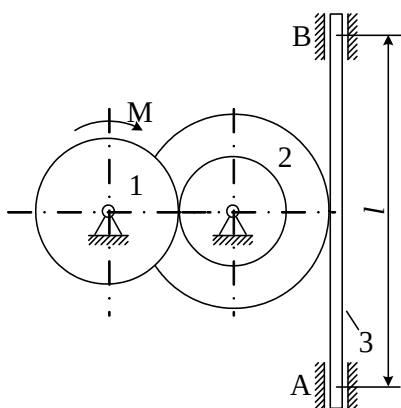
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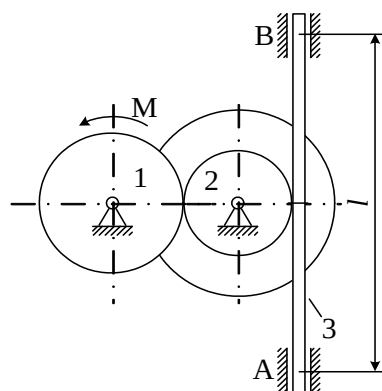
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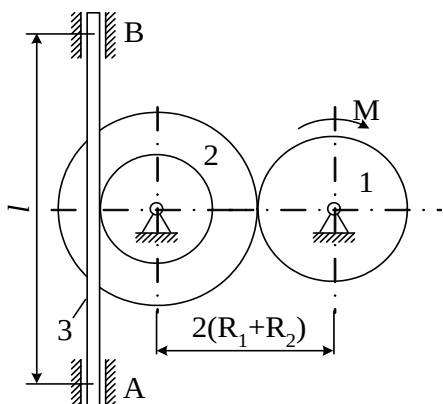
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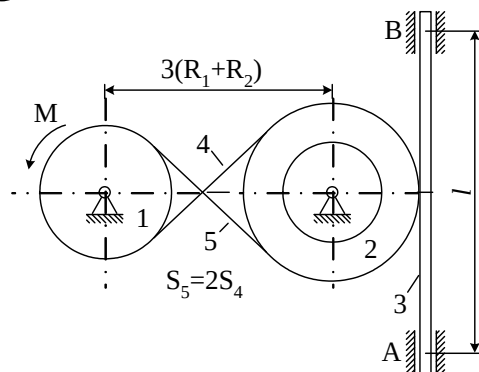
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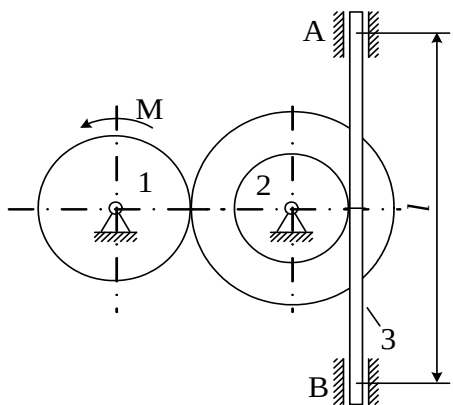
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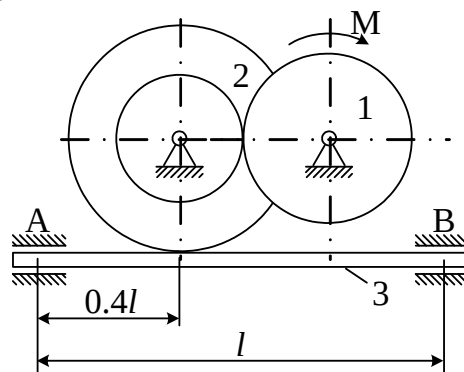
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4.6 Example of completing the task for D4

The material system (Fig. 4.19) moves under the action of the moment M acting on body 1. The axes of bodies 1 and 2 are horizontal. There is no sliding at the points of contact of the bodies and the belt. Neglect the mass of the body. Body 1 is a uniform cylinder.

Determine the acceleration of body 3 and the angular accelerations of bodies 1 and 2, if:

$R_1 = 0,25\text{m}$; $R_2 = 0,45\text{m}$; $r_2 = 0,15\text{m}$; $i_2 = 0,4\text{m}$; $l = 0,7\text{m}$; $m_1 = 0,5\text{kg}$; $m_2 = 5\text{ kg}$; $m_3 = 4\text{ kg}$;
 $M = 3\text{t}^3\text{N}\cdot\text{m}$; $t_2 = 2\text{s}$.

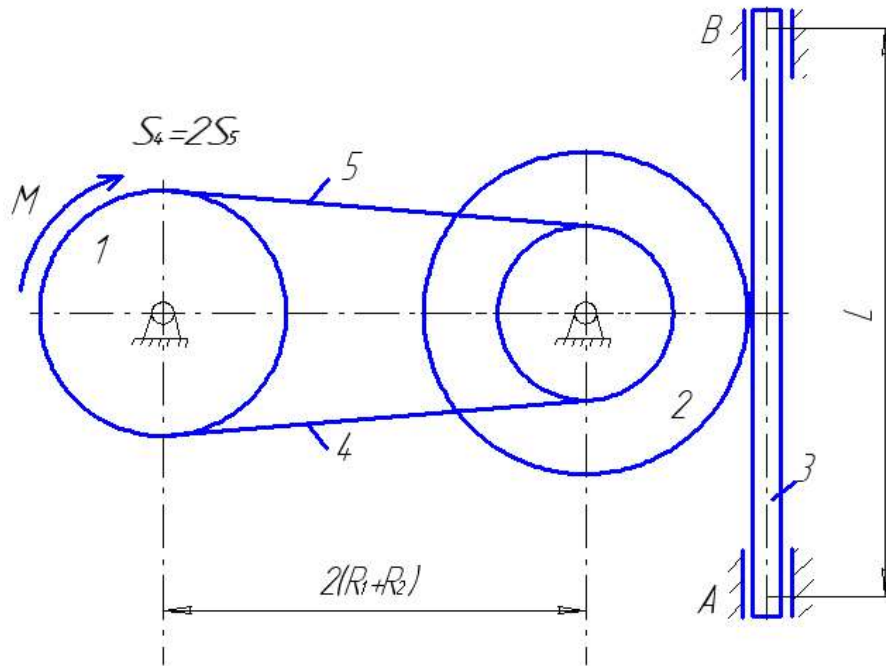


Figure 4.19

The solution

To study the motion of a material system (Fig. 4.19), we will use the theorem on the change in kinetic energy in differential form:

$$\frac{dT}{dt} = N^e + N^i, \quad (4.11)$$

where T – kinetic energy of the system,

N^e – power of external forces of the system,

N^i – power of internal forces of the system,

$N^i = 0$ – the bodies are rigid, and the belt is completely flexible and inextensible.

The kinetic energy of a system consists of the kinetic energy of the bodies included in the system $T = T_1 + T_2 + T_3$.

Bodies 1 and 2 rotate around fixed horizontal axes and their kinetic energy is found by the formulas:

$$T_1 = \frac{1}{2} I_1 \cdot \omega_1^2, T_2 = \frac{1}{2} I_2 \omega_2^2, \quad (4.12)$$

where $I_1 = \frac{1}{2} m_1 R_1^2$, $I_2 = m_2 i_2^2$ – moments of inertia of bodies 1 and 2,

ω_1, ω_2 – angular velocities of bodies.

Body 3 moves forward with a speed V_3 , then

$$T_3 = \frac{1}{2} m_3 V_3^2. \quad (4.13)$$

The relationship between the kinematic characteristics of the motion of bodies (Fig.. 4.20)

$$\omega_2 = \frac{V_3}{R_2}, \quad \omega_1 = \omega_2 \frac{r_2}{R_1} = V_3 \frac{r_2}{R_1 \cdot R_2}. \quad (4.14)$$

Let us write down the kinetic energy of the system, taking into account (4.12), (4.13) and (4.14), as a function of velocity V_3 of the body 3

$$T = \frac{V_3^2}{4R_2^2} (m_1 r_2^2 + 2m_2 i_2^2 + 2m_3 R_2^2). \quad (4.15)$$

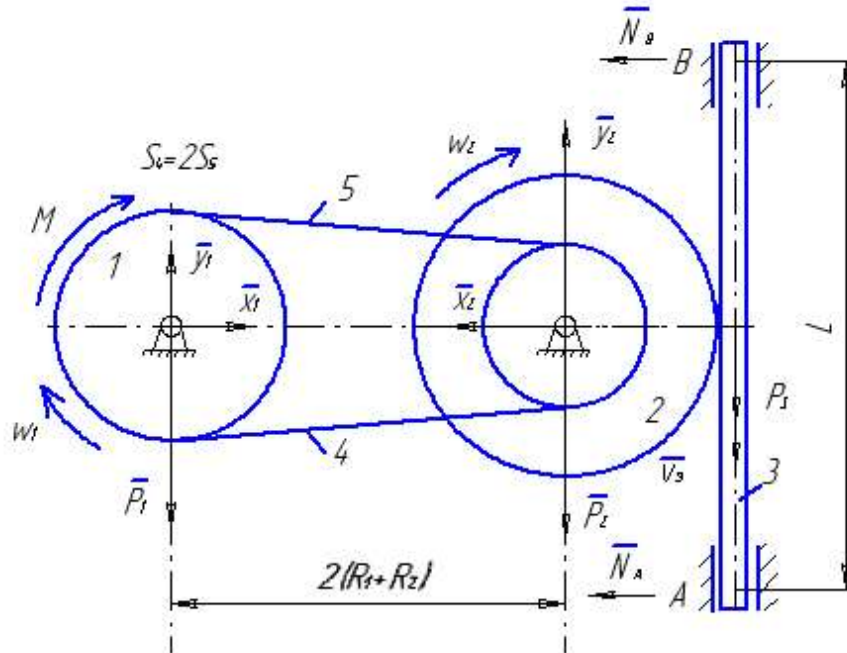


Figure 4.20

Let us find the power of the external forces (Fig. 4.20) of the material system: gravitational forces $P_1 = m_1 g$, $P_2 = m_2 g$, $P_3 = m_3 g$, moment M ; reactions of fixed (cylindrical) hinges $X_1, Y_1, X_2, Y_2, N_A, N_B$.

Power of forces $X_1, Y_1, P_1, X_2, Y_2, P_2, N_A$ and N_B is zero because the points of application of the forces do not move. Then the power N^e external forces will consist of the moment power M and the gravitational force of the body 3 – P_3 .

$$N^e = N(M) + N(\bar{P}_3)$$

where $N(M) = M \cdot \omega_1 = MV_3 \frac{r_2}{R_1 R_2}$,

$$N(\bar{P}_3) = P_3 \cdot V_3 = m_3 g V_3.$$

or

$$N^e = V_3 \left(m_3 g + M \frac{r_2}{R_1 \cdot R_2} \right). \quad (4.16)$$

The theorem on the change in kinetic energy of a material system, taking into account (4.15) and (4.16), can be written as:

$$\frac{1}{4R_2^2} (m_2 r_2^2 + 2m_2 i_2^2 + 2m_3 R_2^2) \frac{dV_3^2}{dt} = V_3 (m_3 g + M \frac{r_2}{R_1 \cdot R_2})$$

Since $\frac{dV_3^2}{dt} = 2V_3 \cdot a_3$, then

$$a_3 = \frac{4R_2^2 (m_3 g + M \frac{r_2}{R_1 \cdot R_2})}{m_1 r_2^2 + 2m_2 i_2^2 + 2m_3 R_2^2}. \quad (4.17)$$

Angular accelerations of bodies

$$\ddot{\varphi}_2 = \frac{a_3}{R_2}, \quad \ddot{\varphi}_1 = a_3 \frac{r_2}{R_1 \cdot R_2}.$$

Substituting these conditions into the problem, we obtain :

$$a_3 = (4,93 + 0,50t^3) \frac{m}{s^2},$$

$$\ddot{\varphi}_2 = (10,96 + 1,11t^3) \frac{1}{s^2},$$

$$\ddot{\varphi}_1 = (6,57 + 0,67t^3) \frac{1}{s^2}.$$

When $t_1 = 2s$, $a_3 = 8,93 \frac{m}{s^2}$, $\ddot{\varphi}_2 = 19,84 \frac{1}{s^2}$, $\ddot{\varphi}_1 = 11,9 \frac{1}{s^2}$.

4.7 Calculation, graphic and control tasks

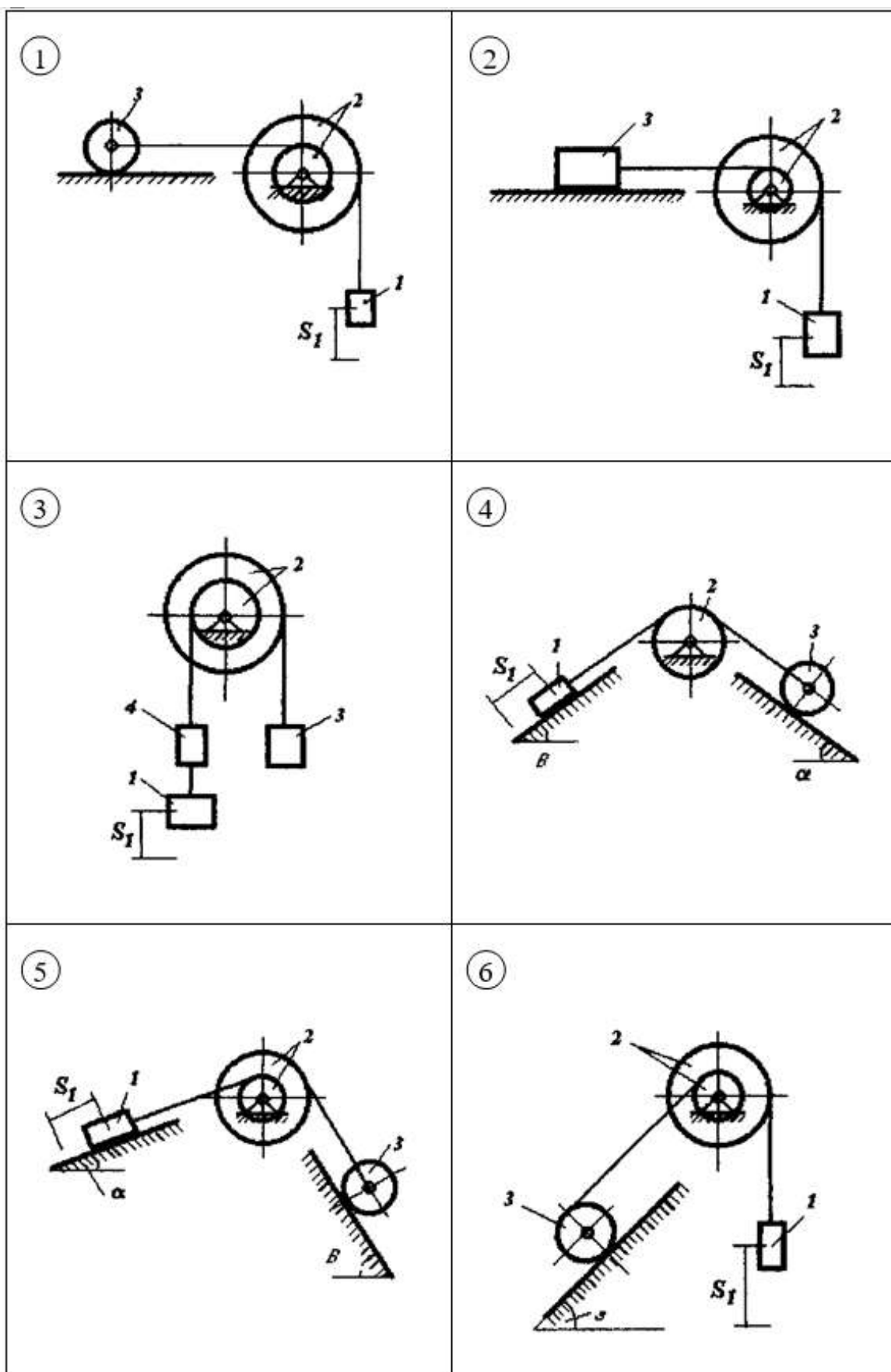
Д5 Using the Kinetic Energy Change Theorem to Study the Motion of a Material System

A material system (pp. 121 – 125) moves from rest under the action of gravity. Find the acceleration and velocity of body 1 at the time when it has traveled a distance S . Neglect the masses of the cords and the resistance forces in the joints. The bodies roll on surfaces without slipping.

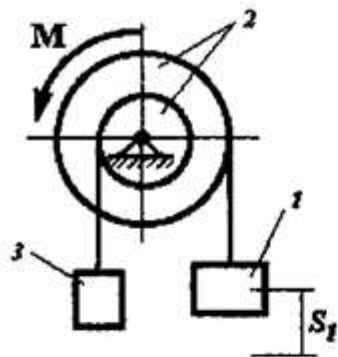
The values for the calculations are given in Table 4.2 where the following notations are used: m_1, m_2, m_3 — mass of bodies 1, 2, 3; R_2, r_2, R_3, r_3 — the largest and smallest sizes of stepped pulleys of bodies 2 and 3; ρ_{2x}, ρ_{3x} — radii of inertia of stepped pulleys 2 and 3 relative to the axis of rotation; α, β the angles of inclination of the planes to the horizon. If the bodies (pulleys) 2 or 3 are homogeneous, then in the calculations take R_2, R_3 . The chords above inclined planes are parallel to these planes. If necessary, take: $S_1 = 0,1 \text{ m}$, $\beta = 90^\circ - \alpha$, $\varphi = 90^\circ - \alpha$, $M = 2 \text{ N}\cdot\text{m}$.

Table 4.2

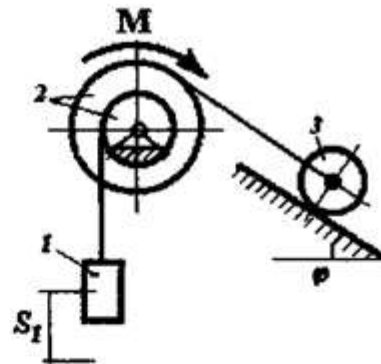
Variant	m_1 , kg	m_2 , kg	m_3 , kg	R_2 , m	r_2 , m	R_3 , m	r_3 , m	ρ_{2x} , m	ρ_{3x} , m	$\alpha, ^\circ$
1	6	1	2	0.6	0.5	0.5	0.3	0.6	0.4	30
2	8	2	2	0.8	0.6	0.7	0.4	0.7	0.6	45
3	10	3	3	0.65	0.5	0.6	0.4	0.5	0.5	60
4	5	0.5	1.5	0.5	0.4	0.45	0.2	0.45	0.3	30
5	4	0.75	1.25	0.4	0.3	0.3	0.15	0.35	0.2	45
6	9	4	1	0.4	0.35	0.35	0.25	0.3	0.3	60
7	3	0.3	1	0.3	0.2	0.25	0.15	0.2	0.2	30
8	2	0.2	0.3	0.2	0.15	0.25	0.2	0.2	0.25	45
9	11	4	2	0.7	0.6	0.6	0.3	0.65	0.5	60
0	12	5	1	0.75	0.65	0.65	0.45	0.7	0.5	30



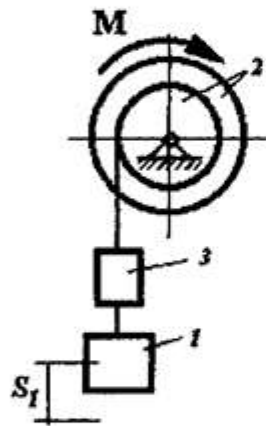
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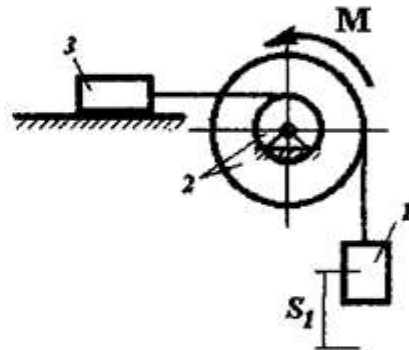
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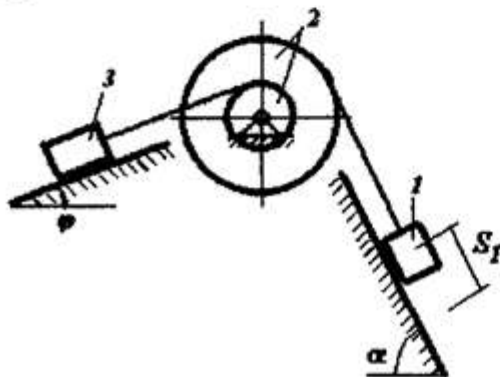
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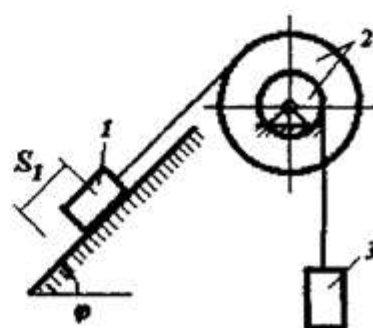
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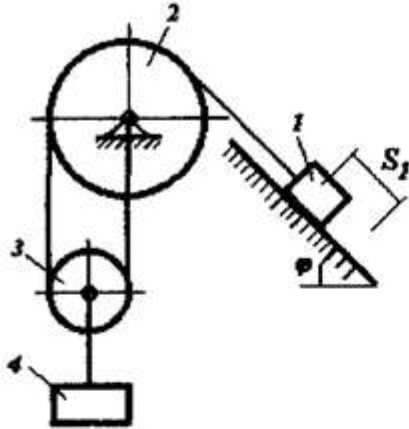
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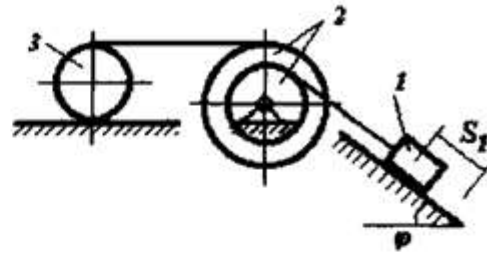
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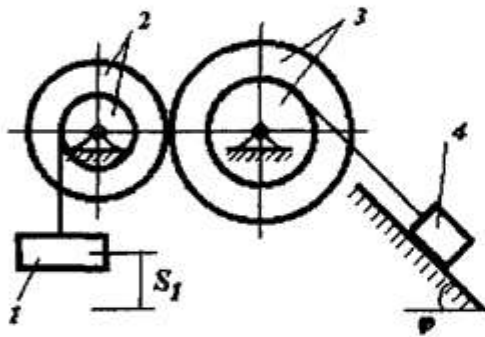
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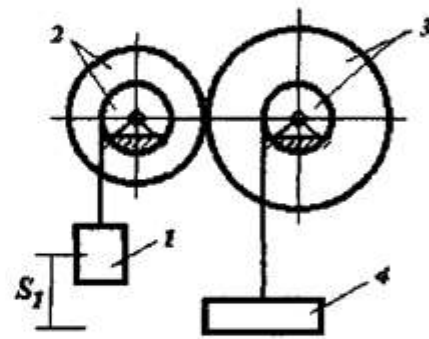
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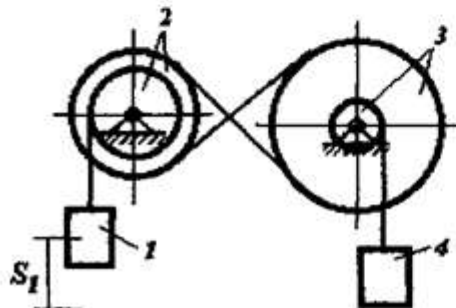
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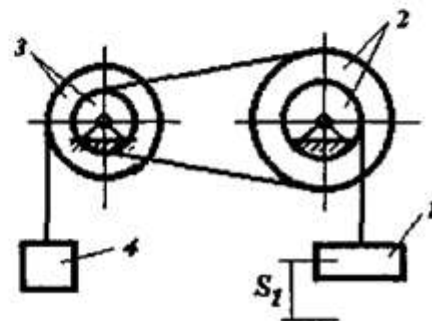
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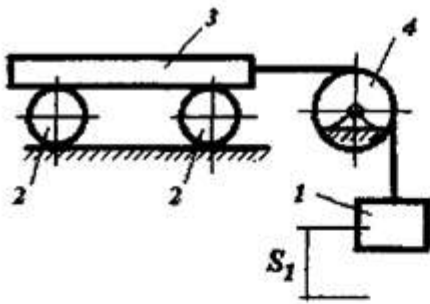
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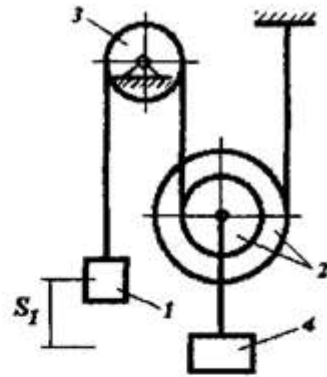
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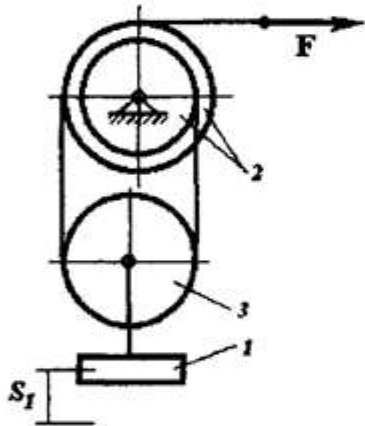
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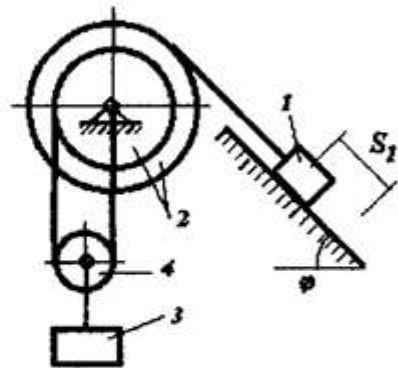
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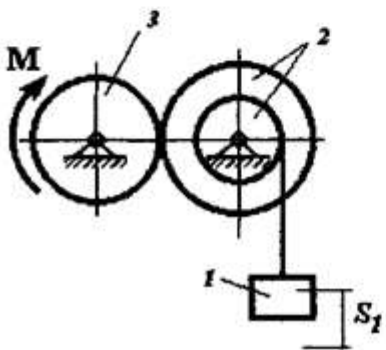
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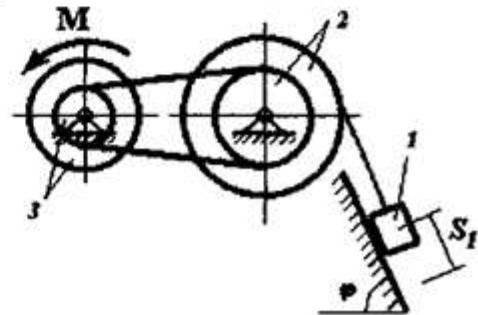
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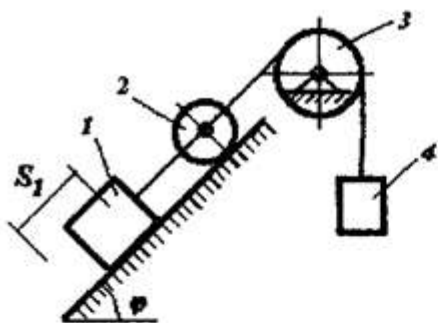
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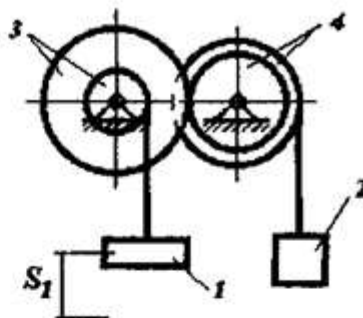
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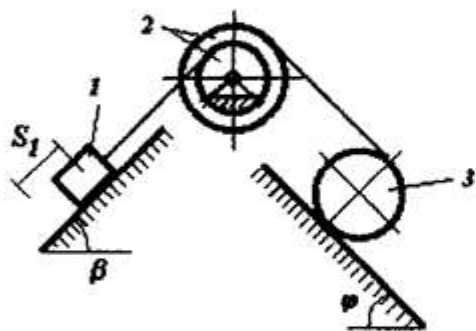
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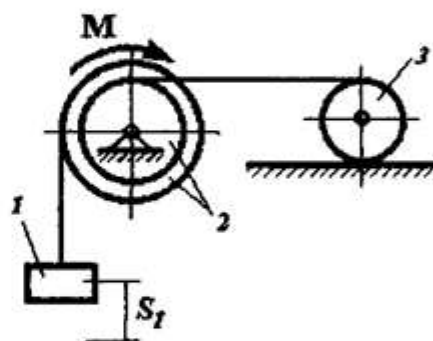
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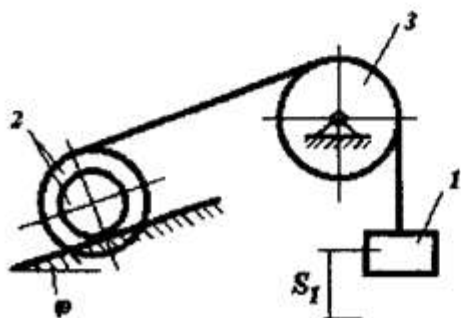
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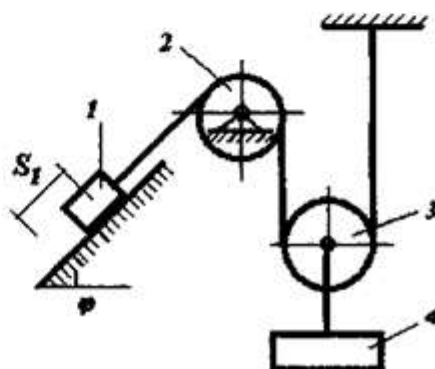
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4.8 Example of completing the task for Д5

Determine the acceleration and velocity of the center of mass of body 1 at the time when it travels the path S_1 , if the material system (Fig. 4.21) starts to move from rest. Neglect the masses of the cords. Bodies 1 and 3 move without slipping. Given: $m_1=10$ kg; $m_2=2$ kg; $m_3=1$ kg; $R_2=0.4$ m; $r_2=0.3$ m; $R_3=0.3$ m; $r_3=0.2$ m; $\rho_2 = 0.35$ m; $\rho_3 = 0.25$ m; $\alpha=30^\circ$; $\beta=45^\circ$; $S_1=0.4$ m.

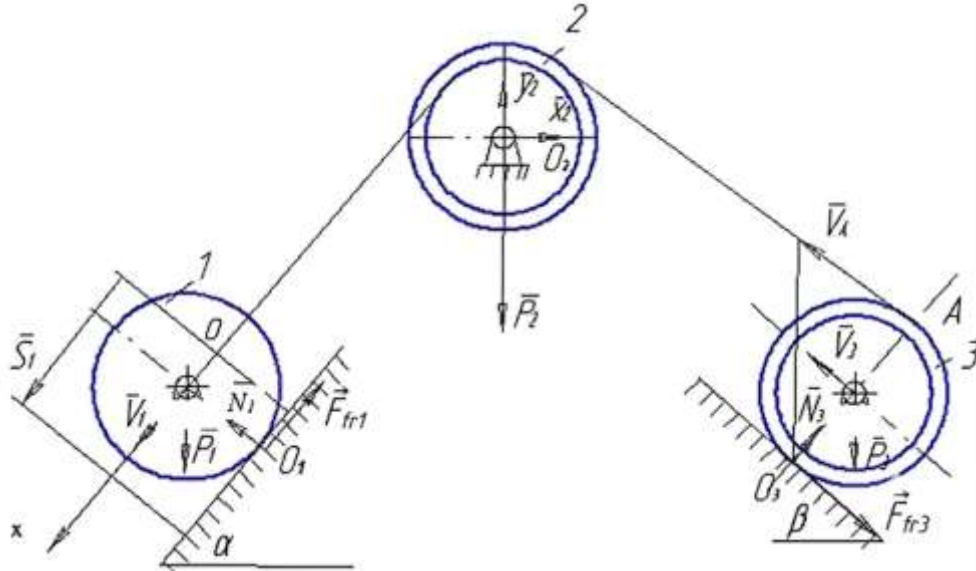


Figure 4.21

The solution

To study the motion of material systems (Fig. 4.21), we will apply the theorem on the change in kinetic energy of a mechanical system in differential form.

$$\frac{dT}{dt} = \sum N^e + \sum N^i \quad (4.18)$$

where T – kinetic energy of the system at $0 < x \leq S_1$; $\sum N^i$ and $\sum N^e$ – the sum of the powers of the internal and external forces of the system. The material system (Fig. 4.21) consists of rigid bodies and inextensible cords, then

$$\sum N^i = 0. \quad (4.19)$$

The differential equation (4.18) takes the form

$$\frac{dT}{dt} = \sum N^e. \quad (4.20)$$

Let us find the kinematic relations between the velocities of points and the angular velocities of bodies, writing them through the velocity V_1 of the center of mass of body 1.

The angular velocity of body 1, given that the instantaneous center of velocity of body 1 is located at the point O_1 (fig.4.21)

$$\omega_1 = \frac{V_1}{R_1} \quad (4.21)$$

where R_1 – radius of a uniform solid disk (body)1.

Angular velocity of the body 2

$$\omega_2 = \frac{V_1}{r_2} \quad (4.22)$$

The velocity of point A, given that bodies 2 and 3 are connected by an inextensible cord.

$$V_A = \omega_2 \cdot R_2. \quad (4.23)$$

Since point O_3 – the instantaneous center of velocity of the body 3, then

$$V_A = \omega_3 \cdot 2R_3 \quad (4.24)$$

From (4.23) and (4.24) determine the angular velocity of the body 3

$$\omega_3 = \omega_2 \frac{R_2}{2R_3} = V_1 \frac{R_2}{2r_2 R_3}. \quad (4.25)$$

Speed V_3 of the center of mass of the body 3

$$V_3 = \omega_3 \cdot R_3 = V_1 \frac{R_2}{2r_2}. \quad (4.26)$$

Let's find the displacement of the center of mass of the body 3.

Since

$$V_1 = \frac{dS_1}{dt}, V_3 = \frac{dS_3}{dt} \quad (4.27)$$

and, considering (4.26), we receive

$$\frac{dS_3}{dt} = \frac{dS_1}{dt} \cdot \frac{R_2}{2r_2}$$

or

$$dS_3 = \frac{R_2}{2r_2} \cdot dS_1. \quad (4.28)$$

when $t = 0$, $S_1 = 0$, and $S_3 = 0$ and after integration (4.28) we receive

$$S_3 = S_1 \cdot \frac{R_2}{2r_2}. \quad (4.29)$$

Let us find the kinetic energy of the material system as the sum of the kinetic energies of bodies 1, 2 and 3.

$$T = T_1 + T_2 + T_3. \quad (4.30)$$

Kinetic energy of body 1 moving parallel to the plane,

$$T_1 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} I_1 \cdot \omega_1^2$$

where $I_3 = \frac{1}{2} m_3 R_3^2$ – moment of inertia of body 1 about the central axis. Then, taking into account (4.21),

$$T_1 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} \cdot \frac{1}{2} m_1 R_1^2 \cdot \frac{V_1^2}{R_1^2} = \frac{3}{4} m_1 V_1^2. \quad (4.31)$$

Body 2 rotates around a horizontal axis and the kinetic energy is found by the formula

$$T_2 = \frac{1}{2} I_2 \cdot \omega_2$$

where $I_2 = m_2 \rho_2^2$ – moment of inertia of body 3 about the main central axis, ω_2 – angular velocity of the body 2 (4.22). Then

$$T_2 = \frac{1}{2} m_2 V_1^2 \frac{\rho_2^2}{r_2^2}. \quad (4.32)$$

Kinetic energy of a body 3

$$T_3 = \frac{1}{2} m_3 V_3^2 + \frac{1}{2} I_3 \cdot \omega_3^2$$

where $I_3 = m_3 \rho_3^2$ – moment of inertia of body 3 relative to the main central axis ,

$$\omega_3 = V_1 \frac{R_2}{2r_2 R_3} - (4.25),$$

$V_3 = \omega_3 \cdot R_3$ – speed of the center of mass of the body 3.

$$T_3 = \frac{1}{8} m_3 V_1^2 \frac{R_2^2}{r_2^2} + \frac{1}{8} m_3 V_1^2 \cdot \frac{R_2^2 \cdot \rho_3^2}{r_2^2 R_3^2}. \quad (4.33)$$

Now, taking into account (4.31) – (4.33), we determine the kinetic energy (4.30) of the system by the formula

$$T = V_1^2 \left(\frac{3}{4} m_1 + \frac{1}{2} m_2 \frac{\rho_2^2}{r_2^2} + \frac{1}{8} m_3 \frac{R_2^2}{r_2^2} \left(1 + \frac{\rho_3^2}{R_3^2} \right) \right). \quad (4.34)$$

The power of external forces under the action of which the material system moves (Fig. 4.21)

$$\sum N^e = N(\bar{P}_1) + N(\bar{P}_3),$$

where $N(\bar{P}_1) = P_1 \cdot V_1 \cdot \sin \alpha$; $N(\bar{P}_3) = -P_3 V_3 \cdot \sin \beta$; $P_1 = m_1 g$; $P_3 = m_3 g$.

Then, considering that (4.26)

$$V_3 = V_1 \frac{R_2}{2r_2}$$

we receive

$$\sum N^e = V_1 (m_1 \sin \alpha - m_3 \frac{R_2}{2r_2} \sin \beta) \cdot g. \quad (4.35)$$

Substituting (4.34) i (4.35) in (4.20), we receive

$$\left(\frac{3}{4} m_1 + \frac{1}{2} m_2 \frac{\rho_2^2}{r_2^2} + \frac{1}{8} m_3 \frac{R_2^2}{r_2^2} \left(1 + \frac{\rho_3^2}{R_3^2} \right) \right) \cdot \frac{dV_1^2}{dt} = V_1 (m_1 \sin \alpha - m_3 \frac{R_2}{2r_2} \sin \beta) \cdot g. \quad (4.36)$$

Since $\frac{dV_1^2}{dt} = 2V_1 \cdot a_1$ (a_1 – acceleration of the body's center of mass 1), then

$$a_1 = \sqrt{\frac{m_1 \sin \alpha - m_3 \frac{R_2}{2r_2} \sin \beta}{\frac{3}{2} m + m_2 \frac{\rho_2^2}{r_2^2} + \frac{1}{4} m_3 \frac{R_2^2}{r_2^2} \left(1 + \frac{\rho_3^2}{R_3^2} \right)}} \cdot g$$

Or, substituting the given conditions of the problem,

$$a_1 = \sqrt{\frac{10 \cdot \sin 30^\circ - 1 \cdot \frac{0.4}{2 \cdot 0.3} \cdot \sin 45^\circ}{\frac{3}{2} \cdot 10 + 2 \cdot \frac{0.35^2}{0.3^2} + \frac{1}{4} \cdot 1 \cdot \frac{0.4^2}{0.3^2} \left(1 + \frac{0.25^2}{0.3^2} \right)}} \cdot 9.81 = 1.5 \frac{m}{s^2}$$

Let's find the velocity of the center of mass of the body 1.

Equation (4.36) can be written in the form:

$$\left(\frac{3}{4} m_1 + \frac{1}{2} m_2 \frac{\rho_2^2}{r_2^2} + \frac{1}{8} m_3 \frac{R_2^2}{r_2^2} \left(1 + \frac{\rho_3^2}{R_3^2} \right) \right) \cdot dV_1^2 = g (m_1 \sin \alpha - m_3 \frac{R_2}{2r_2} \sin \beta) \cdot V_1 \cdot dt. \quad (4.37)$$

When $t=0$, $V_{10}=0$, $S_{10}=0$.

When $t=\tau$, $V_{1\tau}=V_1$, $S_{1\tau}=S_1$. (4.38)

where τ – the time it takes for the center of mass of body 1 to travel the path S_1 .

Integrating the equation (4.37) in terms of (4.38), we determine the velocity of the center of mass of body 1 per time τ .

$$V_1 = \sqrt{\frac{S_1 g (m_1 \sin \alpha - m_3 \frac{R_2}{2r_2} \sin \beta)}{\frac{3}{4} m_1 + \frac{1}{2} m_2 \frac{\rho_2^2}{r_2^2} + \frac{1}{8} m_3 \frac{R_2^2}{r_2^2} (1 + \frac{\rho_3^2}{R_3^2})}},$$

or substituting numerical data, we get:

$$V_1 = \sqrt{\frac{0.4 \cdot 9.81 (10 \cdot \sin 30^\circ - 1 \cdot \frac{0.4}{2 \cdot 0.3} \sin 45^\circ)}{\frac{3}{4} \cdot 10 + 2 \cdot \frac{0.35^2}{0.3^2} + \frac{1}{4} \cdot 1 \cdot \frac{0.4^2}{0.3^2} (1 + \frac{0.25^2}{0.3^2})}} = 1.3 \frac{m}{s}.$$

Self-awareness questions

1. What is the essence of the theorem on the motion of the center of mass of a mechanical system and what consequences follow from it?
2. What is the quantity of motion of a material point called?
3. What is the quantity of motion of a system of material points called?
4. What is the impulse of force called?
5. What is the essence of the theorem on the change in the quantity of motion of a system of material points called and what consequences follow from it?
6. What is the moment of the quantity of motion of a point relative to some center called?
7. What is the kinetic moment of a system of material points called or the principal moment of the quantity of motion of the system relative to some center called?
8. What is the kinetic moment of a rigid body relative to a fixed axis of rotation? Чому суть теореми про зміну моменту кількості руху механічної системи і які наслідки впливають із неї?
9. How is the differential equation of rotation of a rigid body around a fixed axis written?
10. What is called the kinetic energy of a material point?
11. What is called the kinetic energy of a system of material points?
12. What is called elementary work?
13. What is called power?
14. What is the essence of the theorem on the change in kinetic energy of a mechanical system?

5 ELEMENTS OF ANALYTICAL MECHANICS

The equations of motion of material objects (material points, solids and mechanical systems) considered above and the general theorems that are conclusions from them, proceed directly from the laws of Galileo and Newton.

But there are many such engineering problems of dynamics, the solution of which by the methods of Newtonian mechanics requires very great efforts, and sometimes is impossible at all. In such cases, methods of another, so-called **analytical mechanics**, are used, which are based on principles different from Newtonian ones. The founder of this branch of mechanics was Leibniz.

Analytical mechanics is a large, rich branch of mechanics. Its principles extend to areas such as relativity and quantum mechanics, where the laws of Galileo and Newton cannot be applied. For simplicity, and in order not to go beyond Newtonian mechanics, we will consider these principles as consequences of the laws of classical mechanics with the addition of the axiom of liberation from bonds. At the same time, it is necessary to understand that these are not theorems that are proved using the laws of Galileo and Newton, but rather principles. If we take them as a basis, we can obtain both Newton's laws and all of Newtonian mechanics..

The developers of the basic principles of analytical mechanics were the French scientists d'Alembert (1717–1783) and Lagrange (1736–1813).

D'Alembert's principle allows all problems related to the motion of bodies to be reduced to a simpler problem of equilibrium.

5.1 Kinetostatics method

The modern interpretation of d'Alembert's principle using the concept of *inertial force*, which was introduced into mechanics at the beginning of the 19th century, is the foundation of an important method of technical mechanics – the *method of kinetostatics*.

Kinetostatics– a branch of technical mechanics whose goal is to apply statics methods to solving dynamic problems in the theory of machines and mechanisms.

The method of kinetostatics is a formal technique that makes it possible to write the equations of motion in the form of equilibrium equations.

5.1.1 D'Alembert's principle for a material point

The principle of D'Alembert allows you to balance a system of forces.

For a non-free moving material point, at each moment of time the vector sum of the active forces applied to the point, the reactions of the joints and the forces of inertia is equal to zero:

$$\bar{F} + \bar{R} + \bar{\Phi} = 0,$$

where $\bar{F}$, $\bar{\Phi}$ – the resultant of active forces and inertia forces applied to a point;

$\bar{R}$ – connection reaction.

That is, the equation of motion is written in the form of a static equilibrium condition.

Force of inertia $\overline{\Phi}$ of a material point is called a vector that is equal in magnitude to the product of the mass of the point and its acceleration and is directed in the direction opposite to the acceleration of the point: $\overline{\Phi} = -m \cdot \overline{a}$. It is called the d'Alembert force of inertia.

With the coordinate method of specifying motion in the reference system $Oxyz$, the vector equation turns into a system of scalar equations:

$$F_x + R_x + \Phi_x = 0;$$

$$F_y + R_y + \Phi_y = 0;$$

$$F_z + R_z + \Phi_z = 0.$$

If the motion of the point is given in a natural way, then we will have a system of such equations:

$$F_\tau + R_\tau + \Phi_\tau = 0;$$

$$F_n + R_n + \Phi_n = 0;$$

$$F_b + R_b + \Phi_b = 0.$$

5.1.2 D'Alembert's principle for a material system

For a non-free moving material system, at each moment of time the vector sums of the principal vectors of active forces, reactions of bonds and inertia forces, and the principal moments of these same forces relative to the chosen center are equal to zero..

According to the D'Alembert's principle:

$$\sum_{i=1}^n \overline{F}_i + \sum_{i=1}^n \overline{R}_i + \sum_{i=1}^n \overline{\Phi}_i = 0,$$

$$\sum_{i=1}^n \overline{M}_0(\overline{F}_i) + \sum_{i=1}^n \overline{M}_0(\overline{R}_i) + \sum_{i=1}^n \overline{M}_0(\overline{\Phi}_i) = 0.$$

D'Alembert's principle provides a single method for composing differential equations of motion for constrained systems.

The vector equations correspond to six algebraic equations in coordinate form:

$$\sum_1^n (F_{kx} + R_{kx} + \Phi_{kx}) = 0; \quad \sum_1^n \left(M_x(\overline{F}_k) + M_x(\overline{R}_k) + M_x(\overline{\Phi}_k) \right) = 0;$$

$$\sum_1^n (F_{ky} + R_{ky} + \Phi_{ky}) = 0; \quad \sum_1^n \left(M_y(\overline{F}_k) + M_y(\overline{R}_k) + M_y(\overline{\Phi}_k) \right) = 0;$$

$$\sum_1^n (F_{kz} + R_{kz} + \Phi_{kz}) = 0; \quad \sum_1^n \left(M_z(\overline{F}_k) + M_z(\overline{R}_k) + M_z(\overline{\Phi}_k) \right) = 0.$$

Since the principal vector and principal moment of internal forces are equal to zero, there are no internal forces in the equations.

5.1.3 Reduction of inertia forces

The inertia forces for each body of the system are given separately.

The principal vector of inertial forces $\bar{\Phi}$ of a rigid body is equal to the product of the mass of the body M and the acceleration of its center of mass $\bar{a}_C$ and is directed in the direction opposite to this acceleration $\bar{\Phi} = -M \cdot \bar{a}_C$.

a) translational motion – inertia forces are reduced to $\bar{\Phi}_C$, which is applied at the center of mass of the body;

b) planar motion of a rigid body (the body has a plane of material symmetry that coincides with the plane of motion) – inertial forces are reduced to the main vector

$\bar{\Phi}_C = -M \cdot \bar{a}_C$ ($\bar{a}_C$ – acceleration of the center of mass) and the main moment $\bar{M}_c^{ih} = -I_{zc} \cdot \bar{\varepsilon}$. Sign « $-$ » means that the moment of inertia of the force pair is directed in the direction opposite to the angular acceleration of the body $\bar{\varepsilon}$.

c) rotational motion of a rigid body:

main moment $\bar{M}^i = -I_z \cdot \bar{\varepsilon}$,

where I_z – moment of inertia about the axis of rotation.

If the center of mass of the system lies on the axis of rotation, then the inertia forces are reduced only to a pair $\bar{M}^i$.

Example. Application of d'Alembert's principle to study the rotational motion of a rigid body

A system of bodies consisting of a rod 1 with weight P , a uniform disk 2 and a ring 3, having weights respectively Q_2 and Q_3 and radius R , rotates around a horizontal axis O (fig. 5.1) under the action of a pair of forces with a moment M . Find the angular acceleration, angular velocity, and force at the axis supports when the material system is rotated by an angle α , if at the initial moment of time the system was at rest.

Accept: $P = 10 \text{ N}$; $Q_2 = 20 \text{ N}$; $M = 30 \text{ N}\cdot\text{m}$; $Q_3 = 10 \text{ N}$; $R = 0,2 \text{ m}$; $AO = 0,8 \text{ m}$; $\alpha = 60^\circ$.

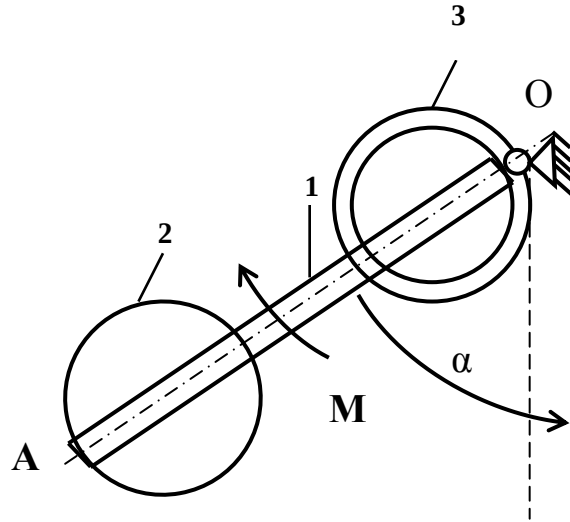


Figure 5.1

The solution

To study the system, we will use **the method of kinetostatics** in projections on the x, y axes for a plane arbitrary system of forces .

$$\begin{aligned} F_x + R_x + F_x^{in} &= 0, \\ F_y + R_y + F_y^{in} &= 0, \\ M_z^R + M_z^F + M_z^{in} &= 0. \end{aligned} \quad (5.1)$$

Determining the angular acceleration ε of a system of bodies

We will find the angular acceleration of bodies based on the third equation of the system (5.1)

$$M_z^R + M_z^F + M_z^{in} = 0. \quad (5.2)$$

Active forces act on a system of bodies $\overline{Q_1}, \overline{Q_2}, \overline{P}$ and a force couple with a moment M (fig. 5.1); the movement of bodies is prevented by connection: a cylindrical hinge O, the action of which on the body, based on the axiom of liberation from ligatures, is replaced by the reactions of the connections $\overline{X}_o$ and $\overline{Y}_o$.

Let's find the principal moment of inertia forces M_z^{in}

$$M_z^{in} = I_z \cdot \varepsilon. \quad (5.3)$$

Moment of inertia of bodies I_z , relative to the axis Z, we find as the sum of the moments of inertia of bodies 1, 2, 3 relative to the axis Z:

$$I_z = I_{z1} + I_{z2} + I_{z3}.$$

Moment of inertia of disk 2 relative to the axis Z:

$$I_{z2} = \frac{1}{2} m_2 R^2 + m_2 (OA - R)^2 = \frac{Q_2}{g} \left[\frac{1}{2} R^2 + (OA - R)^2 \right] =$$

$$= \frac{20}{9,81} \left[\frac{1}{2} 0,2^2 + (0,8 - 0,2)^2 \right] = 0,775 \text{ kg} \cdot \text{m}^2.$$

Moment of inertia of ring 3 relative to the axis Z:

$$I_{z3} = m_3 R^2 + m_3 R^2 = 2 \frac{Q_3}{g} R^2 = 2 \cdot \frac{10}{9,81} \cdot 0,2^2 = 0,082 \text{ kg} \cdot \text{m}^2.$$

Moment of inertia of rod 1 relative to the axis Z:

$$I_{z1} = \frac{1}{3} m_1 \cdot OA^2 = \frac{1}{3} \frac{P_1}{g} \cdot OA^2 = \frac{1}{3} \cdot 9,81 \cdot 0,8^2 = 0,217 \text{ kg} \cdot \text{m}^2.$$

Then $I_z = 0,775 + 0,082 + 0,217 = 1,07 \text{ kg} \cdot \text{m}^2$.

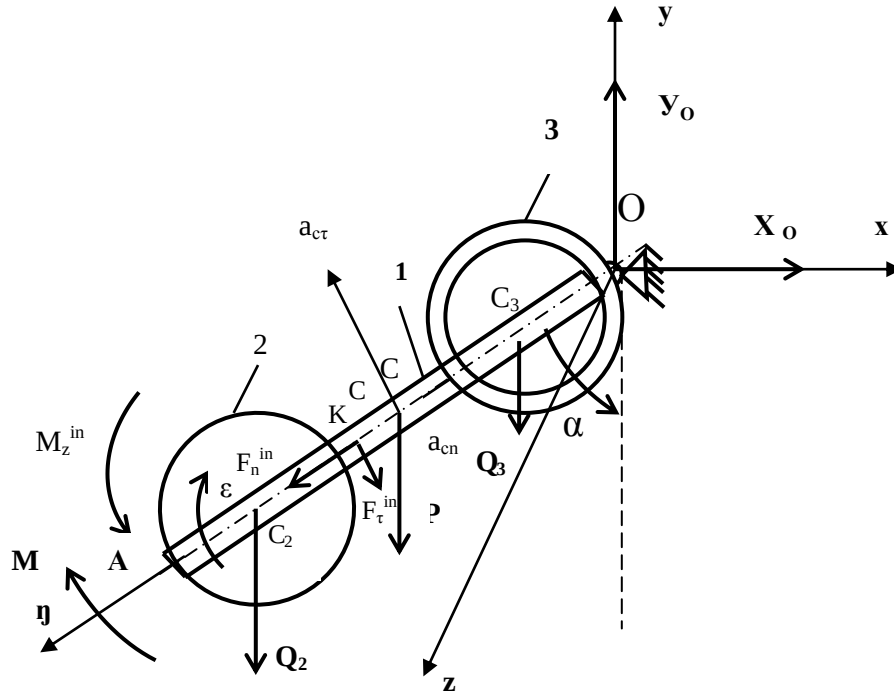


Figure 5.2

The main moment of inertia forces M_z^{in} is directed in the direction opposite to the angular acceleration ϵ . Let us write equation (5.2) for the system of forces applied to bodies 1, 2, 3 (Fig. 5.2).

$$-M + Q_2 \cdot OC_2 \sin \alpha + P \cdot OC_1 \sin \alpha + Q_3 \cdot OC_3 \sin \alpha + M_z^{in} = 0$$

$$I_z \epsilon = M - (Q_2 \cdot OC_2 + P \cdot OC_1 + Q_3 \cdot OC_3) \sin \alpha.$$

From which:

$$\begin{aligned}\varepsilon &= \frac{M}{I_z} - \frac{Q_2 \cdot OC_2 \sin \alpha + P \cdot OC_1 \sin \alpha + Q_3 \cdot OC_3}{I_z} \cdot \sin \alpha, \\ \varepsilon &= \frac{30}{1,074} - \frac{20 \cdot 0,6 + 10 \cdot 0,4 + 10 \cdot 0,2}{1,074} \cdot \sin \alpha = 27,93 - 16,76 \cdot \sin \alpha.\end{aligned}\quad (5.4)$$

when $\alpha = 60^\circ$, $\varepsilon = 27,93 - 16,76 \cdot \frac{\sqrt{3}}{2} = 13,4 \text{ } 1/c^2$.

Determining angular velocity

Since

$$\varepsilon = \frac{d\omega}{dt} = \frac{d\omega}{dt} \cdot \frac{d\alpha}{d\alpha} = \frac{d\alpha}{dt} \cdot \frac{d\omega}{d\alpha} = \omega \frac{d\omega}{d\alpha},$$

then expression (5.4) can be written as

$$\omega \frac{d\omega}{d\alpha} = 27,93 - 16,76 \cdot \sin \alpha. \quad (5.5)$$

When $t = 0$; $\omega = \omega_0 = 0$; $\alpha = 0$,

When $t = t_1$; $\omega_1 = \omega_\alpha$; $\alpha = \frac{\pi}{3}$.

Integrating the equation (5.5)

$$\begin{aligned}\int_0^{\omega_\alpha} \omega d\omega &= \int_0^{\frac{\pi}{3}} 27,93 d\alpha - \int_0^{\frac{\pi}{3}} 16,76 \sin \alpha d\alpha, \\ \frac{\omega_\alpha^2}{2} &= 27,93 \cdot \frac{\pi}{3} + 16,76 \cos \frac{\pi}{3} - 16,76, \\ \omega_\alpha &= \sqrt{2 \cdot (27,93 \frac{3,14}{3} + 16,76 \frac{1}{2} - 16,76)} = 6,46 \frac{1}{s}, \\ \omega_\alpha &= 6,46 \frac{1}{s}.\end{aligned}$$

Determination of support reactions

Let's find the coordinates of the center of mass of the system of bodies.

$$\eta_c = \frac{\sum_{i=1}^n m_i \eta_i}{\sum_{i=1}^n m_i} = \frac{m_1 \cdot OC_1 + m_2 \cdot OC_2 + m_3 \cdot OC_3}{m_1 + m_2 + m_3};$$

$$\eta_c = \frac{P \cdot 0,4 + Q_2 \cdot 0,6 + Q_3 \cdot 0,2}{P + Q_2 + Q_3} = \frac{10 \cdot 0,4 + 20 \cdot 0,6 + 10 \cdot 0,2}{10 + 20 + 10} = 0,45 \text{ m.}$$

Projections of the acceleration of the center of mass at $\alpha = \frac{\pi}{3}$ on natural coordinate axes:

$$a_{c\tau} = \varepsilon \cdot \eta_c = \varepsilon \cdot OC = 13,4 \cdot 0,45 = 6,03 \text{ m/s}^2,$$

$$a_{cn} = \omega_\alpha^2 \cdot \eta_c = \omega_\alpha^2 \cdot OC = 6,46 \cdot 0,45 = 18,78 \text{ m/s}^2.$$

Let us write the first two equations of system (5.1) for the forces applied to bodies 1, 2, 3 (Fig. 5.2).

$$X_0 - F_n^{in} \sin \alpha + F_\tau^{in} \cos \alpha = 0,$$

$$Y_0 - P - Q_2 - Q_3 - F_n^{in} \cos \alpha - F_\tau^{in} \sin \alpha = 0,$$

where:

$$F_n^{in} = (m_1 + m_2 + m_3) \cdot a_{cn} = \frac{P + Q_2 + Q_3}{g} \cdot a_{cn} = \frac{10 + 20 + 10}{9,81} \cdot 18,78 = 76,57 \text{ N};$$

$$F_\tau^{in} = (m_1 + m_2 + m_3) \cdot a_{c\tau} = \frac{P + Q_2 + Q_3}{g} \cdot a_{c\tau} = \frac{10 + 20 + 10}{9,81} \cdot 6,03 = 24,59 \text{ N}.$$

Inertia forces $\bar{F}_n^{in}$ та $\bar{F}_\tau^{in}$ applied at point K of the rod OA.

$$OK = \frac{I_z \cdot \varepsilon}{F_\tau^{in}} = \frac{13,4 \cdot 1,074}{24,59} = 0,59 \text{ m.}$$

From here we find the reactions of the supports X_0 and Y_0 .

$$X_0 = \bar{F}_n^{in} \sin \alpha - \bar{F}_\tau^{in} \cos \alpha = 76,57 \sin 60^\circ - 24,59 \cos 60^\circ,$$

$$X_0 = 54,02 \text{ N}.$$

$$Y_0 = P + Q_2 + Q_3 + \bar{F}_n^{in} \cos \alpha + \bar{F}_\tau^{in} \sin \alpha,$$

$$Y_0 = 10 + 20 + 10 + 76,57 \cos 60^\circ + 24,59 \sin 60^\circ,$$

$$Y_0 = 99,58 \text{ N}.$$

Answer: $\omega_\alpha = 6,46 \frac{1}{s}; \varepsilon = 13,4 \frac{1}{s^2}; X_0 = 54,02 \text{ N}; Y_0 = 99,58 \text{ N}.$

Self-awareness questions

1. What is the method of kinetostatics called?
2. How is d'Alembert's principle formulated for a material point?
3. What is the force of inertia of a material point called?
4. How is d'Alembert's principle formulated for a material system?

5.2 Analytical statics

Actual and possible displacements. Number of degrees of freedom

Actual moving are movements of points of the system that do not contradict the constraints and occur under the action of external forces. These movements correspond to the actual motion of the system.

Possible moving are called such elementary imaginary movements of points of the system that do not contradict the constraints and occur at a fixed moment of time.

Possible displacements of a mechanical system are determined by any set of possible displacements of individual points of the system. Possible displacements should not be confused with actual displacement. For example, if a point lies on a plane, then its possible displacements are infinitesimal displacements in any direction along this plane. The actual displacement of a point can be one of these possible ones, namely the one that is determined not only by the nature of the connection, but also by the forces acting on the point and the initial conditions of motion.

Unlike the real elementary displacements, which we denoted by $d\vec{r}$ or $d\vec{S}$, we will denote possible movements by $\delta\vec{r}$, $\delta\vec{S}$ etc.

The number of independent possible movements of a mechanical system is called the number of **degrees of freedom** of this system..

A point lying on a surface can be given two independent possible movements, for example, along two axes (Fig. 5.3). Any third possible movement can be expressed in terms of these two, that is, it has two degrees of freedom.

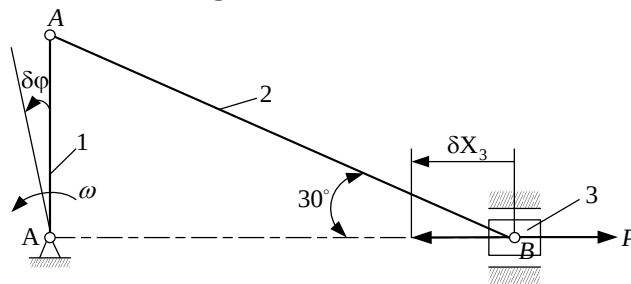


Figure 5.3

The system (Fig. 5.3) has one degree of freedom, because $\delta\varphi$ depends on δx_3 and vice versa.

The concept of ideal connections

Ideal are called connections, the algebraic sum of the elementary work of the reactions of which at any possible displacements of the points of the system is equal to

zero:
$$\sum_{i=1}^n R_i \cdot \delta \bar{r}_i = 0.$$

Ideal connections: hinge, surface, weightless rod, ideal thread – without taking into account friction forces.

Forces that are not reactions of the muscles are called active forces.

5.2.1 Principle of possible displacements (Lagrange's Principle)

The principle of possible displacements is the most general principle of analytical statics. From it, one can derive the equilibrium conditions of any mechanical system. The principle of possible displacements was established by J. Lagrange in 1788.

The principle of possible movements: For the equilibrium of a system subject to restraining, holonomic, ideal, and stationary constraints, it is necessary and sufficient that the sum of the elementary works of the active forces on any possible displacement of the system from the considered equilibrium position is equal to zero, provided that the system is stationary at the initial moment:

$$\sum_{i=1}^n \delta A_i^a = \sum_{i=1}^n \bar{F}_i \delta \bar{S}_i = 0,$$

or
$$\sum_{i=1}^n F_i \delta S_i \cdot \cos(\bar{F}_i, \delta S_i) = 0,$$

or
$$\sum_{i=1}^n (F_{xi} \cdot \delta x_i + F_{yi} \cdot \delta y_i + F_{zi} \cdot \delta z_i) = 0.$$

The principle of possible displacements gives in general form the equilibrium conditions for any mechanical system, while the equilibrium conditions of statics consider the equilibrium of each body of the system separately.

Methodology for solving problems based on the principle of possible displacements

When applying the principle of possible displacements to solving specific problems, it is recommended to adhere to the following sequence of actions:

1. Determine the system of material points or bodies whose motion must be considered.
2. Determine the number of degrees of freedom of this system.

3. Determine the nature of the constraints imposed on this material system, i.e. determine whether these constraints are ideal or not. In the latter case, as indicated above, the friction forces should be attributed to the given forces.

4. If some of the reactions of the joints, which are mostly excluded from consideration due to their ideality, need to be determined, then in this case, by imaginary discarding the joint, its reaction is replaced and, transferring the reaction to the category of given forces, the principle of possible displacements is applied. In this case, the joints whose reactions need to be determined are discarded one by one so that only one unknown force enters the equation. If it is necessary to determine the reaction of the hinge, then it is decomposed along the directions of the coordinate axes and after that one component is determined first, and then the other. To determine the horizontal component, the hinge should be replaced by a rod on the horizontal plane, and not to discard the ligature completely, because the reaction also has a vertical component, which does not create work on the horizontal movement possible in this case and, thus, will be excluded from the corresponding equation. After that, the vertical component is determined in a similar way, replacing the hinge with a rod on the vertical plane.

5. Draw a diagram of the given forces applied to the points of the material system.

6. Give the system one of the possible displacements. When choosing this displacement, if the system has several degrees of freedom, care should be taken to ensure that the elements that need to be determined are not excluded from the equation, and that the equation has the simplest form.

7. Show the directions of displacements of individual points of a material system to which given forces are applied.

8. Determine the work of given forces on the corresponding possible displacements and compose equations based on the principle of possible displacements. It is obvious that such equations can be composed as many as the degrees of freedom of a given material system.

9. Establish the relationship between the possible displacements of the points of the system and thus determine the possible displacements of all points of the system as a function of the possible displacements independent of each other, based on the considerations given above.

10. As a result, a system of equations is obtained, the number of which corresponds to the number of degrees of freedom of the material system. By excluding from these equations the possible displacements independent of each other due to their arbitrariness, the required forces or other quantities can be determined.

5.3 Analytical dynamics

5.3.1 General equation of dynamics (D'Alembert-Lagrange principle)

The general equation of dynamics is obtained by combining d'Alembert's principle with the principle of possible Lagrangian displacements: *When a material system moves with ideal, restraining, holonomic constraints, the sum of the work of all active forces and inertial forces of all points of the system at each moment of time, at any of its possible displacements, is equal to zero, i.e.*

$$\sum_{i=1}^n \delta A(\bar{F}_i^a) + \sum_{i=1}^n \delta A(\bar{\Phi}_i) = 0,$$

$$\text{or } \sum_{i=1}^n [(F_{ix}^a + \Phi_{ix})\delta x_i + (F_{iy}^a + \Phi_{iy})\delta y_i + (F_{iz}^a + \Phi_{iz})\delta z_i] = 0,$$

$$\text{or } \sum_{i=1}^n [(F_{ix} - m_i \ddot{x}_i)\delta x_i + (F_{iy} - m_i \ddot{y}_i)\delta y_i + (F_{iz} - m_i \ddot{z}_i)\delta z_i] = 0,$$

where F_{ix}, F_{iy}, F_{iz} – projections of active forces on the Cartesian coordinate axis,

$\ddot{x}, \ddot{y}, \ddot{z}$ – projection of the acceleration of the i-th point of the system;

$\delta x_i, \delta y_i, \delta z_i$ – projection of virtual displacements of a point onto the same coordinate axes.

Example

The material system (Fig. 5.4) starts to move from rest under the action of the moment M applied to body 1. The axes of bodies 1 and 2 are horizontal. The coefficient of sliding friction is f . There is no sliding at the points of contact of the bodies. Neglect the mass of the belt. Body 1 is a uniform cylinder.

Determine the acceleration of body 3, the tension S_5 in the driven part of the belt 5 and the tension S_4 in the leading part 4 (accept $S_4 = 2S_5$) and the force at the point of contact of bodies 1 and 2 at $t = t_1$.

Accept : $R_1 = 0,25$ m; $R_2 = 0,45$ m; $r_2 = 0,15$ m; $i_2 = 0,4$ m; $L = 0,7$ m; $m_1 = 0,5$ kg; $m_2 = 5$ kg; $m_3 = 4$ kg; $M = 3t^3$ N·m; $t_1 = 2$ s; $f = 0,4$.

The solution

To study the motion of a material system (Fig. 5.6), we apply the general equation of dynamics

$$\sum_{\kappa=1}^n \bar{F}_{\kappa} \cdot \delta \bar{r}_{\kappa} + \sum_{\kappa=1}^n \bar{F}_{\kappa}^{in} \cdot \delta \bar{r}_{\kappa} = 0. \quad (5.6)$$

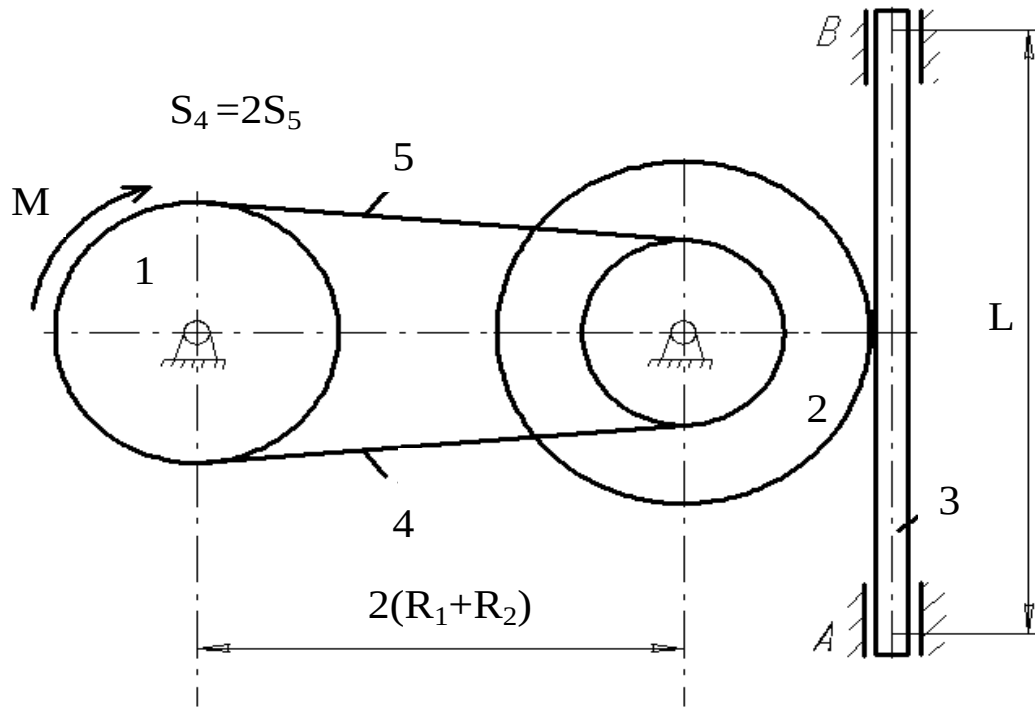


Figure 5.4

Determining the acceleration of a body 3

A homogeneous body 1 rotates around a horizontal axis under the action of a moment M (Fig. 5.4) with an angular acceleration ε_1 . The force from body 1 to body 2 is transmitted by a string, the acceleration of the points of which on straight sections is $\bar{a}_n$:

$$a_n = \varepsilon_1 R_1 = \varepsilon_2 r_2. \quad (5.7)$$

In order for the movement of a material system to occur without impacts, the acceleration of the body 3 $\bar{a}_3$ must be equal to the tangential acceleration of the points of body 2, located at a distance R_2 from the axis of rotation:

$$a_3 = \varepsilon_2 R_2. \quad (5.8)$$

Active forces are applied to the material system (Fig. 5.5): moment M , weight $\bar{P}_1, \bar{P}_2, \bar{P}_3$, ($\bar{P}_1 = m_1 g, \bar{P}_2 = m_2 g, \bar{P}_3 = m_3 g$) respectively the bodies 1, 2 and 3. The center of mass of bodies 1 and 2 is located on the axis of rotation, therefore the principal vectors of inertial forces of bodies 1 and 2 are equal to zero.

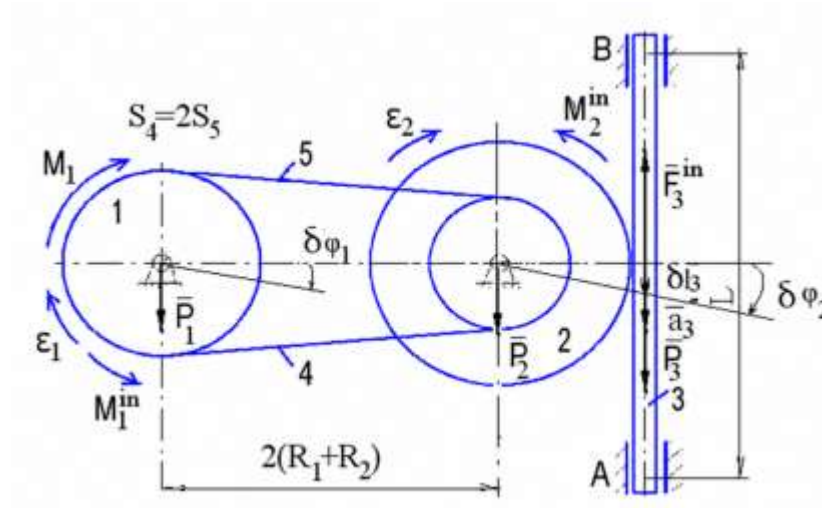


Figure 5.5

The main moments of inertia of the body 1 M_1^{in} and 2 M_2^{in} directed opposite to the angular acceleration of the bodies ϵ_1 and ϵ_2 .

$$M_1^{in} = I_1 \epsilon_1, \quad M_2^{in} = I_2 \epsilon_2, \quad (5.9)$$

where $I_1 = \frac{1}{2} m_1 R_1^2$ – moment of inertia of body 1 relative to the axis of rotation;

$I_2 = m_2 i_2^2$ – moment of inertia of body 2 relative to the axis of rotation.

The main vector of inertia forces $\bar{F}_3^{in}$ of the body 3 is directed in the opposite direction of acceleration $\bar{a}_3$ (fig. 5.5).

$$\bar{F}_3^{in} = m_3 \bar{a}_3. \quad (5.10)$$

If body 3 is moved virtually $\delta \bar{r}_3$, then body 2 will rotate at an angle $\delta \varphi_2$, and body 1 – at an angle $\delta \varphi_1$ (fig. 5.5).

Let's find the relationship between the virtual movements of the bodies of the system:

$$\begin{aligned} \delta r_3 &= \delta \varphi_2 R_2, \\ \delta \varphi_2 R_2 &= \delta \varphi_1 R_1. \end{aligned} \quad (5.11)$$

Let us write the general equation of dynamics (5.6) for the material system under study (fig. 5.5):

$$P_3 \cdot \delta r_3 - F_3^{in} \cdot \delta r_3 - M_2^{in} \cdot \delta \varphi_2 - M_1^{in} \cdot \delta \varphi_1 + M \cdot \delta \varphi_1 = 0. \quad (5.12)$$

Or, considering (5.7) – (5.11):

$$m_3 g \delta r_3 - m_3 a_3 \delta r_3 - m_2 i_2^2 a_3 \frac{1}{R_2^2} \delta r_3 - \frac{1}{2} m_1 R_1^2 a_3 \frac{r_2^2}{R_2^2 R_1^2} \delta r_3 + M \frac{r_2}{R_2 R_1} \delta r_3 = 0.$$

Since $\delta r_3 \neq 0$, we have:

$$m_3 g - m_3 a_3 - m_2 i_2^2 a_3 \frac{1}{R_2^2} - \frac{1}{2} m_1 R_1^2 a_3 \frac{r_2^2}{R_2^2 R_1^2} + M \frac{r_2}{R_2 R_1} = 0.$$

Where do we find acceleration a_3 of the body 3.

$$a_3 = \frac{m_3 g + M \frac{r_2}{R_2 R_1}}{m_3 + m_2 \frac{i_2^2}{R_2^2} + \frac{1}{2} m_1 \frac{r_2^2}{R_2^2}}. \quad (5.13)$$

Substituting these conditions into the problem, we get:

$$a_3 = 4,92 + 0,5t^3. \quad (5.14)$$

when $t_1 = 2s$, $a_3 = 8,92 \frac{m}{s^2}$.

Determination of tension in belts and forces between bodies

Consider the movement of the body 3. Forces $\bar{N}_2$ and $\bar{S}_2$, with which body 2 affects the body 3, we attribute to active forces (Fig. 5.6).

We give body 3 virtual movement $\delta \bar{r}_3$ and we find the work of active forces $\bar{P}_3, \bar{S}_2, \bar{N}_2$ and inertia forces $\bar{F}_3^{in}$ on this displacement (5.6).

$$P_3 \cdot \delta r_3 + S_2 \cdot \delta r_3 - F_3^{in} \cdot \delta r_3 = 0.$$

Since $\delta \bar{r}_3 \neq 0$, then

$$P_3 + S_2 - F_3^{in} = 0.$$

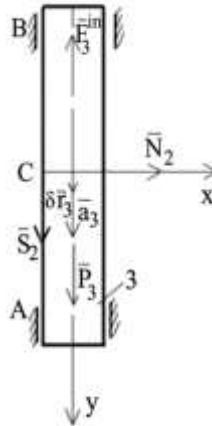


Figure 5.6

Whence, taking into account (5.8) and (5.13), we find:

$$S_2 = m_3(4.92 + 0.5t^3) - m_3g.$$

when $t_1 = 2s$, $S_2 = -3.6N$.

According to the Amonton-Coulomb law: $S_2 = |fN_2|$, тоді $N_2 = \left| \frac{S_2}{f} \right|$.

When $t_1 = 2s$, $N_2 = 9N$.

To determine the forces in the belts, consider the movement of body 2 (Fig. 5.7).

Connections reactions $\bar{S}_2'$, $\bar{N}_2'$, $\bar{S}_5$, $\bar{S}_4$ are attributed to active forces. Moreover

$$\bar{S}_2' = -\bar{S}_2, \bar{N}_2' = -\bar{N}_2.$$

The main moment of inertia of the body 2, M_2^{in} directed in the opposite direction of the angular acceleration ε_2 of the body (fig. 5.7). We give body 2 a virtual movement $\delta\varphi_2$ and we write the general equation of dynamics:

$$-S_2'R_2 \cdot \delta\varphi_2 - M_2^{in} \cdot \delta\varphi_2 - S_5r_2 \cdot \delta\varphi_2 + S_4r_2 \cdot \delta\varphi_2 = 0.$$

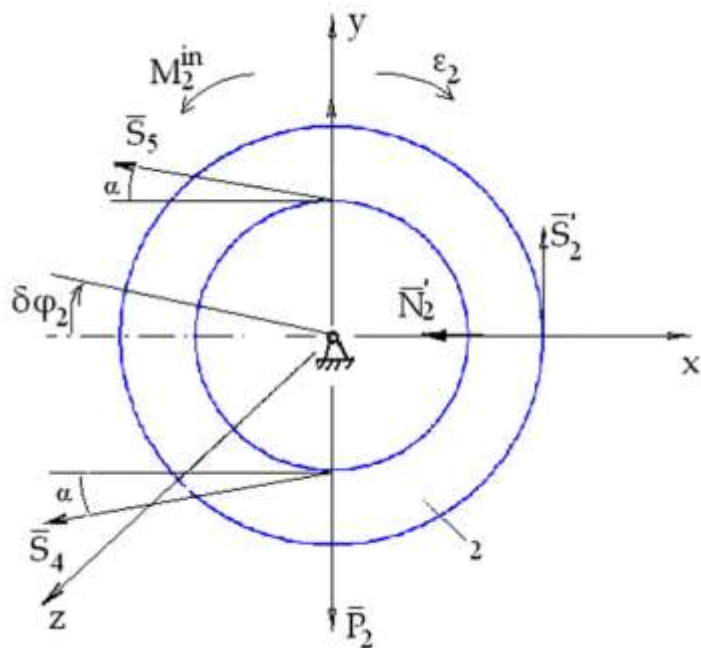


Figure 5.7.

Considering that $\delta\varphi_2 \neq 0$ i $S_4 = 2S_5$, we find:

$$S_5 = 12t^3 - 0,083a_3.$$

When $t_1 = 2s$, $S_5 = 95,26N$, $S_4 = 190,52N$.

Answer: $a_3 = 8,92 \frac{m}{s^2}$, $S_2 = -3,6N$, $N_2 = 9N$, $S_4 = 2 \cdot S_5 = 190,52N$.

Self-awareness questions

1. What is the commonality between actual and possible displacements and how do they differ?
2. What are called actual displacements?
3. What are called possible displacements?
4. What are called ideal connections?
5. How is the principle of possible displacements (Lagrange's principle) formulated?
6. How is the general equation of dynamics (d'Alembert-Lagrange's principle) formulated?

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