Ofer Aluf

Advance Elements of Optoisolation Circuits

Nonlinearity Applications in Engineering

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Preface

This book on advance optoisolation circuits and nonlinearity applications in engineering covers and deals two separate engineering and scientific areas and what happens in between. It is a continuation of the first book “Optoisolation Circuits Nonlinearity Applications in Engineering”. It gives advance analysis methods for optoisolation circuits which represent many applications in engineering. Optoisolation circuits come in many topological structures and represent many specific implementations which stand the target engineering features. Optoisolation circuits includes phototransistor and light-emitting diode (optocouplers), both coupled together. There are many other semiconductors which include optoisolation topologies like photo SCR, SSR, photodiode, etc. The basic optocoupler can be characterized by Ebers–Moll model and the associated equations. The optoisolation circuits include optocouplers and peripheral elements (capacitors, inductors, resistors, switches, operational amplifiers, etc.). The optoisolation circuits analyze as linear and nonlinear dynamical systems and their limit cycles, bifurcation, and limit cycle stability analysis by using Floquet theory. This book is aimed at newcomers to linear and nonlinear dynamics and chaos advance optoisolation circuits. Many advance optoisolation circuits exhibit limit cycle behavior. A limit cycle is a closed trajectory (system phase space $V_1(t)$, $V_2(t)$ voltages in time are coordinates); this means that its neighboring trajectories are not closed—they spiral either towards or away from the limit cycles. Thus, limit cycles can only occur in those optoisolation circuits which exhibit nonlinearity (nonlinear systems). Optoisolation circuits exhibit many kinds of bifurcation behaviors. The basic definition of bifurcation describes the qualitative alterations that occur in the orbit structure of a dynamical system as the parameters on which the optoisolation system depends are varied. There are many bifurcations we discuss related to optoisolation systems: cusp-catastrophe, Bautin bifurcation, Andronov–Hopf bifurcation, Bogdanov–Takens (BT) bifurcation, fold Hopf bifurcation, Hopf–Hopf bifurcation, Torus bifurcation (Neimark–Sacker bifurcation), and saddle–loop or homoclinic bifurcation. Floquet theory helps us to analyze advance optoisolation systems. Floquet theory is the study of the stability of linear periodic systems in continuous time. Another way to describe Floquet theory: it is the study of linear systems of differential equations with periodic coefficients. Floquet theory can be used for anything foe which you would use linear stability analysis, when dealing with optoisolation periodic systems. Many optoisolation systems are periodic and continue in time; therefore Floquet theory is an ideal way for behavior and stability analysis. There are many optoisolation systems which contain periodic forcing source and can be analyzed by this theory. Floquet theory is also implemented in OptoNDR circuit analysis. Optoisolation circuits with periodic limit cycle solutions orbital stability is part of advances system. Optoisolation circuits bifurcation culminating with periodic limit cycle oscillation. The route by which chaos arises from mixed-mode periodic states in optoisolation systems. The optoisolation system displays a rich variety of dynamical behavior including simple oscillations, quasiperiodicity, bi-stability between periodic states, complex periodic oscillations (including the mixed-mode type), and chaos. The route to chaos in this optoisolation system involves a torus attractor which becomes destabilized and breaks up into a fractal object, a strange attractor. Optoisolation systems exhibit sequence leading from a mixed-mode periodic state to a chaotic one in the staircase region and find a familiar cascade of periodic-doubling bifurcations, which finally culminate in chaos. Optoisolation circuits with forced van der Pol sources exhibit a variety of implementations in many engineering areas. Optoisolation circuit averaging analysis and perturbation from geometric viewpoint.

In this book we try to provide the reader with explicit advance procedures for application of general optoisolation circuits mathematical representations to particular advance research problems. Special attention is given to numerical and analytical implementation of the developed techniques.

Let us briefly characterize the content of each chapter.

Chapter 1. Optoisolation Circuits with Limit Cycles. In this chapter optoisolation circuits with limit cycles are described and analyzed. A limit cycle is an closed trajectory (system phase space $V_1(t)$, $V_2(t)$ voltages in time are coordinates); this means that its neighboring trajectories are not closed—they spiral either towards or away from the limit cycle. Thus, limit cycles can only occur in those optoisolation circuits which exhibit nonlinearity (nonlinear systems). In contrast a linear system exhibiting oscillations closed trajectories are neighbored by other closed trajectories (Example: $d\theta/dt = f(\theta)$). A stable limit cycle is one which attracts all neighboring trajectories. A system with a stable limit cycle can exhibit self-sustained oscillations. Neighboring trajectories are repelled from unstable limit cycles. Half-stable limit cycles are ones which attract trajectories from the one side and repel those on the other. If all neighboring trajectories approach the limit cycle, the limit cycle is stable or attracting. Otherwise the limit cycle is unstable, or in exceptional cases, half stable. An optoisolation circuit which exhibits stable limit cycles oscillate even in the absence of external periodic forcing (wave generator $f(t)$). Since limit cycles are nonlinear phenomena; they cannot occur in linear systems. A limit cycle oscillations are determined by the structure of the system itself. We use in nonlinear dynamics some acronyms: Stable limit cycle (SLC), half-stable limit cycle (HLC), unstable limit cycle (ULC), unstable equilibrium point (UEP), stable equilibrium point (SEP), half-stable equilibrium point (HEP).

Chapter 2. Optoisolation Circuits Bifurcation Analysis (I). In this chapter we discuss various bifurcations which exhibit by optoisolation circuits. The first is cusp-catastrophe which occurs in a one-dimensional state space (n = 1) and two-dimensional parameter space (p =2). It has an equilibrium manifold in R^2 x R: $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2)$ $V \in \mathbb{R}^{n=1}$; $\{\Gamma_1, \Gamma_2\} \in \mathbb{R}^{p=2}$; $M = \{(\Gamma_1, \Gamma_2, V) | \Gamma_1 + \Gamma_2 \cdot V - V^3 = 0\}$, where Γ_1 and Γ_2 are two control parameters and V is the system state variable. A two-parameter system near a triple equilibrium point known as cusp bifurcation (equilibrium structure). We can consider system with a higher dimensional state space and parameter space. The Bautin bifurcation is equilibrium in a two-parameter family of system's autonomous ODEs at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov–Hopf bifurcation vanishes. This phenomenon is also called the generalized Hopf (GH) bifurcation. Bogdanov–Takens (BT) bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs, critical equilibrium has a zero eigenvalue of multiplicity two.

Chapter 3. Optoisolation Circuits Bifurcation Analysis (II). In this chapter we discuss various bifurcations which exhibit by optoisolation circuits. The fold Hopf bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues. The Hopf–Hopf bifurcation is when the critical equilibrium has two pairs of purely imaginary eigenvalues in system's two-parameter family of autonomous differential equations. Torus bifurcations (Neimark–Sacker bifurcations) of the limit cycles generated by the Hopf bifurcations. This curve of torus bifurcations is transversal to the saddle–node and Andronov–Hopf bifurcation curves. The torus bifurcation generates an invariant two-dimensional torus. The invariant torus disappears via either a "heteroclinic destruction" or a "blow-up". Saddle–loop or homoclinic bifurcation is when part of a limit cycle moves closer and closer to a saddle point. At the homoclinic bifurcation the cycle touches the saddle point and becomes a homoclinic orbit (infinite period bifurcation).

Chapter 4. Optoisolation Circuits Analysis Floquet Theory. In this chapter optoisolation circuits are analyzed by using Floquet theory. Floquet theory is the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points. Another way to describe Floquet theory: it is the study of linear systems of differential equations with periodic coefficients. Floquet theory can be used for anything for which you would use linear stability analysis, when dealing with a periodic system. Although Floquet theory is a linear theory, nonlinear models can be linearized near limit cycle solutions to enable the use of Floquet theory. Floquet theory deals with continuous-time systems. The theory of periodic discrete-time systems is closely analogous. In that case, one can multiply the T transition matrices together to determine how a perturbation changes over a period, which is similar to finding the fundamental matrix. One limitation of Floquet theory is that it applies only to periodic systems. Although many systems experience periodic forcing, others experience stochastic or chaotic forcing. In these cases, more general Lyapunov exponents are described which play the role of Floquet exponents. Conceptually similar to Floquet exponents, Lyapunov exponents are more challenging to compute numerically because, instead of calculating how a perturbation grows or shrinks over one period, this must be done in the limit at $T \to \infty$.

Chapter 5. Optoisolation NDR Circuits Behavior Investigation by Using Floquet Theory. In this chapter we discuss optoisolation NDR circuits behavior analysis by using Floquet theory. Floquet theory is the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points. We consider an OptoNDR circuit with storage elements (variable capacitor and variable inductance) in the output port. OptoNDR circuits' nonlinear models is linearized near limit cycle solutions to enable the use of Floquet theory. We analyze Chua's circuit and find fixed points and stability. We replace Chua's diode by OptoNDR element in Chua's circuit. The circuit contains three energy storage elements and exhibits different and unique variety of bifurcations and chaos among the chaos-producing mechanism with OptoNDR element. We consider an OptoNDR circuit with two storage elements and find behavior, fixed points, and stability. OptoNDR circuit's two variables are simulated and analyzed by using Floquet theory. We use Floquet theory to test the stability of optoisolation circuits limit cycle solution.

Chapter 6. Optoisolation Circuits with Periodic Limit-Cycle Solutions Orbital Stability. In this chapter we implement systems with periodic limit cycle by using optoisolation circuits. We discuss periodic and solutions orbital stability. Periodic orbits are nonequilibrium trajectories $X(t)$ that satisfy $X(T) = X(0)$ for some $T > 0$. The basic behavior of planar cubic vector field and van der Pol equation is analyzed and discussed for limit cycle and stability. Van der Pol system is implemented by using OptoNDR element and the limit cycle solution is discussed. Glycolytic oscillation is the repetitive fluctuation of in the concentrations of metabolites. The modeling glycolytic oscillation is presented and discussed in control theory and dynamical systems. The model explains sustained oscillations in the yeast glycolytic system. We discuss the differential equations model of glycolytic oscillator and find fixed points, stability, and limit cycle. We present and discuss optoisolation glycolytic circuits limit cycle solution.

Chapter 7. Optoisolation Circuits Poincare Maps and Periodic Orbit. In this chapter we deal with optoisolation circuits and investigate periodic orbit and Poincare map. A Poincare map is a discrete dynamical system with a state space that is one dimension smaller than the original continuous dynamical system. We use it for analyzing the original system. We use Poincare maps to study the flow near a periodic orbit and the flow in some chaotic system. System physical behavior in time is described by a dynamical system. A dynamical system is a function of $\phi(t, X)$, defined for all $t \in \mathbb{R}$ and $X \in E \in \mathbb{R}^n$. A Poincare map is a very useful tool to study the stability and bifurcations of system periodic orbit. Optoisolation van der Pol oscillator circuit with parallel capacitor C_2 dynamical behavior is inspected by using a Poincare map. Li autonomous system with toroidal chaotic attractors is a typical dynamical system. The Li system global Poincare surface of section has two disjoint components. A similarity transformation in the phase space emphasizes symmetry of the attractor. One way for the production of oscillations in L-C networks is to overcome circuit losses through the use of designed-in positive feedback or generation. Negative resistance oscillator with OptoNDR device has better performance characteristics cancellation in terms of resistive losses in oscillators than oscillators that are not based on NDR devices. We investigate it by using Poincare map and find periodic orbit.

Chapter 8. Optoisolation Circuits Averaging Analysis and Perturbation from Geometric Viewpoint. In this chapter we discuss optoisolation circuits averaging analysis and perturbation from geometric viewpoint. In many perturbed systems, we start with a system which includes known solutions and add small perturbations to it. The solutions of unperturbed and perturbed systems are different and system with small perturbation has different structure of solutions. In perturbation theory we use approximate solution by discussing of the exact solution. An approximation of the full solution A, a series in the small parameters (ε), like the following solution $A = \sum_{k=0}^{\infty} A_k \cdot \varepsilon^k$. A_0 is the known solution to the exactly soluble initial problem and $A_1, A_2, \ldots$ represent the higher order terms. Higher order terms become successively smaller for small ε. In every dynamical system we need to use the global existence theorems. The system solution is altered for non-zero but small ε. We analyze OptoNDR circuit van der Pol by using perturbation method and we use multiple scale analysis for construction uniform or global approximate solutions for both small and large values of independent variables of OptoNDR circuit van der Pol. OptoNDR circuit forced van der Pol is analyzed by using perturbation method time scale. We consider forced van der Pol system with forcing term $F \cdot \cos(\omega \cdot t)$. If “F” is small (weak excitation), its effect depends on whether or not ω is close to the natural frequency. If it is, then the oscillation might be generated which is a perturbation of the limit cycle. If “F” is not small (hard excitation), then the “natural oscillation” might be extinguished.

Chapter 9. Optoisolation Advance Circuits—Investigation, Comparison and Conclusions. In this chapter, we summarized the main topics regarding optoisolation advance circuits; inspect behavior, dynamics, stability, comparison, and conclusions. Optoisolation advance circuits are an integral part of every industrial system. An optoisolation circuit can have limit cycles which we can analyze. Additionally there are many bifurcations that can characterize optoisolation circuits. Floquet theory is analyzed in many systems which include optoisolation circuits. Optoisolation NDR circuits behavior is investigated by using Floquet theory and periodic limit cycle solutions orbital stability is discussed. We present optoisolation circuits by Poincare maps and periodic orbit. Averaging analysis and perturbation from geometric viewpoint are implemented in our circuits.

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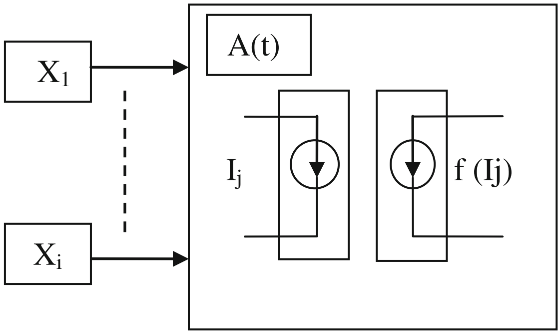
Introduction

Optoisolation products are used in a wide area of many engineering applications which can be characterized by bifurcation behaviors. Among optoelectronics isolation products can be optocouplers (LEDs and bipolar phototransistor or photo MOSFET), isolation amplifier, digital isolators, solid-state relays (MOSFET), etc. Many topological optoisolation products schematics give a variety of nonlinear and bifurcation behaviors which can be implemented in many engineering areas. Each Optoisolation circuit can be represented by a set of differential equations which depend on optoisolation variable parameters. The investigation of optoisolation circuit's differential equation bifurcation theory, the study of possible changes in the structure of the orbits of a differential equation depending on variable parameters. The book illustrates certain optoisolation circuits observations and analyze advance bifurcations behaviors (cusp bifurcation, Bautin bifurcation, Bogdanov–Takens (double-zero) bifurcation, Fold–Hopf bifurcation, Hopf–Hopf bifurcation, torus bifurcation, and homoclinic bifurcation, etc.). Since the implicit function theorem is the main ingredient used in these generalizations, a precise statement of this theorem is included. Additionally the bifurcations of an optoisolation product's differential equation on the periodic limit cycle solution orbital stability are analyzed. Implementation of Floquet theory in many optoisolation circuits and existence of a Hopf bifurcation culminates in periodic limit cycle oscillation. The periodic limit-cycle solution orbital stability. All of that for optimization of advance optoisolation circuits (parameters optimization) is to get the best engineering performance. Floquet theory is the mathematical theory of linear, periodic systems of ordinary differential equation (ODEs) and the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points. We can test optoisolation system stability of limit cycle solutions. Optocouplers can be represented as couple which includes LED/LEDs (input control circuit) and phototransistor/photodiode/photo MOSFET (Output controlled circuit). The influence between the two optically separated elements can be represented as current source which depends on input current. The functionality is nonlinear behavior and depends on many coupling parameters between LEDs and phototransistor $f(I_j) = \sum_{k=1}^{\infty} a_{jk} \cdot I_j^k$. We consider optoisolation system which can be described by a set of linear, homogeneous, time periodic differential equations $\frac{dX}{dt} = A(t) \cdot X$, ($X = x_{1,\ldots,}x_i$). X is a n-dimensional vector and $A(t)$ is an n x n matrix with minimal period T.



Although optoisolation system parameters $A(t)$ vary periodically, the solutions are typically not periodic, and despite its linearity, closed form solutions typically cannot be found. The general solution of Floquet optoisolation system equations takes the form $x(t) = \sum_{i}^{n} c_i \cdot e^{\mu_i \cdot t} \cdot p_i(t)$. Where c_i are constants that depend on optoisolation system initial conditions, $p_i(t)$ are vector valued functions with periodic T, and μ_i are complex numbers called characteristic or Floquet exponents. Characteristic or Floquet multipliers are related to the Floquet exponents by the relationship $\rho_i = e^{\mu_i \cdot T}$. The optoisolation system fold–Hopf bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues. This phenomenon is also called the zero–Hopf (ZH) bifurcation, saddle–node Hopf bifurcation or Gavrilov–Guckenheimer bifurcation. The bifurcation point in the parameter plane lies at a tangential intersection of curves of saddle–node bifurcations and Andronov–Hopf bifurcations. Depending on the system, a branch of torus bifurcations can emanate from the ZH-point. In such cases, other bifurcations occur for nearby parameter values, including saddle–node bifurcations of periodic orbits on the invariant torus, torus breakdown, and bifurcations of Shil'nikov homoclinic orbits to saddle-foci and heteroclinic orbits connecting equilibria. This bifurcation, therefore, can imply a local birth of "chaos". Optoisolation system which is characterized by Neimark–Sacker bifurcation is the birth of a closed invariant curve from a fixed point in dynamical systems with discrete time (iterated maps), when the fixed point changes stability via a pair of complex eigenvalues with unit modulus. The bifurcation can be supercritical or subcritical, resulting in a stable or unstable (within an invariant two-dimensional manifold) closed invariant curve, respectively. When it happens in the Poincare map of a limit cycle, the bifurcation generates an invariant two-dimensional torus in the corresponding ODE. Neimark–Sacker bifurcation in optoisolation system occurs in the Poincare map of a limit cycle in ODE, the fixed point corresponding to the limit cycle has a pair of simple eigenvalue $\lambda_{1,2} = e^{\pm i \cdot \theta_0}$. A unique two-dimensional invariant torus bifurcates from the cycle, while it changes stability. The intersection of the torus with the Poincare section corresponds to the closed invariant curve. The torus bifurcation is sometimes called the secondary Hopf bifurcation. The torus bifurcation can occur near the fold–Hopf bifurcation and is always present near the Hopf–Hopf bifurcation of equilibria in ODEs. A supercritical torus bifurcation: a stable *spiraling* periodic orbit becomes repelling surrounded by a stable 2D torus. We now consider an optoisolation autonomous system of ordinary differential equations (ODEs) $\frac{dx}{dt} = f(x, \alpha)$, $x \in R^n$ depending on two parameters $\alpha \in R^2$, where f is smooth. Suppose that at $\alpha = 0$ the system has an equilibrium x = 0. Assume that it is Jacobian matrix $A = f_x(0,0)$ has a zero eigenvalue $\lambda_1 = 0$ a pair of purely imaginary eigenvalues $\lambda_{2,3} = +/-j \cdot \omega$ with $\omega > 0$. This codimension two bifurcation is characterized by the conditions $\lambda_1 = 0$ and Re $\lambda_{2,3} = 0$ and appears in open sets of two-parameter families of smooth ODEs. Generically, $\alpha = 0$ lies at a tangential intersection of curves of saddle–node bifurcation curve and Andronov–Hopf bifurcation curve within the two–parameter family. In a small fixed neighborhood of x = 0 for optoisolation system parameter values sufficiently close to $\alpha = 0$, the system has at most two equilibria, which can collide and disappear via a saddle–node bifurcation or undergo an Andronov–Hopf bifurcation producing a limit cycle. Additional curves of codimension one bifurcations accumulate at $\alpha = 0$ in the parameter plane, where codimension one bifurcations appear to depend largely upon the quadratic Taylor coefficients of f(x, 0). The most interested case is the appearance of a branch of torus bifurcations (Neimark–Sacker bifurcations) of the limit cycles generated by the Hopf bifurcations in optoisolation system. This curve of torus bifurcations is transversal to the saddle–node and Andronov–Hopf bifurcation curves. The torus bifurcation generates an invariant two-dimensional torus, i.e. "interaction of the saddle–node and Andronov–Hopf bifurcations can lead to tori". The invariant torus disappears via either a "heteroclinic destruction" or a "blow-up". In some cases, homoclinic and heteroclinic orbits connecting the two equilibria appear and disappear, while in other cases, the torus hits the boundary of any small fixed neighborhood of x = 0. The dynamics on the torus can be either periodic or quasiperiodic, and the torus can lose its smoothness before disappearance. The complete bifurcation scenario is unknown. The optoisolation system bifurcations of quasiperiodic oscillations and transitions to chaos via their destruction are one of interesting problems of nonlinear dynamics. There are well-known mechanisms, such as the Landau–Hopf scenario, the Ruelle–Takens scenario, and the Afraimovich–Shilnikov scenario, that describe the details of transitions to chaos through multifrequency oscillations. It is analyzed in many optoisolation systems examples where a wide class of real and model dynamical systems are used. The regimes of quasiperiodic oscillations are most frequently observed in periodically driven discrete-time and differential systems. We develop the optoisolation simplest autonomous differential system that can generate a solution in the form of stable two frequency oscillations and demonstrate the basic bifurcation mechanisms of their destruction including period doubling bifurcations. A two-dimensional torus can be realized in a three-dimensional autonomous dissipative system as it has been demonstrated in the book.

Chapter 1
Optoisolation Circuits with Limit Cycles

Many advanced optoisolation circuits exhibit limit cycle behavior. A limit cycle is a closed trajectory (system phase space $V_1(t)$, $V_2(t)$ voltages in time are coordinates); this means that its neighboring trajectories are not closed—they spiral either toward or away from the limit cycle. Thus, limit cycles can only occur in those optoisolation circuits which exhibit nonlinearity (nonlinear systems). In contrast, a linear system exhibiting oscillations closed trajectories is neighbored by other closed trajectories [Example: $d\theta/dt = f(\theta)$]. A stable limit cycle is the one which attracts all neighboring trajectories. A system with a stable limit cycle can exhibit self-sustained oscillations. Neighboring trajectories are repelled from unstable limit cycles. Half-stable limit cycles are the ones which attract trajectories from one side and repel those on other. If all neighboring trajectories approach the limit cycle, the limit cycle is stable or attracting. Otherwise the limit cycle is unstable, or in exceptional cases, half stable. Optoisolation circuits which exhibit stable limit cycles oscillate even in the absence of external periodic forcing [wave generator $f(t)$]. Since limit cycles are nonlinear phenomena; they cannot occur in linear systems. A limit cycle oscillations are determined by the structure of the system itself. We use in nonlinear dynamics some acronyms like: Stable limit cycle (SLC), half-stable limit cycle (HLC), unstable limit cycle (ULC), unstable equilibrium point (UEP), stable equilibrium point (SEP), half-stable equilibrium point (HEP) [7, 8].

1.1 Optoisolation Circuits with Limit Cycles

In electronics, an optoisolator (or optical isolator, optocoupler, photocoupler, or photoMOS) is a device that uses a short optical transmission path to transfer a signal between elements of a circuit, typically a transmitter and a receiver, while keeping them electrically isolated. One of the famous systems which demonstrates limit cycle is van der Pol oscillator. First, we define van der Pol system block

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diagram with main variable (X) which describes the dynamic of the system. Then, we perform the desired transformation to voltage (V) main system variable and implement it by using optoisolation devices and other elements. Investigation of system behavior when parameters values varied is a crucial stage in our analysis. Oscillations of van der Pol are called relaxation oscillations because the charge that builds up slowly is relaxed during a sudden discharge in a strongly nonlinear limit ($\mu \gg 1$).

$$\frac{d^2X}{dt^2} + \mu \cdot \frac{dX}{dt} \cdot (X^2 - 1) + X = 0$$

Let us describe our Optoisolation van der Pol system block diagram (Fig. 1.1).

$$X = \gamma \cdot \left[-\beta \cdot \frac{d^2X}{dt^2} + \alpha \cdot \frac{dX}{dt} - X^2 \cdot \alpha \cdot \frac{dX}{dt}\right] \Rightarrow X = -\gamma \cdot \beta \cdot \frac{d^2X}{dt^2} + \gamma \cdot \alpha \cdot \frac{dX}{dt} - X^2 \cdot \gamma \cdot \alpha \cdot \frac{dX}{dt}$$

$$X + \gamma \cdot \beta \cdot \frac{d^2X}{dt^2} - \gamma \cdot \alpha \cdot \frac{dX}{dt} + X^2 \cdot \gamma \cdot \alpha \cdot \frac{dX}{dt} = 0 \Rightarrow \gamma \cdot \beta \cdot \frac{d^2X}{dt^2} + \frac{dX}{dt} \cdot [X^2 \cdot \gamma \cdot \alpha - \gamma \cdot \alpha] + X = 0$$

$$\gamma \cdot \beta \cdot \frac{d^2X}{dt^2} + \frac{dX}{dt} \cdot [X^2 - 1] \cdot \gamma \cdot \alpha + X = 0 \Rightarrow \frac{d^2X}{dt^2} + \frac{dX}{dt} \cdot [X^2 - 1] \cdot \frac{\alpha}{\beta} + \frac{1}{\beta \cdot \gamma} \cdot X = 0$$

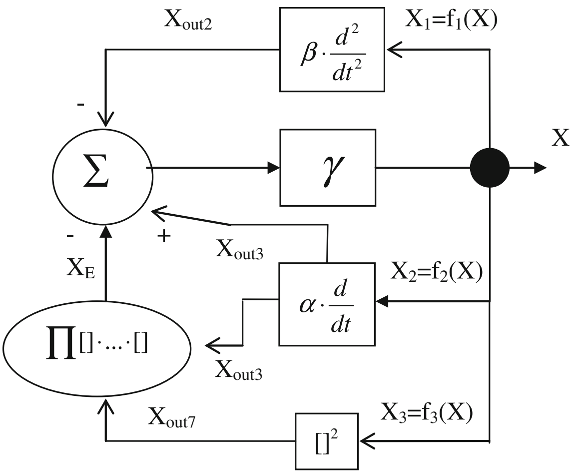


Fig. 1.1 Optoisolation van der Pol system block diagram

$$\mu = \frac{\alpha}{\beta};\ \frac{1}{\beta\cdot\gamma} = 1 \Rightarrow \beta\cdot\gamma = 1 \Rightarrow \frac{\alpha}{\mu}\cdot\gamma = 1 \Rightarrow \mu = \alpha\cdot\gamma;\ \alpha = \mu\cdot\beta \Rightarrow \beta = \frac{\alpha}{\mu} \Rightarrow \gamma = \frac{1}{\beta}$$

We perform system variable transformation for the output variable (V), where $X \to V$.

$$\frac{d^2V}{dt^2} + \mu\cdot\frac{dV}{dt}\cdot(V^2-1) + V = 0 \Rightarrow \frac{d^2V}{dt^2} + \frac{\alpha}{\beta}\cdot\frac{dV}{dt}\cdot(V^2-1) + \frac{1}{\beta\cdot\gamma}\cdot V = 0$$

This is a van der Pol oscillator system equation and block diagram. We need to implement it with optoisolation devices, op-amps, resistors, capacitors, diodes, etc., $\mu \geq 0 \to (\alpha/\beta) \geq 0 \to (1)\ \alpha \geq 0; \beta > 0\ (2)\ \alpha \leq 0; \beta < 0$. System equation is a simple harmonic oscillator with a nonlinear damping term $\mu\cdot\frac{dV}{dt}\cdot(V^2-1)$. This term acts like positive damping for $|V| > 1$ and like negative damping for $|V| < 1$. We have systems which have large output voltage amplitude oscillations decay but if they are small, pumps back up. The van der Pol system settles into a self-sustained oscillation. Our van der Pol system has a unique, stable limit cycle for each $\mu > 0 \to (\alpha/\beta) > 0 \to (1)\ \alpha > 0; \beta > 0\ (2)\ \alpha < 0; \beta < 0$. To portrait our system phase space, we define out (V, dV/dt) phase space and plot the solution.

Results Limit cycle is not a circle and the stable waveform is not a sine wave. We define new two variables V_1, V_2 of our van der Pol system.

$$V_1 = V;\ V_2 = \frac{dV}{dt};\ \frac{d^2V}{dt^2} + \mu\cdot\frac{dV}{dt}\cdot(V^2-1) = \frac{d}{dt}\left(\frac{dV}{dt} + \mu\cdot\left[\frac{1}{3}\cdot V^3 - V\right]\right)$$

$$F(V) = \frac{1}{3}\cdot V^3 - V;\ z = \frac{dV}{dt} + \mu\cdot F(V) = \frac{dV}{dt} + \frac{\alpha}{\beta}\cdot F(V);\ \frac{d^2V}{dt^2} + \mu\cdot\frac{dV}{dt}\cdot(V^2-1) = \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{d^2V}{dt^2} + \mu\cdot\frac{dV}{dt}\cdot(V^2-1) = -V;\ \frac{dV}{dt} = z - \mu\cdot F(V);\ \frac{dz}{dt} = -V \Rightarrow \frac{dV_1}{dt} = z - \mu\cdot F(V_1)$$

$$\frac{dz}{dt} = -V_1;\ \text{Variable change } V_2 = \frac{z}{\mu} \Rightarrow \frac{dV_1}{dt} = \mu\cdot[V_2 - F(V_1)];\frac{dV_2}{dt} = -\frac{1}{\mu}\cdot V_1$$

We get typical trajectories in the (V_1, V_2) phase plane. The trajectory zaps horizontally onto the cubic nullcline $V_2 = F(V_1)$.

$\frac{dV_1}{dt} = 0 \Rightarrow \mu\cdot[V_2 - F(V_1)] = 0 \Rightarrow V_2 = F(V_1)$. Stability investigation of van der Pol system is very important. We are back to our optoisolation basic van der Pol system equation: $\frac{d^2V}{dt^2} + \mu\cdot\frac{dV}{dt}\cdot(V^2-1) + V = 0$. $V = V_1$; $dV/dt = V_2 \to dV_1/dt = V_2$; $d^2V/dt^2 = dV_2/dt$. Then we get our van der Pol system nonlinear

differential equations : $\frac{dV_1}{dt} = V_2$; $\frac{dV_2}{dt} = -V_1 - V_2 \cdot (V_1^2 - 1) \cdot \mu$. Our system fixed point is calculated by setting $\frac{dV_1}{dt} = 0$ and $\frac{dV_2}{dt} = 0$ which yield to $(V_1^*, V_2^*) = (0, 0)$. We define $f_1(V_1, V_2) = V_2$; $f_2(V_1, V_2) = -V_1 - V_2 \cdot (V_1^2 - 1) \cdot \mu$ and calculate the related partial derivatives of f_1 and f_2 respect to V_1 and V_2. $\frac{\partial f_1}{\partial V_1} = 0$; $\frac{\partial f_1}{\partial V_2} = 1$
$\frac{\partial f_2}{\partial V_1} = -1 - 2 \cdot \mu \cdot V_1 \cdot V_2 = -[1 + 2 \cdot \mu \cdot V_1 \cdot V_2]$; $\frac{\partial f_2}{\partial V_2} = -\mu \cdot (V_1^2 - 1)$. The matrix A is called the Jacobian matrix at the fixed points: $(V_1^*, V_2^*) = (0, 0)$.

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{pmatrix}\right|_{(V_1^*, V_2^*) = (0,0)} = \left.\begin{pmatrix} 0 & 1 \\ -[1 + 2 \cdot \mu \cdot V_1 \cdot V_2] & -\mu \cdot (V_1^2 - 1) \end{pmatrix}\right|_{(V_1^*, V_2^*) = (0,0)}$$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{pmatrix}\right|_{(V_1^*, V_2^*) = (0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \lambda = \begin{pmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot \mathrm{I})$, where I is the identity matrix.

$$A - \lambda \cdot I = \left.\begin{pmatrix} \frac{\partial f_1}{\partial V_1} - \lambda & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} - \lambda \end{pmatrix}\right|_{(V_1^*, V_2^*) = (0,0)} = \left.\begin{pmatrix} -\lambda & 1 \\ -[1 + 2 \cdot \mu \cdot V_1 \cdot V_2] & -\mu \cdot (V_1^2 - 1) - \lambda \end{pmatrix}\right|_{(V_1^*, V_2^*) = (0,0)} = \begin{pmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{pmatrix}$$

$$-\lambda \cdot (\mu - \lambda) + 1 = 0 \Rightarrow \lambda^2 - \lambda \cdot \mu + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} = \frac{1}{2} \cdot \mu \pm \frac{1}{2} \cdot \sqrt{\mu - 2} \cdot \sqrt{\mu + 2}$$

$$\mu \gg 1 \Rightarrow \mu - 2 \approx \mu; \quad \mu + 2 \approx \mu \Rightarrow \lambda_1 = 0, \quad \lambda_2 = \mu; \quad \Delta = \lambda_1 \cdot \lambda_2 = 0; \quad \lambda_1 + \lambda_2 = \mu$$

$\Delta = 0$, the origin is not isolated fixed point . There is either a whole line of fixed points or a plane of fixed point, if $A = 0$. Without the assumption $\mu \gg 1$, we need to find our system behavior.

$$\Delta = \lambda_1 \cdot \lambda_2 = \frac{1}{2} \cdot (\mu + \sqrt{\mu^2 - 4}) \cdot \frac{1}{2} \cdot (\mu - \sqrt{\mu^2 - 4}) = \frac{1}{4} \cdot (\mu^2 - (\mu^2 - 4)) = 1$$

$$\tau = \lambda_1 + \lambda_2 = \frac{1}{2} \cdot (\mu + \sqrt{\mu^2 - 4}) + \frac{1}{2} \cdot (\mu - \sqrt{\mu^2 - 4}) = \mu \Rightarrow \Delta = 1, \quad \tau = \mu; \ \mu \geq 0.$$

Case I : $\mu = 0 \rightarrow \Delta = 1;\ \tau = 0$, Case II : $\mu > 0\ \Delta = 1;\ \tau = \mu$. In both cases $\Delta > 0$, so the eigenvalues are either real with the same sign (node), or complex conjugate (spiral and centers). Nodes satisfy $\tau^2 - 4 \cdot \Delta > 0$, Case I ($\mu = 0$) $0 - 4 \cdot 1 > 0$ no exist, Case II ($\mu > 0$) $\mu^2 - 4 > 0 \rightarrow \mu > 2$ or $\mu < -2$ (no exist). Spiral satisfies $\tau^2 - 4 \cdot \Delta < 0$, Case I ($\mu = 0$) $0 - 4 \cdot 1 < 0$ exist, Case II ($\mu > 0$) $\mu^2 - 4 < 0 \rightarrow -2 < \mu < 2 \rightarrow 0 < \mu < 2$. The parabola $\tau^2 - 4 \cdot \Delta = 0$, Case I ($\mu = 0$) $-4 = 0$ (no exist), Case II ($\mu > 0$) $\mu^2 - 4 = 0 \rightarrow \mu = \pm 2$. The parabola $\mu^2 - 4 = 0$ is borderline between nodes and spiral; star nodes degenerate nodes live on this parabola. The stability of the nodes and spiral is determined by τ, in our case μ. μ is the parameter which establishes the stability of the nodes and spiral in our case. Since $\tau > 0 \Rightarrow \mu > 0$ we have the case of unstable spirals and nodes. Neutrally stable centers live on the borderline $\tau = 0 \Rightarrow \mu = 0$, where the eigenvalues are purely imaginary.

$\mu = 0 \Rightarrow \frac{d^2 V}{dt^2} + V = 0 \Rightarrow V = V_1;\ \frac{dV_1}{dt} = V_2$ and we get the system : $\frac{dV_1}{dt} = V_2$; $\frac{dV_2}{dt} = -V_1 \Rightarrow (A - \lambda \cdot I) = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm j$. In our case, $\lambda_{1,2} = \alpha \pm j \cdot \omega;\ \alpha = 0;\ \omega = \frac{1}{2} \cdot \sqrt{4 \cdot \Delta - \tau^2} \Rightarrow \omega = \frac{1}{2} \cdot \sqrt{4 \cdot 1 - 0} = 1$. All solutions are periodic with period $T = 2\pi/\omega = 2\pi$. The oscillations have fixed amplitude and the fixed point $(V_1^*, V_2^*) = (0, 0)$ in a center. The conclusion is that there is a nontrivial periodic orbit for all values of the scalar parameters μ. When $\mu = 0$, our system becomes the linear harmonic oscillator. The case $\mu < 0$ can be reduced to the case of $\mu > 0$ by reversing time. The system phase space origin is the only equilibrium point and the eigenvalues of the linearized equations at the origin have positive real parts, no orbits, except the origin itself, can have the origin as its ω-limit set. This implies that there must be a periodic orbit inside the positive invariant region. We now consider little bit different van der Pol system equation:

$$\frac{d^2 V}{dt^2} - \frac{dV}{dt} \cdot (2 \cdot \Gamma - V^2) + V = 0 \Rightarrow \frac{dV_1}{dt} = V_2;\ \frac{dV_2}{dt} = -V_1 + 2 \cdot \Gamma \cdot V_2 - V_1^2 \cdot V_2$$

The eigenvalues of the linearization of above van der Pol equations about the equilibrium point at the origin are $\Gamma \pm i \cdot \sqrt{1 - \Gamma^2}$. For $\Gamma < 0$, the origin is asymptotically stable because the real parts of the eigenvalues are negative. At $\Gamma = 0$, the origin is still asymptotically stable. For $\Gamma > 0$, the real parts of the eigenvalues become positive and thus the origin is unstable. Plotting V_2 against V_1 by using MATLAB script.

$$x\,(1) \rightarrow V_1;\; x\,(2) \rightarrow V_2;\; a\,(\text{Mui}) \rightarrow \mu$$

```
function g=vanderpol(t,x,a)
g=zeros(2,1);
g(1)=x(2);%x(1) represent the van der pol V1 variable
g(2)=-x(1)-x(2)*(x(1)*x(1)-1)*a;%x(2) represent van der pol V1
variable

function h=vanderpol1 (a,V10,V20)
[t,x]=ODE45(@vanderpol,[0,10],[V10,V20],[],a);
%plot(t,x);
plot(x(:,1),x(:,2))% V1 against V2 at time increase phase plan plot
```

Van der Pol's equation provides an example of an oscillator with nonlinear damping, energy being dissipated at large amplitudes and generated at low amplitudes. Our system typically possesses limit cycles; sustained oscillations around a state at which energy generation and dissipation balances. Single degree of freedom limit cycle oscillators, similar to the unforced van der Pol system. We can write van der Pol equation in a different way:

$$\frac{d^2V}{dt^2} + \alpha \cdot \phi(V) \cdot \frac{dV}{dt} + V = \beta \cdot p(t);\; \alpha = \mu;\; \phi(V) = V^2 - 1$$

Unforced van der Pol system: $\beta = 0 \Rightarrow \frac{d^2V}{dt^2} + \alpha \cdot \phi(V) \cdot \frac{dV}{dt} + V = 0$, where $\phi\,(V)$ is even and $\phi\,(V) < 0$ for $|V| < 1$, $\phi\,(V) > 0$ for $|V| > 1$. $P(t)$ is T periodic and α, β are nonnegative parameters. It is convenient to rewrite van der Pol system as an autonomous system (Fig. 1.2).

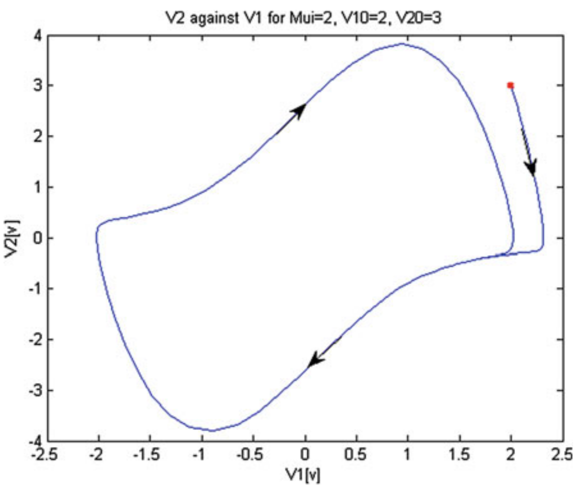


Fig. 1.2 Van der Pol system as an autonomous system

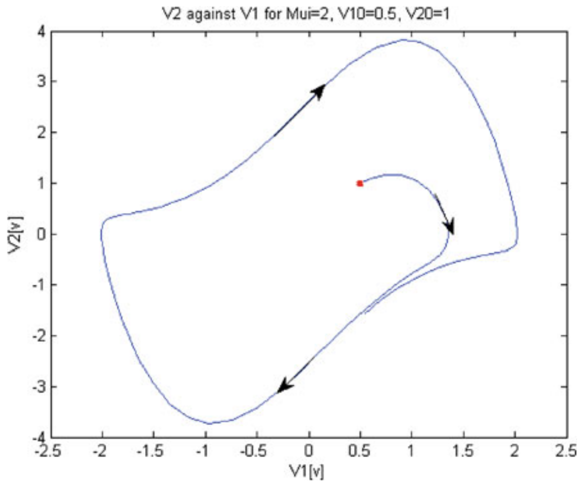


Fig. 1.2 (continued)

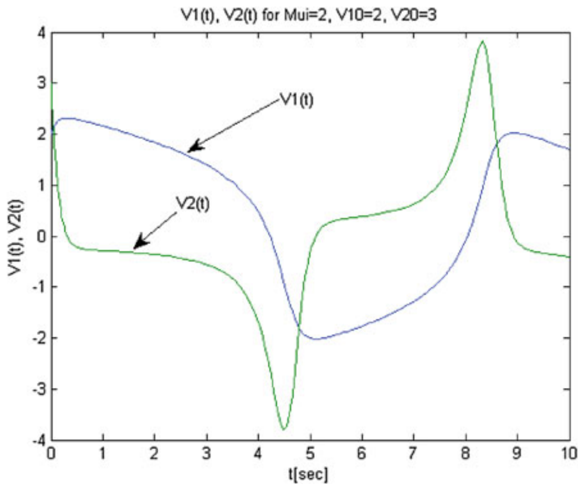


Fig. 1.2 (continued)

$$\frac{dV_1}{dt} = V_2 - \alpha \cdot \phi(V); \ \frac{dV_2}{dt} = -V_1 + \beta \cdot p(\theta); \ \frac{d\theta}{dt} = 1; \ (V_1, V_2, \theta) \in \mathbb{R}^2 \times S^1$$

We need now to implement our van der Pol system by using discrete components: Optocouplers, op-amps, resistors, capacitors, etc., we need to do variables terminology transformation for our van der Pol system block diagram [9] (Figs. 1.3, 1.4 and 1.5).

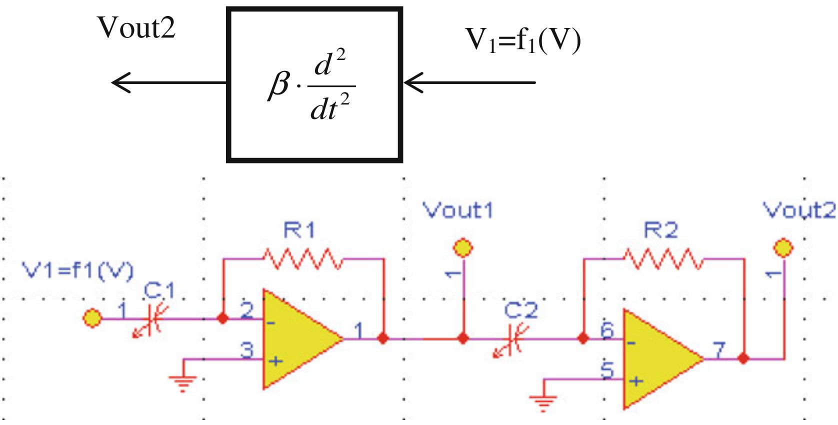


Fig. 1.3 Van der Pol system by using discrete components: Optocouplers, op-amps, resistors, capacitors, etc. (V_{out2})

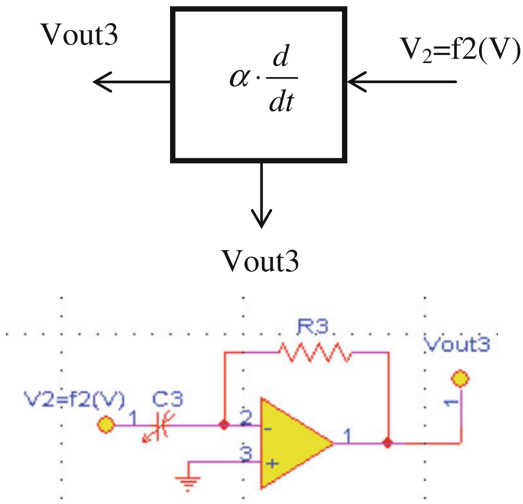
$$X_1 = f_1(X) \to V_1 = f_1(V);\ X_2 = f_2(X) \to V_2 = f_2(V);\ X_3 = f_3(X) \to V_3 = f_3(V)$$
$$X_{\text{out2}} \to V_{\text{out2}};\ X_{\text{out3}} \to V_{\text{out3}};\ X_{\text{out7}} \to V_{\text{out7}};\ X_E \to V_E$$

$$V_{\text{out1}} = -R_1 \cdot C_1 \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t} = -R_1 \cdot C_1 \cdot \frac{\mathrm{d}f_1(V)}{\mathrm{d}t};$$
$$V_{\text{out2}} = -R_2 \cdot C_2 \cdot \frac{\mathrm{d}V_{\text{out1}}}{\mathrm{d}t} = R_1 \cdot C_1 \cdot R_2 \cdot C_2 \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2}$$
$$R_1 \cdot C_1 \cdot R_2 \cdot C_2 = \beta \Rightarrow V_{\text{out2}} = \beta \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2}$$

$$V_{\text{out3}} = -R_3 \cdot C_3 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} = -R_3 \cdot C_3 \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t};\ R_3 \cdot C_3 = \alpha \Rightarrow V_{\text{out3}} = -\alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}$$

$$V_{\text{out4}} = -V_t \cdot \ln\left\{\frac{V_3}{I_s \cdot R_4}\right\};\ V_{\text{out5}} = -V_t \cdot \ln\left\{\frac{V_3}{I_s \cdot R_5}\right\} \Rightarrow V_{\text{out6}} = -R_8 \cdot \left\{\frac{V_{\text{out4}}}{R_6} + \frac{V_{\text{out5}}}{R_7}\right\}$$
$$R_8 = R_6 = R_7 \Rightarrow V_{\text{out6}} = -(V_{\text{out4}} + V_{\text{out5}}) = V_t \cdot \left[\ln\left\{\frac{V_3}{I_s \cdot R_4}\right\} + \ln\left\{\frac{V_3}{I_s \cdot R_5}\right\}\right]$$
$$V_{\text{out6}} = V_t \cdot \ln\left[\frac{V_3^2}{I_s^2 \cdot R_4 \cdot R_5}\right];\ V_{\text{out7}} = -I_s \cdot R_9 \cdot \exp\left[\frac{1}{V_t} \cdot V_{\text{out6}}\right]$$
$$V_{\text{out7}} = -I_s \cdot R_9 \cdot \exp\left[\frac{1}{V_t} \cdot V_{\text{out6}}\right] = -I_s \cdot R_9 \cdot \exp\left\{\frac{1}{V_t} \cdot V_t \cdot \ln\left[\frac{V_3^2}{I_s^2 \cdot R_4 \cdot R_5}\right]\right\}$$
$$V_{\text{out7}} = -I_s \cdot R_9 \cdot \frac{V_3^2}{I_s^2 \cdot R_4 \cdot R_5} = -\frac{R_9}{I_s \cdot R_4 \cdot R_5} \cdot V_3^2 = -\frac{R_9}{I_s \cdot R_4 \cdot R_5} \cdot f_3^2(V)$$
$$R_9 = R_4 = R_5 \Rightarrow V_{\text{out7}} = -\frac{1}{I_s \cdot R_4} \cdot f_3^2(V);\ I_s \cdot R_4 \approx 1 \Rightarrow V_{\text{out7}} = -f_3^2(V)$$

Fig. 1.4 Van der Pol system by using discrete components: Optocouplers, op-amps, resistors, capacitors, etc. (V_{out3})



We can summarize our results for this stage (Fig. 1.6):

$$V_{out2} = \beta \cdot \frac{d^2 f_1(V)}{dt^2};\; V_{out3} = -\alpha \cdot \frac{df_2(V)}{dt};\; V_{out7} = -f_3^2(V)$$

The circuit implementation for the above system multiplication block (Fig. 1.7):

$$V_A = -V_t \cdot \ln\left[\frac{V_{out3}}{I_s \cdot R_{10}}\right];\; V_B = -V_t \cdot \ln\left[\frac{V_{out7}}{I_s \cdot R_{11}}\right];\; V_C = -R_{14} \cdot \left[\frac{V_A}{R_{12}} + \frac{V_B}{R_{13}}\right]$$

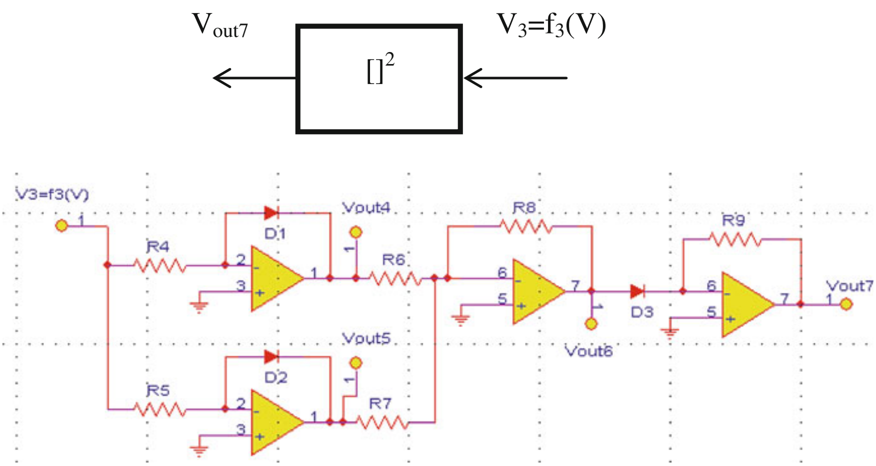


Fig. 1.5 Van der Pol system by using discrete components: Optocouplers, op-amps, resistors, capacitors, etc. (V_{out7})

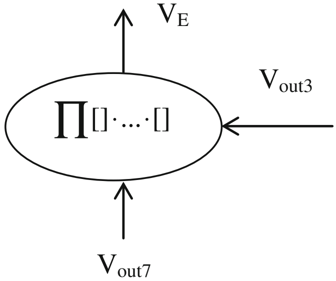


Fig. 1.6 System multiplication block

$$R_{14} = R_{12} = R_{13} \Rightarrow V_C = -(V_A + V_B);$$
$$V_C = -\left(-V_t \cdot \ln\left[\frac{V_{\text{out3}}}{I_s \cdot R_{10}}\right] - V_t \cdot \ln\left[\frac{V_{\text{out7}}}{I_s \cdot R_{11}}\right]\right)$$

$$V_C = V_t \cdot \ln\left[\frac{V_{\text{out3}} \cdot V_{\text{out7}}}{I_s^2 \cdot R_{10} \cdot R_{11}}\right];\ I_s^2 \cdot R_{10} \cdot R_{11} \approx 1 \Rightarrow V_C = V_t \cdot \ln\left[V_{\text{out3}} \cdot V_{\text{out7}}\right]$$

$$V_D = -I_s \cdot R_{15} \cdot \exp\left[\frac{1}{V_t} \cdot V_C\right] = -I_s \cdot R_{15} \cdot \exp\left\{\frac{1}{V_t} \cdot V_t \cdot \ln\left[V_{\text{out3}} \cdot V_{\text{out7}}\right]\right\}$$
$$= -I_s \cdot R_{15} \cdot V_{\text{out3}} \cdot V_{\text{out7}}$$

$$I_s \cdot R_{15} \approx 1 \Rightarrow V_D = -V_{\text{out3}} \cdot V_{\text{out7}} = -\left\{-\alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t} \cdot \left[-f_3^2(V)\right]\right\}$$
$$= -\alpha \cdot \left[f_3^2(V)\right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}$$

$$R_{20} = R_{21} \Rightarrow V_E = -V_D = \alpha \cdot \left[f_3^2(V)\right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}$$

We can summarize our results for this stage (Fig. 1.8):

$$V_{\text{out2}} = \beta \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2};\ V_{\text{out3}} = -\alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t};\ V_E = \alpha \cdot \left[f_3^2(V)\right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}$$

Circuit implementation (Fig. 1.9):

$$V = -R_{19} \cdot \left[\frac{V_{\text{out2}}}{R_{16}} + \frac{V_E}{R_{17}} + \frac{V_{\text{out3}}}{R_{18}}\right];\ R_{16} = R_{17} = R_{18} = R^+;\ R_{19} = R^-;\ \frac{R^-}{R^+} = \gamma$$

$$V = -\gamma \cdot \left[V_{\text{out2}} + V_E + V_{\text{out3}}\right]$$
$$= -\gamma \cdot \left[\beta \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2} + \alpha \cdot \left[f_3^2(V)\right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t} - \alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}\right]$$

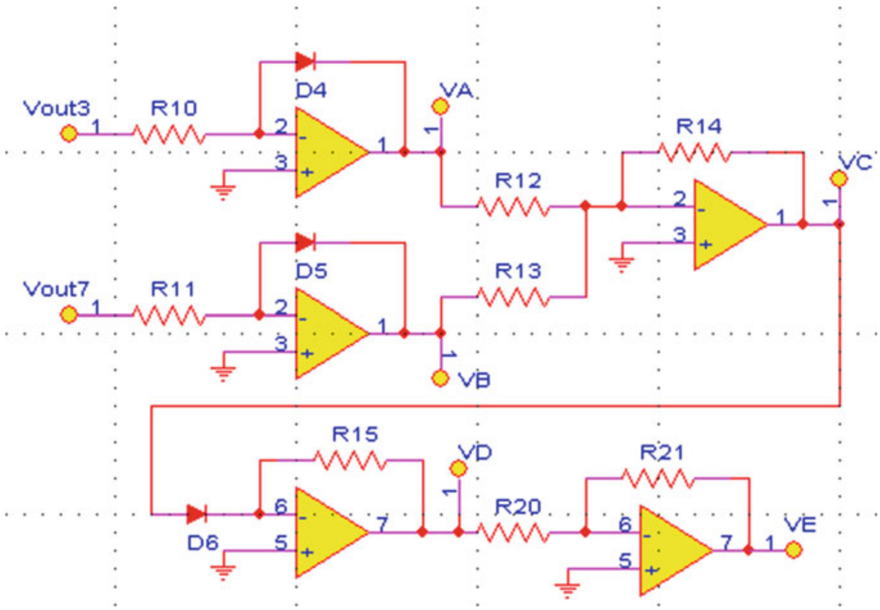
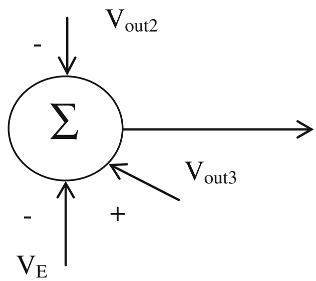


Fig. 1.7 The circuit implementation for the system multiplication block

Fig. 1.8 System summation block



$$f_1(V) \sim V;\ f_2(V) \sim V;\ f_3(V) \sim V \Rightarrow V = \gamma \cdot \left[\beta \cdot \frac{d^2V}{dt^2} + \alpha \cdot [V^2] \cdot \frac{dV}{dt} - \alpha \cdot \frac{dV}{dt}\right]$$

$$\begin{aligned} V &= -\gamma \cdot \left[\beta \cdot \frac{d^2V}{dt^2} + \alpha \cdot [V^2] \cdot \frac{dV}{dt} - \alpha \cdot \frac{dV}{dt}\right] \\ &= -\gamma \cdot \beta \cdot \frac{d^2V}{dt^2} - \gamma \cdot \alpha \cdot [V^2] \cdot \frac{dV}{dt} + \gamma \cdot \alpha \cdot \frac{dV}{dt} \end{aligned}$$

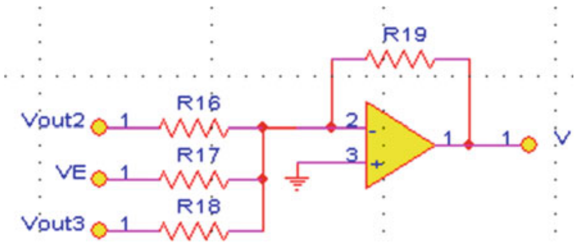


Fig. 1.9 Circuit implementation

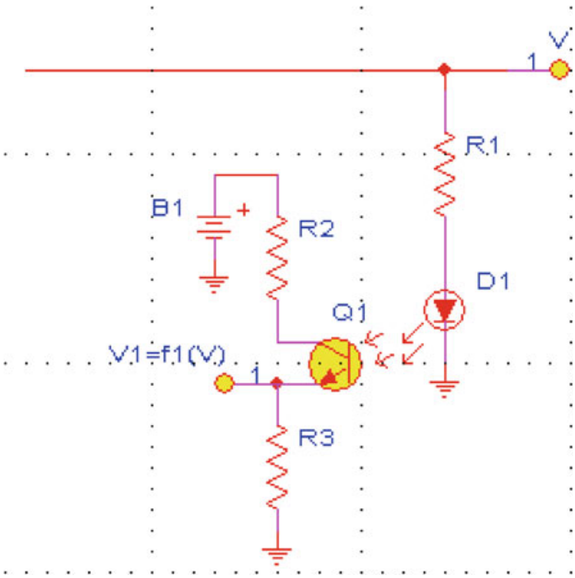


Fig. 1.10 Circuit describing optoisolation feedback loop by using optocoupler unit

$$V + \gamma \cdot \beta \cdot \frac{d^2V}{dt^2} + \gamma \cdot \alpha \cdot V^2 \cdot \frac{dV}{dt} - \gamma \cdot \alpha \cdot \frac{dV}{dt} = 0$$
$$\Rightarrow V + \gamma \cdot \beta \cdot \frac{d^2V}{dt^2} + \gamma \cdot \alpha \cdot \frac{dV}{dt} \cdot [V^2 - 1] = 0$$

$$V \cdot \frac{1}{\gamma \cdot \beta} + \frac{1}{\gamma \cdot \beta} \cdot \gamma \cdot \beta \cdot \frac{d^2V}{dt^2} + \frac{1}{\gamma \cdot \beta} \cdot \gamma \cdot \alpha \cdot \frac{dV}{dt} \cdot [V^2 - 1] = 0$$

$$V \cdot \frac{1}{\gamma \cdot \beta} + \frac{d^2V}{dt^2} + \frac{\alpha}{\beta} \cdot \frac{dV}{dt} \cdot [V^2 - 1] = 0 \Rightarrow \frac{d^2V}{dt^2} + \frac{dV}{dt} \cdot [V^2 - 1] \cdot \frac{\alpha}{\beta} + V \cdot \frac{1}{\gamma \cdot \beta} = 0$$

$$\mu = \frac{\alpha}{\beta}; \ \frac{1}{\beta \cdot \gamma} = 1 \Rightarrow \beta \cdot \gamma = 1 \Rightarrow \frac{\alpha}{\mu} \cdot \gamma = 1 \Rightarrow \mu = \alpha \cdot \gamma; \ \alpha = \mu \cdot \beta \Rightarrow \beta = \frac{\alpha}{\mu}$$
$$\Rightarrow \gamma = \frac{1}{\beta}$$

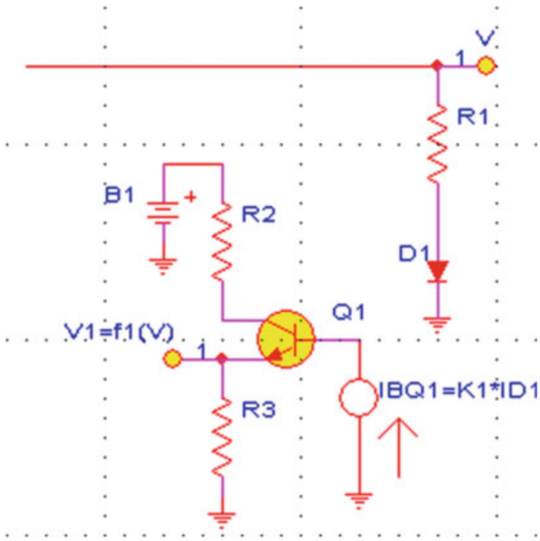


Fig. 1.11 Optoisolation feedback loop by using optocoupler unit equivalent circuit

We now need to implement the system feedback loops by using optoisolation circuits. The below circuit describes optoisolation feedback loop by using optocoupler unit. We perform feedback loop No. 1 but it is the same for all three system feedback loops [15, 16] (Fig. 1.10).

Now we proceed to the circuit model for analytical analysis. The optical coupling between the LED ($D1$) to Phototransistor ($Q1$) is represented as transistor dependent base current on LED ($D1$) current. $I_{BQ1} = k_1 \cdot I_D$ (Fig. 1.11).

$$V = f_1(V) = I_{EQ1} \cdot R_3;\ V_{B1} = I_{CQ1} \cdot R_2 + V_{CEQ1} + I_{EQ1} \cdot R_3 \Rightarrow$$
$$V_{CEQ1} = V_{B1} - I_{CQ1} \cdot R_2 - I_{EQ1} \cdot R_3$$
$$I_{EQ1} = I_{BQ1} + I_{CQ1};\ I_{BQ1} = k_1 \cdot I_{D1} \Rightarrow I_{EQ1} = k_1 \cdot I_{D1} + I_{CQ1} \Rightarrow I_{CQ1} = I_{EQ1} - k_1 \cdot I_{D1}$$
$$V_{CEQ1} = V_{B1} - I_{CQ1} \cdot R_2 - I_{EQ1} \cdot R_3 = V_{B1} - [I_{EQ1} - k_1 \cdot I_{D1}] \cdot R_2 - I_{EQ1} \cdot R_3$$
$$V_{CEQ1} = V_{B1} - [I_{EQ1} - k_1 \cdot I_{D1}] \cdot R_2 - I_{EQ1} \cdot R_3 = V_{B1} - I_{EQ1} \cdot R_2 + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot R_3$$
$$V_{CEQ1} = V_{B1} - I_{EQ1} \cdot R_2 + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot R_3 = V_{B1} - I_{EQ1} \cdot [R_2 + R_3] + k_1 \cdot I_{D1} \cdot R_2$$

$$(*)\ V_{CEQ1} = V_{B1} + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot [R_2 + R_3]$$

The mathematical analysis is based on the basic transistor Ebers–Moll equations. We need to implement the regular Ebers–Moll model to the above optocoupler circuit.

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right];$$
$$V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = V_{CBQ1} + V_{BEQ1},\ \text{but}\ V_{CBQ1} = -V_{BCQ1},\ \text{then}\ V_{CEQ1} = V_{BEQ1} - V_{BCQ1}$$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right)$$

Since $I_{sc} - I_{se} \to \varepsilon \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.

(**) $V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right]$

(*) = (**) $\to V_{B1} + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot [R_2 + R_3] \simeq Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right]$

$$\alpha r \cdot I_{CQ1} - I_{EQ1} = \alpha r \cdot [I_{EQ1} - k_1 \cdot I_{D1}] - I_{EQ1} = I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f = I_{EQ1} - k_1 \cdot I_{D1} - I_{EQ1} \cdot \alpha f = I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}$$

$$\begin{aligned} &V_{B1} + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot [R_2 + R_3] \\ &\quad \simeq Vt \cdot \ln\left[\frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right] \end{aligned}$$

$$I_{D1} = g_1(V);\ V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_o} + 1\right];\ V = I_{D1} \cdot R_1 + V_{D1};$$

$$V = I_{D1} \cdot R_1 + V_t \cdot \ln\left[\frac{I_{D1}}{I_o} + 1\right]$$

To get $I_{D1}(V)$ we need to develop $\ln\left(\frac{I_{D1}}{I_0} + 1\right)$, Taylor series around zero (0); also known as the Mercator series.

$$\begin{aligned} \ln\left(\frac{I_{D1}}{I_0} + 1\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{I_{D1}}{I_0}\right)^n \\ &= \frac{I_{D1}}{I_0} - \frac{1}{2} \cdot \left(\frac{I_{D1}}{I_0}\right)^2 + \frac{1}{3} \cdot \left(\frac{I_{D1}}{I_0}\right)^3 - \ldots\ \forall \left|\frac{I_{D1}}{I_0}\right| \leq 1 \end{aligned}$$

Unless $\frac{I_{D1}}{I_0} = -1$. The Taylor polynomials for $\ln\left(\frac{I_{D1}}{I_0} + 1\right)$ only provide accurate approximations in the range $-1 \leq \frac{I_{D1}}{I_0} \leq 1$, for $\frac{I_{D1}}{I_0} > 1$ the Taylor polynomial of higher degree are worse approximations. The low degree approximation gives

$$\ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx \frac{I_{D1}}{I_0};\ V = I_{D1} \cdot R_1 + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx I_{D1} \cdot R_1 + V_t \cdot \frac{I_{D1}}{I_0}$$

$$V \approx I_{D1} \cdot \left[R_1 + V_t \cdot \frac{1}{I_0}\right];\ I_{D1} \approx \left[\frac{I_0}{I_0 \cdot R_1 + V_t}\right] \cdot V \Rightarrow g_1(V) = \left[\frac{I_0}{I_0 \cdot R_1 + V_t}\right] \cdot V$$

For the case $\frac{I_{D1}}{I_0} > 1$, we can consider it as $I_{D1}(V)$ and the exact expression can be achieved by numerical calculation and algebraic function. For simplicity, let us consider: (Taylor expansion)

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x-1)^n = (x-1) - \frac{1}{2} \cdot (x-1)^2 + \frac{1}{3} \cdot (x-1)^3 - \cdots$$

for $|x-1| \leq 1$ unless $x = 0$. For our expression, consider $\ln(x) \approx x - 1$. Then

$$Vt \cdot \ln\left[\frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right]$$
$$\approx Vt \cdot \left[\frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)} - 1\right]$$

By using the above assumption we get:

$$V_{B1} + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot [R_2 + R_3]$$
$$\simeq Vt \cdot \left[\frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)} - 1\right]$$

$$\frac{1}{Vt} \cdot \{V_{B1} + k_1 \cdot I_{D1} \cdot R_2 - I_{EQ1} \cdot [R_2 + R_3]\} + 1$$
$$\simeq \frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}$$

$$\frac{1}{Vt} \cdot \{V_{B1} + k_1 \cdot I_{D1} \cdot R_2\} - \frac{1}{Vt} \cdot I_{EQ1} \cdot [R_2 + R_3] + 1$$
$$\simeq \frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}$$

$$\frac{1}{Vt} \cdot \{V_{B1} + k_1 \cdot I_{D1} \cdot R_2\} + 1 - I_{EQ1} \cdot \frac{[R_2 + R_3]}{Vt}$$
$$\simeq \frac{(I_{EQ1} \cdot (\alpha r - 1) - \alpha r \cdot k_1 \cdot I_{D1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{EQ1} \cdot (1 - \alpha f) - k_1 \cdot I_{D1}) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}$$

For simplicity, we define the following functions:

$$g_1(V) = \frac{1}{Vt} \cdot \{V_{B1} + k_1 \cdot I_{D1} \cdot R_2\} + 1;\ g_2(V) = -\alpha r \cdot k_1 \cdot I_{D1} + I_{se} \cdot (\alpha r \cdot \alpha f - 1)$$

$g_3(V) = -k_1 \cdot I_{D1} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)$ and we get the expressions:

$$g_1(V) = \frac{1}{Vt} \cdot \left\{ V_{B1} + k_1 \cdot \left[\frac{I_0}{I_0 \cdot R_1 + V_t} \right] \cdot V \cdot R_2 \right\} + 1;$$
$$g_2(V) = -\alpha r \cdot k_1 \cdot \left[\frac{I_0}{I_0 \cdot R_1 + V_t} \right] \cdot V + I_{se} \cdot (\alpha r \cdot \alpha f - 1);$$
$$g_3(V) = -k_1 \cdot \left[\frac{I_0}{I_0 \cdot R_1 + V_t} \right] \cdot V + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)$$

By using the above functions, we get the expression:

$$g_1(V) - I_{EQ1} \cdot \frac{[R_2 + R_3]}{Vt} \simeq \frac{I_{EQ1} \cdot (\alpha r - 1) + g_2(V)}{I_{EQ1} \cdot (1 - \alpha f) + g_3(V)}$$

$$I_{EQ1} \cdot (\alpha r - 1) + g_2(V) = \left\{ g_1(V) - I_{EQ1} \cdot \frac{[R_2 + R_3]}{Vt} \right\} \cdot \left\{ I_{EQ1} \cdot (1 - \alpha f) + g_3(V) \right\}$$

$$\begin{aligned} I_{EQ1} \cdot (\alpha r - 1) + g_2(V) \\ &= g_1(V) \cdot I_{EQ1} \cdot (1 - \alpha f) + g_1(V) \cdot g_3(V) \\ &\quad - I_{EQ1}^2 \cdot \frac{[R_2 + R_3]}{V_t} \cdot (1 - \alpha f) - I_{EQ1} \cdot \frac{[R_2 + R_3]}{V_t} \cdot g_3(V) \end{aligned}$$

$$\begin{aligned} I_{EQ1}^2 \cdot \frac{[R_2 + R_3]}{Vt} \cdot (1 - \alpha f) + I_{EQ1} \cdot \left\{ (\alpha r - 1) - g_1(V) \cdot (1 - \alpha f) + \frac{[R_2 + R_3]}{Vt} \cdot g_3(V) \right\} \\ + g_2(V) - g_1(V) \cdot g_3(V) = 0 \end{aligned}$$

We define the following functions: $A_1(V)$, $A_2(V)$, $A_3(V)$.

$$A_1(V) = \frac{[R_2 + R_3]}{Vt} \cdot (1 - \alpha f); \; A_2 = (\alpha r - 1) - g_1(V) \cdot (1 - \alpha f) + \frac{[R_2 + R_3]}{Vt} \cdot g_3(V)$$
$$A_3(V) = g_2(V) - g_1(V) \cdot g_3(V)$$

Finally, we get the cubic optoisolation equation:

$$\begin{aligned} I_{EQ1}^2 \cdot A_1(V) + I_{EQ1} \cdot A_2(V) + A_3(V) = 0 \Rightarrow I_{EQ1}(V) \\ = \frac{-A_2(V) \pm \sqrt{[A_2(V)]^2 - 4 \cdot A_1(V) \cdot A_3(V)}}{2 \cdot A_1(V)} \end{aligned}$$

$$V_1 = f_1(V) = I_{EQ1}(V) \cdot R_3 = R_3 \cdot \left\{ \frac{-A_2(V) \pm \sqrt{[A_2(V)]^2 - 4 \cdot A_1(V) \cdot A_3(V)}}{2 \cdot A_1(V)} \right\}$$

We need to plot the graph of $V_1 = f_1(V)$. First we consider optoisolation circuit parameters values (Table 1.1).

$$\begin{aligned} g_1(V) &= \frac{1}{Vt} \cdot \left\{ V_{B1} + k_1 \cdot \left[\frac{I_0}{I_0 \cdot R_1 + V_t} \right] \cdot V \cdot R_2 \right\} + 1 \\ &= \frac{1}{0.026} \cdot \left\{ 15 + 0.024 \cdot \left[\frac{10^{-6}}{10^{-6} \cdot 10^3 + 0.026} \right] \cdot V \cdot 3 \cdot 10^3 \right\} + 1 \\ &= 577.9 + V \cdot 0.099 \\ g_2(V) &= -\alpha r \cdot k_1 \cdot \left[\frac{I_0}{I_0 \cdot R_1 + V_t} \right] \cdot V + I_{se} \cdot (\alpha r \cdot \alpha f - 1) \\ &= -0.5 \cdot 0.024 \cdot \left[\frac{10^{-6}}{10^{-6} \cdot 10^3 + 0.026} \right] \cdot V + 10^{-6} \cdot (0.5 \cdot 0.98 - 1) \\ &= -10^{-6} \cdot [0.44 \cdot V + 0.51] \\ g_3(V) &= -k_1 \cdot \left[\frac{I_0}{I_0 \cdot R_1 + V_t} \right] \cdot V + I_{sc} \cdot (\alpha r \cdot \alpha f - 1) \\ &= -0.024 \cdot \left[\frac{10^{-6}}{10^{-6} \cdot 10^3 + 0.026} \right] \cdot V + 2 \cdot 10^{-6} \cdot [0.5 \cdot 0.98 - 1] \\ &= -10^{-6} \cdot [0.88 \cdot V + 1] \end{aligned}$$

$$\begin{aligned} A_1(V) &= \frac{[R_2 + R_3]}{Vt} \cdot (1 - \alpha f) = \left[\frac{5 \cdot 10^3}{0.026} \right] \cdot (1 - 0.98) = 3846.14 \\ A_2(V) &= (\alpha r - 1) - g_1(V) \cdot (1 - \alpha f) + \frac{[R_2 + R_3]}{Vt} \cdot g_3(V) \\ &= -0.5 - 0.02 \cdot g_1(V) + 192307.6 \cdot g_3(V) \\ A_3(V) &= g_2(V) - g_1(V) \cdot g_3(V) \end{aligned}$$

Table 1.1 Optoisolation circuit parameters values

V_t	0.026	α_f	0.98
I_0	1E−6	β_f	49
k1	0.024	α_r	0.5
k2,k3	0.028	β_r	1
I_{se}	1 μA	R_3	2 kΩ
I_{sc}	2 μA	R_2	3 kΩ
R_1	1 kΩ	V_{B1}	15 V

We get to $V_1 = f(V)$ options: the (+) and (−) of root function in the below expression:

$$\begin{aligned} V_1 &= f_1(V) = I_{EQ1}(V) \cdot R_3 \\ &= R_3 \cdot \left\{ \frac{-[0.5 - 0.02 \cdot g_1(V) + 192307.6 \cdot g_3(V)]}{2.3846.14} \right\} \\ &\quad \pm R_3 \cdot \left\{ \frac{\sqrt{[-0.5 - 0.02 \cdot g_1(V) + 192307.6 \cdot g_3(V)]^2 - 4.3846.14 \cdot [g_2(V) - g_1(V) \cdot g_3(V)]}}{2.3846.14} \right\} \end{aligned}$$

MATLAB Script

```
EDU≫V=0:0.1:10;    g1=577.9+0.099*V;    g2=-0.000001*(0.44*V+0.51);
g3=-0.000001*(0.88*V+1);
EDU≫A1=3846.14; A2=-0.5-0.02*g1+192307.6*g3; A3=g2-g1.*g3;
EDU≫u=(-A2±(A2.^2-4*A1.*A3).^0.5)./(2*A1);
EDU≫u1=u*2000;
EDU≫plot(V,u1,‘r’),grid (Fig. 1.12)
```

The results graphs are mathematic plot, actually our system behave according to only one option. We choose (+) option for our system analysis.

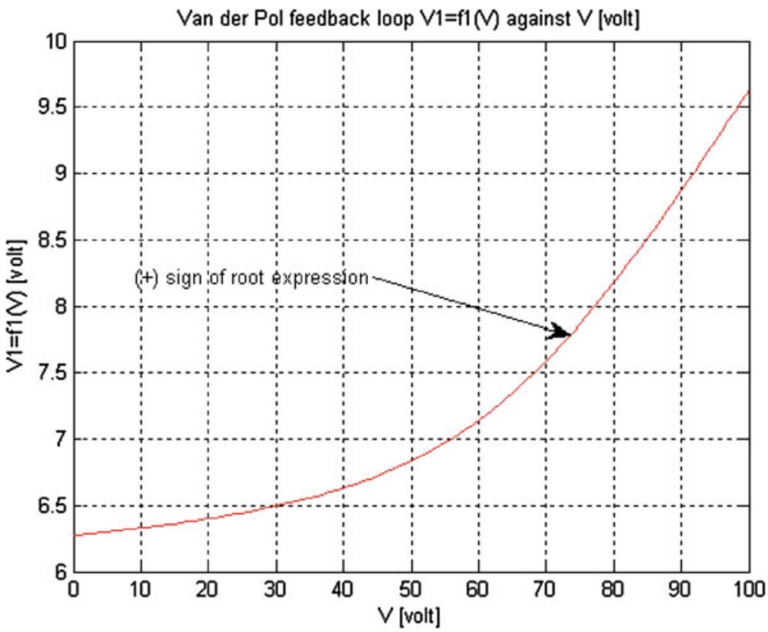


Fig. 1.12 Van der Pol feedback loop $V_1 = f_1(V)$ against V graph ("+" and "−" sign of root expression)

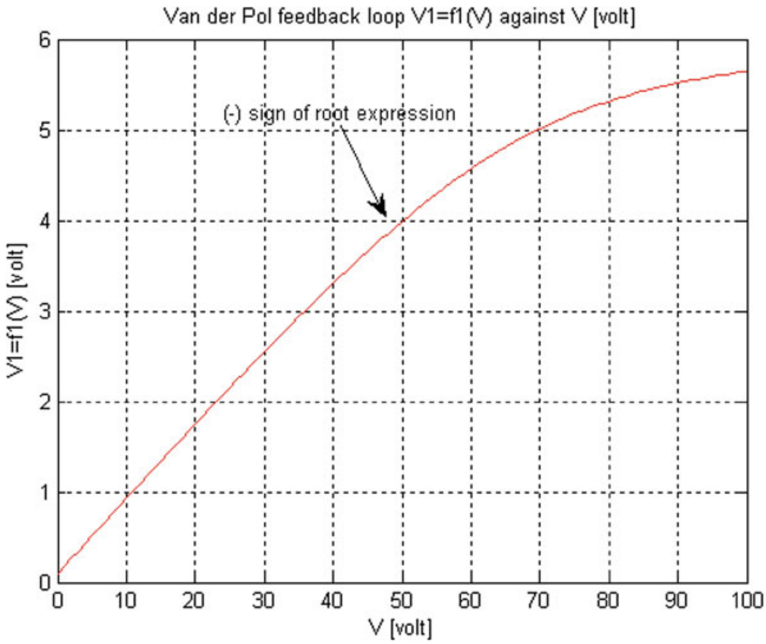


Fig. 1.12 (continued)

We need to inspect our van der Pol system with the right optoisolation feedback loop functions $f_1(V)$, $f_2(V)$, $f_3(V)$. We consider all f's functions are the same [16].

$$\begin{aligned} V &= -\gamma \cdot [V_{\text{out2}} + V_E + V_{\text{out3}}] \\ &= -\gamma \cdot \left[\beta \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2} + \alpha \cdot \left[f_3^2(V)\right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t} - \alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}\right] \end{aligned}$$

$$\frac{\partial g_1(V)}{\partial V} = \frac{1}{V_t} \cdot k_1 \cdot \left[\frac{I_0}{(I_0 \cdot R_1 + V_t)}\right] \cdot R_2$$

$$\frac{\partial g_2(V)}{\partial V} = -\alpha_r \cdot k_1 \cdot \left[\frac{I_0}{(I_0 \cdot R_1 + V_t)}\right]$$

$$\frac{\partial g_3(V)}{\partial V} = -k_1 \cdot \left[\frac{I_0}{(I_0 \cdot R_1 + V_t)}\right]; \; \frac{\partial A_1(V)}{\partial V} = 0;$$

$$\frac{\partial A_2(V)}{\partial V} = -\frac{\partial g_1(V)}{\partial V} \cdot (1 - \alpha_f) + \frac{R_2 + R_3}{V_t} \cdot \frac{\partial g_3(V)}{\partial V}$$

$$\frac{\partial A_3(V)}{\partial V} = \frac{\partial g_2(V)}{\partial V} - \frac{\partial g_1(V)}{\partial V} \cdot g_3(V) - g_1(V) \cdot \frac{\partial g_3(V)}{\partial V}$$

$$V_1 = f_1(V) = I_{EQ1}(V) \cdot R_3$$
$$= R_3 \cdot \left\{ -\frac{1}{2} \cdot \frac{A_2(V)}{A_1(V)} \pm \frac{1}{2} \cdot \frac{\left\{[A_2(V)]^2 - 4 \cdot A_1(V) \cdot A_3(V)\right\}^{\frac{1}{2}}}{A_1(V)} \right\}$$

$$f(V) = f_1(V) = f_2(V) = f_3(V) = I_{EQ1}(V) = I_{EQ2}(V) = I_{EQ3}(V)$$

We consider $k_1 = k_2 = k_3 = k$. We get the expression for $\partial f(V)/\partial t$:

$$\frac{\partial f(V)}{\partial t} = \frac{1}{2} \cdot R_3 \cdot \left\{ -\frac{\frac{\partial A_2(V)}{\partial t} \cdot A_1(V) - A_2(V) \cdot \frac{\partial A_1(V)}{\partial t}}{[A_1(V)]^2} \right.$$
$$\pm \left[\frac{\frac{1}{2} \cdot \left\{[A_2(V)]^2 - 4 \cdot A_1(V) \cdot A_3(V)\right\}^{-\frac{1}{2}} \cdot \left\{2 \cdot A_2(V) \cdot \frac{\partial A_2(V)}{\partial t} - 4 \cdot \left[\frac{\partial A_1(V)}{\partial t} \cdot A_3(V) + A_1(V) \cdot \frac{\partial A_3(V)}{\partial t}\right]\right\} \cdot A_1(V)}{[A_1(V)]^2} \right.$$
$$\left.\left. - \frac{\left\{[A_2(V)]^2 - 4 \cdot A_1(V) \cdot A_3(V)\right\}^{\frac{1}{2}} \cdot \frac{\partial A_1(V)}{\partial t}}{[A_1(V)]^2} \right] \right\}$$

There are two solutions for $f(v)$ function: (+) root and (−) root. We define $f(V)$ function with (+) root as $f(V) = P^{\dagger}_{(+)}$ and (−) root as $f(V) = P^{\dagger}_{(-)}$. We take out from van der Pol feedback loop $V_1 = f_1(V)$ against V graph ("+" sign of root expression) some dots which denote the data points (V_{1_i}, V_i), while the red curve shows the interpolation polynomial. The numerical connection between $V_1 = f_1(V)$ against V is described in Table 1.2.

We need to construct the interpolation polynomial $P^{\dagger}_{(-)}$. Suppose that the interpolation polynomial is in the form $P^{\dagger}_{(+)}(V) = \sum_{i=0}^{N} \eta_i \cdot V^i$. $N = 6$ for our above six data points (V_{1_i}, V_i). We have a set of six data points (V_{1_i}, V_i), where no two V_i are the same. We are looking for a polynomial $f(V) = P^{\dagger}_{(+)}$ of degree at most five with property $P^{\dagger}_{(+)}(V_i) = V1_i;\ \ i = 1, \ldots, 5$. According to unisolvence theorem such a polynomial $P^{\dagger}_{(+)}$ exists and can be approved by the Vandermonde matrix. For $n + 1 = (6)$ interpolation nodes (our case) V_i, polynomial interpolation

Table 1.2 The numerical connection between $V_1 = f_1(V)$ against V

V_1 (V)	V (V)
$V_{1_0} = 6.3$	$V_0 = 0$
$V_{1_1} = 6.5$	$V_1 = 30$
$V_{1_2} = 6.8$	$V_2 = 50$
$V_{1_3} = 7.7$	$V_3 = 70$
$V_{1_4} = 8.8$	$V_4 = 90$
$V_{1_5} = 9.7$	$V_5 = 100$

defines a linear bijection $L_n : \mathrm{K}^{n+1} \to \prod_n$, where $\prod_n$ is the vector space of polynomials (defined on any interval containing the nodes) of degree at most n. Suppose that the interpolation polynomial is in the form $P^{\dagger}_{(+)}(V) = \eta_5 \cdot V^5 + \cdots + \eta_0$. The statement that $P^{\dagger}_{(+)}$ interpolates the data points means that $P^{\dagger}_{(+)}(V_i) = V_{1_i}\ \forall\ i \in \{0, 1, \ldots, 5\}$. If we substitute the interpolation polynomial in here, we get a system of linear equations in the coefficients η_k. The system in matrix-vector form leads:

$$\begin{pmatrix} V_0^5 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ V_5^5 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_5 \\ \vdots \\ \eta_0 \end{pmatrix} = \begin{pmatrix} V_{1_0} \\ \vdots \\ V_{1_5} \end{pmatrix};$$

We get Vandermonde matrix $\begin{pmatrix} V_0^5 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ V_5^5 & \cdots & 1 \end{pmatrix}$

$$0 \cdot \eta_5 + \cdots + 0 \cdot \eta_1 + \eta_0 = 6.3;\ 30^5 \cdot \eta_5 + \cdots + 30 \cdot \eta_1 + \eta_0 = 6.5;$$
$$50^5 \cdot \eta_5 + \cdots + 50 \cdot \eta_1 + \eta_0 = 6.8$$

$$70^5 \cdot \eta_5 + \cdots + 70 \cdot \eta_1 + \eta_0 = 7.7;\ 90^5 \cdot \eta_5 + \cdots + 90 \cdot \eta_1 + \eta_0 = 8.8;$$
$$100^5 \cdot \eta_5 + \cdots + 100 \cdot \eta_1 + \eta_0 = 9.7$$

We get η_0 = 6.3 and set of five polynomials:

$$\eta_0 = 6.3;\ 30^5 \cdot \eta_5 + \cdots + 30 \cdot \eta_1 = 0.2;\ 50^5 \cdot \eta_5 + \cdots + 50 \cdot \eta_1 = 0.5$$

$$70^5 \cdot \eta_5 + \cdots + 70 \cdot \eta_1 = 1.4;\ 90^5 \cdot \eta_5 + \cdots + 90 \cdot \eta_1 = 2.5;$$
$$100^5 \cdot \eta_5 + \cdots + 100 \cdot \eta_1 = 3.4$$

We use Mathematica software to find η_5, …, η_1 values. $\eta_i \leftrightarrow a_i$ and we get the following results:

$$\begin{aligned} \mathrm{NSolve}\big[\{&30^5 * \mathrm{a5} + 30^4 * \mathrm{a4} + 30^3 * \mathrm{a3} + 30^2 * \mathrm{a2} + 30 * \mathrm{a1} \\ &= 00.2, 50^5 * \mathrm{a5} + 50^4 * \mathrm{a4} + 50^3 * \mathrm{a3} + 50^2 * \mathrm{a2} + 50 * \mathrm{a1} \\ &= 00.5, 70^5 * \mathrm{a5} + 70^4 * \mathrm{a4} + 70^3 * \mathrm{a3} + 70^2 * \mathrm{a2} + 70 * \mathrm{a1} \\ &= 10.4, 90^5 * \mathrm{a5} + 90^4 * \mathrm{a4} + 90^3 * \mathrm{a3} + 90^2 * \mathrm{a2} + 90 * \mathrm{a1} \\ &= 20.5, 100^5 * \mathrm{a5} + 100^4 * \mathrm{a4} + 100^3 * \mathrm{a3} + 100^2 * \mathrm{a2} + 100 * \mathrm{a1} \\ &= 3.4\},\ \{\mathrm{a1}, \mathrm{a2}, \mathrm{a3}, \mathrm{a4}, \mathrm{a5}\}\big] \end{aligned}$$

Out: `{{a1→0.0871111,a2→-0.00602688,a3→0.000152735, a4→ -1.54392 · 10^-6, a5→6.66138· 10^-9}}`

Results

$$\eta_0 = 6.3;\ \eta_1 = 0.0871111;\ \eta_2 = -0.00602688$$
$$\eta_3 = 0.000152735;\ \eta_4 = -1.54392\ 10^{-6};\ \eta_5 = 5.66138\ 10^{-9}$$

Double check when $V \to x$ and $V_1 \to u$

x=30; u=6.3+0.0871111*x-0.00602688*x^2+0.000152735*x^3-1.54392*10^-6*x^4+5.66138*10^-9*x^5; u→u=6.5000.
x=50; *u*=6.3+0.0871111*x-0.00602688*x^2+0.000152735*x^3-1.54392*10^-6*x^4+5.66138*10^-9*x^5;u→u=6.7999.
x=70;u=6.3+0.0871111*x-0.00602688*x^2+0.000152735*x^3-1.54392*10^-6*x^4+5.66138*10^-9*x^5;u→u=7.6997.
x=90;u=6.3+0.0871111*x-0.00602688*x^2+0.000152735*x^3-1.54392*10^-6*x^4+5.66138*10^-9*x^5;u→u=8.7994.
x=100;u=6.3+0.0871111*x-0.00602688*x^2+0.000152735*x^3-1.54392*10^-6*x^4+5.66138*10^-9*x^5;u→u=9.6991.

Back to our original system differential equations. We consider $f_1(V) = f_2(V) = f_3(V) = f(V)$, then we get:

$$\begin{aligned} V &= -\gamma \cdot [V_{\text{out2}} + V_E + V_{\text{out3}}] \\ &= -\gamma \cdot \left[\beta \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2} + \alpha \cdot \left[f_3^2(V)\right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t} - \alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t}\right] \end{aligned}$$

$$f(V) = P^{\dagger}_{(+)}(V) = \eta_5 \cdot V^5 + \cdots + \eta_0 = \sum_{k=0}^{5} \eta_k \cdot V^k;$$
$$\sum_{k=0}^{5} \eta_k \cdot k \cdot V^{k-1} = \sum_{k=1}^{5} \eta_k \cdot k \cdot V^{k-1};\ \frac{\partial}{\partial t} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\begin{aligned} \frac{\partial f(V)}{\partial t} &= \frac{\partial P^{\dagger}_{(+)}(V)}{\partial t} \\ &= \eta_1 \cdot \frac{\partial V}{\partial t} + 2 \cdot \eta_2 \cdot V \cdot \frac{\partial V}{\partial t} + 3 \cdot \eta_3 \cdot V^2 \cdot \frac{\partial V}{\partial t} + 4 \cdot \eta_4 \cdot V^3 \cdot \frac{\partial V}{\partial t} \\ &\quad + 5 \cdot \eta_5 \cdot V^4 \cdot \frac{\partial V}{\partial t} \end{aligned}$$

$$\frac{\partial f(V)}{\partial t} = \frac{\partial P^{\dagger}_{(+)}(V)}{\partial t} = \frac{\partial V}{\partial t} \cdot \left\{\eta_1 + 2 \cdot \eta_2 \cdot V + 3 \cdot \eta_3 \cdot V^2 + 4 \cdot \eta_4 \cdot V^3 + 5 \cdot \eta_5 \cdot V^4\right\}$$

$$\frac{\partial f(V)}{\partial t} = \frac{\partial P^{\dagger}_{(+)}(V)}{\partial t} = \frac{\partial V}{\partial t} \cdot \left\{ \sum_{k=1}^{5} k \cdot \eta_k \cdot V^{k-1} \right\}$$

$$\frac{\partial^2 f(V)}{\partial t^2} = \frac{\partial^2 P^{\dagger}_{(+)}(V)}{\partial t^2}$$
$$= \frac{\partial^2 V}{\partial t^2} \cdot \left\{ \sum_{k=1}^{5} k \cdot \eta_k \cdot V^{k-1} \right\} + \left[\frac{\partial V}{\partial t} \right]^2 \cdot \left\{ \sum_{k=1}^{5} k \cdot (k-1) \cdot \eta_k \cdot V^{k-2} \right\}$$

We define the following functions for simplicity:

$$\chi_1(V) = \sum_{k=1}^{5} k \cdot \eta_k \cdot V^{k-1}; \chi_2(V) = \sum_{k=1}^{5} k \cdot (k-1) \cdot \eta_k \cdot V^{k-2};$$
$$\frac{\partial f(V)}{\partial t} = \frac{\partial V}{\partial t} \cdot \chi_1(V)$$

$$\chi_1(V) = \sum_{k=1}^{5} k \cdot \eta_k \cdot V^{k-1}; \ \frac{\partial \chi_1(V)}{\partial t} = \frac{\partial V}{\partial t} \cdot \sum_{k=1}^{5} k \cdot (k-1) \cdot \eta_k \cdot V^{k-2}; \ \frac{\partial \chi_1(V)}{\partial t}$$
$$= \frac{\partial V}{\partial t} \cdot \chi_2(V)$$

$$\frac{\partial^2 f(V)}{\partial t^2} = \frac{\partial^2 V}{\partial t^2} \cdot \chi_1(V) + \frac{\partial V}{\partial t} \cdot \frac{\partial \chi_1(V)}{\partial t} = \frac{\partial^2 V}{\partial t^2} \cdot \chi_1(V) + \left[\frac{\partial V}{\partial t} \right]^2 \cdot \chi_2(V)$$

$$f(V) = \chi_3(V) = \sum_{k=1}^{6} \eta_{k-1} \cdot V^{k-1} \Rightarrow f^2 = \chi_4(V) = [\chi_3(V)]^2 = \left[\sum_{k=1}^{6} \eta_{k-1} \cdot V^{k-1} \right]^2$$

Back to our system equation:

$$V = -\gamma \cdot [V_{\text{out2}} + V_E + V_{\text{out3}}]$$
$$= -\gamma \cdot \left[\beta \cdot \frac{\mathrm{d}^2 f_1(V)}{\mathrm{d}t^2} + \alpha \cdot \left[f_3^2(V) \right] \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t} - \alpha \cdot \frac{\mathrm{d}f_2(V)}{\mathrm{d}t} \right]$$

$f_1(V) = f_2(V) = f_3(V) = f(V)$, then we get:

$$V = -\gamma \cdot \left[\beta \cdot \frac{\mathrm{d}^2 f(V)}{\mathrm{d}t^2} + \alpha \cdot [f^2(V)] \cdot \frac{\mathrm{d}f(V)}{\mathrm{d}t} - \alpha \cdot \frac{\mathrm{d}f(V)}{\mathrm{d}t} \right]$$

$$V = -\gamma \cdot \left[\beta \cdot \left\{ \frac{\partial^2 V}{\partial t^2} \cdot \chi_1(V) + \left[\frac{\partial V}{\partial t} \right]^2 \cdot \chi_2(V) \right\} + \alpha \cdot \chi_4(V) \cdot \frac{\partial V}{\partial t} \cdot \chi_1(V) - \alpha \cdot \frac{\partial V}{\partial t} \cdot \chi_1(V) \right]$$

$$-\frac{1}{\gamma}V = \beta \cdot \frac{\partial^2 V}{\partial t^2} \cdot \chi_1(V) + \beta \cdot \left[\frac{\partial V}{\partial t}\right]^2 \cdot \chi_2(V) + \alpha \cdot \chi_4(V) \cdot \frac{\partial V}{\partial t} \cdot \chi_1(V) - \alpha \cdot \frac{\partial V}{\partial t} \cdot \chi_1(V)$$

$$-\frac{1}{\gamma}V = \beta \cdot \frac{\partial^2 V}{\partial t^2} \cdot \chi_1(V) + \beta \cdot \left[\frac{\partial V}{\partial t}\right]^2 \cdot \chi_2(V) + \alpha \cdot \chi_1(V) \cdot \frac{\partial V}{\partial t} \cdot [\chi_4(V) - 1]$$

To get our system new differential equations with optoisolation feedback polynomial, we define:

$$\frac{\partial}{\partial t} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t};\ \frac{\partial^2 V}{\partial t^2} = \frac{\mathrm{d}^2 V}{\mathrm{d}t^2} = \frac{\mathrm{d}V_2}{\mathrm{d}t};$$
$$V_2 = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V_1}{\mathrm{d}t};\ V_1 = V;\ V \to V_1;\ \frac{\mathrm{d}V}{\mathrm{d}t} \to V_2;$$
$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = V_2$$

$$-\frac{1}{\gamma}V = \beta \cdot \frac{\mathrm{d}^2 V}{\mathrm{d}t^2} \cdot \chi_1(V) + \beta \cdot \left[\frac{\mathrm{d}V}{\mathrm{d}t}\right]^2 \cdot \chi_2(V) + \alpha \cdot \chi_1(V) \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot [\chi_4(V) - 1]$$

$$-\frac{1}{\gamma}V_1 = \beta \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \chi_1(V_1) + \beta \cdot V_2^2 \cdot \chi_2(V_1) + \alpha \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]$$

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \beta \cdot \chi_1(V_1) = -\frac{1}{\gamma}V_1 - \beta \cdot V_2^2 \cdot \chi_2(V_1) - \alpha \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]$$

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \frac{-\frac{1}{\gamma}V_1 - \beta \cdot V_2^2 \cdot \chi_2(V_1) - \alpha \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\beta \cdot \chi_1(V_1)}$$

We can summarize our system differential equations under optoisolation feedback loop polynomial connection:

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = V_2$$
$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \frac{-\frac{1}{\gamma}V_1 - \beta \cdot V_2^2 \cdot \chi_2(V_1) - \alpha \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\beta \cdot \chi_1(V_1)}$$

In the above system differential equations, we interest to inspect which kind of limit cycle we have and how overall parameters influence it. We can define two functions: $\frac{\mathrm{d}V_1}{\mathrm{d}t} = g_1(V_1, V_2)$; $\frac{\mathrm{d}V_2}{\mathrm{d}t} = g_2(V_1, V_2)$

$$g_1(V_1, V_2) = V_2; \; g_2(V_1, V_2) = \frac{-\frac{1}{\gamma} V_1 - \beta \cdot V_2^2 \cdot \chi_2(V_1) - \alpha \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\beta \cdot \chi_1(V_1)}$$

We consider as before for α, β, γ, the relationship:

$$\mu = \frac{\alpha}{\beta}; \; \frac{1}{\beta \cdot \gamma} = 1 \Rightarrow \beta \cdot \gamma = 1 \Rightarrow \frac{\alpha}{\mu} \cdot \gamma = 1 \Rightarrow \mu = \alpha \cdot \gamma;$$
$$\alpha = \mu \cdot \beta \Rightarrow \beta = \frac{\alpha}{\mu} \Rightarrow \gamma = \frac{1}{\beta}$$

Then we get the following set of differential equations:

$$\frac{dV_1}{dt} = V_2$$
$$\frac{dV_2}{dt} = \frac{-\frac{1}{\gamma \cdot \beta} V_1 - V_2^2 \cdot \chi_2(V_1) - \frac{\alpha}{\beta} \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\chi_1(V_1)}$$

For the above notation we get:

$$\frac{dV_1}{dt} = V_2$$
$$\frac{dV_2}{dt} = \frac{-V_1 - V_2^2 \cdot \chi_2(V_1) - \mu \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\chi_1(V_1)}$$

Accordingly we have g_1, g_2 functions:

$$g_1(V_1, V_2) = V_2; \; g_2(V_1, V_2) = \frac{-V_1 - V_2^2 \cdot \chi_2(V_1) - \mu \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\chi_1(V_1)}$$

$$\chi_1(V_1) = \sum_{k=1}^{5} k \cdot \eta_k \cdot V_1^{k-1} = \eta_1 + 2 \cdot \eta_2 \cdot V_1 + 3 \cdot \eta_3 \cdot V_1^2 + 4 \cdot \eta_4 \cdot V_1^3 + 5 \cdot \eta_5 \cdot V_1^4$$

$$\chi_2(V_1) = \sum_{k=1}^{5} k \cdot (k-1) \cdot \eta_k \cdot V_1^{k-2}$$
$$= 2 \cdot \eta_2 + 6 \cdot \eta_3 \cdot V_1 + 12 \cdot \eta_4 \cdot V_1^2 + 20 \cdot \eta_5 \cdot V_1^3$$

$$\chi_4(V_1) = \left[\sum_{k=1}^{6} \eta_{k-1} \cdot V_1^{k-1}\right]^2 = \left[\eta_0 + \eta_1 \cdot V_1 + \cdots + \eta_5 \cdot V_1^5\right]^2$$

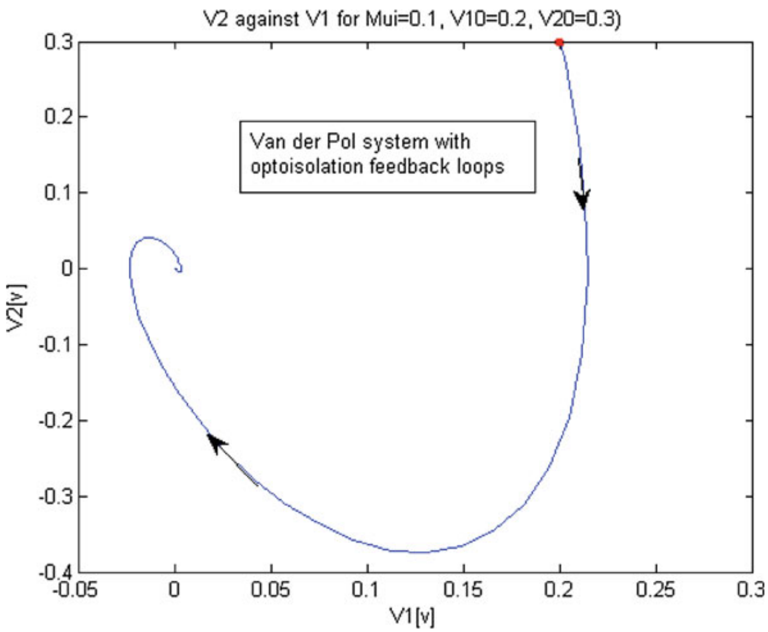


Fig. 1.13 Van der Pol system with optoisolation feedback loops

We choose V_1, V_2 initial values (V_{10}, V_{20}), 0.2, 0.3, respectively and change our system μ (Mui) parameter value from 0.1 down to 0.0001 (0.1, 0.01, 0.001, 0.0001).

The related MATLAB script to get plot of V_2 against V_1 phase space.

$x(1) \rightarrow V_1$; $x(2) \rightarrow V_2$; a(Mui) $\rightarrow \mu$ (Fig. 1.13).

```
function g=vanderpolopto(t,x,a)
g=zeros(2,1);
g(1)=x(2);%x(1) represent the van der pol V1 variable
m1=0.0871-0.012*x(1)+0.000456*(x(1)).^2-
6.172*0.000001*(x(1)).^3+28.3*0.000000001*(x(1)).^4;
m2=-0.012+0.000912*x(1)-
18.516*0.000001*(x(1)).^2+113.2*0.000000001*(x(1)).^3;
m4=(6.3+0.0871*x(1)-0.006*(x(1)).^2+0.000152*(x(1)).^3-
1.543*0.000001*(x(1)).^4+5.66*0.000000001*(x(1)).^5).^2;
g(2)=(-x(1)-(x(2)).^2.*m2-a*m1.*x(2).*(m4-1))./m1;
```

```
function h=vanderpol2 (a,V10,V20)
[t,x]=ODE45(@vanderpolopto,[0,10],[V10,V20],[],a);
%plot(t,x);
plot(x(:,1),x(:,2))% V1 against V2 at time increase phase plan plot
```

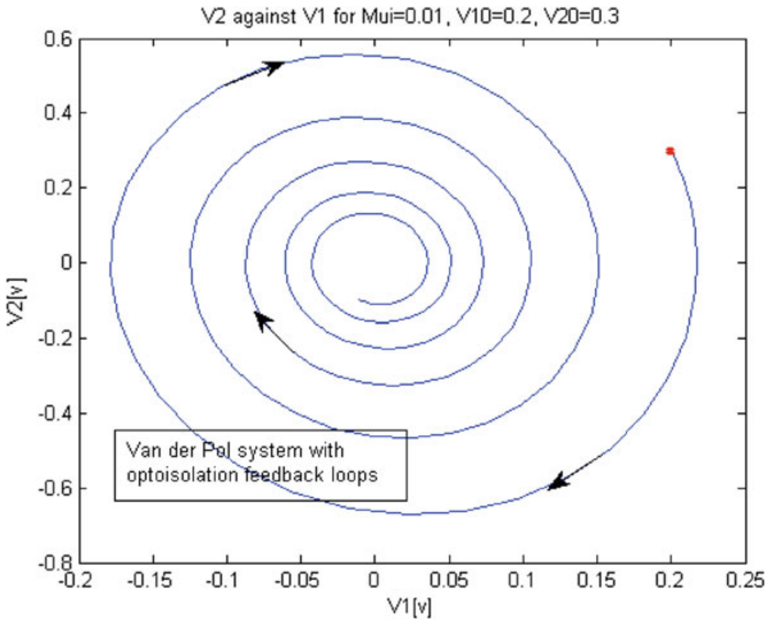


Fig. 1.13 (continued)

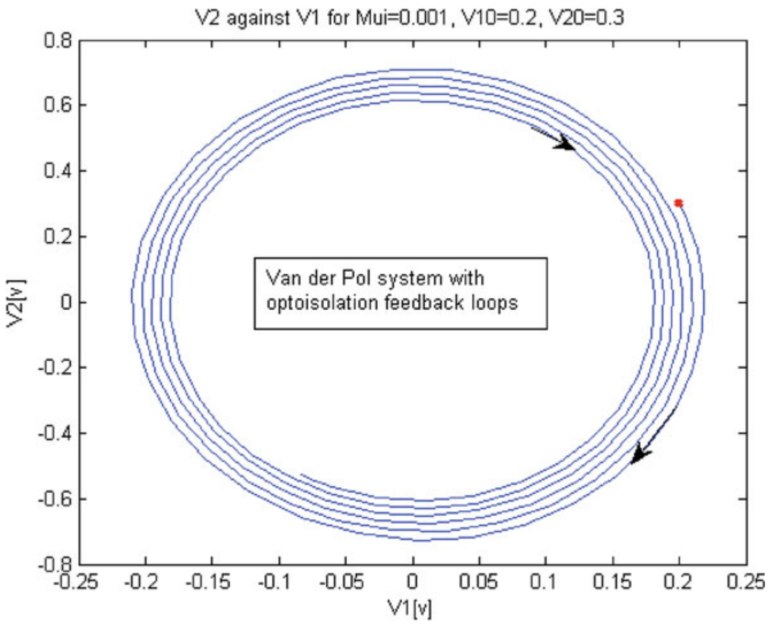


Fig. 1.13 (continued)

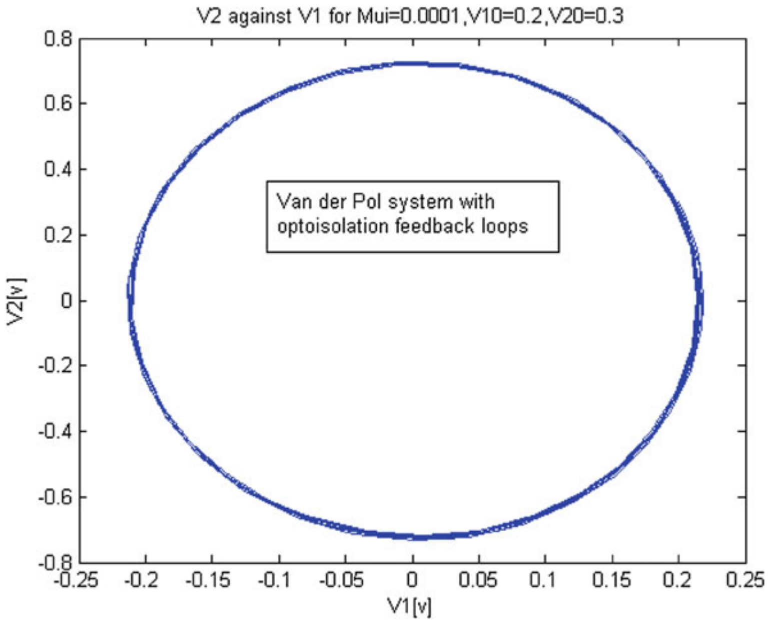


Fig. 1.13 (continued)

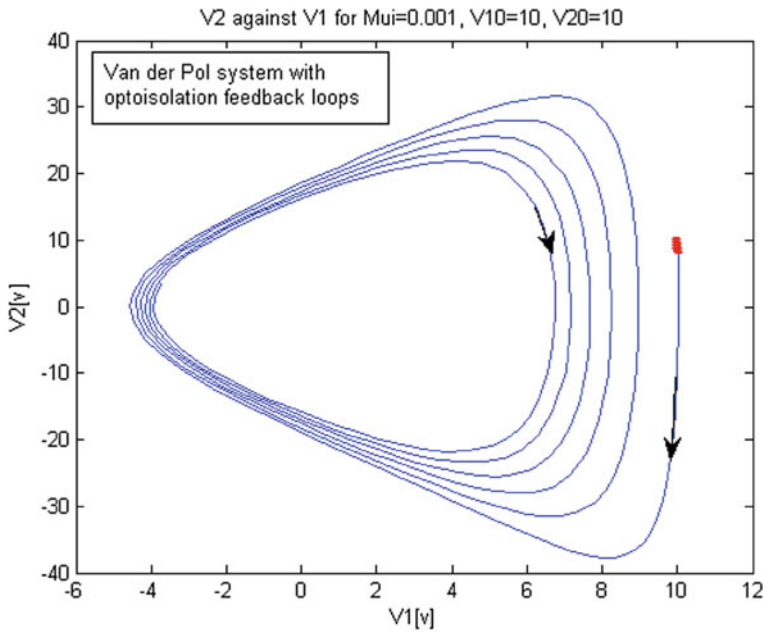


Fig. 1.13 (continued)

1.2 Optoisolation Circuits with Limit Cycles Stability Analysis

In the last analysis, we discuss our van der Pol system with optoisolation feedback loop . We need to analyze and find fixed points , stability under parameters values change [5, 6].

Our system differential equations:

$$\frac{dV_1}{dt} = g_1(V_1, V_2);\ \frac{dV_2}{dt} = g_2(V_1, V_2)$$

Accordingly we have g_1, g_2 functions:

$$g_1(V_1, V_2) = V_2;\ g_2(V_1, V_2) = \frac{-V_1 - V_2^2 \cdot \chi_2(V_1) - \mu \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\chi_1(V_1)}$$

To find fixed points we set

$$\frac{dV_1}{dt} = 0;\ \frac{dV_2}{dt} = 0 \Rightarrow g_1(V_1, V_2) = 0;\ g_2(V_1, V_2) = 0 \Rightarrow V_2 = 0$$

$$g_2(V_1, V_2 = 0) = \frac{-V_1}{\chi_1(V_1)} = 0 \Rightarrow \chi_1(V_1) = \sum_{k=1}^{5} k \cdot \eta_k \cdot V^{k-1} \neq 0\ ;\ V_1 = 0$$

We get the result, our system fixed point : $(V_1^*, V_2^*) = (0, 0)$ then $g_1(V_1^*, V_2^*) = 0$; $g_2(V_1^*, V_2^*) = 0$, let $u = V_1 - V_1^*$; $v = V_2 - V_2^*$ denote the components of a small disturbance from the van der Pol optoisolation fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v. $\frac{du}{dt} = \frac{dV_1}{dt} = g_1(V_1^* + u, V_2^* + v)$ since V_1^* is a constant. By using Taylor series expansion and since $g_1(V_1^*, V_2^*) = 0$, we get the following equation:

$$\frac{du}{dt} = \frac{dV_1}{dt} = g_1(V_1^*, V_2^*) + u \cdot \frac{\partial g_1}{\partial V_1} + v \cdot \frac{\partial g_1}{\partial V_2} + O(u^2, v^2, u \cdot v)\Big|_{g_1(V_1^*, V_2^*)=0}$$
$$= u \cdot \frac{\partial g_1}{\partial V_1} + v \cdot \frac{\partial g_1}{\partial V_2} + O(u^2, v^2, u \cdot v)$$

Since $\frac{\partial g_1}{\partial V_1}$ and $\frac{\partial g_1}{\partial V_2}$ are evaluated at the fixed points $(V_1^*, V_2^*) = (0, 0)$ thus they are numbers, not functions. $O(u^2, v^2, u \cdot v)$ denotes quadratic terms in u and v (very small). Similarly we find:

$$\frac{dv}{dt} = \frac{dV_2}{dt} = g_2(V_1^*, V_2^*) + u \cdot \frac{\partial g_2}{\partial V_1} + v \cdot \frac{\partial g_2}{\partial V_2} + O(u^2, v^2, u \cdot v)\Big|_{g_2(V_1^*, V_2^*)=0}$$
$$= u \cdot \frac{\partial g_2}{\partial V_1} + v \cdot \frac{\partial g_2}{\partial V_2} + O(u^2, v^2, u \cdot v)$$

The disturbance (u, v) evolves according to

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \text{Quadratic term;}$$

$$\text{Jacobian matrix } A = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} \end{pmatrix}_{(V_1^*, V_2^*)}$$

The quadratic terms are tiny and we neglect them and obtain linearized system. $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$. We classify our system fixed points as stable (spiral, Node) or Unstable (spiral, Node), which is done by inspection of the system characteristic equation.

$$A - \lambda \cdot I = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} \end{pmatrix}_{(V_1^*, V_2^*)} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \lambda = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} - \lambda & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} - \lambda \end{pmatrix}_{(V_1^*, V_2^*)}$$

$$\det|A - \lambda \cdot I| = 0 \Rightarrow \det\left|\begin{pmatrix} \frac{\partial g_1}{\partial V_1} - \lambda & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} - \lambda \end{pmatrix}_{(V_1^*, V_2^*)}\right| = 0$$

$$\Rightarrow \left\{ \left[\frac{\partial g_1}{\partial V_1} - \lambda\right] \cdot \left[\frac{\partial g_2}{\partial V_2} - \lambda\right] - \frac{\partial g_2}{\partial V_1} \cdot \frac{\partial g_1}{\partial V_2} \right\}_{(V_1^*, V_2^*)} = 0$$

$$g_1(V_1, V_2) = V_2;\ g_2(V_1, V_2) = \frac{-V_1 - V_2^2 \cdot \chi_2(V_1) - \mu \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]}{\chi_1(V_1)}$$

$$\left.\frac{\partial g_1(V_1, V_2)}{\partial V_1}\right|_{(V_1^*, V_2^*)=(0,0)} = 0;\ \left.\frac{\partial g_1(V_1, V_2)}{\partial V_2}\right|_{(V_1^*, V_2^*)=(0,0)} = 1$$

$$\frac{\partial g_2(V_1, V_2)}{\partial V_1} = \frac{\left\{-1 - V_2^2 \cdot \frac{\partial \chi_2(V_1)}{\partial V_1} - \mu \cdot V_2 \cdot \frac{\partial \chi_1(V_1)}{\partial V_1} - \mu \cdot \chi_1(V_1) \cdot V_2 \cdot \frac{\partial \chi_4(V_1)}{\partial V_1}\right\} \cdot \chi_1(V_1)}{[\chi_1(V_1)]^2} + \frac{\frac{\partial \chi_1(V_1)}{\partial V_1} \cdot \left\{V_1 + V_2^2 \cdot \chi_2(V_1) + \mu \cdot \chi_1(V_1) \cdot V_2 \cdot [\chi_4(V_1) - 1]\right\}}{[\chi_1(V_1)]^2}$$

$$\frac{\partial g_2}{\partial V_2} = \frac{\{-2 \cdot V_2 \cdot \chi_2(V_1) - \mu \cdot \chi_1(V_1) \cdot [\chi_4(V_1) - 1]\} \cdot \chi_1(V_1)}{[\chi_1(V_1)]^2}$$

$$\chi_1(V_1) = \sum_{k=1}^{5} k \cdot \eta_k \cdot V_1^{k-1} \Rightarrow \chi_1(V_1 = 0) = \eta_1$$

$$\chi_2(V_1) = \sum_{k=1}^{5} k \cdot (k-1) \cdot \eta_k \cdot V_1^{k-2} \Rightarrow \chi_2(V_1 = 0) = 2 \cdot \eta_2$$

$$\chi_4(V_1) = \left[\sum_{k=1}^{6} \eta_{k-1} \cdot V_1^{k-1}\right]^2 = [\eta_0 + \eta_1 \cdot V_1 + \cdots + \eta_5 \cdot V_1^5]^2 \Rightarrow \chi_4(V_1 = 0) = \eta_0^2$$

$$\left.\frac{\partial g_2}{\partial V_1}\right|_{(V_1^*, V_2^*)} = -\frac{1}{\eta_1^2};\ \left.\frac{\partial g_2}{\partial V_2}\right|_{(V_1^*, V_2^*)} = -\mu \cdot [\eta_0^2 - 1]$$

$$\det|A - \lambda \cdot I| = 0 \Rightarrow \det\left|\begin{pmatrix} \frac{\partial g_1}{\partial V_1} - \lambda & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} - \lambda \end{pmatrix}_{(V_1^*, V_2^*)}\right|$$

$$= \det\left|\begin{pmatrix} -\lambda & 1 \\ -\frac{1}{\eta_1^2} & -\mu \cdot [\eta_0^2 - 1] - \lambda \end{pmatrix}\right| = 0$$

$$\Rightarrow \{\lambda \cdot \{\mu \cdot [\eta_0^2 - 1] + \lambda\} + \frac{1}{\eta_1^2} = 0$$

$$\Rightarrow \lambda^2 + \lambda \cdot \mu \cdot [\eta_0^2 - 1] + \frac{1}{\eta_1^2} = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{-\mu \cdot [\eta_0^2 - 1] \pm \sqrt{\mu^2 \cdot [\eta_0^2 - 1]^2 - 4 \cdot \frac{1}{\eta_1^2}}}{2}$$

$$\lambda_1 = -\frac{1}{2} \cdot \mu \cdot [\eta_0^2 - 1] + \frac{1}{2} \cdot \sqrt{\mu^2 \cdot [\eta_0^2 - 1]^2 - 4 \cdot \frac{1}{\eta_1^2}};$$

$$\lambda_2 = -\frac{1}{2} \cdot \mu \cdot [\eta_0^2 - 1] - \frac{1}{2} \cdot \sqrt{\mu^2 \cdot [\eta_0^2 - 1]^2 - 4 \cdot \frac{1}{\eta_1^2}}$$

We need to classify our optoisolation linear system. The V_1 and V_2 axes played a crucial geometric role. They determined the direction of the trajectories as $t \to \pm\infty$. They also contained special straight line trajectories which started on one of the coordinate axes, stayed on that axis forever, and exhibited simple exponential growth or decay along it. Table 1.3 summarizes our system stability classification based on eigenvalues.

We already found η_0, η_1 parameters circuit values: $\eta_0 = 6.3$, $\eta_1 = 0.087111$.

For this values $\eta_0^2 - 1 = 38.69$; $[\eta_0^2 - 1]^2 = 1496.9$; $\frac{1}{\eta_1^2} = \frac{1}{[0.087111]^2} = 131.92$.

Table 1.3 System stability classification based on eigenvalues

Eigenvalues	Meaning
$\lambda_1 > 0, \lambda_2 > 0$	Unstable node
$\lambda_1 < 0, \lambda_2 < 0$	Stable node
$\lambda_1 < 0$ & $\lambda_2 > 0$ or $\lambda_1 > 0$ & $\lambda_2 < 0$ $\lambda_1 \cdot \lambda_2 < 0$	Saddle point
λ_1, λ_2 complex then $\lambda_{1,2} = \alpha \pm j\omega$ $\alpha = \mathrm{Re}(\lambda_{1,2}) < 0$ then decaying oscillations spiral	Stable point spiral
λ_1, λ_2 complex then $\lambda_{1,2} = \alpha \pm j\omega$ $\alpha = \mathrm{Re}(\lambda_{1,2}) > 0$ then growing oscillations Spiral	Unstable point spiral
λ_1, λ_2 complex then $\lambda_{1,2} = \alpha \pm j\omega$ and λ_1, λ_2 are pure imaginary $\alpha = 0$	Solutions are periodic solution with period $T = \frac{2\cdot\pi}{\omega}$

$$\lambda_{1,2} = -\mu \cdot 19.34 \pm \sqrt{\mu^2 \cdot 374.22 - 131.92}; \text{ we get two roots:}$$
$$\lambda_1 = -\mu \cdot 19.34 + \sqrt{\mu^2 \cdot 374.22 - 131.92};$$
$$\lambda_2 = -\mu \cdot 19.34 - \sqrt{\mu^2 \cdot 374.22 - 131.92}$$

Unstable Node

$$\lambda_1 = \left\{-\mu \cdot 19.34 + \sqrt{\mu^2 \cdot 374.22 - 131.92}\right\} > 0;$$
$$\lambda_2 = \left\{-\mu \cdot 19.34 - \sqrt{\mu^2 \cdot 374.22 - 131.92}\right\} > 0$$

Stable Node:

$$\lambda_1 = \left\{-\mu \cdot 19.34 + \sqrt{\mu^2 \cdot 374.22 - 131.92}\right\} < 0;$$
$$\lambda_2 = \left\{-\mu \cdot 19.34 - \sqrt{\mu^2 \cdot 374.22 - 131.92}\right\} < 0$$

Saddle Point:

$$\lambda_1 \cdot \lambda_2 < 0 \Rightarrow \lambda_1 \cdot \lambda_2 = \left\{-\mu \cdot 19.34 + \sqrt{\mu^2 \cdot 374.22 - 131.92}\right\} \cdot \left\{-\mu \cdot 19.34 - \sqrt{\mu^2 \cdot 374.22 - 131.92}\right\} < 0$$
$$\lambda_1 \cdot \lambda_2 = \mu^2 \cdot 19.34^2 - (\mu^2 \cdot 374.22 - 131.92) < 0$$
$$\Rightarrow \mu > 26.74 \quad \text{or} \quad \mu < -26.74$$

Stable Point Spiral

$$\lambda_{1,2} = \alpha \pm j \cdot \omega;\ \alpha = \mathrm{Re}\,(\lambda_{1,2}) < 0 \Rightarrow \{\mu^2 \cdot 374.22 - 131.92\} < 0 \Rightarrow -26.74 < \mu < +26.74$$
$$\alpha = \mathrm{Re}\,(\lambda_{1,2}) = -\mu \cdot 19.34 < 0 \Rightarrow \mu > 0 \Rightarrow 0 < \mu < +26.74$$

Unstable Point Spiral

$$\lambda_{1,2} = \alpha \pm j \cdot \omega;\ \alpha = \mathrm{Re}\,(\lambda_{1,2}) > 0 \Rightarrow \{\mu^2 \cdot 374.22 - 131.92\} < 0 \Rightarrow -26.74 < \mu < +26.74$$
$$\alpha = \mathrm{Re}\,(\lambda_{1,2}) = -\mu \cdot 19.34 > 0 \Rightarrow \mu < 0 \Rightarrow -26.74 < \mu < 0$$

Solutions Are Periodic with Period

$$\lambda_{1,2} = \alpha \pm j \cdot \omega; \alpha = \mathrm{Re}\,(\lambda_{1,2}) = 0$$
$$\Rightarrow \{\mu^2 \cdot 374.22 - 131.92\} < 0 \Rightarrow -26.74 < \mu < +26.74$$
$$\alpha = \mathrm{Re}\,(\lambda_{1,2}) = -\mu \cdot 19.34 = 0$$
$$\Rightarrow \mu = 0 \Rightarrow \sqrt{\mu^2 \cdot 374.22 - 131.92} = j \cdot 11.48 \Rightarrow \omega = 11.48$$

We get a periodic behavior with period: $T = \frac{2\cdot\pi}{\omega} = \frac{2\cdot\pi}{11.48} = \frac{\pi}{5.74}$.

We want to classify our system stability statuses more deeply based on parameters values μ, η_0, η_1. We define our system trajectory of the form $V(t) = e^{\lambda \cdot t} \cdot \Psi$, where $\Psi \neq 0$ is some fixed vector to be determined, and λ is a growth rate. If the solutions form exist, they correspond to exponential motion along the line spanned by the vector Ψ. To find the conditions on Ψ and λ, we substitute $V(t) = e^{\lambda \cdot t} \cdot \Psi$ into $\mathrm{d}V/\mathrm{d}t = A \cdot V$ $\left(V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}; \frac{\mathrm{d}V}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}V_1}{\mathrm{d}t} \\ \frac{\mathrm{d}V_2}{\mathrm{d}t} \end{pmatrix} \right)$ and obtain $\lambda \cdot e^{\lambda \cdot t} \cdot \Psi = e^{\lambda \cdot t} \cdot A \cdot \Psi$. When we cancel the nonzero scalar factor $e^{\lambda \cdot t}$ yields $A \cdot \Psi = \lambda \cdot \Psi$. The solutions exist if Ψ is an eigenvector of A with corresponding eigenvalue λ. $V(t) = e^{\lambda \cdot t} \cdot \Psi$ is an eigensolution . The eigenvalues of a linearized matrix A are given by the characteristic equation $\det(A - \lambda \cdot I) = 0$, where I is the identity matrix. We already found our linearized matrix A.

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} \end{pmatrix}_{(V_1^*, V_2^*)} = \begin{pmatrix} \left.\frac{\partial g_1}{\partial V_1}\right|_{(V_1^*, V_2^*)} & \left.\frac{\partial g_1}{\partial V_2}\right|_{(V_1^*, V_2^*)} \\ \left.\frac{\partial g_2}{\partial V_1}\right|_{(V_1^*, V_2^*)} & \left.\frac{\partial g_2}{\partial V_2}\right|_{(V_1^*, V_2^*)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\eta_1^2} & -\mu \cdot [\eta_0^2 - 1] \end{pmatrix}$$

For simplicity, we define four parameters a, b, c, d and build our A matrix by those parameters.

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial V_1} & \frac{\partial g_1}{\partial V_2} \\ \frac{\partial g_2}{\partial V_1} & \frac{\partial g_2}{\partial V_2} \end{pmatrix}_{(V_1^*, V_2^*)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\eta_1^2} & -\mu \cdot [\eta_0^2 - 1] \end{pmatrix}$$

The characteristic equation becomes:

$$\det\left|\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right| = 0 \Rightarrow \lambda^2 - \tau\cdot\lambda + \Delta = 0;$$
$$\tau = \text{trace}\,(A) = a+d;\ \Delta = \det\,(A) = a\cdot d - b\cdot c$$

Then $\lambda_1 = \frac{\tau+\sqrt{\tau^2-4\cdot\Delta}}{2}$; $\lambda_1 = \frac{\tau-\sqrt{\tau^2-4\cdot\Delta}}{2}$. The eigenvalues depend only on the trace and determinant of the matrix A. Typical solution $\lambda_1 \neq \lambda_2$ and vectors Ψ_1, Ψ_2 are linearly independent and span the entire plane. Any initial condition V_0 can be written as a linear combination of eigenvectors $V_0 = c_1 \bullet \psi_1 + c_2 \bullet \psi_2$ and the general solution for $V(t)$:

$$V(t) = c_1\cdot e^{\lambda_1\cdot t}\cdot\psi_1 + c_2\cdot e^{\lambda_2\cdot t}\cdot\psi_2$$

$$V(t) = c_1\cdot e^{\left\{-\frac{1}{2}\cdot\mu\cdot\left[\eta_0^2-1\right]+\frac{1}{2}\cdot\sqrt{\mu^2\cdot\left[\eta_0^2-1\right]^2-4\cdot\frac{1}{\eta_1^2}}\right\}\cdot t}\cdot\psi_1 + c_2 \cdot e^{\left\{-\frac{1}{2}\cdot\mu\cdot\left[\eta_0^2-1\right]-\frac{1}{2}\cdot\sqrt{\mu^2\cdot\left[\eta_0^2-1\right]^2-4\cdot\frac{1}{\eta_1^2}}\right\}\cdot t}\cdot\psi_2$$

$$\Delta = \lambda_1\cdot\lambda_2 = \frac{1}{4}\cdot\left\{\mu^2\cdot\left[\eta_0^2-1\right]^2 - \left\{\mu^2\cdot\left[\eta_0^2-1\right]^2 - 4\cdot\frac{1}{\eta_1^2}\right\}\right\} = \frac{1}{\eta_1^2};$$
$$\tau = \lambda_1+\lambda_2 = -\mu\cdot\left[\eta_0^2-1\right]$$

$(\lambda-\lambda_1)\cdot(\lambda-\lambda_2) = \lambda^2 - \tau\cdot\lambda + \Delta = 0$; $\Delta < 0$, the eigenvalues are real and have opposite signs; hence the fixed point is a saddle point. $\Delta > 0$, the eigenvalues are either real with the same sign (nodes), or complex conjugate which is spirals and centers. Nodes satisfy $\tau^2 - 4\cdot\Delta > 0$ and spirals satisfy $\tau^2 - 4\cdot\Delta < 0$. The parabola $\tau^2 - 4\cdot\Delta = 0$ is the borderline between nodes and spiral; star nodes and degenerate nodes live on this parabola. When $\tau < 0$, both eigenvalues have negative real parts and the fixed point is stable. When $\tau > 0$, then unstable spiral and node behavior. Neutrally stable centers live on the borderline $\tau = 0$, when the eigenvalues are purely imaginary. If $\Delta = 0$, at least one of the eigenvalues is zero and the origin is not an isolated fixed point. There is either a whole line of fixed points.

1.3 Poincare–Bendixson Stability and Limit Cycle Analysis

We have optoisolation system and we need to establish to which parameters values the closed orbits exist in our optoisolation particular system. We define Poincare–Bendixson theorem which proposes that our optoisolation system resides in the plane. When R is a closed subset of the plane, dV/dt = $f(V)$ is a continuously differentiable vector field on an open set containing R. When R does not contain any fixed point and there exist trajectories C that is confined in R, in the sense that it starts in R and stays in R for all future time. C is a closed orbit, or it spirals toward a close orbit as $t \to \infty$ R contains a closed orbit. We define our system equations as:

$$\frac{dr}{dt} = r\cdot(1-r^2) + \mu\cdot r\cdot\cos(\theta);\ \frac{d\theta}{dt} = 1;\ f(\mu, r) = \frac{\frac{dr}{dt} - r\cdot(1-r^2)}{\mu\cdot r} = \cos(\theta)$$

We define θ and r as system global variables. μ is system main parameter.

$$\theta = \arccos[f(\mu, r)] = \arccos\left[\frac{\frac{dr}{dt} - r\cdot(1-r^2)}{\mu\cdot r}\right];\ \frac{d\theta}{dt} = \frac{d}{dt}\{\arccos[f(\mu, r)]\} = 1$$

$$\frac{d}{dt}\{\arccos[f(\mu, r)]\} = 1 \Rightarrow \frac{d}{dt}\left\{\arccos\left[\frac{\frac{dr}{dt} - r\cdot(1-r^2)}{\mu\cdot r}\right]\right\} = 1;$$

$$U(f(\mu, r)) = \arccos[f(\mu, r)]$$

$$\frac{dU}{dt} = \frac{dU}{df}\cdot\frac{df}{dt} \Rightarrow \frac{dU}{dt} = \frac{d}{df}\{\arccos[f(\mu, r)]\}\cdot\frac{d}{dt}\left[\frac{\frac{dr}{dt} - r\cdot(1-r^2)}{\mu\cdot r}\right]$$

$$\frac{d}{df}\{\arccos[f(\mu, r)]\} = -\frac{1}{\sqrt{1-[f(\mu, r)]^2}} \Rightarrow \frac{dU}{dt} = -\frac{1}{\sqrt{1-[f(\mu, r)]^2}}\cdot\frac{d}{dt}\left[\frac{\frac{dr}{dt} - r\cdot(1-r^2)}{\mu\cdot r}\right]$$

$$\frac{df(\mu, r)}{dt} = \frac{\left[\frac{d^2r}{dt^2} - \frac{dr}{dt}\cdot(1-r^2) - r\cdot\left(-2\cdot r\cdot\frac{dr}{dt}\right)\right]\cdot\mu\cdot r - \mu\cdot\frac{dr}{dt}\cdot\left[\frac{dr}{dt} - r\cdot(1-r^2)\right]}{[\mu\cdot r]^2}$$

$$\frac{df(\mu, r)}{dt} = \frac{1}{\mu}\cdot\left\{\frac{\left[\frac{d^2r}{dt^2} - \frac{dr}{dt} + 3\cdot\frac{dr}{dt}\cdot r^2\right]\cdot r - \frac{dr}{dt}\cdot\left[\frac{dr}{dt} - r + r^3\right]}{r^2}\right\}$$

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{1}{\sqrt{1-[f(\mu,r)]^2}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\frac{\mathrm{d}r}{\mathrm{d}t} - r\cdot(1-r^2)}{\mu\cdot r}\right]$$
$$= -\frac{1}{\sqrt{1-\left[\frac{\frac{\mathrm{d}r}{\mathrm{d}t}-r\cdot(1-r^2)}{\mu\cdot r}\right]^2}} \cdot \frac{1}{\mu}$$
$$\cdot \left\{\frac{\left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - \frac{\mathrm{d}r}{\mathrm{d}t} + 3\cdot\frac{\mathrm{d}r}{\mathrm{d}t}\cdot r^2\right]\cdot r - \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\left[\frac{\mathrm{d}r}{\mathrm{d}t} - r + r^3\right]}{r^2}\right\}$$

$$-\mu\cdot r^2\cdot\sqrt{1-\left[\frac{\frac{\mathrm{d}r}{\mathrm{d}t} - r\cdot(1-r^2)}{\mu\cdot r}\right]^2} = \left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - \frac{\mathrm{d}r}{\mathrm{d}t} + 3\cdot\frac{\mathrm{d}r}{\mathrm{d}t}\cdot r^2\right] \cdot r - \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\left[\frac{\mathrm{d}r}{\mathrm{d}t} - r + r^3\right]$$

We do variable transformation and get our system set of differential equations.

$$X = \frac{\mathrm{d}r}{\mathrm{d}t};\ \frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2};\ Y = r \Rightarrow X = \frac{\mathrm{d}Y}{\mathrm{d}t};\ \frac{\mathrm{d}r}{\mathrm{d}t} \to X;\ \frac{\mathrm{d}^2r}{\mathrm{d}t^2} \to \frac{\mathrm{d}X}{\mathrm{d}t};\ r \to Y$$

$$-\mu\cdot Y\cdot\sqrt{1-\left[\frac{X - Y\cdot(1-Y^2)}{\mu\cdot Y}\right]^2} = \frac{\mathrm{d}X}{\mathrm{d}t} - X + 3\cdot X\cdot Y^2 - \frac{X}{Y}\cdot[X - Y + Y^3]$$

We get set of two differential equations:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\mu\cdot Y\cdot\sqrt{1-\left[\frac{X - Y\cdot(1-Y^2)}{\mu\cdot Y}\right]^2} + X - 3\cdot X\cdot Y^2 + \frac{X}{Y}\cdot[X - Y + Y^3]$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = X$$

We define two functions:

$$f(X,Y) = -\mu\cdot Y\cdot\sqrt{1-\left[\frac{X - Y\cdot(1-Y^2)}{\mu\cdot Y}\right]^2} + X - 3\cdot X\cdot Y^2 + \frac{X}{Y}\cdot[X - Y + Y^3];\ \mathrm{g}\,(X,Y) = X$$

Accordingly we get the set: $\frac{\mathrm{d}X}{\mathrm{d}t} = f(X,Y);\ \frac{\mathrm{d}Y}{\mathrm{d}t} = g\,(X,Y)$

To our system fixed points , we define:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 0 \Rightarrow f(X^*, Y^*) = 0\,;\ \frac{\mathrm{d}Y}{\mathrm{d}t} = 0 \Rightarrow g(X^*, Y^*) = 0 \Rightarrow X^* = 0$$

$$g(X^* = 0, Y^*) = 0 \Rightarrow f(X^* = 0, Y^*) = 0 \Rightarrow f(X^* = 0, Y^*)$$
$$= -\mu \cdot Y^* \cdot \sqrt{1 - \left[\frac{\left(1 - [Y^*]^2\right)}{\mu}\right]^2} = 0$$

$$X^* = 0 \Rightarrow Y^{(0)} = 0;\ Y^{(1)} = +\sqrt{1+\mu};$$
$$Y^{(2)} = -\sqrt{1+\mu};\ Y^{(3)} = +\sqrt{1-\mu};\ Y^{(4)} = -\sqrt{1-\mu}$$

$$\sqrt{1 - \left[\frac{(1 - [Y^*]^2)}{\mu}\right]^2} = 0 \Rightarrow \left[\frac{(1 - [Y^*]^2)}{\mu}\right]^2 = 1$$
$$\Rightarrow \frac{(1 - [Y^*]^2)}{\mu} = \pm 1 \Rightarrow (1 - [Y^*]^2) = \pm\mu$$

$$(1 - [Y^*]^2) = \pm\mu \Rightarrow [Y^*]^2 = 1 \pm \mu \Rightarrow Y^* = \pm\sqrt{1 \pm \mu}$$

We get five fixed points in our system:

$$E^{(0)}(X^{(0)}, Y^{(0)}) = (0, 0);\ E^{(1)}(X^{(1)}, Y^{(1)}) = (0, +\sqrt{1+\mu});$$
$$E^{(2)}(X^{(2)}, Y^{(2)}) = (0, -\sqrt{1+\mu})$$

$$E^{(3)}(X^{(3)}, Y^{(3)}) = (0, +\sqrt{1-\mu});\ E^{(4)}(X^{(4)}, Y^{(4)}) = (0, -\sqrt{1-\mu})$$

We can find our fixed points directly from our system original equation.

$$-\mu \cdot r^2 \cdot \sqrt{1 - \left[\frac{\frac{\mathrm{d}r}{\mathrm{d}t} - r \cdot (1 - r^2)}{\mu \cdot r}\right]^2} = \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{\mathrm{d}r}{\mathrm{d}t} + 3 \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot r^2\right]$$
$$\cdot r - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \left[\frac{\mathrm{d}r}{\mathrm{d}t} - r + r^3\right]$$

The condition to find our system fixed points: $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$; $\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = 0$

$$-\mu\cdot[r^*]^2\cdot\sqrt{1-\left[\frac{[r^*]^2-1}{\mu}\right]^2}=0\Rightarrow r^*=0\quad\text{or}\quad 1-\left[\frac{[r^*]^2-1}{\mu}\right]^2=0\Rightarrow\left[\frac{[r^*]^2-1}{\mu}\right]^2=1$$

$$\left[\frac{[r^*]^2-1}{\mu}\right]^2=1\Rightarrow\left[\frac{[r^*]^2-1}{\mu}\right]=\pm1\Rightarrow[r^*]^2-1=\pm\mu\Rightarrow[r^*]^2=1\pm\mu\Rightarrow r^*\pm\sqrt{1\pm\mu}$$

To classify our system fixed points stability, we first need to find the Jacobian matrix at the fixed points $(X^{(i)}, Y^{(i)})$; i = 0, 1, 2, 3, 4. Linearization technique.

$$A=\begin{pmatrix}\frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y}\\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y}\end{pmatrix}_{(X^{(i)},Y^{(i)})}$$

$$\frac{\partial f}{\partial X}=\frac{1}{\sqrt{1-\left\{\frac{X-Y\cdot(1-Y^2)}{Y\cdot\mu}\right\}^2}}\cdot\frac{[X-Y\cdot(1-Y^2)]}{Y\cdot\mu}-2\cdot Y^2+2\cdot\frac{X}{Y}$$

In all our system, fixed points $X^{(i)}=0$ for i = 0, 1, 2, 3, 4. We get

$$\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(i)})}=\frac{1}{\sqrt{1-\left\{\frac{([Y^{(i)}]^2-1)}{\mu}\right\}^2}}\cdot\frac{([Y^{(i)}]^2-1)}{\mu}-2\cdot[Y^{(i)}]^2$$

We need to calculate $\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(i)})}$ for our system Jacobian matrix.

$$Y^{(i=0)}=0\Rightarrow\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(0)}=0)}=-\frac{1}{\mu\cdot\sqrt{1-\frac{1}{\mu^2}}}$$

$$Y^{(i=1)}=\sqrt{1+\mu}\Rightarrow\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(1)}=\sqrt{1+\mu})}\to\infty$$

$$Y^{(i=2)}=-\sqrt{1+\mu}\Rightarrow\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(2)}=-\sqrt{1+\mu})}\to\infty$$

$$Y^{(i=3)}=\sqrt{1-\mu}\Rightarrow\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(3)}=\sqrt{1-\mu})}\to\infty$$

$$Y^{(i=4)}=-\sqrt{1-\mu}\Rightarrow\left.\frac{\partial f}{\partial X}\right|_{(X^{(i)}=0,Y^{(4)}=-\sqrt{1-\mu})}\to\infty$$

We need to calculate $\left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0,Y^{(i)})}$ for our system Jacobian matrix.

$$\frac{\partial f}{\partial Y} = -\mu \cdot \sqrt{1 - \left\{\frac{X - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right\}^2} + \frac{X - Y \cdot (1 - Y^2)}{\sqrt{1 - \left\{\frac{X - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right\}^2}} \cdot \left\{\frac{2Y^3 - X}{\mu Y^2}\right\}$$
$$- 6XY - \left[\frac{X}{Y}\right]^2 + 2 \cdot X \cdot Y$$

In all our system fixed points $X^{(i)} = 0$ for $i = 0, 1, 2, 3, 4$. We get

$$\left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0, Y^{(i)})} = -\mu \cdot \sqrt{1 - \left\{\frac{([Y^{(i)}]^2 - 1)}{\mu}\right\}^2} + \frac{([Y^{(i)}]^2 - 1)}{\sqrt{1 - \left\{\frac{([Y^{(i)}]^2 - 1)}{\mu}\right\}^2}} \cdot \left\{\frac{2 \cdot [Y^{(i)}]^2}{\mu}\right\}$$

$$Y^{(i=0)} = 0 \Rightarrow \left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0, Y^{(0)}=0)} = -\mu \cdot \sqrt{1 - \frac{1}{\mu^2}}$$

$$Y^{(i=1)} = \sqrt{1 + \mu} \Rightarrow \left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0, Y^{(1)}=\sqrt{1+\mu})} \to \infty$$

$$Y^{(i=2)} = -\sqrt{1 + \mu} \Rightarrow \left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0, Y^{(2)}=-\sqrt{1+\mu})} \to \infty$$

$$Y^{(i=3)} = \sqrt{1 - \mu} \Rightarrow \left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0, Y^{(3)}=\sqrt{1-\mu})} \to \infty$$

$$Y^{(i=4)} = -\sqrt{1 - \mu} \Rightarrow \left.\frac{\partial f}{\partial Y}\right|_{(X^{(i)}=0, Y^{(4)}=-\sqrt{1-\mu})} \to \infty$$

$$\left.\frac{\partial g}{\partial X}\right|_{(X^{(i)}, Y^{(i)})} = 1; \quad \left.\frac{\partial g}{\partial Y}\right|_{(X^{(i)}, Y^{(i)})} = 0$$

We get our Jacobian matrix elements. There is only meaning to the first fixed point since all other fixed points Jacobian matrix elements ($\frac{\partial f}{\partial X}$; $\frac{\partial f}{\partial Y}$) go to infinite [8, 9].

To classify our system fixed points stability we first need to find the Jacobian matrix at the fixed points $(X^{(i)}, Y^{(i)})$; $i = 0$. Linearization technique.

$$A = \begin{pmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{pmatrix}_{(X^{(i=0)}, Y^{(i=0)})} = \begin{pmatrix} -\frac{1}{\mu \cdot \sqrt{1 - \frac{1}{\mu^2}}} & -\mu \cdot \sqrt{1 - \frac{1}{\mu^2}} \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{pmatrix}_{(X^{(i=0)}, Y^{(i=0)})} - \lambda \cdot I = \begin{pmatrix} -\frac{1}{\mu \cdot \sqrt{1-\frac{1}{\mu^2}}} - \lambda & -\mu \cdot \sqrt{1 - \frac{1}{\mu^2}} \\ 1 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \left(-\frac{1}{\mu \cdot \sqrt{1 - \frac{1}{\mu^2}}} - \lambda \right) \cdot (-\lambda) + \mu \cdot \sqrt{1 - \frac{1}{\mu^2}} = 0$$

$$\text{Characteristic equation: } \left(\frac{1}{\mu \cdot \sqrt{1 - \frac{1}{\mu^2}}} + \lambda \right) \cdot \lambda + \mu \cdot \sqrt{1 - \frac{1}{\mu^2}} = 0; \, \Gamma = \mu \cdot \sqrt{1 - \frac{1}{\mu^2}}$$

$$\Gamma = \mu \cdot \sqrt{1 - \frac{1}{\mu^2}} \Rightarrow \left(\frac{1}{\Gamma} + \lambda \right) \cdot \lambda + \Gamma = 0 \Rightarrow \lambda^2 + \lambda \cdot \frac{1}{\Gamma} + \Gamma = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{1}{2} \cdot \left\{ -\frac{1}{\Gamma} \pm \sqrt{\frac{1}{\Gamma^2} - 4 \cdot \Gamma} \right\}$$

$$\lambda_{1,2} = \frac{1}{2 \cdot \Gamma} \cdot \left\{ -1 \pm \sqrt{1 - 4 \cdot \Gamma^3} \right\};$$

$$\Gamma = \mu \cdot \sqrt{1 - \frac{1}{\mu^2}} \Rightarrow \Gamma = \mu \cdot \sqrt{\frac{\mu^2 - 1}{\mu^2}} = \sqrt{\mu^2 - 1}$$

$$\mu = 0 \Rightarrow \Gamma = \sqrt{\mu^2 - 1}_{\mu=0} = i; \quad \frac{1}{\Gamma_{\mu=0}} = -i; \, \Gamma^3_{\mu=0} = -i$$

$$\lambda_{1,2|\mu=0} = \frac{1}{2 \cdot \Gamma} \cdot \left\{ -1 \pm \sqrt{1 - 4 \cdot \Gamma^3} \right\}_{\mu=0} = -\frac{i}{2} \cdot \left\{ -1 \pm \sqrt{1 + 4 \cdot i} \right\};$$

$$\sqrt{1 + 4 \cdot i} = m_1 + i \cdot m_2$$

$$\sqrt{1 + 4 \cdot i} = m_1 + i \cdot m_2 \Rightarrow 1 + 4 \cdot i = m_1^2 - m_2^2 + 2 \cdot i \cdot m_1 \cdot m_2$$

$$\Rightarrow m_1^2 - m_2^2 = 1; \; m_1 \cdot m_2 = 2$$

$$m_1^2 - m_2^2 = 1 \; \& \; m_1 \cdot m_2 = 2 \Rightarrow m_1 = \frac{2}{m_2} \Rightarrow \frac{4}{m_2^2} - m_2^2 = 1$$

$$\Rightarrow \frac{4 - m_2^4}{m_2^2} = 1 \Rightarrow m_2^4 + m_2^2 - 4 = 0$$

$$m_2^4 + m_2^2 - 4 = 0 \Rightarrow n_2^2 = m_2^4; \; n_2 = m_2^2 \Rightarrow n_2^2 + n_2 - 4 = 0 \Rightarrow n_2 = \frac{-1 \pm \sqrt{17}}{2}$$

$$n_2 = m_2^2 \Rightarrow m_2 = \pm\sqrt{n_2} = \pm\sqrt{\frac{-1 \pm \sqrt{17}}{2}};$$

$$m_1 = \frac{2}{m_2} = \frac{2}{\pm\sqrt{\frac{-1\pm\sqrt{17}}{2}}} = \pm 2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}}$$

$$m_1 = \pm 2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}};$$

$$m_2 = \pm\sqrt{\frac{-1 \pm \sqrt{17}}{2}} \Rightarrow \sqrt{1 + 4 \cdot i} = m_1 + i \cdot m_2$$

$$= \pm 2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}} \pm i \cdot \sqrt{\frac{-1 \pm \sqrt{17}}{2}}$$

$$\lambda_{1,2|\mu=0} = \frac{1}{2 \cdot \Gamma} \cdot \left\{-1 \pm \sqrt{1 - 4 \cdot \Gamma^3}\right\}_{\mu=0} = -\frac{i}{2} \cdot \{-1 \pm \sqrt{1 + 4 \cdot i}\}$$

$$= -\frac{i}{2} \cdot \left\{-1 \pm \left[\pm 2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}} \pm i \cdot \sqrt{\frac{-1 \pm \sqrt{17}}{2}}\right]\right\}$$

$$\lambda_{1,2|\mu=0} = \frac{1}{2 \cdot \Gamma} \cdot \left\{-1 \pm \sqrt{1 - 4 \cdot \Gamma^3}\right\}_{\mu=0}$$

$$= \left\{\frac{i}{2} \mp \frac{i}{2} \cdot \left[\pm 2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}} \pm i \cdot \sqrt{\frac{-1 \pm \sqrt{17}}{2}}\right]\right\}$$

$$= \left\{\frac{i}{2} \mp \frac{i}{2} \cdot \left[2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}} + i \cdot \sqrt{\frac{-1 \pm \sqrt{17}}{2}}\right]\right\}$$

$$\lambda_{1,2|\mu=0} = \frac{1}{2 \cdot \Gamma} \cdot \left\{-1 \pm \sqrt{1 - 4 \cdot \Gamma^3}\right\}_{\mu=0}$$

$$= \left\{\frac{i}{2} \mp \frac{i}{2} \cdot \left[2 \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}} + i \cdot \sqrt{\frac{-1 \pm \sqrt{17}}{2}}\right]\right\}$$

$$= \frac{i}{2} \mp \left[i \cdot \sqrt{\frac{2}{-1 \pm \sqrt{17}}} - \frac{1}{2} \cdot \sqrt{\frac{-1 \pm \sqrt{17}}{2}}\right]$$

We can summarize our intermediate result in Tables 1.4 and 1.5:

We need to verify for our system, when there is a stable limit cycle at $r = 1$ and the close orbit still exist for $\mu > 0$. If we start from initial radius variable value r_0

Table 1.4 Intermediate result

m_1	m_2	$m_1 + i \cdot m_2 = \sqrt{1 + 4 \cdot i}$
$2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}}$	$\sqrt{\frac{-1+\sqrt{17}}{2}}$	$2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} + i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}$
$2 \cdot \sqrt{\frac{2}{-1-\sqrt{17}}} = 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}}$	$\sqrt{\frac{-1-\sqrt{17}}{2}} = i \cdot \sqrt{\frac{1+\sqrt{17}}{2}}$	$-\sqrt{\frac{1+\sqrt{17}}{2}} + 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}}$
$-2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}}$	$-\sqrt{\frac{-1+\sqrt{17}}{2}}$	$-2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} - i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}$
$-2 \cdot \sqrt{\frac{2}{-1-\sqrt{17}}} = -2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}}$	$-\sqrt{\frac{-1-\sqrt{17}}{2}} = -i \cdot \sqrt{\frac{1+\sqrt{17}}{2}}$	$\sqrt{\frac{1+\sqrt{17}}{2}} - 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}}$

then the equation for $r(t)$ is $r(t) = r_0 + [\mathrm{d}r/\mathrm{d}t] \cdot \mathrm{d}t$. We define $\mathrm{d}r/\mathrm{d}t$ as the change rate respect to time of radius vector $r(t)$. We define two main cases: in the inner circle and outer circle. We have three kinds of possible limit cycle in our system: SLC, HLC, and USL. For each limit cycle, we need to establish the $r_{\min}$ and $r_{\max}$ values for $\mathrm{d}r/\mathrm{d}t$ conditions in the inner and outer circle. We need to find $r_{\min}$ and $r_{\max}$ values ($r > 0$) then Poincare–Bendixson theorem implies existence of a closed orbit. Table 1.6 summarizes each limit cycle type and the conditions for $\mathrm{d}r/\mathrm{d}t$ in the inner and outer circle [6, 7].

Option I: dr/dt > *0*

$$\frac{\mathrm{d}r}{\mathrm{d}t} > 0 \Rightarrow r \cdot (1 - r^2) + \mu \cdot r \cdot \cos\theta > 0;\ \cos\theta \geq -1;\ \cos\theta = -1$$
$$\Rightarrow r \cdot (1 - r^2) - \mu \cdot r > 0$$

$$r \cdot (1 - r^2) - \mu \cdot r > 0 \Rightarrow r \cdot [1 - r^2 - \mu] > 0;$$
$$r > 0 \Rightarrow [1 - r^2 - \mu] > 0 \Rightarrow [(1 - \mu) - r^2] > 0$$

$[(1 - \mu) - r^2] > 0 \Rightarrow \left\{\sqrt{(1-\mu)} - r\right\} \cdot \left\{\sqrt{(1-\mu)} + r\right\} > 0$; we have two cases:

Case I: $\left\{\sqrt{(1-\mu)} - r\right\} > 0 \Rightarrow r < \sqrt{(1-\mu)}$; $\left\{\sqrt{(1-\mu)} + r\right\} > 0 \Rightarrow r > -\sqrt{(1-\mu)}$

$-\sqrt{(1-\mu)} < r < \sqrt{(1-\mu)}$ as long as $\mu < 1$ the square root make sense.

Case II: $\left\{\sqrt{(1-\mu)} - r\right\} < 0 \Rightarrow r > \sqrt{(1-\mu)}$; $\left\{\sqrt{(1-\mu)} + r\right\} < 0 \Rightarrow r < -\sqrt{(1-\mu)}$

Not possible!.

$$\frac{\mathrm{d}r}{\mathrm{d}t} > 0 \Rightarrow r \cdot (1 - r^2) + \mu \cdot r \cdot \cos\theta > 0;\ \cos\theta \geq -1;$$
$$\cos\theta = 1 \Rightarrow r \cdot (1 - r^2) + \mu \cdot r > 0$$

Table 1.5 Intermediate result

$m_1 + i \cdot m_2 = \sqrt{1 + 4 \cdot i}$	$\lambda_{1,2\|\mu=0} = -\frac{i}{2} \cdot \left\{-1 \pm \sqrt{1 + 4 \cdot i}\right\}$
$2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} + i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}$	$\lambda_{1,2\|\mu=0} = -\frac{i}{2} \cdot \left\{-1 \pm \left[2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} + i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}\right]\right\}$ $\lambda_{1\|\mu=0} = -\frac{i}{2} \cdot \left\{-1 + 2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} + i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}\right\}$ $\lambda_{1\|\mu=0} = \frac{1}{2} \cdot \sqrt{\frac{-1+\sqrt{17}}{2}} + i \cdot \left\{\frac{1}{2} - \sqrt{\frac{2}{-1+\sqrt{17}}}\right\}$ $\lambda_{2\|\mu=0} = -\frac{i}{2} \cdot \left\{-1 - 2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} - i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}\right\}$ $\lambda_{2\|\mu=0} = \frac{i}{2} \cdot \left\{1 + 2 \cdot \sqrt{\frac{2}{-1+\sqrt{17}}} + i \cdot \sqrt{\frac{-1+\sqrt{17}}{2}}\right\}$ $\lambda_{2\|\mu=0} = -\frac{1}{2} \cdot \sqrt{\frac{-1+\sqrt{17}}{2}} + i \cdot \left\{\frac{1}{2} + \sqrt{\frac{2}{-1+\sqrt{17}}}\right\}$
$m_1 + i \cdot m_2 = \sqrt{1 + 4 \cdot i}$	$\lambda_{1,2\|\mu=0} = -\frac{i}{2} \cdot \left\{-1 \pm \sqrt{1 + 4 \cdot i}\right\}$

(continued)

Table 1.5 (continued)

$-\sqrt{\frac{1+\sqrt{17}}{2}}+2\cdot i\cdot\sqrt{\frac{2}{1+\sqrt{17}}}$	$\lambda_{1,2\|\mu=0}=-\frac{i}{2}\cdot\left\{-1\pm\left[-\sqrt{\frac{1+\sqrt{17}}{2}}+2\cdot i\cdot\sqrt{\frac{2}{1+\sqrt{17}}}\right]\right\}$ $\lambda_{1\|\mu=0}=-\frac{i}{2}\cdot\left\{-1-\sqrt{\frac{1+\sqrt{17}}{2}}+2\cdot i\cdot\sqrt{\frac{2}{1+\sqrt{17}}}\right\}$ $\lambda_{1\|\mu=0}=\frac{i}{2}\cdot\left\{1+\sqrt{\frac{1+\sqrt{17}}{2}}-2\cdot i\cdot\sqrt{\frac{2}{1+\sqrt{17}}}\right\}$ $\lambda_{1\|\mu=0}=\sqrt{\frac{2}{1+\sqrt{17}}}+\frac{i}{2}\cdot\left\{1+\sqrt{\frac{1+\sqrt{17}}{2}}\right\}$ $\lambda_{2\|\mu=0}=-\frac{i}{2}\cdot\left\{-1+\sqrt{\frac{1+\sqrt{17}}{2}}-2\cdot i\cdot\sqrt{\frac{2}{1+\sqrt{17}}}\right\}$ $\lambda_{2\|\mu=0}=-\sqrt{\frac{2}{1+\sqrt{17}}}+\frac{i}{2}\cdot\left\{1-\sqrt{\frac{1+\sqrt{17}}{2}}\right\}$

(continued)

Table 1.5 (continued)

$-\left\{2\cdot\sqrt{\frac{2}{-1+\sqrt{17}}}+i\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}\right\}$	$\lambda_{1,2\mid\mu=0}=-\frac{i}{2}\cdot\left\{-1\pm\left[-\left\{2\cdot\sqrt{\frac{2}{-1+\sqrt{17}}}+i\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}\right\}\right]\right\}$ $\lambda_{1\mid\mu=0}=-\frac{i}{2}\cdot\left\{-1-2\cdot\sqrt{\frac{2}{-1+\sqrt{17}}}-i\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}\right\}$ $\lambda_{1\mid\mu=0}=\frac{i}{2}\cdot\left\{1+2\cdot\sqrt{\frac{2}{-1+\sqrt{17}}}+i\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}\right\}$ $\lambda_{1\mid\mu=0}=i\cdot\left[\frac{1}{2}+\sqrt{\frac{2}{-1+\sqrt{17}}}\right]-\frac{1}{2}\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}$ $\lambda_{1\mid\mu=0}=-\frac{1}{2}\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}+i\cdot\left[\frac{1}{2}+\sqrt{\frac{2}{-1+\sqrt{17}}}\right]$ $\lambda_{2\mid\mu=0}=-\frac{i}{2}\cdot\left\{-1+\left\{2\cdot\sqrt{\frac{2}{-1+\sqrt{17}}}+i\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}\right\}\right\}$ $\lambda_{2\mid\mu=0}=i\cdot\left[\frac{1}{2}-\sqrt{\frac{2}{-1+\sqrt{17}}}\right]+\frac{1}{2}\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}$ $\lambda_{2\mid\mu=0}=\frac{1}{2}\cdot\sqrt{\frac{-1+\sqrt{17}}{2}}+i\cdot\left[\frac{1}{2}-\sqrt{\frac{2}{-1+\sqrt{17}}}\right]$
$m_1+i\cdot m_2=\sqrt{1+4\cdot i}$	$\lambda_{1,2\mid\mu=0}=-\frac{i}{2}\cdot\left\{-1\pm\sqrt{1+4\cdot i}\right\}$

(continued)

Table 1.5 (continued)

$\sqrt{\frac{1+\sqrt{17}}{2}} - 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}}$	$\lambda_{1,2\|\mu=0} = -\frac{i}{2} \cdot \left\{ -1 \pm \left[\sqrt{\frac{1+\sqrt{17}}{2}} - 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}} \right] \right\}$ $\lambda_{1\|\mu=0} = -\frac{i}{2} \cdot \left\{ -1 + \left[\sqrt{\frac{1+\sqrt{17}}{2}} - 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}} \right] \right\}$ $\lambda_{1\|\mu=0} = -\frac{i}{2} \cdot \left\{ -1 + \sqrt{\frac{1+\sqrt{17}}{2}} - 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}} \right\}$ $\lambda_{1\|\mu=0} = \frac{i}{2} \cdot \left[1 - \sqrt{\frac{1+\sqrt{17}}{2}} \right] - \sqrt{\frac{2}{1+\sqrt{17}}}$ $\lambda_{2\|\mu=0} = -\frac{i}{2} \cdot \left\{ -1 - \left[\sqrt{\frac{1+\sqrt{17}}{2}} - 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}} \right] \right\}$ $\lambda_{2\|\mu=0} = -\frac{i}{2} \cdot \left\{ -1 - \sqrt{\frac{1+\sqrt{17}}{2}} + 2 \cdot i \cdot \sqrt{\frac{2}{1+\sqrt{17}}} \right\}$ $\lambda_{2\|\mu=0} = \frac{i}{2} \cdot \left[1 + \sqrt{\frac{1+\sqrt{17}}{2}} \right] + \sqrt{\frac{2}{1+\sqrt{17}}}$

Table 1.6 Limit cycle type and the conditions for d*r*/d*t* in the inner and outer circle

Limit cycle type	d*r*/d*t* in the inner circle	d*r*/d*t* in the outer circle
Stable limit cycle (SLC)	$dr/dt > 0$	$dr/dt < 0$
Half-stable limit cycle (HLC)	$dr/dt < 0$	$dr/dt < 0$
Unstable limit cycle (ULC)	$dr/dt < 0$	$dr/dt > 0$

$$r \cdot (1 - r^2) + \mu \cdot r > 0 \Rightarrow r \cdot [1 - r^2 + \mu] > 0;$$
$$r > 0 \Rightarrow [1 - r^2 + \mu] > 0 \Rightarrow [(1 + \mu) - r^2] > 0$$

$[(1+\mu) - r^2] > 0 \Rightarrow \left\{\sqrt{(1+\mu)} - r\right\} \cdot \left\{\sqrt{(1+\mu)} + r\right\} > 0$; we have two cases:

Case I: $\{\sqrt{(1+\mu)} - r\} > 0 \Rightarrow r < \sqrt{(1+\mu)}$; $\{\sqrt{(1+\mu)} + r\} > 0 \Rightarrow r > -\sqrt{(1+\mu)}$

$-\sqrt{(1+\mu)} < r < \sqrt{(1+\mu)}$ as long as $\mu < 1$ the square root make sense. *Case II:* $\left\{\sqrt{(1+\mu)} - r\right\} < 0 \Rightarrow r > \sqrt{(1+\mu)}$; $\left\{\sqrt{(1+\mu)} + r\right\} < 0 \Rightarrow r < -\sqrt{(1+\mu)}$

Not possible!

Option II: dr/dt < *0*

$$\frac{dr}{dt} < 0 \Rightarrow r \cdot (1 - r^2) + \mu \cdot r \cdot \cos\theta < 0;\ \cos\theta \geq -1;$$
$$\cos\theta = -1 \Rightarrow r \cdot (1 - r^2) - \mu \cdot r < 0$$
$$r \cdot (1 - r^2) - \mu \cdot r < 0 \Rightarrow r \cdot [1 - r^2 - \mu] < 0;$$
$$r > 0 \Rightarrow [1 - r^2 - \mu] < 0 \Rightarrow [(1 - \mu) - r^2] < 0$$

$[(1-\mu) - r^2] < 0 \Rightarrow \left\{\sqrt{(1-\mu)} - r\right\} \cdot \left\{\sqrt{(1-\mu)} + r\right\} < 0$; we have two cases:

Case I: $\left\{\sqrt{(1-\mu)} - r\right\} > 0 \Rightarrow r < \sqrt{(1-\mu)}$; $\left\{\sqrt{(1-\mu)} + r\right\} < 0 \Rightarrow r < -\sqrt{(1-\mu)}$

$r < -\sqrt{(1-\mu)}$ as long as $\mu < 1$ the square root make sense.

Case II: $\left\{\sqrt{(1-\mu)} - r\right\} < 0 \Rightarrow r > \sqrt{(1-\mu)}$; $\left\{\sqrt{(1-\mu)} + r\right\} > 0 \Rightarrow r > -\sqrt{(1-\mu)}$

$r > \sqrt{(1-\mu)}$ as long as $\mu < 1$ the square root makes sense.

$$\frac{dr}{dt} < 0 \Rightarrow r \cdot (1 - r^2) + \mu \cdot r \cdot \cos\theta < 0;$$
$$\cos\theta \geq -1; \ \cos\theta = 1 \Rightarrow r \cdot (1 - r^2) + \mu \cdot r < 0$$
$$r \cdot (1 - r^2) + \mu \cdot r < 0 \Rightarrow r \cdot [1 - r^2 + \mu] < 0;$$
$$r > 0 \Rightarrow [1 - r^2 + \mu] < 0 \Rightarrow [(1 + \mu) - r^2] < 0$$

$[(1+\mu) - r^2] < 0 \Rightarrow \{\sqrt{(1+\mu)} - r\} \cdot \{\sqrt{(1+\mu)} + r\} < 0$; We have two cases:

Case I: $\{\sqrt{(1+\mu)} - r\} > 0 \Rightarrow r < \sqrt{(1+\mu)}$; $\{\sqrt{(1+\mu)} + r\} < 0 \Rightarrow r < -\sqrt{(1+\mu)}$

Then $r < -\sqrt{(1+\mu)}$.

Case II: $\{\sqrt{(1+\mu)} - r\} < 0 \Rightarrow r > \sqrt{(1+\mu)}$; $\{\sqrt{(1+\mu)} + r\} > 0 \Rightarrow r > -\sqrt{(1+\mu)}$

Then $r > \sqrt{(1+\mu)}$

We need to plot phase portrait for our Poincare–Bendixson system. We choose different μ parameter values and X_0, Y_0 initial values.

Our system differential equations are

$$\frac{dX}{dt} = -\mu \cdot Y \cdot \sqrt{1 - \left[\frac{X - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2} + X - 3 \cdot X \cdot Y^2 + \frac{X}{Y} \cdot [X - Y + Y^3]$$

$$\frac{dY}{dt} = X$$

$x(1) \rightarrow X$, $x(2) \rightarrow Y$, $a(\text{Mui}) \rightarrow \mu$

MATLAB script (Fig. 1.14):

```
function g=poincarebendixson (t,x,a)
g=zeros (2,1);
g(1)=-a.*x(2).*(1-((x(1)-x(2).*(1-
x(2).^2))./(a.*x(2))).^2).^0.5+x(1)-
3.*x(1).*x(2).^2+(x(1)./x(2)).*(x(1)-x(2)+x(2).^3);
g(2)=x(1);
```

```
function h=poincarebendixson1 (a, X0, Y0)
[t,x]=ODE45 (@poincarebendixson, [0,10] , [X0,Y0] , [] , a);
%plot(t,x);
plot(x(:,1),x(:,2)) % X against Y at time increase phase plan plot
```

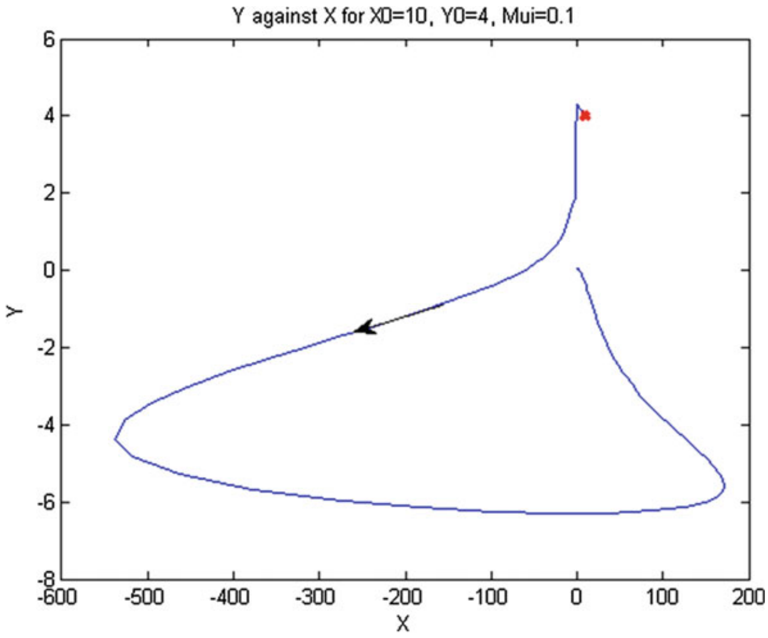


Fig. 1.14 Poincare–Bendixson system

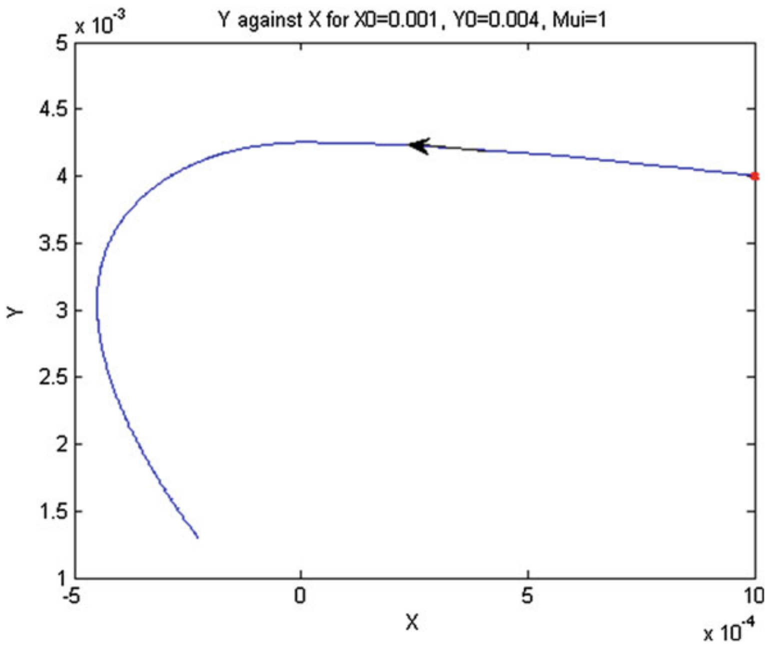


Fig. 1.14 (continued)

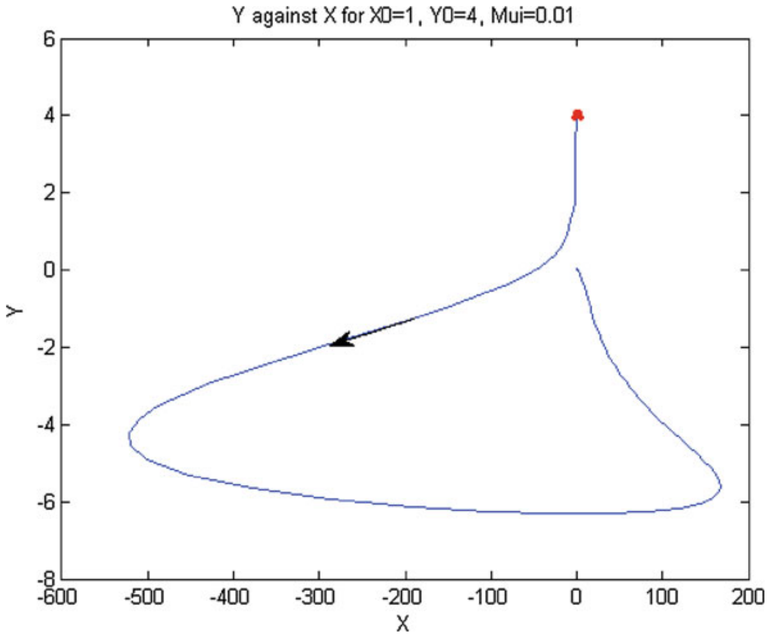


Fig. 1.14 (continued)

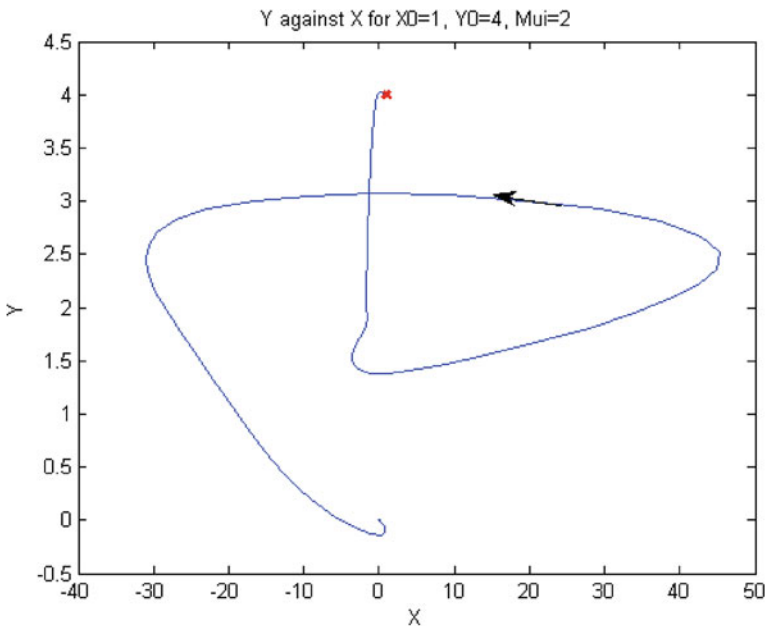


Fig. 1.14 (continued)

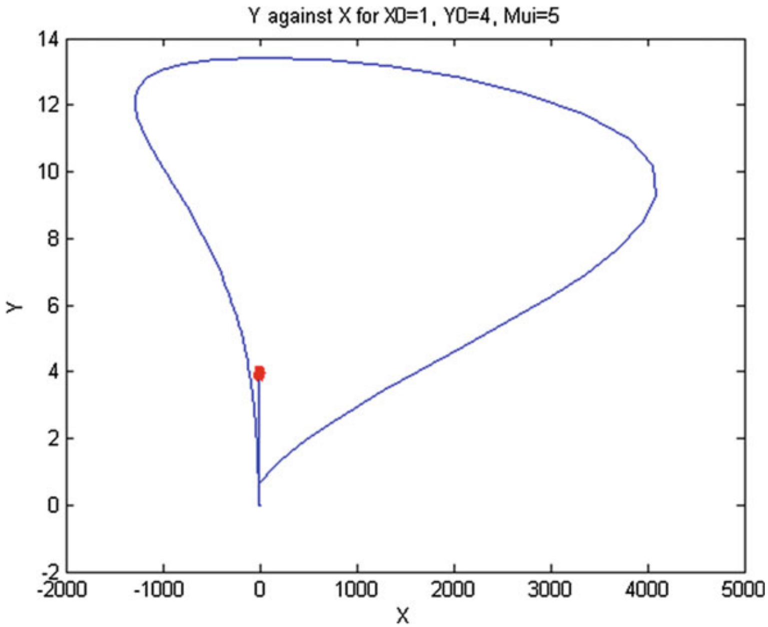


Fig. 1.14 (continued)

1.4 Optoisolation Circuits Poincare–Bendixson Analysis

We need to implement our Poincare–Bendixson system by using optoisolation circuits.

Our system differential equations are

$$\frac{dX}{dt} = -\mu \cdot Y \cdot \sqrt{1 - \left[\frac{X - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2} + X - 3 \cdot X \cdot Y^2 + \frac{X}{Y} \cdot \left[X - Y + Y^3\right]$$

$$\frac{dY}{dt} = X$$

We get by transformation $\frac{dY}{dt} \rightarrow X$ one system differential equation.

$$\frac{d^2Y}{dt^2} = -\mu \cdot Y \cdot \sqrt{1 - \left[\frac{\frac{dY}{dt} - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2} + \frac{dY}{dt} - 3 \cdot \frac{dY}{dt} \cdot Y^2 + \frac{dY}{dt} \cdot \frac{1}{Y} \cdot \left[\frac{dY}{dt} - Y + Y^3\right]$$

$$\frac{d^2Y}{dt^2} = -\mu \cdot Y \cdot \sqrt{1 - \left[\frac{\frac{dY}{dt} - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2} + \frac{dY}{dt} - 3 \cdot \frac{dY}{dt} \cdot Y^2 + \left[\frac{dY}{dt}\right]^2 \cdot \frac{1}{Y} - \frac{dY}{dt} + \frac{dY}{dt} \cdot Y^2$$

$$\frac{d^2Y}{dt^2} = -\mu \cdot Y \cdot \sqrt{1 - \left[\frac{\frac{dY}{dt} - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2} - 2 \cdot \frac{dY}{dt} \cdot Y^2 + \left[\frac{dY}{dt}\right]^2 \cdot \frac{1}{Y}$$

$$\frac{d^2Y}{dt^2} + 2 \cdot \frac{dY}{dt} \cdot Y^2 - \left[\frac{dY}{dt}\right]^2 \cdot \frac{1}{Y} = -\mu \cdot Y \cdot \sqrt{1 - \left[\frac{\frac{dY}{dt} - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2}$$

$$1 - \left[\frac{\frac{dY}{dt} - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2 = 1 - \frac{\left[\frac{dY}{dt} - Y \cdot (1 - Y^2)\right]^2}{[\mu \cdot Y]^2} = \frac{[\mu \cdot Y]^2 - \left[\frac{dY}{dt} - Y \cdot (1 - Y^2)\right]^2}{[\mu \cdot Y]^2}$$

$$\sqrt{1 - \left[\frac{\frac{dY}{dt} - Y \cdot (1 - Y^2)}{\mu \cdot Y}\right]^2} = \sqrt{\frac{[\mu \cdot Y]^2 - \left[\frac{dY}{dt} - Y \cdot (1 - Y^2)\right]^2}{[\mu \cdot Y]^2}} = \frac{1}{\mu \cdot Y} \cdot \sqrt{[\mu \cdot Y]^2 - \left[\frac{dY}{dt} - Y \cdot (1 - Y^2)\right]^2}$$

Back to the main equation:

$$\frac{d^2Y}{dt^2} + 2 \cdot \frac{dY}{dt} \cdot Y^2 - \left[\frac{dY}{dt}\right]^2 \cdot \frac{1}{Y} = -\sqrt{[\mu \cdot Y]^2 - \left[\frac{dY}{dt} - Y \cdot (1 - Y^2)\right]^2}$$

$$\left\{\frac{d^2Y}{dt^2} + 2 \cdot \frac{dY}{dt} \cdot Y^2 - \left[\frac{dY}{dt}\right]^2 \cdot \frac{1}{Y}\right\}^2 = [\mu \cdot Y]^2 - \left[\frac{dY}{dt} - Y \cdot (1 - Y^2)\right]^2$$

We use the formula: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2 \cdot a \cdot b + 2 \cdot a \cdot c + 2 \cdot b \cdot c$

$$\left[\frac{d^2Y}{dt^2}\right]^2 + 4\cdot\left[\frac{dY}{dt}\right]^2\cdot Y^4 + \left[\frac{dY}{dt}\right]^4\cdot\frac{1}{Y^2} + 4\cdot\frac{d^2Y}{dt^2}\cdot\frac{dY}{dt}\cdot Y^2$$
$$-2\cdot\frac{d^2Y}{dt^2}\cdot\left[\frac{dY}{dt}\right]^2\cdot\frac{1}{Y} - 4\cdot\frac{dY}{dt}\cdot Y\cdot\left[\frac{dY}{dt}\right]^2$$
$$= [\mu\cdot Y]^2 - \left[\frac{dY}{dt}\right]^2 + 2\cdot\frac{dY}{dt}\cdot Y\cdot(1-Y^2) - Y^2\cdot(1-Y^2)^2$$

Different terminology: $\frac{d^2Y}{dt^2}\to\ddot{Y};\ \frac{dY}{dt}\to\dot{Y};\ Y\to Y$

$$[\ddot{Y}]^2 + 4\cdot[\dot{Y}]^2\cdot Y^4 + [\dot{Y}]^4\cdot\frac{1}{Y^2} + 4\cdot\ddot{Y}\cdot\dot{Y}\cdot Y^2$$
$$-2\cdot\ddot{Y}\cdot[\dot{Y}]^2\cdot\frac{1}{Y} - 4\cdot\dot{Y}\cdot Y\cdot[\dot{Y}]^2$$
$$= [\mu\cdot Y]^2 - [\dot{Y}]^2 + 2\cdot\dot{Y}\cdot Y\cdot(1-Y^2) - Y^2\cdot(1-Y^2)^2$$

$$[\ddot{Y}]^2 + 4\cdot[\dot{Y}]^2\cdot Y^4 + [\dot{Y}]^4\cdot\frac{1}{Y^2} + 4\cdot\ddot{Y}\cdot\dot{Y}\cdot Y^2$$
$$-2\cdot\ddot{Y}\cdot[\dot{Y}]^2\cdot\frac{1}{Y} - 4\cdot Y\cdot[\dot{Y}]^3$$
$$= [\mu\cdot Y]^2 - [\dot{Y}]^2 + 2\cdot\dot{Y}\cdot Y - 2\cdot\dot{Y}\cdot Y^3$$
$$-Y^2\cdot(1-2\cdot Y^2+Y^4)$$

$$[\ddot{Y}]^2 + 4\cdot[\dot{Y}]^2\cdot Y^4 + [\dot{Y}]^4\cdot\frac{1}{Y^2}$$
$$+4\cdot\ddot{Y}\cdot\dot{Y}\cdot Y^2 - 2\cdot\ddot{Y}\cdot[\dot{Y}]^2\cdot\frac{1}{Y} - 4\cdot Y\cdot[\dot{Y}]^3$$
$$= [\mu\cdot Y]^2 - [\dot{Y}]^2 + 2\cdot\dot{Y}\cdot Y - 2\cdot\dot{Y}\cdot Y^3 - Y^2 + 2\cdot Y^4 - Y^6$$

Final equation:

$$[\ddot{Y}]^2 + 4\cdot[\dot{Y}]^2\cdot Y^4 + [\dot{Y}]^4\cdot\frac{1}{Y^2}$$
$$+4\cdot\ddot{Y}\cdot\dot{Y}\cdot Y^2 - 2\cdot\ddot{Y}\cdot[\dot{Y}]^2\cdot\frac{1}{Y} - 4\cdot Y\cdot[\dot{Y}]^3$$
$$-[\mu\cdot Y]^2 + [\dot{Y}]^2 - 2\cdot\dot{Y}\cdot Y$$
$$+2\cdot\dot{Y}\cdot Y^3 + Y^2 - 2\cdot Y^4 + Y^6 = 0$$

We represent above equation by multiplication of two functions:

$$\prod_{i=1}^{2} \xi_i([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = \xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) \cdot \xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0 \quad \forall\, k, l, m \in 0, 1, 2, 3, \ldots$$

The solution can be for two main cases:

$\xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0$ or $\xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0$ at least.
We try to implement the first equation ξ_1 by using optoisolation circuits.

The optoisolation circuit includes one optocoupler, two capacitors (C1 is initially charged to specific voltage and C2 can be initially charged or not), switch and resistors. The circuit has voltage double element which doubles the voltage in his input. We consider, voltage double element's series resistance is infinite (current which flow through the voltage double element is very small). We define: $R_{\text{doubler}} \to \infty$; $I_{\text{doubler}} \to \varepsilon$. At $t = 0$, we move switch S1 from OFF to ON state, and need to investigate the circuit dynamics [8, 9] (Fig. 1.15).

$$(*)\; C_1 \cdot \frac{dV_1}{dt} = I_{CQ1}(V_2) + C_2 \cdot \frac{dV_2}{dt};\; I_{R1} = I_{C1};\; V_2 = V_{CEQ1};$$
$$I_{R1} = I_{C1} = I_{CQ1} + I_{C2} + I_{\text{doubler}}$$

$$I_{doubler} \to \varepsilon \Rightarrow I_{R1} = I_{C1} = I_{CQ1} + I_{C2} + \varepsilon \Rightarrow I_{R1} = I_{C1} \approx I_{CQ1} + I_{C2}$$

$$V_2^2 = I_{D1} \cdot R_2 + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right);\; \text{Taylor Approx} \Rightarrow \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx \frac{I_{D1}}{I_0}$$

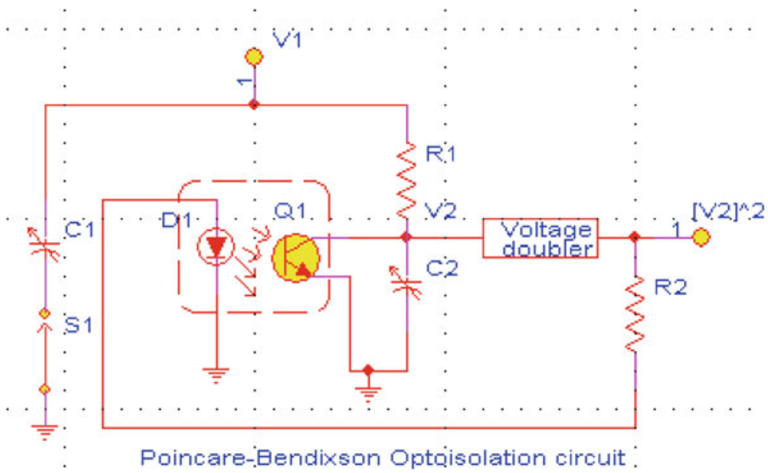


Fig. 1.15 Poincare–Bendixson optoisolation circuit

$$V_2^2 = I_{D1} \cdot R_2 + V_t \cdot \frac{I_{D1}}{I_0} \Rightarrow I_{D1} = \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2;$$

$$I_{BQ1} = k \cdot I_{D1} = k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2$$

We consider the optical coupling between LED D1 and phototransistor $Q1$ as $Q1$'s basis dependent current source ($I_{BQ1} = k \cdot I_{D1}$). k is the coupling coefficient.

$$I_{R1} = I_{C1} = C_1 \cdot \frac{dV_1}{dt};$$

$$V_1 = V_2 + I_{R1} \cdot R_1 = V_2 + R_1 \cdot C_1 \cdot \frac{dV_1}{dt} \Rightarrow V_2 = V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}$$

$$\frac{dV_2}{dt} = \frac{d}{dt}\left\{V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}\right\} \Rightarrow \frac{dV_2}{dt} = \frac{dV_1}{dt} - R_1 \cdot C_1 \cdot \frac{d^2V_1}{dt^2}$$

$$\frac{dV_2}{dt} \Rightarrow (*) \Rightarrow C_1 \cdot \frac{dV_1}{dt} = I_{CQ1}(V_2) + C_2 \cdot \left\{\frac{dV_1}{dt} - R_1 \cdot C_1 \cdot \frac{d^2V_1}{dt^2}\right\}$$

$$C_1 \cdot \frac{dV_1}{dt} = I_{CQ1}(V_2) + C_2 \cdot \frac{dV_1}{dt} - R_1 \cdot C_1 \cdot C_2 \cdot \frac{d^2V_1}{dt^2} \Rightarrow R_1 \cdot C_1 \cdot C_2 \cdot \frac{d^2V_1}{dt^2} = I_{CQ1}(V_2) + (C_2 - C_1) \cdot \frac{dV_1}{dt}$$

Using BJT transistor Ebers–Moll equations:

$$V_2 = V_{CEQ1} = V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$\frac{I_{sc}}{I_{se}} \approx 1 \Rightarrow \ln\left[\frac{I_{sc}}{I_{se}}\right] \to \varepsilon \Rightarrow V_2 = V_{CEQ1} \approx V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$(**)\ V_2 = V_{CEQ1} \approx V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1} = k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{CQ1}$$

$$\begin{aligned}\alpha_r \cdot I_{CQ1} - I_{EQ1} &= \alpha_r \cdot I_{CQ1} - k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 - I_{CQ1} \\ &= I_{CQ1} \cdot (\alpha_r - 1) - k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2\end{aligned}$$

$$\begin{aligned}I_{CQ1} - I_{EQ1} \cdot \alpha_f &= I_{CQ1} - \left\{k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{CQ1}\right\} \cdot \alpha_f \\ &= I_{CQ1} \cdot (1 - \alpha_f) - \frac{k \cdot \alpha_f}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2\end{aligned}$$

$$(**) \Rightarrow e^{\left[\frac{V_2}{V_t}\right]} = e^{\left[\frac{V_{CEQ1}}{V_t}\right]} = \frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)};\ e^{\left[\frac{V_2}{V_t}\right]} \approx 1 + \frac{V_2}{V_t}$$

$$1 + \frac{V_2}{V_t} = \frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

$$1 + \frac{V_2}{V_t} = \frac{I_{CQ1} \cdot (\alpha_r - 1) - k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{I_{CQ1} \cdot (1 - \alpha_f) - k \cdot \frac{\alpha_f}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

$$\begin{aligned}&I_{CQ1} \cdot (1 - \alpha_f) - k \cdot \frac{\alpha_f}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \\ &\quad + \frac{V_2}{V_t} \cdot I_{CQ1} \cdot (1 - \alpha_f) - \frac{V_2}{V_t} \cdot k \cdot \frac{\alpha_f}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + \frac{V_2}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \\ &= I_{CQ1} \cdot (\alpha_r - 1) - k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)\end{aligned}$$

$$I_{CQ1} \cdot (1-\alpha_f) \cdot \left(1+\frac{V_2}{V_t}\right) + \frac{V_2}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)$$
$$- k \cdot \frac{\alpha_f}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 - \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^3$$
$$+ I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) = I_{CQ1} \cdot (\alpha_r - 1)$$
$$- k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$I_{CQ1} \cdot \left\{(1-\alpha_f) \cdot (1+\frac{V_2}{V_t}) - (\alpha_r - 1)\right\}$$
$$= \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^3 + k \cdot \frac{\alpha_f}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2 - k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2$$
$$- \frac{V_2}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + (I_{se} - I_{sc}) \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$I_{CQ1} \cdot \left\{(1-\alpha_f) \cdot \left(1+\frac{V_2}{V_t}\right) - (\alpha_r - 1)\right\} = \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^3 + k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot V_2^2$$
$$- \frac{V_2}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + (I_{se} - I_{sc}) \cdot (\alpha_r \cdot \alpha_f - 1)$$

For simplicity, we define the following:

$$A_1 = \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]};\ A_2 = k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]};\ A_3 = -\frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)$$
$$A_4 = (I_{se} - I_{sc}) \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$I_{CQ1} \cdot \left\{(1-\alpha_f) \cdot \left(1+\frac{V_2}{V_t}\right) - (\alpha_r - 1)\right\} = A_1 \cdot V_2^3 + A_2 \cdot V_2^2 + A_3 \cdot V_2 + A_4$$

$$I_{CQ1}(V_2) = \frac{A_1 \cdot V_2^3 + A_2 \cdot V_2^2 + A_3 \cdot V_2 + A_4}{(1-\alpha_f) \cdot \left(1+\frac{V_2}{V_t}\right) - (\alpha_r - 1)} = \frac{\sum_{i=1}^{4} A_i \cdot V_2^{4-i}}{(1-\alpha_f) \cdot \left(1+\frac{V_2}{V_t}\right) - (\alpha_r - 1)}$$

$$R_1 \cdot C_1 \cdot C_2 \cdot \frac{\mathrm{d}^2 V_1}{\mathrm{d}t^2} = \frac{\sum_{i=1}^{4} A_i \cdot V_2^{4-i}}{(1-\alpha_f) \cdot \left(1+\frac{V_2}{V_t}\right) - (\alpha_r - 1)} + (C_2 - C_1) \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t};$$
$$V_2 = V_1 - R_1 \cdot C_1 \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t}$$

$$R_1 \cdot C_1 \cdot C_2 \cdot \frac{d^2 V_1}{dt^2} = \frac{\sum_{i=1}^{4} A_i \cdot \left[V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}\right]^{4-i}}{(1-\alpha_f) \cdot \left(1 + \frac{V_2}{V_t}\right) - (\alpha_r - 1)} + (C_2 - C_1) \cdot \frac{dV_1}{dt}$$

$$R_1 \cdot C_1 \cdot C_2 \cdot \frac{d^2 V_1}{dt^2} = \frac{\sum_{i=1}^{4} A_i \cdot \left[V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}\right]^{4-i}}{(1-\alpha_f) + (1-\alpha_f) \cdot \frac{V_2}{V_t} - (\alpha_r - 1)} + (C_2 - C_1) \cdot \frac{dV_1}{dt}$$

$$m(V_2) = (1-\alpha_f) - (\alpha_r - 1) + (1-\alpha_f) \cdot \frac{V_2}{V_t} = 2 - \alpha_f - \alpha_r + (1-\alpha_f) \cdot \frac{V_2}{V_t}$$

$$V_2 = V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt} \Rightarrow m(V_2) = 2 - \alpha_f - \alpha_r + (1-\alpha_f) \cdot \frac{1}{V_t} \cdot \left\{V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}\right\}$$

$$m\left(V_1, \frac{dV_1}{dt}\right) = 2 - \alpha_f - \alpha_r + \frac{(1-\alpha_f)}{V_t} \cdot V_1 - \frac{(1-\alpha_f) \cdot R_1 \cdot C_1}{V_t} \cdot \frac{dV_1}{dt}$$

We define $B_1 = 2 - \alpha_f - \alpha_r;\ B_2 = \frac{(1-\alpha_f)}{V_t};\ B_3 = -\frac{(1-\alpha_f) \cdot R_1 \cdot C_1}{V_t}$

$$m\left(V_1, \frac{dV_1}{dt}\right) = B_1 + B_2 \cdot V_1 + B_3 \cdot \frac{dV_1}{dt}$$

$$R_1 \cdot C_1 \cdot C_2 \cdot \frac{d^2 V_1}{dt^2} = \frac{\sum_{i=1}^{4} A_i \cdot \left[V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}\right]^{4-i}}{B_1 + B_2 \cdot V_1 + B_3 \cdot \frac{dV_1}{dt}} + (C_2 - C_1) \cdot \frac{dV_1}{dt};$$

$$B_4 = R_1 \cdot C_1 \cdot C_2;\ B_5 = C_2 - C_1$$

$$B_4 \cdot \frac{d^2 V_1}{dt^2} = \frac{\sum_{i=1}^{4} A_i \cdot \left[V_1 - R_1 \cdot C_1 \cdot \frac{dV_1}{dt}\right]^{4-i}}{B_1 + B_2 \cdot V_1 + B_3 \cdot \frac{dV_1}{dt}} + B_5 \cdot \frac{dV_1}{dt}$$

We define $\frac{d^2 V_1}{dt^2} = \ddot{Y};\ \frac{dV_1}{dt} = \dot{Y};\ V_1 = Y;\ B_4 \cdot \ddot{Y} = \frac{\sum_{i=1}^{4} A_i \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}]^{4-i}}{B_1 + B_2 \cdot Y + B_3 \cdot \dot{Y}} + B_5 \cdot \dot{Y}$

$$B_4 \cdot \ddot{Y} \cdot \left\{B_1 + B_2 \cdot Y + B_3 \cdot \dot{Y}\right\} = \sum_{i=1}^{4} A_i \cdot \left[Y - R_1 \cdot C_1 \cdot \dot{Y}\right]^{4-i} + B_5 \cdot \dot{Y} \cdot \left\{B_1 + B_2 \cdot Y + B_3 \cdot \dot{Y}\right\}$$

$$B_4 \cdot \ddot{Y} \cdot \left\{B_1 + B_2 \cdot Y + B_3 \cdot \dot{Y}\right\} - B_5 \cdot \dot{Y} \cdot \left\{B_1 + B_2 \cdot Y + B_3 \cdot \dot{Y}\right\} = \sum_{i=1}^{4} A_i \cdot \left[Y - R_1 \cdot C_1 \cdot \dot{Y}\right]^{4-i}$$

$$B_4 \cdot B_1 \cdot \ddot{Y} + B_4 \cdot B_2 \cdot Y \cdot \ddot{Y} + B_3 \cdot B_4 \cdot \ddot{Y} \cdot \dot{Y} - B_5 \cdot B_1 \cdot \dot{Y} - B_5 \cdot B_2 \cdot Y \cdot \dot{Y} - B_5 \cdot B_3 \cdot [\dot{Y}]^2$$
$$= \sum_{i=1}^{4} A_i \cdot \left[Y - R_1 \cdot C_1 \cdot \dot{Y}\right]^{4-i}$$

$$\sum_{i=1}^{4} A_i \cdot \left[Y - R_1 \cdot C_1 \cdot \dot{Y}\right]^{4-i} = A_1 \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}]^3 + A_2 \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}]^2$$
$$+ A_3 \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}] + A_4$$

$$B_4 \cdot B_1 \cdot \ddot{Y} + B_4 \cdot B_2 \cdot Y \cdot \ddot{Y} + B_3 \cdot B_4 \cdot \ddot{Y} \cdot \dot{Y} - B_5 \cdot B_1 \cdot \dot{Y} - B_5 \cdot B_2 \cdot Y \cdot \dot{Y} - B_5 \cdot B_3 \cdot [\dot{Y}]^2$$
$$= A_1 \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}]^3 + A_2 \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}]^2 + A_3 \cdot [Y - R_1 \cdot C_1 \cdot \dot{Y}] + A_4$$

We use the formula: $(a - b)^3 = a^3 - 3 \cdot a^2 \cdot b + 3 \cdot a \cdot b^2 - b^3$

$$B_4 \cdot B_1 \cdot \ddot{Y} + B_4 \cdot B_2 \cdot Y \cdot \ddot{Y} + B_3 \cdot B_4 \cdot \ddot{Y} \cdot \dot{Y} - B_5 \cdot B_1 \cdot \dot{Y} - B_5 \cdot B_2 \cdot Y \cdot \dot{Y} - B_5 \cdot B_3 \cdot [\dot{Y}]^2$$
$$- A_1 \cdot Y^3 + 3 \cdot A_1 \cdot R_1 \cdot C_1 \cdot \dot{Y} \cdot Y^2 - 3 \cdot A_1 \cdot R_1^2 \cdot C_1^2 \cdot Y \cdot [\dot{Y}]^2 + A_1 \cdot R_1^3 \cdot C_1^3 \cdot [\dot{Y}]^3$$
$$- A_2 \cdot Y^2 + 2 \cdot A_2 \cdot R_1 \cdot C_1 \cdot Y \cdot \dot{Y} - A_2 \cdot R_1^2 \cdot C_1^2 \cdot [\dot{Y}]^2 - A_3 \cdot Y + A_3 \cdot R_1 \cdot C_1 \cdot \dot{Y} - A_4 = 0$$

$$B_4 \cdot B_1 \cdot \ddot{Y} + B_4 \cdot B_2 \cdot Y \cdot \ddot{Y} + B_3 \cdot B_4 \cdot \ddot{Y} \cdot \dot{Y} + \dot{Y} \cdot (A_3 \cdot R_1 \cdot C_1 - B_5 \cdot B_1)$$
$$+ Y \cdot \dot{Y} \cdot (2 \cdot A_2 \cdot R_1 \cdot C_1 - B_5 \cdot B_2) - [\dot{Y}]^2 \cdot (B_5 \cdot B_3 + A_2 \cdot R_1^2 \cdot C_1^2)$$
$$- A_1 \cdot Y^3 + 3 \cdot A_1 \cdot R_1 \cdot C_1 \cdot \dot{Y} \cdot Y^2 - 3 \cdot A_1 \cdot R_1^2 \cdot C_1^2 \cdot Y \cdot [\dot{Y}]^2$$
$$+ A_1 \cdot R_1^3 \cdot C_1^3 \cdot [\dot{Y}]^3 - A_2 \cdot Y^2 - A_3 \cdot Y - A_4 = 0$$

Back to our Poincare–Bendixson final equation:

$$[\ddot{Y}]^2 + 4 \cdot [\dot{Y}]^2 \cdot Y^4 + [\dot{Y}]^4 \cdot \frac{1}{Y^2} + 4 \cdot \ddot{Y} \cdot \dot{Y} \cdot Y^2 - 2 \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y} - 4 \cdot Y \cdot [\dot{Y}]^3$$
$$- [\mu \cdot Y]^2 + [\dot{Y}]^2 - 2 \cdot \dot{Y} \cdot Y + 2 \cdot \dot{Y} \cdot Y^3 + Y^2 - 2 \cdot Y^4 + Y^6 = 0$$

We represent above equation by multiplication of two functions:

$$\prod_{i=1}^{2} \xi_i([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = \xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m)$$
$$\cdot \xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0 \quad \forall\, k, l, m \in 0, 1, 2, 3, \ldots..$$

We define ξ_1 as our Poincare–Bendixson optoisolation equation [6, 7].

$$\xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m, \alpha_i \text{ [i = 1,2,..., 13]}) = \alpha_1 \cdot \ddot{Y} + \alpha_2 \cdot Y \cdot \ddot{Y} + \alpha_3 \cdot \ddot{Y} \cdot \dot{Y} + \alpha_4 \cdot \dot{Y} + \alpha_5 \cdot Y \cdot \dot{Y} + \alpha_6 \cdot [\dot{Y}]^2 + \alpha_7 \cdot Y^3 + \alpha_8 \dot{Y} \cdot Y^2 + \alpha_9 \cdot Y \cdot [\dot{Y}]^2 + \alpha_{10} \cdot [\dot{Y}]^3 + \alpha_{11} \cdot Y^2 + \alpha_{12} \cdot Y + \alpha_{13} = 0$$

$$\begin{aligned}\xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = {} & B_4 \cdot B_1 \cdot \ddot{Y} + B_4 \cdot B_2 \cdot Y \cdot \ddot{Y} \\ & + B_3 \cdot B_4 \cdot \ddot{Y} \cdot \dot{Y} + \dot{Y} \cdot (A_3 \cdot R_1 \cdot C_1 - B_5 \cdot B_1) \\ & + Y \cdot \dot{Y} \cdot (2 \cdot A_2 \cdot R_1 \cdot C_1 - B_5 \cdot B_2) \\ & - [\dot{Y}]^2 \cdot (B_5 \cdot B_3 + A_2 \cdot R_1^2 \cdot C_1^2) - A_1 \cdot Y^3 \\ & + 3 \cdot A_1 \cdot R_1 \cdot C_1 \cdot \dot{Y} \cdot Y^2 \\ & - 3 \cdot A_1 \cdot R_1^2 \cdot C_1^2 \cdot Y \cdot [\dot{Y}]^2 + A_1 \cdot R_1^3 \cdot C_1^3 \cdot [\dot{Y}]^3 \\ & - A_2 \cdot Y^2 - A_3 \cdot Y - A_4 = 0\end{aligned}$$

Equal parameters expressions between two above equations:

$$\begin{aligned}&\alpha_1 = B_1 \cdot B_4;\ \alpha_2 = B_2 \cdot B_4; \alpha_3 = B_3 \cdot B_4; \\ &\alpha_4 = (A_3 \cdot R_1 \cdot C_1 - B_5 \cdot B_1);\ \alpha_5 = 2 \cdot A_2 \cdot R_1 \cdot C_1 - B_5 \cdot B_2 \\ &\alpha_6 = -(B_5 \cdot B_3 + A_2 \cdot R_1^2 \cdot C_1^2);\ \alpha_7 = -A_1;\ \alpha_8 = 3 \cdot A_1 \cdot R_1 \cdot C_1; \\ &\alpha_9 = -3 \cdot A_1 \cdot R_1^2 \cdot C_1^2;\ \alpha_{10} = A_1 \cdot R_1^3 \cdot C_1^3; \\ &\alpha_{11} = -A_2;\ \alpha_{12} = -A_3; \alpha_{13} = -A_4\end{aligned}$$

We define $\xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m, \beta_i \text{ [i = 1,2,..., 6]})$

$$\xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m, \beta_i \text{ [i = 1,2,..., 6]}) = \beta_1 \cdot \ddot{Y} + \beta_2 \cdot Y^3 + \beta_3 \cdot \frac{\dot{Y}}{Y^2} + \beta_4 \cdot \dot{Y} + \beta_5 \cdot \frac{\dot{Y}}{Y} + \beta_6$$

$$\begin{aligned}\prod_{i=1}^{2} \xi_i\Big([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\Big) &= \xi_1\Big([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\Big) \cdot \xi_2\Big([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\Big) \\ &= \big\{\alpha_1 \cdot \ddot{Y} + \alpha_2 \cdot Y \cdot \ddot{Y} + \alpha_3 \cdot \ddot{Y} \cdot \dot{Y} + \alpha_4 \cdot \dot{Y} \\ &\quad + \alpha_5 \cdot Y \cdot \dot{Y} + \alpha_6 \cdot [\dot{Y}]^2 \\ &\quad + \alpha_7 \cdot Y^3 + \alpha_8 \dot{Y} \cdot Y^2 + \alpha_9 \cdot Y \cdot [\dot{Y}]^2 \\ &\quad + \alpha_{10} \cdot [\dot{Y}]^3 + \alpha_{11} \cdot Y^2 + \alpha_{12} \cdot Y + \alpha_{13}\big\} \\ &\quad \cdot \left\{\beta_1 \cdot \ddot{Y} + \beta_2 \cdot Y^3 + \beta_3 \cdot \frac{\dot{Y}}{Y^2} + \beta_4 \cdot \dot{Y} + \beta_5 \cdot \frac{\dot{Y}}{Y} + \beta_6\right\} \\ &= 0 \quad \forall\, k, l, m \in 0, 1, 2, 3, \ldots\end{aligned}$$

The final multiplication of ξ_1, ξ_2:

$$
\begin{aligned}
\xi_1 \cdot \xi_2 = {} & \alpha_1 \cdot \beta_1 \cdot [\ddot{Y}]^2 + \ddot{Y} \cdot Y^3 \cdot (\alpha_1 \cdot \beta_2 + \beta_1 \cdot \alpha_7) + \alpha_1 \cdot \beta_3 \cdot \ddot{Y} \cdot \frac{\dot{Y}}{Y^2} \\
& + Y \cdot \ddot{Y} \cdot (\alpha_1 \cdot \beta_4 + \beta_6 \cdot \alpha_2 + \alpha_{12} \cdot \beta_1) \\
& + \ddot{Y} \cdot \dot{Y} \cdot \frac{1}{Y} \cdot (\alpha_1 \cdot \beta_5 + \beta_3 \cdot \alpha_2) + \ddot{Y} \cdot (\alpha_1 \cdot \beta_6 + \alpha_{13} \cdot \beta_1) \\
& + [\ddot{Y}]^2 \cdot Y \cdot \alpha_2 \cdot \beta_1 + Y^4 \cdot \ddot{Y} \cdot \alpha_2 \cdot \beta_2 \\
& + \ddot{Y} \cdot Y^2 \cdot (\beta_4 \cdot \alpha_2 + \alpha_{11} \cdot \beta_1) + \ddot{Y} \cdot \dot{Y} \cdot (\beta_5 \cdot \alpha_2 + \alpha_4 \cdot \beta_1) \\
& + \alpha_3 \cdot \beta_1 \cdot [\ddot{Y}]^2 \cdot \dot{Y} + \alpha_3 \cdot \beta_2 \cdot \ddot{Y} \cdot \dot{Y} \cdot Y^3 \\
& + \alpha_3 \cdot \beta_3 \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y^2} + \ddot{Y} \cdot \dot{Y} \cdot Y \cdot (\alpha_3 \cdot \beta_4 + \alpha_5 \cdot \beta_1) + \alpha_3 \cdot \beta_5 \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y} \\
& + \alpha_3 \cdot \beta_6 \cdot \ddot{Y} \cdot \dot{Y} + \dot{Y} \cdot Y^3 \cdot (\alpha_4 \cdot \beta_2 + \alpha_8 \cdot \beta_4) + \alpha_4 \cdot \beta_3 \cdot [\dot{Y}]^2 \cdot \frac{1}{Y^2} \\
& + \dot{Y} \cdot Y \cdot (\beta_4 \cdot \alpha_4 + \alpha_7 \cdot \beta_3 + \alpha_5 \cdot \beta_6 + \alpha_{11} \cdot \beta_5) \\
& + \frac{[\dot{Y}]^2}{Y} \cdot (\beta_5 \cdot \alpha_4 + \alpha_5 \cdot \beta_3) + \dot{Y} \cdot (\beta_6 \cdot \alpha_4 + \alpha_{11} \cdot \beta_3 + \alpha_{12} \cdot \beta_5) \\
& + \alpha_5 \cdot \beta_2 \cdot Y^4 \cdot \dot{Y} + \dot{Y} \cdot Y^2 \cdot (\alpha_5 \cdot \beta_4 + \alpha_7 \cdot \beta_5 + \alpha_8 \cdot \beta_6) \\
& + [\dot{Y}]^2 \cdot (\beta_5 \cdot \alpha_5 + \beta_6 \cdot \alpha_6 + \alpha_8 \cdot \beta_3) \\
& + \alpha_6 \cdot \beta_1 \cdot [\dot{Y}]^2 \cdot \ddot{Y} + \alpha_6 \cdot \beta_2 \cdot [\dot{Y}]^2 \cdot Y^3 + \alpha_6 \cdot \beta_3 \cdot [\dot{Y}]^3 \cdot \frac{1}{Y^2} \\
& + [\dot{Y}]^2 \cdot Y \cdot (\beta_4 \cdot \alpha_6 + \beta_5 \cdot \alpha_8 + \alpha_9 \cdot \beta_6) + \frac{[\dot{Y}]^3}{Y} \cdot (\beta_5 \cdot \alpha_6 + \beta_3 \cdot \alpha_9) \\
& + \beta_2 \cdot \alpha_7 \cdot Y^6 + Y^4 \cdot (\beta_4 \cdot \alpha_7 + \beta_2 \cdot \alpha_{12}) \\
& + Y^3 \cdot (\beta_6 \cdot \alpha_7 + \beta_4 \cdot \alpha_{11} + \alpha_{13} \cdot \beta_2) \\
& + \alpha_8 \cdot \beta_1 \cdot Y^2 \cdot \dot{Y} \cdot \ddot{Y} + \alpha_8 \cdot \beta_2 \cdot \dot{Y} \cdot Y^5 \\
& + \beta_1 \cdot \alpha_9 \cdot Y \cdot [\dot{Y}]^2 \cdot \ddot{Y} \alpha_9 \cdot \beta_2 \cdot Y^4 \cdot [\dot{Y}]^2 \\
& + \alpha_9 \cdot \beta_4 \cdot Y^2 \cdot [\dot{Y}]^2 + \alpha_9 \cdot \beta_5 \cdot [\dot{Y}]^3 \\
& + \alpha_{10} \cdot \beta_1 \cdot [\dot{Y}]^3 \cdot \ddot{Y} + \alpha_{10} \cdot \beta_2 \cdot [\dot{Y}]^3 \cdot Y^3 \\
& + \alpha_{10} \cdot \beta_3 \cdot [\dot{Y}]^4 \cdot \frac{1}{Y^2} + \alpha_{10} \cdot \beta_4 \cdot Y \cdot [\dot{Y}]^3 + \frac{[\dot{Y}]^4}{Y} \cdot \beta_5 \cdot \alpha_{10} \\
& + \beta_6 \cdot \alpha_{10} \cdot [\dot{Y}]^3 + \alpha_{11} \cdot \beta_2 \cdot Y^5 Y^2 \cdot (\alpha_{11} \cdot \beta_6 + \alpha_{12} \cdot \beta_4) \\
& + \frac{\dot{Y}}{Y} \cdot (\alpha_{12} \cdot \beta_3 + \beta_5 \cdot \alpha_{13}) + Y \cdot (\alpha_{12} \cdot \beta_6 + \beta_4 \cdot \alpha_{13}) \\
& + \beta_3 \cdot \alpha_{13} \cdot \frac{\dot{Y}}{Y^2} + \alpha_{13} \cdot \beta_6 = 0
\end{aligned}
$$

We can separate the above multiplication result into two additive functions.

$$\prod_{i=1}^{2} \xi_i\left([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\right) = \xi_1\left([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\right) \cdot \xi_2\left([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\right)$$
$$= \Gamma_1\left([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\right) + \Gamma_x\left([\ddot{Y}]^k, [\dot{Y}]^l, Y^m\right) = 0$$

We define the above multiplication result as a summation of Poincare–Bendixson final equation Γ_1 and additional function Γ_x.

$$\Gamma_1 = [\ddot{Y}]^2 + 4 \cdot [\dot{Y}]^2 \cdot Y^4 + [\dot{Y}]^4 \cdot \frac{1}{Y^2} + 4 \cdot \ddot{Y} \cdot \dot{Y} \cdot Y^2 - 2 \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y} - 4 \cdot Y \cdot [\dot{Y}]^3$$
$$- [\mu \cdot Y]^2 + [\dot{Y}]^2 - 2 \cdot \dot{Y} \cdot Y + 2 \cdot \dot{Y} \cdot Y^3 + Y^2 - 2 \cdot Y^4 + Y^6 = 0$$

We need to find Γ_1 ($\Gamma_1 = 0$) and Γ_x ($\Gamma_x = 0$) in terms of optoisolation circuit parameters.

$$\Gamma_1 = \alpha_1 \cdot \beta_1 \cdot [\ddot{Y}]^2 + \alpha_3 \cdot \beta_5 \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y} + \dot{Y} \cdot Y^3 \cdot (\alpha_4 \cdot \beta_2 + \alpha_8 \cdot \beta_4)$$
$$+ \dot{Y} \cdot Y \cdot (\beta_4 \cdot \alpha_4 + \alpha_7 \cdot \beta_3 + \alpha_5 \cdot \beta_6 + \alpha_{11} \cdot \beta_5)$$
$$+ [\dot{Y}]^2 \cdot (\beta_5 \cdot \alpha_5 + \beta_6 \cdot \alpha_6 + \alpha_8 \cdot \beta_3)$$
$$+ \beta_2 \cdot \alpha_7 \cdot Y^6 + Y^4 \cdot (\beta_4 \cdot \alpha_7 + \beta_2 \cdot \alpha_{12}) + \alpha_8 \cdot \beta_1 \cdot Y^2 \cdot \dot{Y} \cdot \ddot{Y}$$
$$+ \alpha_9 \cdot \beta_2 \cdot Y^4 \cdot [\dot{Y}]^2$$
$$+ \alpha_{10} \cdot \beta_3 \cdot [\dot{Y}]^4 \cdot \frac{1}{Y^2} + \alpha_{10} \cdot \beta_4 \cdot Y \cdot [\dot{Y}]^3 + Y^2 \cdot (\alpha_{11} \cdot \beta_6 + \alpha_{12} \cdot \beta_4)$$

$$\alpha_1 \cdot \beta_1 = 1; \alpha_9 \cdot \beta_2 = 4; \alpha_{10} \cdot \beta_3 = 1;\ \alpha_8 \cdot \beta_1 = 4;$$
$$\alpha_3 \cdot \beta_5 = -2;\ \alpha_{10} \cdot \beta_4 = -4$$
$$(\alpha_{11} \cdot \beta_6 + \alpha_{12} \cdot \beta_4) = 1 - \mu^2;\ (\beta_5 \cdot \alpha_5 + \beta_6 \cdot \alpha_6 + \alpha_8 \cdot \beta_3) = 1;$$
$$(\beta_4 \cdot \alpha_4 + \alpha_7 \cdot \beta_3 + \alpha_5 \cdot \beta_6 + \alpha_{11} \cdot \beta_5) = -2$$
$$(\alpha_4 \cdot \beta_2 + \alpha_8 \cdot \beta_4) = 2;\ (\beta_4 \cdot \alpha_7 + \beta_2 \cdot \alpha_{12}) = -2;\ \beta_2 \cdot \alpha_7 = 1$$

We get the restriction on Poincare–Bendixson optoisolation circuit's parameters related on Poincare–Bendixson final equation.

$$\alpha_1 \cdot \beta_1 = B_1 \cdot B_4 \cdot \beta_1 = \left(2 - \alpha_f - \alpha_r\right) \cdot R_1 \cdot C_1 \cdot C_2 \cdot \beta_1 = 1 \Rightarrow \beta_1$$
$$= \frac{1}{\left(2 - \alpha_f - \alpha_r\right) \cdot R_1 \cdot C_1 \cdot C_2}$$

$$\alpha_9 \cdot \beta_2 = -3 \cdot A_1 \cdot R_1^2 \cdot C_1^2 \cdot \beta_2 = -3 \cdot \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot R_1^2 \cdot C_1^2 \cdot \beta_2$$
$$= 4 \Rightarrow \beta_2 = -\frac{4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f \cdot R_1^2 \cdot C_1^2}$$

$$\alpha_{10} \cdot \beta_3 = A_1 \cdot R_1^3 \cdot C_1^3 \cdot \beta_3 = \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot R_1^3 \cdot C_1^3 \cdot \beta_3$$
$$= 1 \Rightarrow \beta_3 = \frac{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3}$$

$$\alpha_8 \cdot \beta_1 = 3 \cdot A_1 \cdot R_1 \cdot C_1 \cdot \beta_1 = 3 \cdot \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot R_1 \cdot C_1 \cdot \beta_1$$
$$= 4 \Rightarrow \beta_1 = \frac{4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f \cdot R_1 \cdot C_1}$$

We get intermediate result:

$$\beta_1 = \frac{1}{(2 - \alpha_f - \alpha_r) \cdot R_1 \cdot C_1 \cdot C_2} \;\&\; \beta_1 = \frac{4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f \cdot R_1 \cdot C_1}$$
$$\Rightarrow \frac{1}{(2 - \alpha_f - \alpha_r) \cdot C_2} = \frac{4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f} \Rightarrow 3 \cdot k \cdot \alpha_f$$
$$= 4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right] \cdot (2 - \alpha_f - \alpha_r) \cdot C_2$$

Next results:

$$\alpha_3 \cdot \beta_5 = B_3 \cdot B_4 \cdot \beta_5 = -\frac{(1 - \alpha_f) \cdot R_1 \cdot C_1}{V_t} \cdot R_1 \cdot C_1 \cdot C_2 \cdot \beta_5$$
$$= -2 \Rightarrow \beta_5 = \frac{2 \cdot V_t}{(1 - \alpha_f) \cdot R_1^2 \cdot C_1^2 \cdot C_2}$$

$$\alpha_{10} \cdot \beta_4 = A_1 \cdot R_1^3 \cdot C_1^3 \cdot \beta_4 = \frac{k \cdot \alpha_f}{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot R_1^3 \cdot C_1^3 \cdot \beta_4$$
$$= -4 \Rightarrow \beta_4 = \frac{-4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3}$$

$$
\begin{aligned}
(\alpha_{11} \cdot \beta_6 + \alpha_{12} \cdot \beta_4) &= -(A_2 \cdot \beta_6 + A_3 \cdot \beta_4) \\
&= -\left(k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \beta_6 - \frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \beta_4 \right) \\
&= 1 - \mu^2
\end{aligned}
$$

$$
-\left(k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \beta_6 - \frac{1}{V_t} \cdot I_{sc} \cdot \left(\alpha_r \cdot \alpha_f - 1\right) \cdot \beta_4 \right) = 1 - \mu^2
$$

$$
\beta_4 = \frac{-4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \;\&\; -\left(k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \beta_6 - \frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \beta_4 \right) = 1 - \mu^2
$$

$$
\begin{aligned}
&\Rightarrow -\left(k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \beta_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{4 \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right) = 1 - \mu^2 \\
&\Rightarrow k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \beta_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{4 \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} = \mu^2 - 1 \\
&\Rightarrow k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \beta_6 = \left\{ \mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{4 \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right\} \\
&\Rightarrow \beta_6 = \frac{\left\{ \mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{4 \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)}
\end{aligned}
$$

$$
\begin{aligned}
&(\beta_5 \cdot \alpha_5 + \beta_6 \cdot \alpha_6 + \alpha_8 \cdot \beta_3) \\
&\quad = \left(\frac{2 \cdot V_t}{(1 - \alpha_f) \cdot R_1 \cdot C_1^2 \cdot R_1 \cdot C_2} \cdot \alpha_5 \right. \\
&\qquad \left. + \frac{\left\{ \mu^2 - 1 - I_{sc} \cdot \left(\alpha_r \cdot \alpha_f - 1\right) \cdot \frac{4 \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)} \cdot \alpha_6 + \alpha_8 \cdot \frac{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right) \\
&\quad = 1
\end{aligned}
$$

$$\left(\frac{2\cdot V_t}{(1-\alpha_f)\cdot R_1^2\cdot C_1^2\cdot C_2}\cdot\left\{2\cdot k\cdot\frac{(\alpha_f-1)}{\left[R_2+V_t\cdot\frac{1}{I_0}\right]}\cdot R_1\cdot C_1-(C_2-C_1)\cdot\frac{(1-\alpha_f)}{V_t}\right\}\right.$$
$$-\frac{\left\{\mu^2-1-I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)\cdot\frac{4\cdot\left[R_2+V_t\cdot\frac{1}{I_0}\right]}{k\cdot\alpha_f\cdot R_1^3\cdot C_1^3}\right\}\cdot\left[R_2+V_t\cdot\frac{1}{I_0}\right]}{k\cdot(\alpha_f-1)}$$
$$\cdot\left\{(C_1-C_2)\cdot\frac{(1-\alpha_f)\cdot R_1\cdot C_1}{V_t}+k\cdot\frac{(\alpha_f-1)}{\left[R_2+V_t\cdot\frac{1}{I_0}\right]}\cdot R_1^2\cdot C_1^2\right\}$$
$$\left.+3\cdot\frac{k\cdot\alpha_f}{V_t\cdot\left[R_2+V_t\cdot\frac{1}{I_0}\right]}\cdot R_1\cdot C_1\cdot\frac{V_t\cdot\left[R_2+V_t\cdot\frac{1}{I_0}\right]}{k\cdot\alpha_f\cdot R_1^3\cdot C_1^3}\right)=1$$

$$\left(\frac{2\cdot V_t}{(1-\alpha_f)\cdot R_1^2\cdot C_1^2\cdot C_2}\cdot\left\{2\cdot k\cdot\frac{(\alpha_f-1)}{\left[R_2+V_t\cdot\frac{1}{I_0}\right]}\cdot R_1\cdot C_1-(C_2-C_1)\cdot\frac{(1-\alpha_f)}{V_t}\right\}\right.$$
$$-\frac{\left\{\mu^2-1-I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)\cdot\frac{4\cdot[R_2+V_t\cdot\frac{1}{I_0}]}{k\cdot\alpha_f\cdot R_1^3\cdot C_1^3}\right\}\cdot\left[R_2+V_t\cdot\frac{1}{I_0}\right]}{k\cdot(\alpha_f-1)}$$
$$\cdot\left\{(C_1-C_2)\cdot\frac{(1-\alpha_f)\cdot R_1\cdot C_1}{V_t}+k\cdot\frac{(\alpha_f-1)}{\left[R_2+V_t\cdot\frac{1}{I_0}\right]}\cdot R_1^2\cdot C_1^2\right\}$$
$$\left.+3\cdot\frac{1}{R_1^2\cdot C_1^2}\right)=1$$

$$(\beta_4 \cdot \alpha_4 + \alpha_7 \cdot \beta_3 + \alpha_5 \cdot \beta_6 + \alpha_{11} \cdot \beta_5)$$

$$= -2 \quad \Rightarrow \left\{ \frac{-4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right.$$

$$\cdot \left\{ -\frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot R_1 \cdot C_1 - (C_2 - C_1) \cdot \left[2 - \alpha_f - \alpha_r\right] \right\}$$

$$- \frac{1}{R_1^3 \cdot C_1^3} + \left(2 \cdot k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot R_1 \cdot C_1 - (C_2 - C_1) \cdot \frac{(1 - \alpha_f)}{V_t} \right)$$

$$\cdot \left\{ \frac{\left\{ \mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{4 \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3} \right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)} \right\}$$

$$\left. + k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \frac{2 \cdot V_t}{R_1^2 \cdot C_1^2 \cdot C_2} \right\} = -2$$

$$(\alpha_4 \cdot \beta_2 + \alpha_8 \cdot \beta_4) = 2$$

$$\Rightarrow \left(\frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot R_1 \cdot C_1 + (C_2 - C_1) \cdot \left[2 - \alpha_f - \alpha_r\right] \right)$$

$$\cdot \frac{4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f \cdot R_1^2 \cdot C_1^2} - \frac{12}{R_1^2 \cdot C_1^2} = 2$$

$$(\beta_4 \cdot \alpha_7 + \beta_2 \cdot \alpha_{12}) = -2 \Rightarrow \frac{1}{R_1^3 \cdot C_1^3} - \frac{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f \cdot R_1^2 \cdot C_1^2} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) = -\frac{1}{2}$$

$$\beta_2 \cdot \alpha_7 = 1 \Rightarrow \frac{4}{3 \cdot R_1^2 \cdot C_1^2} = 1 \Rightarrow R_1^2 \cdot C_1^2 = \frac{4}{3} \Rightarrow R_1 \cdot C_1 = \frac{2}{\sqrt{3}}$$

We implement our last result $R_1 \cdot C_1 = \frac{2}{\sqrt{3}}$ in all last equations.

$$\beta_1 = \left. \frac{1}{(2 - \alpha_f - \alpha_r) \cdot R_1 \cdot C_1 \cdot C_2} \right|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{(2 - \alpha_f - \alpha_r) \cdot 2 \cdot C_2}$$

$$\beta_2 = -\frac{4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{3 \cdot k \cdot \alpha_f \cdot R_1^2 \cdot C_1^2}\Bigg|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} = -\frac{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f}$$

$$\beta_3 = \frac{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3}\Bigg|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} = \frac{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right] \cdot 3 \cdot \sqrt{3}}{k \cdot \alpha_f \cdot 8}$$

$$\beta_1 = \frac{4 \cdot V_t \cdot [R_2 + V_t \cdot \frac{1}{I_0}]}{3 \cdot k \cdot \alpha_f \cdot R_1 \cdot C_1}\Bigg|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} = \frac{2 \cdot V_t \cdot [R_2 + V_t \cdot \frac{1}{I_0}]}{\sqrt{3} \cdot k \cdot \alpha_f}$$

$$\beta_5 = \frac{2 \cdot V_t}{(1 - \alpha_f) \cdot R_1 \cdot C_1^2 \cdot R_1 \cdot C_2}\Bigg|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} = \frac{3 \cdot V_t}{(1 - \alpha_f) \cdot 2 \cdot C_2}$$

$$\beta_4 = \frac{-4 \cdot V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3}\Bigg|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} = \frac{-V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right] \cdot 3 \cdot \sqrt{3}}{k \cdot \alpha_f \cdot 2}$$

$$\beta_6 = \frac{\left\{\mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{4 \cdot [R_2 + V_t \cdot \frac{1}{I_0}]}{k \cdot \alpha_f \cdot R_1^3 \cdot C_1^3}\right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)}\Bigg|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}}$$

$$= \frac{\left\{\mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{\left[R_2 + V_t \cdot \frac{1}{I_0}\right] \cdot 3 \cdot \sqrt{3}}{k \cdot \alpha_f \cdot 2}\right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)}$$

$$(\beta_5 \cdot \alpha_5 + \beta_6 \cdot \alpha_6 + \alpha_8 \cdot \beta_3) = 1 \Rightarrow |_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}}$$

$$\Rightarrow \left(\frac{3 \cdot V_t}{(1 - \alpha_f) \cdot 2 \cdot C_2} \cdot \left\{k \cdot \frac{(\alpha_f - 1)}{[R_2 + V_t \cdot \frac{1}{I_0}]} \cdot \frac{4}{\sqrt{3}} - (C_2 - C_1) \cdot \frac{(1 - \alpha_f)}{V_t}\right\}\right.$$

$$- \frac{\left\{\mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{\left[R_2 + V_t \cdot \frac{1}{I_0}\right] \cdot 3\sqrt{3}}{k \cdot \alpha_f \cdot 2}\right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)}$$

$$\left. \cdot \left\{(C_1 - C_2) \cdot \frac{(1 - \alpha_f) \cdot \frac{2}{\sqrt{3}}}{V_t} + k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \frac{4}{3}\right\} + \frac{9}{4}\right) = 1$$

$$(\beta_4 \cdot \alpha_4 + \alpha_7 \cdot \beta_3 + \alpha_5 \cdot \beta_6 + \alpha_{11} \cdot \beta_5) = -2 \Rightarrow|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}}$$

$$\Rightarrow \left\{ \frac{-V_t \cdot [R_2 + V_t \cdot \frac{1}{I_0}] \cdot 3 \cdot \sqrt{3}}{k \cdot \alpha_f \cdot 2} \cdot \left\{ -\frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{2}{\sqrt{3}} - (C_2 - C_1) \cdot \left[2 - \alpha_f - \alpha_r\right] \right\} \right.$$
$$- \frac{3 \cdot \sqrt{3}}{8} + (k \cdot \frac{(\alpha_f - 1)}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \frac{4}{\sqrt{3}} - (C_2 - C_1) \cdot \frac{(1 - \alpha_f)}{V_t})$$
$$\cdot \left\{ \frac{\left\{ \mu^2 - 1 - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{\left[R_2 + V_t \cdot \frac{1}{I_0}\right] \cdot 3 \cdot \sqrt{3}}{k \cdot \alpha_f \cdot 2} \right\} \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot (\alpha_f - 1)} \right\}$$
$$\left. + k \cdot \frac{1}{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]} \cdot \frac{3 \cdot V_t}{2 \cdot C_2} \right\} = -2$$

$$(\alpha_4 \cdot \beta_2 + \alpha_8 \cdot \beta_4) = 2 \Rightarrow|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}} \Rightarrow \left(\frac{1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{2}{\sqrt{3}} + (C_2 - C_1) \cdot \left[2 - \alpha_f - \alpha_r\right] \right)$$
$$\cdot \frac{V_t \cdot \left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f} - 9 = 2$$

$$(\beta_4 \cdot \alpha_7 + \beta_2 \cdot \alpha_{12}) = -2 \Rightarrow|_{R_1 \cdot C_1 = \frac{2}{\sqrt{3}}}$$
$$\Rightarrow \frac{3\sqrt{3}}{8} - \frac{\left[R_2 + V_t \cdot \frac{1}{I_0}\right]}{k \cdot \alpha_f \cdot 4} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) = -\frac{1}{2}$$

We need to find the expression for Γ_x.

$$\Gamma_x([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = \prod_{i=1}^{2} \xi_i([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) - \Gamma_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m)$$
$$\Gamma_x([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = \xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) \cdot \xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) - \Gamma_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m)$$

$$\begin{aligned}
\Gamma_x = \xi_1 \cdot \xi_2 - \Gamma_1 &= \ddot{Y} \cdot Y^3 \cdot (\alpha_1 \cdot \beta_2 + \beta_1 \cdot \alpha_7) \\
&+ \alpha_1 \cdot \beta_3 \cdot \ddot{Y} \cdot \frac{\dot{Y}}{Y^2} + Y \cdot \ddot{Y} \cdot (\alpha_1 \cdot \beta_4 + \beta_6 \cdot \alpha_2 + \alpha_{12} \cdot \beta_1) \\
&+ \ddot{Y} \cdot \dot{Y} \cdot \frac{1}{Y} \cdot (\alpha_1 \cdot \beta_5 + \beta_3 \cdot \alpha_2) + \ddot{Y} \cdot (\alpha_1 \cdot \beta_6 + \alpha_{13} \cdot \beta_1) \\
&+ [\ddot{Y}]^2 \cdot Y \cdot \alpha_2 \cdot \beta_1 + Y^4 \cdot \ddot{Y} \cdot \alpha_2 \cdot \beta_2 \\
&+ \dot{Y} \cdot Y^2 \cdot (\beta_4 \cdot \alpha_2 + \alpha_{11} \cdot \beta_1) + \ddot{Y} \cdot \dot{Y} \cdot (\beta_5 \cdot \alpha_2 + \alpha_4 \cdot \beta_1) \\
&+ \alpha_3 \cdot \beta_1 \cdot [\ddot{Y}]^2 \cdot \dot{Y} + \alpha_3 \cdot \beta_2 \cdot \ddot{Y} \cdot \dot{Y} \cdot Y^3 \\
&+ \alpha_3 \cdot \beta_3 \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y^2} + \ddot{Y} \cdot \dot{Y} \cdot Y \cdot (\alpha_3 \cdot \beta_4 + \alpha_5 \cdot \beta_1) \\
&+ \alpha_3 \cdot \beta_6 \cdot \ddot{Y} \cdot \dot{Y} + \alpha_4 \cdot \beta_3 \cdot [\dot{Y}]^2 \cdot \frac{1}{Y^2} \\
&+ \frac{[\dot{Y}]^2}{Y} \cdot (\beta_5 \cdot \alpha_4 + \alpha_5 \cdot \beta_3) \\
&+ \dot{Y} \cdot (\beta_6 \cdot \alpha_4 + \alpha_{11} \cdot \beta_3 + \alpha_{12} \cdot \beta_5) + \alpha_5 \cdot \beta_2 \cdot Y^4 \cdot \dot{Y} \\
&+ \dot{Y} \cdot Y^2 \cdot (\alpha_5 \cdot \beta_4 + \alpha_7 \cdot \beta_5 + \alpha_8 \cdot \beta_6) \\
&+ \alpha_6 \cdot \beta_1 \cdot [\dot{Y}]^2 \cdot \ddot{Y} + \alpha_6 \cdot \beta_2 \cdot [\dot{Y}]^2 \cdot Y^3 + \alpha_6 \cdot \beta_3 \cdot [\dot{Y}]^3 \cdot \frac{1}{Y^2} \\
&+ [\dot{Y}]^2 \cdot Y \cdot (\beta_4 \cdot \alpha_6 + \beta_5 \cdot \alpha_8 + \alpha_9 \cdot \beta_6) + \frac{[\dot{Y}]^3}{Y} \cdot (\beta_5 \cdot \alpha_6 + \beta_3 \cdot \alpha_9) \\
&+ Y^3 \cdot (\beta_6 \cdot \alpha_7 + \beta_4 \cdot \alpha_{11} + \alpha_{13} \cdot \beta_2) \\
&+ \alpha_8 \cdot \beta_2 \cdot \dot{Y} \cdot Y^5 + \beta_1 \cdot \alpha_9 \cdot Y \cdot [\dot{Y}]^2 \cdot \ddot{Y} \\
&+ \alpha_9 \cdot \beta_4 \cdot Y^2 \cdot [\dot{Y}]^2 + \alpha_9 \cdot \beta_5 \cdot [\dot{Y}]^3 + \alpha_{10} \cdot \beta_1 \cdot [\dot{Y}]^3 \cdot \ddot{Y} \\
&+ \alpha_{10} \cdot \beta_2 \cdot [\dot{Y}]^3 \cdot Y^3 + \frac{[\dot{Y}]^4}{Y} \cdot \beta_5 \cdot \alpha_{10} + \beta_6 \cdot \alpha_{10} \cdot [\dot{Y}]^3 \\
&+ \alpha_{11} \cdot \beta_2 \cdot Y^5 + \frac{\dot{Y}}{Y} \cdot (\alpha_{12} \cdot \beta_3 + \beta_5 \cdot \alpha_{13}) \\
&+ Y \cdot (\alpha_{12} \cdot \beta_6 + \beta_4 \cdot \alpha_{13}) + \beta_3 \cdot \alpha_{13} \cdot \frac{\dot{Y}}{Y^2} + \alpha_{13} \cdot \beta_6
\end{aligned}$$

Since $\Gamma_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) + \Gamma_x([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0$ & $\Gamma_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0$ $\Rightarrow \Gamma_x([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0$

We need to find possible optoisolation circuits which implement Γ_x function and fulfill $\xi_1([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) \cdot \xi_2([\ddot{Y}]^k, [\dot{Y}]^l, Y^m) = 0$. We define new Γ_x global parameters γ_i.

$$\gamma_1 = (\alpha_1 \cdot \beta_2 + \beta_1 \cdot \alpha_7);\ \gamma_2 = \alpha_1 \cdot \beta_3;\ \gamma_3 = (\alpha_1 \cdot \beta_4 + \beta_6 \cdot \alpha_2 + \alpha_{12} \cdot \beta_1)$$
$$\gamma_4 = (\alpha_1 \cdot \beta_5 + \beta_3 \cdot \alpha_2);\ \gamma_5 = (\alpha_1 \cdot \beta_6 + \alpha_{13} \cdot \beta_1);\ \gamma_6 = \alpha_2 \cdot \beta_1;\ \gamma_7 = \alpha_2 \cdot \beta_2$$
$$\gamma_8 = (\beta_4 \cdot \alpha_2 + \alpha_{11} \cdot \beta_1);\ \gamma_9 = (\beta_5 \cdot \alpha_2 + \alpha_4 \cdot \beta_1);\ \gamma_{10} = \alpha_3 \cdot \beta_1;\ \gamma_{11} = \alpha_3 \cdot \beta_2$$
$$\gamma_{12} = \alpha_3 \cdot \beta_3;\ \gamma_{13} = (\alpha_3 \cdot \beta_4 + \alpha_5 \cdot \beta_1);\ \gamma_{14} = \alpha_3 \cdot \beta_6;\ \gamma_{15} = \alpha_4 \cdot \beta_3$$
$$\gamma_{16} = (\beta_5 \cdot \alpha_4 + \alpha_5 \cdot \beta_3);\ \gamma_{17} = (\beta_6 \cdot \alpha_4 + \alpha_{11} \cdot \beta_3 + \alpha_{12} \cdot \beta_5);\ \gamma_{18} = \alpha_5 \cdot \beta_2$$
$$\gamma_{19} = (\alpha_5 \cdot \beta_4 + \alpha_7 \cdot \beta_5 + \alpha_8 \cdot \beta_6);\ \gamma_{20} = \alpha_6 \cdot \beta_1;\ \gamma_{21} = \alpha_6 \cdot \beta_2;\ \gamma_{22} = \alpha_6 \cdot \beta_3$$
$$\gamma_{23} = (\beta_4 \cdot \alpha_6 + \beta_5 \cdot \alpha_8 + \alpha_9 \cdot \beta_6);\ \gamma_{24} = (\beta_5 \cdot \alpha_6 + \beta_3 \cdot \alpha_9);\ \gamma_{25} = (\beta_6 \cdot \alpha_7 + \beta_4 \cdot \alpha_{11} + \alpha_{13} \cdot \beta_2)$$
$$\gamma_{26} = \alpha_8 \cdot \beta_2;\ \gamma_{27} = \beta_1 \cdot \alpha_9;\ \gamma_{28} = \alpha_9 \cdot \beta_4;\ \gamma_{29} = \alpha_9 \cdot \beta_5;\ \gamma_{30} = \alpha_{10} \cdot \beta_1$$
$$\gamma_{31} = \alpha_{10} \cdot \beta_2;\ \gamma_{32} = \beta_5 \cdot \alpha_{10};\ \gamma_{33} = \beta_6 \cdot \alpha_{10};\ \gamma_{34} = \alpha_{11} \cdot \beta_2;\ \gamma_{35} = (\alpha_{12} \cdot \beta_3 + \beta_5 \cdot \alpha_{13})$$
$$\gamma_{36} = (\alpha_{12} \cdot \beta_6 + \beta_4 \cdot \alpha_{13});\ \gamma_{37} = \beta_3 \cdot \alpha_{13};\ \gamma_{38} = \alpha_{13} \cdot \beta_6$$

The Γ_x function with γ_i parameters ($\Gamma_x = 0$):

$$\begin{aligned}
\Gamma_x = \xi_1 \cdot \xi_2 - \Gamma_1 = {} & \ddot{Y} \cdot Y^3 \cdot \gamma_1 + \gamma_2 \cdot \ddot{Y} \cdot \frac{\dot{Y}}{Y^2} \\
& + Y \cdot \ddot{Y} \cdot \gamma_3 + \ddot{Y} \cdot \dot{Y} \cdot \frac{1}{Y} \cdot \gamma_4 + \ddot{Y} \cdot \gamma_5 + [\ddot{Y}]^2 \cdot Y \cdot \gamma_6 + Y^4 \cdot \ddot{Y} \cdot \gamma_7 \\
& + \ddot{Y} \cdot Y^2 \cdot \gamma_8 + \ddot{Y} \cdot \dot{Y} \cdot \gamma_9 + \gamma_{10} \cdot [\ddot{Y}]^2 \cdot \dot{Y} + \gamma_{11} \cdot \ddot{Y} \cdot \dot{Y} \cdot Y^3 \\
& + \gamma_{12} \cdot \ddot{Y} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y^2} + \ddot{Y} \cdot \dot{Y} \cdot Y \cdot \gamma_{13} + \gamma_{14} \cdot \ddot{Y} \cdot \dot{Y} \\
& + \gamma_{15} \cdot [\dot{Y}]^2 \cdot \frac{1}{Y^2} + \frac{[\dot{Y}]^2}{Y} \cdot \gamma_{16} + \dot{Y} \cdot \gamma_{17} + \gamma_{18} \cdot Y^4 \cdot \dot{Y} \\
& + \dot{Y} \cdot Y^2 \cdot \gamma_{19} + \gamma_{20} \cdot [\dot{Y}]^2 \cdot \ddot{Y} + \gamma_{21} \cdot [\dot{Y}]^2 \cdot Y^3 + \gamma_{22} \cdot [\dot{Y}]^3 \cdot \frac{1}{Y^2} \\
& + [\dot{Y}]^2 \cdot Y \cdot \gamma_{23} + \frac{[\dot{Y}]^3}{Y} \cdot \gamma_{24} + Y^3 \cdot \gamma_{25} + \gamma_{26} \cdot \dot{Y} \cdot Y^5 \\
& + \gamma_{27} \cdot Y \cdot [\dot{Y}]^2 \cdot \ddot{Y} + \gamma_{28} \cdot Y^2 \cdot [\dot{Y}]^2 + \gamma_{29} \cdot [\dot{Y}]^3 + \gamma_{30} \cdot [\dot{Y}]^3 \cdot \ddot{Y} \\
& + \gamma_{31} \cdot [\dot{Y}]^3 \cdot Y^3 + \frac{[\dot{Y}]^4}{Y} \cdot \gamma_{32} + \gamma_{33} \cdot [\dot{Y}]^3 + \gamma_{34} \cdot Y^5 \\
& + \frac{\dot{Y}}{Y} \cdot \gamma_{35} + Y \cdot \gamma_{36} + \gamma_{37} \cdot \frac{\dot{Y}}{Y^2} + \gamma_{38}
\end{aligned}$$

1.5 Optoisolation Nonlinear Oscillations Lienard Circuits

We need to implement optoisolation oscillation circuits which can be modeled by second-order differential equations: $\frac{d^2V}{dt^2} + f(V) \cdot \frac{dV}{dt} + g(V) = 0$. This is known as Lienard's equation. The equation is a generalization of the van der pole oscillator system which we already discussed $\left(\frac{d^2V}{dt^2} + \mu \cdot (V^2 - 1) \cdot \frac{dV}{dt} + V = 0\right)$.

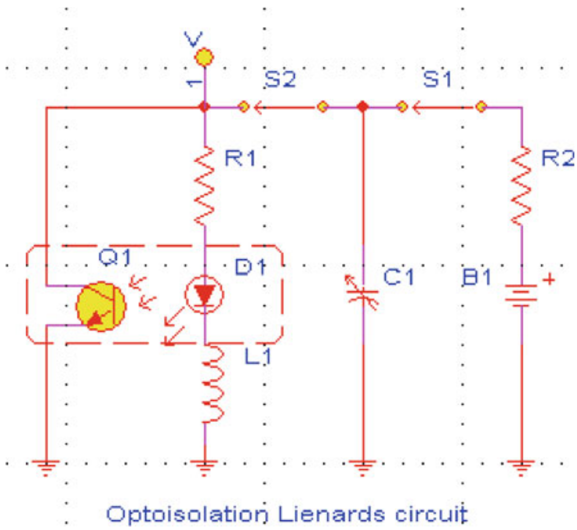
We define $-f(V) \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$ as nonlinear damping force and $-g(V)$ as a nonlinear restoring force. We can define the following new system variables: $Y = \mathrm{d}V/\mathrm{d}t$, $X = V$. Then we can describe Lienard's equation equivalent system [9, 10].

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2} + f(V) \cdot \frac{\mathrm{d}V}{\mathrm{d}t} + g(V) = 0 \Rightarrow \frac{\mathrm{d}X}{\mathrm{d}t} = Y; \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = -g(X) - f(X) \cdot Y$$

The below optoisolation circuit implements Lienard's system. First switch S1 is ON and S2 is OFF, capacitor C1 is charged to V_{B1} voltage ($t \to \infty$). The next step switch S1 is OFF and S2 is ON ($t = t_0$). We consider the initial C1 voltage at $t = t_0$ is greater than LED D1 ON voltage $(V_{C1}(\mathrm{t} = \mathrm{t}_0) \gg V_{D1ON})$. At $t = t_0$ C1 charged capacitor acts as voltage source. We can write the following equation for the initial state: $V(t = t_0) = V_{C1}(t = t_0) = I_{D1} \cdot R_1 + V_{D1}(I_{D1}) + L_1 \cdot \frac{\mathrm{d}I_{D1}}{\mathrm{d}t}$. By switching S1 from OFF to ON state, LED D1 forward current is a continuous rising function with LED D1 forward voltage. If we look on optocoupler's phototransistor collector current, it is rising continuous function of phototransistor collector emitter voltage $V_{CEQ1} = V$. Optocoupler current transfer ratio (CTR) parameter is a continuous rising function of optocoupler's LED forward current up to specific LED forward peak current value. We consider $I_{L1}(t = t_0) = 0$ (Fig. 1.16).

The most important parameter for our optoisolation Lienards system is the optocoupler's transfer efficiency, usually measured in terms of its current transfer ratio (CTR) . This is simply the ratio between a current change in the output phototransistor and the current change in the input LED (D1) which produced it. Typical values of CTR range from 10 to 50% for devices with an output phototransistor and up to 2000% or so for those with a darlington transistor pair in the output. Note, however that in most devices CTR tends to vary with absolute current

Fig. 1.16 Optoisolation Lienards circuit



level. Typically, it peaks at a LED current level of about 10 mA, and falls away at both higher and lower current levels. Current Transfer Ratio (CTR) is the gain of the optocoupler. It is the ratio of the phototransistor collector current to the IRED (D1) forward current $\mathrm{CTR} = \frac{I_{CQ1}}{I_{D1}} \times 100$. It is expressed as a percentage (%). The CTR depends on the current gain (hfe) of the phototransistor, the supply voltage to the phototransistor, the forward current through the IRED (D1) and operating temperature [16].

$$I_{C1} = \frac{\mathrm{d}V_{C1}}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t};\ I_{CQ1} + I_{R1} = I_{C1};\ I_{R1} = I_{D1} = I_{L1};$$

$$V = I_{R1} \cdot R_1 + V_{D1} + V_{L1};\ V_{CEQ1} = V_{C1} = V$$

$$V_{CEQ1} = V_t \cdot \ln\left[\left(\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right) - V_t \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha_f}{I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right) + 1\right];\right.$$

$$V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] \to \varepsilon$$

$$V = V_{CEQ1} \approx V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_r) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$V = R_1 \cdot I_{D1} + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) + V_{L1};\ V_{L1} = L_1 \cdot \frac{\mathrm{d}I_{D1}}{\mathrm{d}t} = L_1 \cdot \frac{\mathrm{d}I_{L1}}{\mathrm{d}t} = L_1 \cdot \frac{\mathrm{d}I_{R1}}{\mathrm{d}t}$$

$$I_{\mathrm{BQ1}} = k \cdot I_{D1};\ I_{EQ1} = I_{\mathrm{BQ1}} + I_{\mathrm{CQ1}};\ I_{EQ1} = k \cdot I_{D1} + I_{\mathrm{CQ1}} = k \cdot I_{D1} + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - I_{D1}$$

$$I_{CQ1} = \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - I_{D1};\ I_{EQ1} = I_{D1} \cdot (k-1) + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1$$

$$\begin{aligned}\alpha_r \cdot I_{CQ1} - I_{EQ1} &= \alpha_r \cdot \left[\frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - I_{D1}\right] - \left[I_{D1} \cdot (k-1) + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1\right] \\ &= \alpha_r \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - \alpha_r \cdot I_{D1} - I_{D1} \cdot (k-1) - \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 \\ &= \alpha_r \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - \alpha_r \cdot I_{D1} - I_{D1} \cdot k + I_{D1} - \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1\end{aligned}$$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = (\alpha_r - 1) \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 + I_{D1} \cdot [1 - k - \alpha_r]$$

$$\begin{aligned}I_{CQ1} - I_{EQ1} \cdot \alpha_f &= \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - I_{D1} - \alpha_f \cdot \left[I_{D1} \cdot (k-1) + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1\right] \\ &= \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1 - I_{D1} - \alpha_f \cdot I_{D1} \cdot (k-1) - \alpha_f \cdot I_{D1} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot C_1\end{aligned}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \frac{dV}{dt} \cdot C_1 \cdot (1 - \alpha_f) + I_{D1} \cdot [\alpha_f \cdot (1 - k) - 1]$$

If we derivative the equation: $V = R_1 \cdot I_{D1} + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) + L_1 \cdot \frac{dI_{D1}}{dt} \Rightarrow \frac{dV}{dt} = R_1 \cdot \frac{dI_{D1}}{dt} + V_t \cdot \frac{1}{\left(\frac{I_{D1}}{I_0} + 1\right)} \cdot \frac{1}{I_0} \cdot \frac{dI_{D1}}{dt} + L_1 \cdot \frac{d^2 I_{D1}}{dt^2}$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = (\alpha_r - 1) \cdot C_1 \cdot \left\{ R_1 \cdot \frac{dI_{D1}}{dt} + V_t \cdot \frac{1}{\left(\frac{I_{D1}}{I_0} + 1\right)} \cdot \frac{1}{I_0} \cdot \frac{dI_{D1}}{dt} + L_1 \cdot \frac{d^2 I_{D1}}{dt^2} \right\} + I_{D1} \cdot [1 - k - \alpha_r]$$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = (\alpha_r - 1) \cdot C_1 \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0}\right] \cdot \frac{dI_{D1}}{dt} + (\alpha_r - 1) \cdot C_1 \cdot L_1 \cdot \frac{d^2 I_{D1}}{dt^2} + I_{D1} \cdot [1 - k - \alpha_r]$$

We define I_{D1} as our global variable. $I = I_{D1}$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = \frac{d^2 I_{D1}}{dt^2} \cdot (\alpha_r - 1) \cdot C_1 \cdot L_1 + \frac{dI_{D1}}{dt} \cdot (\alpha_r - 1) \cdot C_1 \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0}\right] + I_{D1} \cdot [1 - k - \alpha_r]$$

$$\xi_1 = (\alpha_r - 1) \cdot C_1 \cdot L_1;$$
$$\xi_2(I_{D1}) = (\alpha_r - 1) \cdot C_1 \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0}\right]$$
$$\Rightarrow \xi_2(I) = (\alpha_r - 1) \cdot C_1 \cdot \left[R_1 + \frac{V_t}{I + I_0}\right]$$
$$\xi_3 = [1 - k - \alpha_r]$$

We get the following expression:

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = \frac{d^2 I_{D1}}{dt^2} \cdot \xi_1 + \frac{dI_{D1}}{dt} \cdot \xi_2(I_{D1}) + I_{D1} \cdot \xi_3$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \left\{ R_1 \cdot \frac{\mathrm{d}I_{D1}}{\mathrm{d}t} + V_t \cdot \frac{1}{\left(\frac{I_{D1}}{I_0} + 1\right)} \cdot \frac{1}{I_0} \cdot \frac{\mathrm{d}I_{D1}}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I_{D1}}{\mathrm{d}t^2} \right\} \cdot C_1 \cdot (1 - \alpha_f) + I_{D1} \cdot [\alpha_f \cdot (1 - k) - 1]$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = C_1 \cdot (1 - \alpha_f) \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0} \right] \cdot \frac{\mathrm{d}I_{D1}}{\mathrm{d}t} + L_1 \cdot C_1 \cdot (1 - \alpha_f) \cdot \frac{\mathrm{d}^2 I_{D1}}{\mathrm{d}t^2} + I_{D1} \cdot [\alpha_f \cdot (1 - k) - 1]$$

We define I_{D1} as our global variable. $I = I_{D1}$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \frac{\mathrm{d}^2 I_{D1}}{\mathrm{d}t^2} \cdot L_1 \cdot C_1 \cdot (1 - \alpha_f) + \frac{\mathrm{d}I_{D1}}{\mathrm{d}t} \cdot C_1 \cdot (1 - \alpha_f) \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0} \right] + I_{D1} \cdot [\alpha_f \cdot (1 - k) - 1]$$

$$\xi_4 = L_1 \cdot C_1 \cdot (1 - \alpha_f);$$
$$\xi_5(I_{D1}) = C_1 \cdot (1 - \alpha_f) \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0} \right] \Rightarrow \xi_5(I) = C_1 \cdot (1 - \alpha_f) \cdot \left[R_1 + \frac{V_t}{I + I_0} \right]$$
$$\xi_6 = [\alpha_f \cdot (1 - k) - 1]$$

We get the following expression:

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \frac{\mathrm{d}^2 I_{D1}}{\mathrm{d}t^2} \cdot \xi_4 + \frac{\mathrm{d}I_{D1}}{\mathrm{d}t} \cdot \xi_5(I_{D1}) + I_{D1} \cdot \xi_6$$

$$I_{D1} \to I \Rightarrow \alpha_r \cdot I_{CQ1} - I_{EQ1} = \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \cdot \xi_1 + \frac{\mathrm{d}I}{\mathrm{d}t} \cdot \xi_2(I_{D1}) + I \cdot \xi_3$$
$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \cdot \xi_4 + \frac{\mathrm{d}I}{\mathrm{d}t} \cdot \xi_5(I_{D1}) + I \cdot \xi_6$$

$$V = |_{I = I_{D1}} R_1 \cdot I + V_t \cdot \ln\left(\frac{I}{I_0} + 1\right) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t};$$
$$V = V_{\mathrm{CEQ1}} \approx V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$R_1 \cdot I + V_t \cdot \ln\left(\frac{I}{I_0} + 1\right) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} \approx V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$R_1 \cdot I + V_t \cdot \ln\left(\frac{I}{I_0} + 1\right) + L_1 \cdot \frac{dI}{dt} \approx V_t \cdot \ln\left[\frac{\frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

We use Taylor series approximation: $\ln[1+I] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot I^n \approx I$ $\Rightarrow \ln\left(\frac{I}{I_0} + 1\right) \approx \frac{I}{I_0}$

$$R_1 \cdot I + V_t \cdot \frac{I}{I_0} + L_1 \cdot \frac{dI}{dt} \approx V_t \cdot \ln\left[\frac{\frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

We use Taylor series approximation: $\ln[I] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (I-1)^n \Rightarrow$ $\ln[f(I)] \approx f(I) - 1$

$$\ln\left[\frac{\frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right] \approx \left\{\frac{\frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)} - 1\right\}$$

$$R_1 \cdot I + V_t \cdot \frac{I}{I_0} + L_1 \cdot \frac{dI}{dt} \approx V_t \cdot \left[\frac{\frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)} - 1\right]$$

$$\frac{1}{V_t}\left[R_1 \cdot I + V_t \cdot \frac{I}{I_0} + L_1 \cdot \frac{dI}{dt}\right] + 1 \approx \frac{\frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

$$\left\{I \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + \frac{L_1}{V_t} \cdot \frac{dI}{dt} + 1\right\} \cdot \left\{\frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)\right\} \approx \frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\frac{d^2I}{dt^2} \cdot I \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot \xi_4 + \frac{dI}{dt} \cdot I \cdot \xi_5(I) \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + I^2 \cdot \xi_6 \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right]$$
$$+ I \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \frac{d^2I}{dt^2} \cdot \frac{dI}{dt} \cdot \frac{L_1}{V_t} \cdot \xi_4 + \left[\frac{dI}{dt}\right]^2 \cdot \frac{L_1}{V_t} \cdot \xi_5(I)$$
$$+ \frac{dI}{dt} \cdot I \cdot \xi_6 \cdot \frac{L_1}{V_t} + \frac{dI}{dt} \cdot \frac{L_1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \frac{d^2I}{dt^2} \cdot \xi_4 + \frac{dI}{dt} \cdot \xi_5(I) + I \cdot \xi_6$$
$$+ I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \simeq \frac{d^2I}{dt^2} \cdot \xi_1 + \frac{dI}{dt} \cdot \xi_2(I) + I \cdot \xi_3 + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\frac{d^2I}{dt^2} \cdot I \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot \xi_4 + \frac{dI}{dt} \cdot I \cdot \left\{\xi_5(I) \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + \xi_6 \cdot \frac{L_1}{V_t}\right\} + I^2 \cdot \xi_6 \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right]$$
$$+ I \cdot \left\{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_6 - \xi_3\right\} + \frac{d^2I}{dt^2} \cdot \frac{dI}{dt} \cdot \frac{L_1}{V_t} \cdot \xi_4 + \left[\frac{dI}{dt}\right]^2 \cdot \frac{L_1}{V_t} \cdot \xi_5(I)$$
$$+ \frac{dI}{dt} \cdot \left\{\frac{L_1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_5(I) - \xi_2(I)\right\} + \frac{d^2I}{dt^2} \cdot (\xi_4 - \xi_1) + (I_{sc} - I_{se}) \cdot (\alpha_r \cdot \alpha_f - 1) \simeq 0$$

We consider $(I_{sc} - I_{se}) \to \varepsilon \Rightarrow (I_{sc} - I_{se}) \cdot (\alpha_r \cdot \alpha_f - 1) \to \varepsilon$
(*)

$$\frac{d^2I}{dt^2} \cdot I \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot \xi_4 + \frac{dI}{dt} \cdot I \cdot \left\{\xi_5(I) \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + \xi_6 \cdot \frac{L_1}{V_t}\right\} + I^2 \cdot \xi_6 \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right]$$
$$+ I \cdot \left\{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_6 - \xi_3\right\} + \frac{d^2I}{dt^2} \cdot \frac{dI}{dt} \cdot \frac{L_1}{V_t} \cdot \xi_4 + \left[\frac{dI}{dt}\right]^2 \cdot \frac{L_1}{V_t} \cdot \xi_5(I)$$
$$+ \frac{dI}{dt} \cdot \left\{\frac{L_1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_5(I) - \xi_2(I)\right\} + \frac{d^2I}{dt^2} \cdot (\xi_4 - \xi_1) \simeq 0$$

We want to represent the above system differential equation by multiplication of Lienard differential equation $\psi_1\left(\frac{d^2I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I)\right)$ and another differential function $\psi_2\left(\frac{dI}{dt}, I, \Gamma_1, \ldots, \Gamma_3\right)$.

$$\psi_1\left(\frac{d^2I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I)\right) = \frac{d^2I}{dt^2} + \frac{dI}{dt} \cdot \eta_1(I) + \eta_2(I);$$
$$\psi_2\left(\frac{dI}{dt}, I, \Gamma_1, \ldots, \Gamma_3\right) = I \cdot \Gamma_1 + \frac{dI}{dt} \cdot \Gamma_2 + \Gamma_3$$

$$\psi_1\left(\frac{d^2I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I)\right) \cdot \psi_2\left(\frac{dI}{dt}, I, \Gamma_1, \ldots, \Gamma_3\right)$$
$$= \left\{\frac{d^2I}{dt^2} + \frac{dI}{dt} \cdot \eta_1(I) + \eta_2(I)\right\} \cdot \left\{I \cdot \Gamma_1 + \frac{dI}{dt} \cdot \Gamma_2 + \Gamma_3\right\}$$

$$\psi_1\left(\frac{d^2I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I)\right) \cdot \psi_2\left(\frac{dI}{dt}, I, \Gamma_1, \ldots, \Gamma_3\right) = 0$$
$$\Rightarrow \left\{\frac{d^2I}{dt^2} + \frac{dI}{dt} \cdot \eta_1(I) + \eta_2(I)\right\} \cdot \left\{I \cdot \Gamma_1 + \frac{dI}{dt} \cdot \Gamma_2 + \Gamma_3\right\} = 0$$

$$\frac{d^2I}{dt^2} \cdot I \cdot \Gamma_1 + \frac{d^2I}{dt^2} \cdot \frac{dI}{dt} \cdot \Gamma_2 + \frac{d^2I}{dt^2} \cdot \Gamma_3 + \frac{dI}{dt} \cdot I \cdot \eta_1(I) \cdot \Gamma_1 + \left[\frac{dI}{dt}\right]^2 \cdot \eta_1(I) \cdot \Gamma_2$$
$$+ \frac{dI}{dt} \cdot \eta_1(I) \cdot \Gamma_3 + I \cdot \Gamma_1 \cdot \eta_2(I) + \frac{dI}{dt} \cdot \eta_2(I) \cdot \Gamma_2 + \eta_2(I) \cdot \Gamma_3 = 0$$

$$\frac{d^2I}{dt^2} \cdot I \cdot \Gamma_1 + \frac{d^2I}{dt^2} \cdot \frac{dI}{dt} \cdot \Gamma_2 + \frac{d^2I}{dt^2} \cdot \Gamma_3 + \frac{dI}{dt} \cdot I \cdot \eta_1(I) \cdot \Gamma_1 + \left[\frac{dI}{dt}\right]^2 \cdot \eta_1(I) \cdot \Gamma_2$$
$$+ \frac{dI}{dt} \cdot [\eta_1(I) \cdot \Gamma_3 + \eta_2(I) \cdot \Gamma_2] + I \cdot \Gamma_1 \cdot \eta_2(I) + \eta_2(I) \cdot \Gamma_3 = 0$$

We compare the above differential equation's constants expressions to (*) system differential equation and we get the following outcome.

$$\Gamma_1 = \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot \xi_4 = \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot L_1 \cdot C_1 \cdot (1 - \alpha_f)$$

$$\left\{\xi_5(I) \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + \xi_6 \cdot \frac{L_1}{V_t}\right\} = \eta_1(I) \cdot \Gamma_1 \Rightarrow \eta_1(I) = \frac{1}{\Gamma_1} \cdot \left\{\xi_5(I) \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + \xi_6 \cdot \frac{L_1}{V_t}\right\}$$
$$\eta_1(I) = \frac{1}{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot L_1 \cdot C_1 \cdot (1 - \alpha_f)}$$
$$\cdot \left\{C_1 \cdot (1 - \alpha_f) \cdot \left[R_1 + \frac{V_t}{I + I_0}\right] \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] + [\alpha_f \cdot (1 - k) - 1] \cdot \frac{L_1}{V_t}\right\}$$
$$\eta_1(I) = \frac{\left[R_1 + \frac{V_t}{I + I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1 - k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1 - \alpha_f)}$$

$$I^2 \cdot \xi_6 \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] = I \cdot \Gamma_1 \cdot \eta_2(I) \Rightarrow I \cdot \left\{I \cdot \xi_6 \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] - \Gamma_1 \cdot \eta_2(I)\right\} = 0$$

$$I \cdot \xi_6 \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] - \Gamma_1 \cdot \eta_2(I) = 0 \Rightarrow \eta_2(I) = \frac{I \cdot \xi_6}{\Gamma_1} \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right]$$
$$\eta_2(I) = \frac{I \cdot \xi_6}{\Gamma_1} \cdot \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] = \frac{I \cdot [\alpha_f \cdot (1 - k) - 1]}{L_1 \cdot C_1 \cdot (1 - \alpha_f)}$$

$$I \cdot \left\{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_6 - \xi_3\right\} = \eta_2(I) \cdot \Gamma_3$$

$$\eta_2(I) = \frac{I \cdot [\alpha_f \cdot (1-k) - 1]}{L_1 \cdot C_1 \cdot (1-\alpha_f)} \Rightarrow \left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_6 - \xi_3$$
$$= \frac{[\alpha_f \cdot (1-k) - 1]}{L_1 \cdot C_1 \cdot (1-\alpha_f)} \cdot \Gamma_3$$

$$\Gamma_2 = \frac{L_1}{V_t} \cdot \xi_4 = \frac{L_1^2}{V_t} \cdot C_1 \cdot (1-\alpha_f)$$

$$\eta_1(I) \cdot \Gamma_2 = \frac{L_1}{V_t} \cdot \xi_5(I)$$
$$\Rightarrow \left\{ \frac{\left[R_1 + \frac{V_t}{I+I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} \right\} \cdot \frac{L_1^2}{V_t} \cdot C_1 \cdot (1-\alpha_f)$$
$$= \frac{L_1}{V_t} \cdot C_1 \cdot (1-\alpha_f) \cdot \left[R_1 + \frac{V_t}{I_{D1} + I_0}\right]$$

$$\frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} = 0 \Rightarrow \left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f) \neq 0;\ [\alpha_f \cdot (1-k) - 1]$$
$$= 0$$

$$\left\{ \frac{L_1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + \xi_5(I) - \xi_2(I) \right\} = \eta_1(I) \cdot \Gamma_3 + \eta_2(I) \cdot \Gamma_2$$

$$\left\{ \frac{L_1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + C_1 \cdot (1-\alpha_f) \cdot \left[R_1 + \frac{V_t}{I+I_0}\right] - (\alpha_r - 1) \cdot C_1 \cdot \left[R_1 + \frac{V_t}{I+I_0}\right] \right\}$$
$$= \left\{ \frac{\left[R_1 + \frac{V_t}{I+I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} \right\} \cdot C_1 \cdot L_1 \cdot [2 - \alpha_f - \alpha_r]$$
$$+ \frac{I \cdot [\alpha_f \cdot (1-k) - 1]}{L_1 \cdot C_1 \cdot (1-\alpha_f)} \cdot \frac{L_1^2}{V_t} \cdot C_1 \cdot (1-\alpha_f)$$

$$\left\{ \frac{L_1}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + C_1 \cdot \left[R_1 + \frac{V_t}{I+I_0}\right] \cdot \left[2 - \alpha_f - \alpha_r\right] \right\}$$
$$= \left\{ \left[R_1 + \frac{V_t}{I+I_0}\right] \cdot C_1 + \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot (1-\alpha_f)} \right\} \cdot [2 - \alpha_f - \alpha_r]$$
$$+ I \cdot [\alpha_f \cdot (1-k) - 1] \cdot \frac{L_1}{V_t}$$

$$\frac{1}{V_t} \cdot \left\{ I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) - I \cdot [\alpha_f \cdot (1-k) - 1] \right\} = \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0} \right] \cdot (1 - \alpha_f)} \cdot [2 - \alpha_f - \alpha_r]$$

$$\Gamma_3 = \xi_4 - \xi_1 = L_1 \cdot C_1 \cdot (1 - \alpha_f) - (\alpha_r - 1) \cdot C_1 \cdot L_1 = C_1 \cdot L_1 \cdot [2 - \alpha_f - \alpha_r]$$

We can summarize our last result in Table 1.7:
We already discussed our system differential equation:

$$\psi_1 \left(\frac{d^2 I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I) \right) \cdot \psi_2 \left(\frac{dI}{dt}, I, \Gamma_1, \ldots, \Gamma_3 \right) = 0$$
$$\Rightarrow \left\{ \frac{d^2 I}{dt^2} + \frac{dI}{dt} \cdot \eta_1(I) + \eta_2(I) \right\} \cdot \left\{ I \cdot \Gamma_1 + \frac{dI}{dt} \cdot \Gamma_2 + \Gamma_3 \right\} = 0$$

One of the option is $\psi_1 \left(\frac{d^2 I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I) \right) = \left\{ \frac{d^2 I}{dt^2} + \frac{dI}{dt} \cdot \eta_1(I) + \eta_2(I) \right\} = 0$

ψ_1 represents Lienard system differential equation. Lienard's theorem states that our optoisolation system has a unique, stable limit cycle under appropriate hypotheses of η_1, $\eta_{2.}$ We need to check that our system $\eta_1(I)$ and $\eta_2(I)$ satisfy the following statements (Lienard's theorem). Our $\eta_1(I)$ and $\eta_2(I)$ are continuously differentiable for all I values. $\eta_2(-I) = -\eta_2(I)$ for all I values. $\eta_2(I) > 0$ for $I > 0$ only if $\frac{[\alpha_f \cdot (1-k) - 1]}{L_1 \cdot C_1 \cdot (1 - \alpha_f)} > 0 \Rightarrow [\alpha_f \cdot (1-k) - 1] > L_1 \cdot C_1 \cdot (1 - \alpha_f)$. We need to prove under which conditions $\eta_1(-\mathrm{I}) = -\eta_1(I)$ for all I values [17, 18].

Table 1.7 System/variables parameters and expressions

System/variables parameters	System expressions
$\eta_1(I)$	$\frac{\left[R_1 + \frac{V_t}{I + I_0} \right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0} \right] \cdot C_1 \cdot (1 - \alpha_f)}$
$\eta_2(I)$	$\frac{I \cdot [\alpha_f \cdot (1-k) - 1]}{L_1 \cdot C_1 \cdot (1 - \alpha_f)}$
Γ_1	$\left[\frac{R_1}{V_t} + \frac{1}{I_0} \right] \cdot L_1 \cdot C_1 \cdot (1 - \alpha_f)$
Γ_2	$\frac{L_1^2}{V_t} \cdot C_1 \cdot (1 - \alpha_f)$
Γ_3	$C_1 \cdot L_1 \cdot [2 - \alpha_f - \alpha_r]$

$$\eta_1(I) = \frac{\left[R_1 + \frac{V_t}{I + I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}$$
$$\Rightarrow \eta_1(-I) = \frac{\left[R_1 + \frac{V_t}{-I + I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}$$

$$-\eta_1(I) = -\frac{\left[R_1 + \frac{V_t}{I + I_0}\right]}{L_1} - \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}; \eta_1(-I) = -\eta_1(I)$$
$$\Rightarrow \frac{\left[R_1 + \frac{V_t}{-I + I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}$$
$$= -\frac{\left[R_1 + \frac{V_t}{I + I_0}\right]}{L_1} - \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}$$

$$V_t \cdot \left(\frac{1}{I_0 - I} + \frac{1}{I_0 + I}\right) = V_t \cdot \frac{2 \cdot I_0}{I_0^2 - I^2} \Rightarrow 2 \cdot R_1 + V_t \cdot \frac{2 \cdot I_0}{I_0^2 - I^2}$$
$$= -\frac{2 \cdot [\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}$$

Result The condition to get $\eta_1(-I) = -\eta_1(I)$ for all I values:

$$R_1 + V_t \cdot \frac{I_0}{I_0^2 - I^2} + \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} = 0$$

$$R_1 + V_t \cdot \frac{I_0}{I_0^2 - I^2} + \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} = 0$$

$$\frac{I_0}{I^2 - I_0^2} = \frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\};$$

$$I_0 = (I^2 - I_0^2) \cdot \frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}$$

$$I_0 + I_0^2 \cdot \frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}$$
$$= I^2 \cdot \frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}$$

I values must be

$$I = \pm \sqrt{\frac{I_0 \cdot \left[1 + I_0 \cdot \frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}\right]}{\frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}}}$$

Then we get restriction for expressions below the root.

$$\frac{I_0 \cdot \left[1 + I_0 \cdot \frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}\right]}{\frac{1}{V_t} \cdot \left\{ \frac{[\alpha_f \cdot (1-k) - 1] \cdot L_1}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + R_1 \right\}} \geq 0$$

The next condition we check, if the odd function: $F(I) = \int_0^I \eta_1(u) \cdot du$ has exactly one positive zero at $I = a$, is negative for $0 < I < a$, is positive and non-decreasing for $i > a$, and $F(I) \to \infty$ as $I \to \infty$. If all above conditions fulfill our system equation $\psi_1\left(\frac{d^2I}{dt^2}, \frac{dI}{dt}, I, \eta_1(I), \eta_2(I)\right)$ has a unique, stable limit cycle surrounding the origin in the phase plane.

$$\eta_1(I) = \frac{\left[R_1 + \frac{V_t}{I+I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)}$$

$$F(I) = \int_0^I \eta_1(u) \cdot \mathrm{d}u = \int_0^I \left\{ \frac{\left[R_1 + \frac{V_t}{u+I_0}\right]}{L_1} + \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} \right\} \cdot \mathrm{d}u$$

$$F(I) = \int_0^I \eta_1(u) \cdot \mathrm{d}u = \left\{ \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + \frac{R_1}{L_1} \right\} \cdot \int_0^I \mathrm{d}u + \frac{V_t}{L_1} \cdot \int_0^I \frac{\mathrm{d}u}{u+I_0}$$

$$\begin{aligned} F(I) &= \int_0^I \eta_1(u) \cdot \mathrm{d}u \\ &= \left\{ \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + \frac{R_1}{L_1} \right\} \cdot I + \frac{V_t}{L_1} \cdot \{\ln[I+I_0] - \ln[I_0]\} \end{aligned}$$

$$F(I=a) = \left\{ \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + \frac{R_1}{L_1} \right\} \cdot a + \frac{V_t}{L_1} \cdot \ln\left[\frac{a}{I_0} + 1\right]$$

We need to find "a" positive value which fulfills:

$$F(I=a) = 0 \Rightarrow \left\{ \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + \frac{R_1}{L_1} \right\} \cdot a + \frac{V_t}{L_1} \cdot \ln\left[\frac{a}{I_0} + 1\right] = 0$$

$$-\left\{ \frac{[\alpha_f \cdot (1-k) - 1]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot C_1 \cdot (1-\alpha_f)} + \frac{R_1}{L_1} \right\} \cdot \frac{L_1}{V_t} = \frac{1}{a} \cdot \ln\left[\frac{a}{I_0} + 1\right]$$

Exercise to the reader to check if all other conditions fulfill. Back to our primary multiplication functions, we need to check the second option ($\psi_2 = 0$) which is not related to the first option ($\psi_1 = 0$) of Lienard's differential equation [7].

$$\begin{aligned} &\psi_1\left(\frac{\mathrm{d}^2 I}{\mathrm{d}t^2}, \frac{\mathrm{d}I}{\mathrm{d}t}, I, \eta_1(I), \eta_2(I)\right) \cdot \psi_2\left(\frac{\mathrm{d}I}{\mathrm{d}t}, I, \Gamma_1, \ldots, \Gamma_3\right) = 0 \\ &\Rightarrow \left\{\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{\mathrm{d}I}{\mathrm{d}t} \cdot \eta_1(I) + \eta_2(I)\right\} \cdot \left\{I \cdot \Gamma_1 + \frac{\mathrm{d}I}{\mathrm{d}t} \cdot \Gamma_2 + \Gamma_3\right\} = 0 \end{aligned}$$

$$\psi_2\left(\frac{\mathrm{d}I}{\mathrm{d}t}, I, \Gamma_1, \ldots, \Gamma_3\right) = 0 \Rightarrow I \cdot \Gamma_1 + \frac{\mathrm{d}I}{\mathrm{d}t} \cdot \Gamma_2 + \Gamma_3 = 0$$
$$\Rightarrow I(t) = -\frac{\Gamma_3}{\Gamma_1} + \mathrm{Const} \cdot \mathrm{e}^{-\frac{\Gamma_1}{\Gamma_2} \cdot t}$$

Then we get our second option.

$$I(t) = -\frac{C_1 \cdot L_1 \cdot [2 - \alpha_f - \alpha_r]}{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot L_1 \cdot C_1 \cdot (1 - \alpha_f)} + \mathrm{Const} \cdot \mathrm{e}^{-\frac{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot L_1 \cdot C_1 \cdot (1-\alpha_f)}{\frac{L_1^2}{V_t} \cdot C_1 \cdot (1-\alpha_f)} \cdot t}$$

$$I(t) = -\frac{[2 - \alpha_f - \alpha_r]}{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot (1 - \alpha_f)} + \mathrm{Const} \cdot \mathrm{e}^{-\frac{\left[\frac{R_1}{V_t} + \frac{1}{I_0}\right] \cdot V_t}{L_1} \cdot t}$$

1.6 Optoisolation Circuits with Weakly Nonlinear Oscillations

We implement by using optoisolation circuits weakly Nonlinear oscillator [6, 7]. The general equation of nonlinear oscillator of the form:

$$\frac{\mathrm{d}^2 V(t)}{\mathrm{d}t^2} + V(t) + \varepsilon \cdot h\left(V(t), \frac{\mathrm{d}V(t)}{\mathrm{d}t}\right) = 0$$

V(t) is our system main variable, where $0 = < \varepsilon \ll 1$ and $h\left(V(t), \frac{\mathrm{d}V(t)}{\mathrm{d}t}\right)$ is a smooth function. We have small perturbation of the linear oscillator $\frac{\mathrm{d}^2 V(t)}{\mathrm{d}t^2} + V(t) = 0 \Rightarrow V(t) = \cos(t) + \sin(t)$. Then it is called weakly nonlinear oscillator. The below optoisolation circuit demonstrates our weakly nonlinear oscillator (Fig. 1.17).

We consider capacitor C1 is initially charged to specific positive voltage before switching S1 from OFF to ON state. $V_{\mathrm{C1}}(t = t_0) \gg V_{\mathrm{D1ON}}$ and V_{C1} $(t = t_0) > 0$.

$$I_{C1} = C_1 \cdot \frac{\mathrm{d}V_{C1}}{\mathrm{d}t};\ I_{CQ1} + I_{R1} = I_{C1} = I_{L1};\ I_{R1} = I_{D1};$$
$$V = R_1 \cdot I_{D1} + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right)$$

By using Taylor series approximation: $\ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx \frac{I_{D1}}{I_0}$.

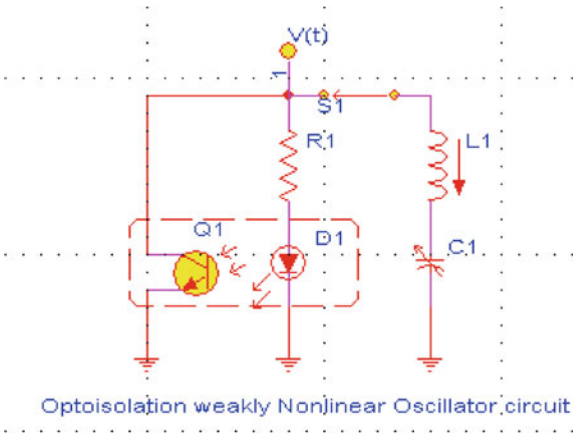


Fig. 1.17 Optoisolation weakly nonlinear oscillator circuit

$$V = R_1 \cdot I_{D1} + V_t \cdot \frac{I_{D1}}{I_0} = I_{D1} \cdot \left(R_1 + V_t \cdot \frac{1}{I_0}\right) \Rightarrow I_{D1} = \frac{V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)}$$

$$V = V_{CEQ1} \approx V_t \cdot \ln[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)};\ I_{\mathrm{BQ1}} = k \cdot I_{D1}$$

$$I_{EQ1} = I_{\mathrm{BQ1}} + I_{CQ1} = \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} + I_{CQ1}$$

Assumption $k > 1$, for keeping the saturation process after breakover happened. $V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] \approx 0$ then we get the expression for Q1 emitter–collector voltage.

$$V = V_{CEQ1} \approx V_t \cdot \ln\left[\frac{\left(\alpha_r \cdot I_{CQ1} - \left[\frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} + I_{CQ1}\right]\right) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\left(I_{CQ1} - \left[\frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} + I_{CQ1}\right] \cdot \alpha_f\right) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$V = V_{CEQ1} \approx V_t \cdot \ln\left[\frac{\left(\alpha_r \cdot I_{CQ1} - \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - I_{CQ1}\right) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\left(I_{CQ1} - \frac{k \cdot V \cdot \alpha_f}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - I_{CQ1} \cdot \alpha_f\right) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$V = V_{CEQ1} \approx V_t \cdot \ln \left[\frac{\left([\alpha_r - 1] \cdot I_{CQ1} - \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} \right) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{\left(-\frac{k \cdot V \cdot \alpha_f}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{CQ1} \cdot [1 - \alpha_f] \right) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)} \right]$$

$$e^{\left[\frac{V}{V_t}\right]} = \frac{[\alpha_r - 1] \cdot I_{CQ1} - \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{-\frac{k \cdot V \cdot \alpha_f}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{CQ1} \cdot [1 - \alpha_f] + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

Taylor series approximation: $e^{\left[\frac{V}{V_t}\right]} \approx \left(1 + \frac{V}{V_t} \right)$

$$\begin{aligned}
&\left(1 + \frac{V}{V_t} \right) \cdot \left\{ -\frac{k \cdot V \cdot \alpha_f}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{CQ1} \cdot [1 - \alpha_f] + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \right\} \\
&\quad = [\alpha_r - 1] \cdot I_{CQ1} - \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \\
&\quad - \frac{k \cdot V \cdot \alpha_f}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{CQ1} \cdot [1 - \alpha_f] + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \\
&\quad - \frac{k \cdot V^2 \cdot \alpha_f}{V_t \cdot \left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + \frac{V}{V_t} \cdot I_{CQ1} \cdot [1 - \alpha_f] \\
&\quad + \frac{V}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) = [\alpha_r - 1] \cdot I_{CQ1} \\
&\quad - \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)
\end{aligned}$$

$$\begin{aligned}
&I_{CQ1} \cdot \left\{ [1 - \alpha_f] + \frac{V}{V_t} \cdot [1 - \alpha_f] - [\alpha_r - 1] \right\} = \frac{k \cdot V \cdot \alpha_f}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} - I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \\
&\quad + \frac{k \cdot V^2 \cdot \alpha_f}{V_t \cdot \left(R_1 + V_t \cdot \frac{1}{I_0} \right)} - \frac{V}{V_t} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) - \frac{k \cdot V}{\left(R_1 + V_t \cdot \frac{1}{I_0} \right)} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)
\end{aligned}$$

$$I_{CQ1}\cdot\left\{[1-\alpha_f]+\frac{V}{V_t}\cdot[1-\alpha_f]-[\alpha_r-1]\right\}=\frac{k\cdot V\cdot\alpha_f}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+\frac{k\cdot V^2\cdot\alpha_f}{V_t\cdot(R_1+V_t\cdot\frac{1}{I_0})}$$
$$-\frac{V}{V_t}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)-\frac{k\cdot V}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+(I_{se}-I_{sc})\cdot(\alpha_r\cdot\alpha_f-1)$$

Since $(I_{se}-I_{sc})\to\varepsilon$ then $(I_{se}-I_{sc})\cdot(\alpha_r\cdot\alpha_f-1)\to\varepsilon$

$$I_{CQ1}\cdot\left\{[1-\alpha_f]+\frac{V}{V_t}\cdot[1-\alpha_f]-[\alpha_r-1]\right\}=\frac{k\cdot V\cdot\alpha_f}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+\frac{k\cdot V^2\cdot\alpha_f}{V_t\cdot\left(R_1+V_t\cdot\frac{1}{I_0}\right)}$$
$$-\frac{V}{V_t}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)-\frac{k\cdot V}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}$$

$$I_{CQ1}=\frac{V\cdot\left\{\frac{k\cdot(\alpha_f-1)}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}-\frac{I_{sc}}{V_t}\cdot(\alpha_r\cdot\alpha_f-1)\right\}+\frac{k\cdot\alpha_f}{V_t\cdot\left(R_1+V_t\cdot\frac{1}{I_0}\right)}\cdot V^2}{[1-\alpha_f]-[\alpha_r-1]+\frac{1}{V_t}\cdot[1-\alpha_f]\cdot V}$$

$$A_1=\frac{k\cdot(\alpha_f-1)}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}-\frac{I_{sc}}{V_t}\cdot(\alpha_r\cdot\alpha_f-1);\ A_2=\frac{k\cdot\alpha_f}{V_t\cdot\left(R_1+V_t\cdot\frac{1}{I_0}\right)};$$
$$A_3=[1-\alpha_f]-[\alpha_r-1]$$

$$A_4=\frac{1}{V_t}\cdot[1-\alpha_f];\ I_{CQ1}(V)=\frac{V\cdot A_1+V^2\cdot A_2}{A_3+A_4\cdot V};\ V=V_{C1}+V_{L1}=V_{C1}+L_1\cdot\frac{\mathrm{d}I_{C1}}{\mathrm{d}t}$$

$$I_{C1}=C_1\cdot\frac{\mathrm{d}V_{C1}}{\mathrm{d}t}\Rightarrow V_{C1}=\frac{1}{C_1}\cdot\int I_{C1}\cdot\mathrm{d}t;$$
$$V=\frac{1}{C_1}\cdot\int I_{C1}\cdot\mathrm{d}t+L_1\cdot\frac{\mathrm{d}I_{C1}}{\mathrm{d}t};\ I_{CQ1}=I_{C1}-I_{D1}$$

$$I_{CQ1}=I_{C1}-I_{D1}=C_1\cdot\frac{\mathrm{d}V_{C1}}{\mathrm{d}t}-\frac{V}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)};\ I_{CQ1}(V)=\frac{V\cdot A_1+V^2\cdot A_2}{A_3+A_4\cdot V}$$

$$I_{C1}=I_{CQ1}+I_{D1}=\frac{V\cdot A_1+V^2\cdot A_2}{A_3+A_4\cdot V}+\frac{V}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)};\ \frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{C_1}\cdot I_{C1}+L_1\cdot\frac{\mathrm{d}^2I_{C1}}{\mathrm{d}t^2}$$

$$\frac{dV}{dt} = \frac{1}{C_1} \cdot I_{C1} + L_1 \cdot \frac{d^2 I_{C1}}{dt^2}$$
$$= \frac{1}{C_1} \cdot \left\{ \frac{V \cdot A_1 + V^2 \cdot A_2}{A_3 + A_4 \cdot V} + \frac{V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} \right\} + L_1 \cdot \frac{d^2 I_{C1}}{dt^2}$$

$$\frac{dV}{dt} = \frac{1}{C_1} \cdot \left\{ \frac{V \cdot A_1 + V^2 \cdot A_2}{A_3 + A_4 \cdot V} + \frac{V}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} \right\} + \frac{d^2 V}{dt^2} \cdot \frac{L_1}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)}$$
$$+ L_1 \cdot \frac{d^2}{dt^2} \left\{ V \cdot \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) \right\}$$

First we need to get the expression for $\frac{dI_{C1}}{dt} = \frac{d}{dt} \left\{ V \cdot \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) \right\}$.

$$\frac{d}{dt} \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) = \frac{dV}{dt} \cdot \left[\frac{A_2 \cdot A_3 - A_1 \cdot A_4}{(A_3 + A_4 \cdot V)^2} \right];$$
$$\frac{d}{dt} \left[\frac{A_2 \cdot A_3 - A_1 \cdot A_4}{(A_3 + A_4 \cdot V)^2} \right] = -2 \cdot (A_2 \cdot A_3 - A_1 \cdot A_4) \cdot A_4 \cdot \frac{\frac{dV}{dt}}{(A_3 + A_4 \cdot V)^3}$$

$$\frac{d}{dt} \left\{ V \cdot \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) \right\} = \frac{dV}{dt} \cdot \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) + V \cdot \frac{dV}{dt} \cdot \left[\frac{A_2 \cdot A_3 - A_1 \cdot A_4}{(A_3 + A_4 \cdot V)^2} \right]$$

$$\frac{d^2}{dt^2} \left\{ V \cdot \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) \right\} = \frac{d^2 V}{dt^2} \cdot \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right) + \frac{dV}{dt} \cdot \frac{d}{dt} \left(\frac{A_1 + V \cdot A_2}{A_3 + A_4 \cdot V} \right)$$
$$+ \left[\frac{dV}{dt} \right]^2 \cdot \left[\frac{A_2 \cdot A_3 - A_1 \cdot A_4}{(A_3 + A_4 \cdot V)^2} \right]$$
$$+ V \cdot \frac{d^2 V}{dt^2} \cdot \left[\frac{A_2 \cdot A_3 - A_1 \cdot A_4}{(A_3 + A_4 \cdot V)^2} \right]$$
$$+ V \cdot \frac{dV}{dt} \cdot \frac{d}{dt} \left[\frac{A_2 \cdot A_3 - A_1 \cdot A_4}{(A_3 + A_4 \cdot V)^2} \right]$$

$$\frac{d^2}{dt^2}\left\{V\cdot\left(\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}\right)\right\}=\frac{d^2V}{dt^2}\cdot\left(\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}\right)+\left[\frac{dV}{dt}\right]^2\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]$$
$$+\left[\frac{dV}{dt}\right]^2\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]$$
$$+V\cdot\frac{d^2V}{dt^2}\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]$$
$$-V\cdot\left[\frac{dV}{dt}\right]^2\cdot\frac{2\cdot(A_2\cdot A_3-A_1\cdot A_4)\cdot A_4}{(A_3+A_4\cdot V)^3}$$

$$\frac{d^2}{dt^2}\left\{V\cdot\left(\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}\right)\right\}=\frac{d^2V}{dt^2}\cdot\left(\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}\right)+2\cdot\left[\frac{dV}{dt}\right]^2\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]$$
$$+V\cdot\frac{d^2V}{dt^2}\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]$$
$$-V\cdot\left[\frac{dV}{dt}\right]^2\cdot\frac{2\cdot(A_2\cdot A_3-A_1\cdot A_4)\cdot A_4}{(A_3+A_4\cdot V)^3}$$

$$\frac{d^2}{dt^2}\left\{V\cdot\left(\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}\right)\right\}=\frac{d^2V}{dt^2}\cdot\left\{\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+V\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]\right\}$$
$$+2\cdot\left[\frac{dV}{dt}\right]^2\cdot\frac{(A_2\cdot A_3-A_1\cdot A_4)}{(A_3+A_4\cdot V)^2}\cdot\left[1-V\cdot\frac{A_4}{(A_3+A_4\cdot V)}\right]$$

$$\frac{dV}{dt}=\frac{1}{C_1}\cdot\left\{\frac{V\cdot A_1+V^2\cdot A_2}{A_3+A_4\cdot V}+\frac{V}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}\right\}+\frac{d^2V}{dt^2}\cdot\frac{L_1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+L_1\cdot\frac{d^2V}{dt^2}\cdot\left\{\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}\right.$$
$$\left.+V\cdot\left[\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right]\right\}+2\cdot L_1\cdot\left[\frac{dV}{dt}\right]^2\cdot\frac{(A_2\cdot A_3-A_1\cdot A_4)}{(A_3+A_4\cdot V)^2}\cdot\left[1-V\cdot\frac{A_4}{(A_3+A_4\cdot V)}\right]\right\}$$

$$\frac{dV}{dt}=\frac{1}{C_1}\cdot V\cdot\left\{\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+\frac{1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}\right\}$$
$$+L_1\cdot\frac{d^2V}{dt^2}\cdot\left[\frac{1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+V\cdot\left\{\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right\}\right]$$
$$+2\cdot L_1\cdot\left[\frac{dV}{dt}\right]^2\cdot\frac{(A_2\cdot A_3-A_1\cdot A_4)}{(A_3+A_4\cdot V)^2}\cdot\left[1-V\cdot\frac{A_4}{(A_3+A_4\cdot V)}\right]$$

We define the following functions for simplicity:

$$f_1(V) = \frac{1}{C_1}\cdot V\cdot\left\{\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+\frac{1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}\right\};\ f_2(V)$$
$$= L_1\cdot\left[\frac{1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+V\cdot\left\{\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right\}\right]$$

$$f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) = 2\cdot L_1\cdot\left[\frac{\mathrm{d}V}{\mathrm{d}t}\right]^2\cdot\frac{(A_2\cdot A_3-A_1\cdot A_4)}{(A_3+A_4\cdot V)^2}\cdot\left[1-V\cdot\frac{A_4}{(A_3+A_4\cdot V)}\right]$$

We get the equation: $\frac{\mathrm{d}V}{\mathrm{d}t} = f_1(V)+\frac{\mathrm{d}^2V}{\mathrm{d}t^2}\cdot f_2(V)+f_3(V,\frac{\mathrm{d}V}{\mathrm{d}t})$

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2}\cdot f_2(V) = \frac{\mathrm{d}V}{\mathrm{d}t}-f_1(V)-f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)\Rightarrow\frac{\mathrm{d}^2V}{\mathrm{d}t^2}$$
$$=\frac{1}{f_2(V)}\cdot\left\{\frac{\mathrm{d}V}{\mathrm{d}t}-f_1(V)-f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)\right\}\frac{\mathrm{d}^2V}{\mathrm{d}t^2}+V$$
$$=\frac{1}{f_2(V)}\cdot\left\{\frac{\mathrm{d}V}{\mathrm{d}t}-f_1(V)-f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)\right\}+V\Rightarrow\frac{\mathrm{d}^2V}{\mathrm{d}t^2}+V$$
$$=\frac{1}{f_2(V)}\cdot\left\{\frac{\mathrm{d}V}{\mathrm{d}t}-f_1(V)-f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)+f_2(V)\cdot V\right\}$$
$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2}+V+\frac{1}{f_2(V)}\cdot\left\{-\frac{\mathrm{d}V}{\mathrm{d}t}+f_1(V)+f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)-f_2(V)\cdot V\right\}=0$$

The general equation of weakly nonlinear system:

$$\frac{\mathrm{d}^2V(t)}{\mathrm{d}t^2}+V(t)+\varepsilon\cdot h\left(V(t),\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right)=0$$
$$\Rightarrow h\left(V(t),\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right)=-\frac{\mathrm{d}V}{\mathrm{d}t}+f_1(V)+f_3\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)-f_2(V)\cdot V$$

$$\varepsilon=\frac{1}{f_2(V)}=\frac{1}{L_1\cdot\left[\frac{1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+V\cdot\left\{\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right\}\right]};f_2(V)\neq 0$$

$$0\leq\varepsilon\ll 1$$

$$\Rightarrow 0\leq\frac{1}{f_2(V)}\ll 1\Rightarrow L_1\cdot\left[\frac{1}{\left(R_1+V_t\cdot\frac{1}{I_0}\right)}+\frac{A_1+V\cdot A_2}{A_3+A_4\cdot V}+V\cdot\left\{\frac{A_2\cdot A_3-A_1\cdot A_4}{(A_3+A_4\cdot V)^2}\right\}\right]\gg 1$$

$$L_1 \cdot \left[\frac{1}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} + \frac{\frac{k\cdot(\alpha_f - 1)}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - \frac{I_{sc}}{V_t} \cdot (\alpha_r \cdot \alpha_f - 1) + V \cdot \left\{ \frac{k \cdot \alpha_f}{V_t \cdot \left(R_1 + V_t \cdot \frac{1}{I_0}\right)} \right\}}{[1 - \alpha_f] - [\alpha_r - 1] + \frac{1}{V_t} \cdot [1 - \alpha_f] \cdot V} \right.$$

$$\left. + V \cdot \left\{ \frac{\left\{ \frac{k \cdot \alpha_f}{V_t \cdot \left(R_1 + V_t \cdot \frac{1}{I_0}\right)} \right\} \cdot \{[1 - \alpha_f] - [\alpha_r - 1]\} - \left\{ \frac{k\cdot(\alpha_f - 1)}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - \frac{I_{sc}}{V_t} \cdot (\alpha_r \cdot \alpha_f - 1) \right\} \cdot \frac{1}{V_t} \cdot [1 - \alpha_f]}{\left([1 - \alpha_f] - [\alpha_r - 1] + \frac{1}{V_t} \cdot [1 - \alpha_f] \cdot V\right)^2} \right\} \right] \gg 1$$

Based on the above equation, we need to find system's C1 initial voltage minimum value.

$$L_1 \cdot \left[\frac{1}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} + \frac{\frac{k \cdot \left\{\alpha_f \cdot \left(1 + \frac{V}{V_t}\right) - 1\right\}}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - \frac{I_{sc}}{V_t} \cdot (\alpha_r \cdot \alpha_f - 1)}{2 - \alpha_f - \alpha_r + \frac{1}{V_t} \cdot [1 - \alpha_f] \cdot V} \right.$$

$$\left. + V \cdot \left\{ \frac{\frac{k \cdot \alpha_f \cdot [2 - \alpha_f - \alpha_r]}{V_t \cdot \left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - \left\{ \frac{k\cdot(\alpha_f - 1)}{\left(R_1 + V_t \cdot \frac{1}{I_0}\right)} - \frac{I_{sc}}{V_t} \cdot (\alpha_r \cdot \alpha_f - 1) \right\} \cdot \frac{1}{V_t} \cdot [1 - \alpha_f]}{\left(2 - \alpha_f - \alpha_r + \frac{1}{V_t} \cdot [1 - \alpha_f] \cdot V\right)^2} \right\} \right] \gg 1$$

1.7 Exercises

1. We have van der Pol system where μ parameter is dependent on overall system variable values $V(t)$; $\mu(V) = f_{\#\#}(V)$. Try to find system block diagram, differential equations, fixed points, phase diagrams for the following cases: (a) $f_{\#\#}(V) = \sin(V)/V$; (b) $f_{\#\#}(V) = V^2 - 1$; (c) $f_{\#\#}(V) = \text{sign}[\sin(V)]$. Our van der Pol system differential equation:

$$\frac{d^2V}{dt^2} + \mu(V) \cdot \frac{dV}{dt} \cdot (V^2 - 1) + V = 0 \Rightarrow \frac{d^2V}{dt^2} + f_{\#\#}(V) \cdot \frac{dV}{dt} \cdot (V^2 - 1) + V = 0.$$

2. Implement van der Pol system with $\mu(V)$ dependent parameter by using optoisolation circuits for $\mu(V) = V^3 - 1$. Use optocouplers, Op-amps, resistors,

capacitors and other discrete components. Find new system differential equations, draw phase plan graphs. Try to find limit cycles.

3. Discuss fixed points stability of van der Pol system in (2) with dependent parameter $\mu(V) = V^3 - 1$.
4. We have system equation: $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r}{1-r^2} + \mu \cdot \mathrm{sign}\,(r) \cdot \cos\,(\theta)$. Find $f(\mu, r)$ for two main cases (sign(r) function). Try to represent system by two main variables X, Y. Find Jacobian matrix, fixed points, eigenvalues, and stability analysis.
5. For system in (4) try to differentiate limit cycle types for dr/dt signs values. Discuss all options and behaviors. Plot X, Y phase plane for different μ parameters values.
6. We have system equation: $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2}{1-r^3} + \mu \cdot \sin\,[\theta + \Gamma(r)]$. $\Gamma(r)$ is phase shift which is dependent on r variable value. We consider the following cases: (a) $\Gamma(r) = r^2 + 1$; (b) $\Gamma(r) = r^3 - 1$; (c) $\Gamma(r) = r/(r^2 - 1)$. Find $f(\mu, r)$. Try to represent the system by two main variables X, Y, find Jacobian matrix, fixed points, eigenvalues, and stability analysis.
7. For system in (6) try to differentiate limit cycle types for dr/dt sign values, discuss all options of behavior. Plot X, Y phase plan for different μ parameter values.
8. Try to implement $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2}{1-r^3} + \mu \cdot \sin\,[\theta + \Gamma(r)]$ system differential equations by using optoisolation circuits. Discuss stability, limit cycle, behavior changes for different $\Gamma(r)$ functions ($\Gamma(r) = r^2/(r - 1)$).
9. We have the following Lienards optoisolation circuit (Fig. 1.18).

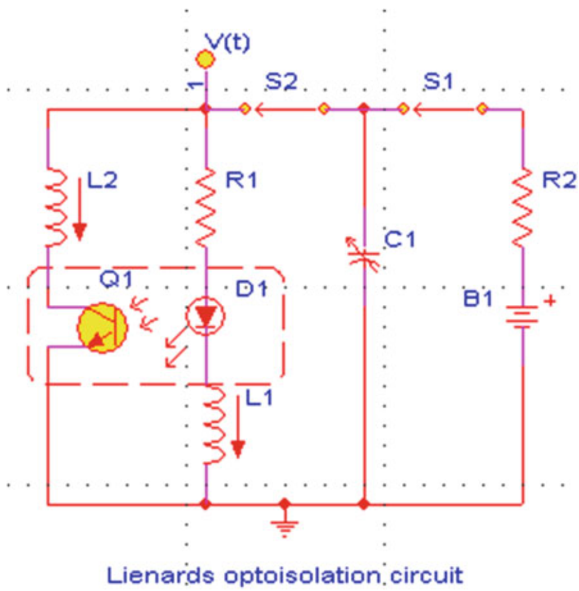


Fig. 1.18 Lienards optoisolation circuit

First switch S1 is ON and S2 is OFF, capacitor C1 is charged to V_{B1} voltage ($t \to \infty$). The next step switch S1 is OFF and S2 is ON ($t = t_0$). We consider the initial C1 voltage at $t = t_0$ is greater than LED D1 ON voltage $(V_{C1}(t = t_0) \gg V_{D1ON})$. At $t = t_0$, C1 capacitor acts as voltage source. Find circuit differential equation, use Taylor series approximation. Find system variables parameters as a function of circuit parameters (R_1, L_1, L_2, I_0, α_r, α_f, k, etc.,). Check if Lienard's theorem satisfies all conditions.

10. We have the following optoisolation weakly nonlinear oscillation circuit (Fig. 1.19).

    We consider that capacitor C1 is initially charged to specific positive voltage before switching S1 from OFF state to ON state. $(V_{C1}(t = t_0) \gg V_{D1ON})$. $V_{C1}(t = t_0) > 0$. $I_{L1}(t = t_0) = 0$, $I_{L2}(t = t_0) = 0$. Find circuit differential equations. Find the expression for ε parameter as a function of $V(t)$. Try to get $V(t)$ extreme values conditions based on the require that $0 \leq \varepsilon \ll 1$. Find $h(V, \mathrm{d}V/\mathrm{d}t)$ smooth function and plot the related graphs. Plot phase space for two variables $\mathrm{d}V/\mathrm{d}t$, $V(t)$.

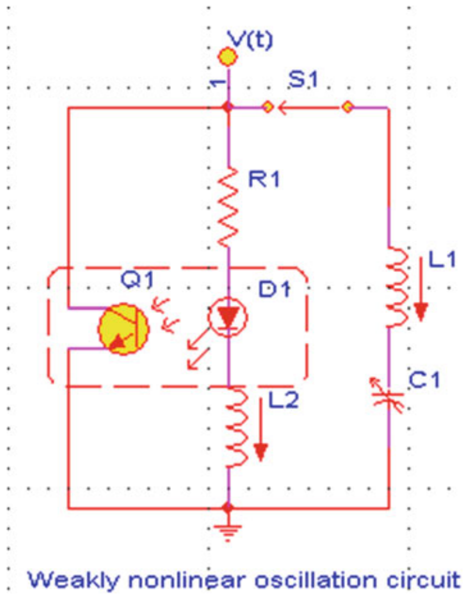


Fig. 1.19 Weakly nonlinear oscillator circuit

Chapter 2
Optoisolation Circuits Bifurcation Analysis (I)

The basic definition of bifurcation describes the qualitative alterations that occur in the orbit structure of a dynamical system as the parameters on which the system depends are varied. In this chapter, we discuss various bifurcations which are exhibited by optoisolation circuits. The first is cusp catastrophe which occurs in a one-dimensional state space ($n = 1$) and two-dimensional parameter space ($p = 2$). It has an equilibrium manifold in $R^2 \times R$: $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2) \quad V \in \mathbb{R}^{n=1}$; $\{\Gamma_1, \Gamma_2\} \in \mathbb{R}^{p=2}$; $M = \{(\Gamma_1, \Gamma_2, V) | \Gamma_1 + \Gamma_2 \cdot V - V^3 = 0\}$. Where Γ_1, Γ_2 are two control parameters and V is the system state variable. A two-parameter system near a triple equilibrium point is known as cusp bifurcation (equilibrium structure). We can consider a system with a higher dimensional state space and parameter space. The Bautin bifurcation is equilibrium in a two-parameter family of system's autonomous ODEs at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov–Hopf bifurcation vanishes. This phenomenon is also called the Generalized Hopf (GH) bifurcation. Bogdanov–Takens (BT) bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs, critical equilibrium has a zero eigenvalue of multiplicity two [6, 7, 71].

2.1 Cusp Bifurcation Analysis System

In electronics, an optoisolator can be implemented in many engineering circuits. Some of them demonstrate cusp bifurcation for specific system voltage band and parameters values. We analyze Cusp catastrophe which occurs in a one-dimensional state space ($n = 1$) and two-dimensional parameter space ($p = 2$). $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2)$ $V \in \mathbb{R}^{n=1}$; $\{\Gamma_1, \Gamma_2\} \in \mathbb{R}^{p=2}$; $M = \{(\Gamma_1, \Gamma_2, V) | \Gamma_1 + \Gamma_2 \cdot V - V^3 = 0\}$. In the above equation Γ_1, Γ_2 are two control parameters and V is the system state variable.

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A two-parameter system near a triple equilibrium point is known as cusp bifurcation (equilibrium structure). This is the simplest degenerate case of the fold bifurcation related to the cusp catastrophe theory with hysteresis bifurcation. The projection of the cusp manifold onto the (Γ_1, Γ_2) plane yields the cusp bifurcation variety, consisting of two algebraic curves in the parameter plane, meeting tangentially at the cusp point (0, 0). For (Γ_1, Γ_2) in the interior of the wedge bounded by the cusp bifurcation variety, there exist three distinct equilibrium points V, while exterior to this wedge there is an unique equilibrium point V. On crossing the bifurcation variety, from the interior to the exterior at any point other than the cusp point (0, 0), two equilibrium points V coalesce and disappear in a fold bifurcation [7–9]. Inside the wedge, the upper and lower equilibrium point V of equation $\frac{dV}{dt} = (\Gamma_1 + \Gamma_2 \cdot V - V^3)$ are stable, while the third equilibrium point lying between them is unstable. This coexistence of two distinct attractors at the same parameter value is called bi-stability. If Γ_1 is varied with fixed $\Gamma_2 > 0$, the system jumps from one stable equilibrium to the other stable equilibrium at the two endpoints of an interval, thus tracing a hysteresis loop. As we increase or decrease Γ_2, the length of this hysteresis interval increases, respectively, and it vanishes at the cusp point (0, 0). $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2)$; $f(V, \Gamma_1, \Gamma_2) = \Gamma_1 + \Gamma_2 \cdot V - V^3$, we wish to find roots of $f(V, \Gamma_1, \Gamma_2)$, which correspond to stationary solutions of the differential equation. Turning points of $f(V, \Gamma_1, \Gamma_2)$ are values of $\frac{df(V,\Gamma_1,\Gamma_2)}{dV}$ satisfying $\frac{df(V,\Gamma_1,\Gamma_2)}{dV} = \Gamma_2 - 3 \cdot V^2 = 0 \Rightarrow V = \pm\sqrt{\frac{\Gamma_2}{3}}$, So $f(V, \Gamma_1, \Gamma_2)$ has turning points only if $\Gamma_2 > 0$. Hence, if $\Gamma_2 < 0$ then $f(V, \Gamma_1, \Gamma_2)$ has a single root, with the root being positive if $\Gamma_1 > 0$ and negative if $\Gamma_1 < 0$.

If $\Gamma_2 > 0$ the value of $f(V, \Gamma_1, \Gamma_2)$ at the turning points is given by $f\left(V = \pm\sqrt{\frac{\Gamma_2}{3}}\right) = \Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}$; $f\left(V = -\sqrt{\frac{\Gamma_2}{3}}\right) = \Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}$. Thus, there will be one real, positive root of $f(V, \Gamma_1, \Gamma_2)$ if $\left\{\Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right\} > 0$, three real roots if $\left\{\Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right\} < 0$ and $\left\{\Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right\} > 0$, one real negative root if $\left\{\Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right\} < 0$. Bifurcation occurs at $\Gamma_1 \pm \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}$, and we see that these bifurcations are probably saddle–node bifurcations. Our system differential equation is $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2)$; $f(V, \Gamma_1, \Gamma_2) = \Gamma_1 + \Gamma_2 \cdot V - V^3$ to find fixed points we set $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2) = 0 \Rightarrow \Gamma_1 + \Gamma_2 \cdot V - V^3$ and we get cubic equation. Every cubic equation with real coefficients has at least one solution V among the real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant, $\Delta = 4 \cdot \Gamma_2^3 - 27 \cdot \Gamma_1^2$.

The following cases need to be considered: if $\Delta > 0$, then the equation has three distinct real roots.

$$\Delta > 0 \Rightarrow 4 \cdot \Gamma_2^3 - 27 \cdot \Gamma_1^2 > 0 \Rightarrow \left(\Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) \cdot \left(\Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) < 0; \quad \Gamma_2 > 0$$

$$\left(\Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) \cdot \left(\Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) < 0$$
$$\Rightarrow -\frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}} < \Gamma_1 < \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}$$

If $\Delta < 0$, then the equation has one real root and a pair of complex conjugate roots.

$$\Delta < 0 \Rightarrow 4 \cdot \Gamma_2^3 - 27 \cdot \Gamma_1^2 < 0 \Rightarrow \left(\Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) \cdot \left(\Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) > 0; \quad \Gamma_2 > 0$$

$$\left(\Gamma_1 - \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) \cdot \left(\Gamma_1 + \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right) > 0$$
$$\Rightarrow \Gamma_1 > \frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}} \quad \text{OR} \quad \Gamma_1 < -\frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}$$

If $\Delta = 0$, then (at least) two roots coincide. It may be that the equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple real root. A possible way to decide between these sub-cases is to compute the resultant of the cubic and its second derivative: a triple root exists if and only if this resultant vanishes.

$$\Delta = 0 \Rightarrow 4 \cdot \Gamma_2^3 - 27 \cdot \Gamma_1^2 = 0 \Rightarrow \left(\Gamma_1 = \pm\frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}\right); \quad \Gamma_2 > 0$$

$\Gamma_1 + \Gamma_2 \cdot V - V^3 = 0 \Rightarrow \Gamma_1 + \Gamma_2 \cdot V - V^3 = (V - a)(V - b)(V - c) = 0$; we need to find a, b, c roots as a function of system overall constant Γ_1, Γ_2.

$$a = \sqrt[3]{\frac{\Gamma_1 + \sqrt{\Gamma_1^2 - \frac{4}{27} \cdot \Gamma_2^3}}{2}} + \sqrt[3]{\frac{\Gamma_1 - \sqrt{\Gamma_1^2 - \frac{4}{27} \cdot \Gamma_2^3}}{2}}$$

$$b = -\frac{(1 + i \cdot \sqrt{3})}{2} \cdot \sqrt[3]{\frac{\Gamma_1 + \sqrt{\Gamma_1^2 - \frac{4}{27} \cdot \Gamma_2^3}}{2}} - \frac{(1 - i\sqrt{3})}{2} \cdot \sqrt[3]{\frac{\Gamma_1 - \sqrt{\Gamma_1^2 - \frac{4}{27} \cdot \Gamma_2^3}}{2}}$$

$$c = -\frac{(1 - i \cdot \sqrt{3})}{2} \cdot \sqrt[3]{\frac{\Gamma_1 + \sqrt{\Gamma_1^2 - \frac{4}{27} \cdot \Gamma_2^3}}{2}} - \frac{(1 + i\sqrt{3})}{2} \cdot \sqrt[3]{\frac{\Gamma_1 - \sqrt{\Gamma_1^2 - \frac{4}{27} \cdot \Gamma_2^3}}{2}}$$

Results Table 2.1 describes our cusp system differential equation behavior. We plot the graphs for stationary case before bifurcation happened.

The graphs related to parameter value $\Gamma_2 > 0$ are plotted for the case of three fixed points (two stable fixed points and one unstable fixed point). All those graphs are for the state before bifurcation. Upon changing Γ_1 parameter value bifurcation happened. The graphs which related to parameter value $\Gamma_2 < 0$ give the result of one stable fixed point. Upon changing Γ_1 parameter value, only the location of the stable fixed point change (no bifurcation). If we plot the fixed points ($V^{(i)}$) above (Γ_1, Γ_2) plane, we get the cusp catastrophe surface. The surface folds over on itself in certain places. The projection of these folds onto the (Γ_1, Γ_2) plane yields the bifurcation curves. The Γ_1, Γ_2 parameters change, the state of the system can be carried over the edge of the upper surface, after which it drops discontinuously to the lower surface. This jump could be truly catastrophic for the equilibrium of the system. We define N variable as the number of fixed points when $\Gamma_2 > 0$. When $\Gamma_1 = \pm\frac{2}{3} \cdot \Gamma_2 \cdot \sqrt{\frac{\Gamma_2}{3}}$ two fixed points (one stable fixed point and the other unstable fixed point) collide to one fixed point (half stable and half unstable fixed point) (Fig. 2.1).

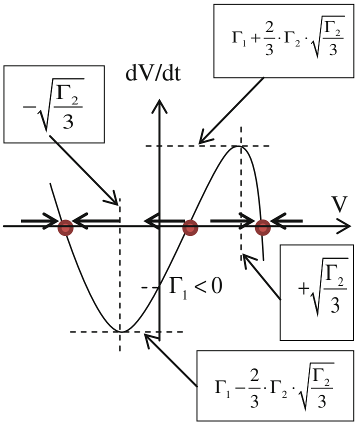
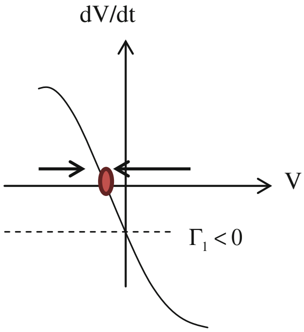
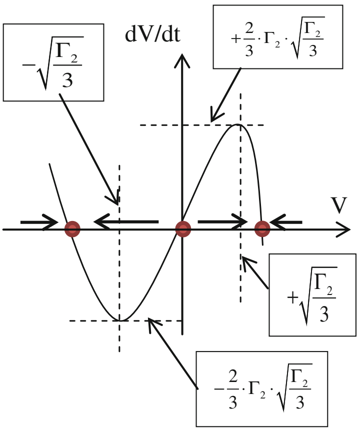
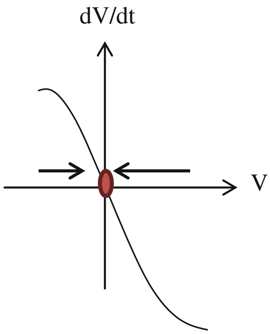
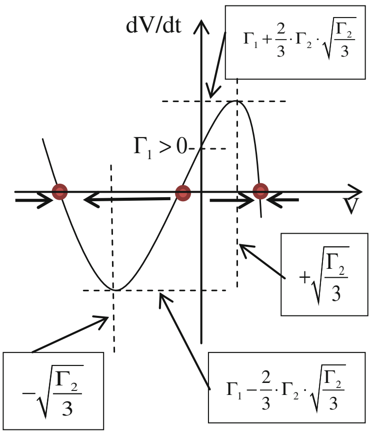
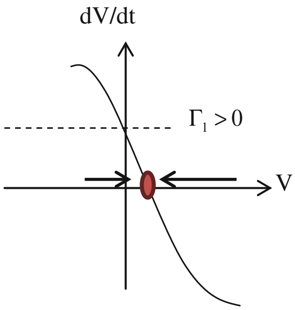
Cusp catastrophe system block diagram which occurs in a one-dimensional state space (n = 1) and two-dimensional parameter space (p = 2) is demonstrated in Fig. 2.2. It has an equilibrium manifold in $R^2 \times R$: $\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2)$ $V \in \mathbb{R}^{n=1}$; $\{\Gamma_1, \Gamma_2\} \in \mathbb{R}^{p=2}$; $M = \{(\Gamma_1, \Gamma_2, V) | \Gamma_1 + \Gamma_2 . V + V^3 = 0\}$. We change the sign of V^3 from (−) to (+). Where Γ_1, Γ_2 are two control parameters and V is the system state variable. A two-parameter system near a triple equilibrium point known as cusp bifurcation (equilibrium structure).

System equation:

$$\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2) = \Gamma_1 + \Gamma_2 \cdot V + V^3$$

$$\frac{dV}{dt} = f(V, \Gamma_1, \Gamma_2) = \Gamma_1 + \Gamma_2 \cdot V + V^3 \Rightarrow V = \frac{1}{\Gamma_2} \cdot \frac{dV}{dt} - \frac{\Gamma_1}{\Gamma_2} - \frac{1}{\Gamma_2} \cdot V^3; \quad \alpha = 1$$

Table 2.1 Cusp system differential equation behavior

<table>
<tr><td></td><td>$\Gamma_2>0$</td><td>$\Gamma_2<0$</td></tr>
<tr><td>$\Gamma_1<0$</td><td>
</td><td>
</td></tr>
<tr><td>$\Gamma_1=0$</td><td>
</td><td>
</td></tr>
<tr><td>$\Gamma_1>0$</td><td>
</td><td>
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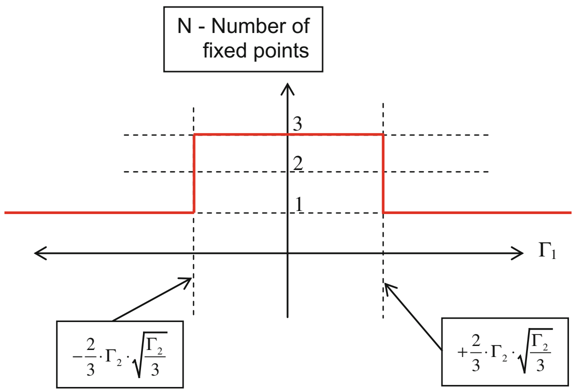

Fig. 2.1 Two fixed points (one stable fixed point and the other unstable fixed point) collide to one fixed point (half stable and half unstable fixed point)

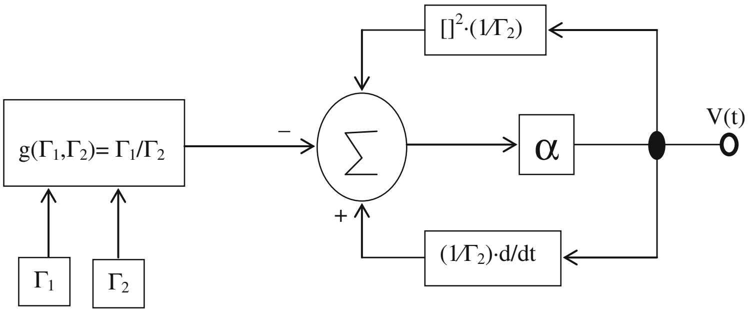

Fig. 2.2 Cusp catastrophe system block diagram

2.2 Optoisolation Circuits Cusp Bifurcation Analysis

We need to implement the above cusp system block diagram by using optoisolation elements, op-amps, resistors, capacitors, etc. Figure 2.3 implements our system.

Remark There is no match between system block diagram feedback loop $[]^2$ operator for V output voltage variable and system equation $-\frac{1}{\Gamma_2}\cdot V^3$ term. We will get the later from optoisolation circuit implementation our feedback loop $[]^3$ operator for V output voltage variable. Gama1 = Γ_1, Gama2 = Γ_2 [15, 16, 18].

At time $t = t_0$ switches S1, S2, and S3 move from OFF state to ON state. The circuit dynamic starts.

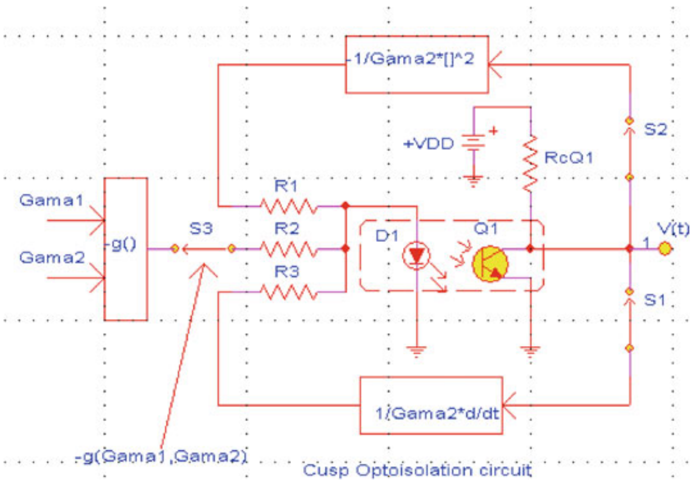

Fig. 2.3 To implement the cusp system block diagram by using optoisolation elements, op-amps, resistors, capacitors

$$I_{D1} = I_{R1} + I_{R2} + I_{R3}; \quad I_{R1} = \frac{-\frac{1}{\Gamma_2} \cdot V^2 - V_{D1}}{R_1};$$

$$I_{R2} = \frac{-\frac{\Gamma_1}{\Gamma_2} - V_{D1}}{R_2}; \quad I_{R3} = \frac{\frac{1}{\Gamma_2} \cdot \frac{dV}{dt} - V_{D1}}{R_3}$$

We consider $V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D1}}{I_0}$ (Taylor series approximation).

$$I_{D1} = \frac{-\frac{1}{\Gamma_2} \cdot V^2 - V_{D1}}{R_1} + \frac{-\frac{\Gamma_1}{\Gamma_2} - V_{D1}}{R_2} + \frac{\frac{1}{\Gamma_2} \cdot \frac{dV}{dt} - V_{D1}}{R_3}$$

$$I_{D1} = \frac{-\frac{1}{\Gamma_2} \cdot V^2 - V_t \cdot \frac{I_{D1}}{I_0}}{R_1} + \frac{-\frac{\Gamma_1}{\Gamma_2} - V_t \cdot \frac{I_{D1}}{I_0}}{R_2} + \frac{\frac{1}{\Gamma_2} \cdot \frac{dV}{dt} - V_t \cdot \frac{I_{D1}}{I_0}}{R_3}$$

$$I_{D1} = -\frac{1}{\Gamma_2 \cdot R_1} \cdot V^2 - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_1} - \frac{\Gamma_1}{\Gamma_2 \cdot R_2} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_2} + \frac{1}{\Gamma_2 \cdot R_3} \cdot \frac{dV}{dt} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_3}$$

$$I_{D1} = \left\{-V_t \cdot \frac{I_{D1}}{I_0 \cdot R_1} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_2} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_3}\right\} - \frac{1}{\Gamma_2 \cdot R_1} \cdot V^2 - \frac{\Gamma_1}{\Gamma_2 \cdot R_2} + \frac{1}{\Gamma_2 \cdot R_3} \cdot \frac{dV}{dt}$$

$$I_{D1} = -\left\{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right\} \cdot \frac{V_t \cdot I_{D1}}{I_0} - \frac{1}{\Gamma_2 \cdot R_1} \cdot V^2 - \frac{\Gamma_1}{\Gamma_2 \cdot R_2} + \frac{1}{\Gamma_2 \cdot R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$I_{D1} + \left\{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right\} \cdot \frac{V_t \cdot I_{D1}}{I_0} = \frac{1}{\Gamma_2} \cdot \left\{-\frac{1}{R_1} \cdot V^2 - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}\right\}$$

$$I_{D1} = \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2} \cdot \left\{-\frac{1}{R_1} \cdot V^2 - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}\right\}$$

For simplicity, we define the following functions:

$$\eta = \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2}; \quad \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right) = -\frac{1}{R_1} \cdot V^2$$
$$-\frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}; \quad I_{D1} = \eta \cdot \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right)$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; \quad I_{BQ1} = k \cdot I_{D1} = k \cdot \eta \cdot \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right);$$
$$I_{EQ1} = k \cdot \eta \cdot \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right) + I_{CQ1}$$

The mathematical analysis is based on the basic Transistor Ebers–Moll equations. We need to implement the regular Ebers-Moll model to the above optocoupler circuit.

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right];$$
$$V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$V_{CEQ1} = V_{CBQ1} + V_{BEQ1}$, but $V_{CBQ1} = -V_{BCQ1}$, then $\mathbf{V_{CEQ1} = V_{BEQ1} - V_{BCQ1}}$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right)$$

Since $I_{sc} - I_{se} \to \varepsilon \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right];$$

$$\psi = \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right)$$

$$\alpha r \cdot I_{CQ1} - I_{EQ1} = \alpha r \cdot I_{CQ1} - \left\{k \cdot \eta \cdot \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right) + I_{CQ1}\right\}$$
$$= I_{CQ1} \cdot (\alpha r - 1) - k \cdot \eta \cdot \psi\left(V \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right)$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f = I_{CQ1} - \left\{k \cdot \eta \cdot \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right) + I_{CQ1}\right\} \cdot \alpha f$$
$$= I_{CQ1} \cdot (1 - \alpha f) - k \cdot \eta \cdot \alpha f \cdot \psi\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, \text{etc.}\right)$$

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{I_{CQ1} \cdot (\alpha r - 1) - k \cdot \eta \cdot \psi + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{I_{CQ1} \cdot (1 - \alpha f) - k \cdot \eta \cdot \alpha f \cdot \psi + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right];$$

$$V_{DD} = I_{CQ1} \cdot R_{CQ1} + V \Rightarrow I_{CQ1} = \frac{V_{DD} - V}{R_{CQ1}}$$

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\left(\frac{V_{DD} - V}{R_{CQ1}}\right) \cdot (\alpha r - 1) - k \cdot \eta \cdot \psi + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{\left(\frac{V_{DD} - V}{R_{CQ1}}\right) \cdot (1 - \alpha f) - k \cdot \eta \cdot \alpha f \cdot \psi + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\frac{V_{DD} \cdot (\alpha r - 1)}{R_{CQ1}} - \frac{V \cdot (\alpha r - 1)}{R_{CQ1}} - k \cdot \eta \cdot \psi + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{\frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} - \frac{V \cdot (1 - \alpha f)}{R_{CQ1}} - k \cdot \eta \cdot \alpha f \cdot \psi + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{-\frac{V \cdot (\alpha r - 1)}{R_{CQ1}} - k \cdot \eta \cdot \psi + \left[\frac{V_{DD} \cdot (\alpha r - 1)}{R_{CQ1}} + I_{se} \cdot (\alpha r \cdot \alpha f - 1)\right]}{-\frac{V \cdot (1 - \alpha f)}{R_{CQ1}} - k \cdot \eta \cdot \alpha f \cdot \psi + \left[\frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)\right]}\right]$$

For simplicity we define new system global parameters: $\xi_1, \xi_2, \xi_3, \xi_4$.

$$\xi_1 = -\frac{(\alpha r - 1)}{R_{CQ1}}; \quad \xi_2 = \frac{V_{DD} \cdot (\alpha r - 1)}{R_{CQ1}} + I_{se} \cdot (\alpha r \cdot \alpha f - 1); \quad \xi_3 = -\frac{(1 - \alpha f)}{R_{CQ1}}$$

$$\xi_4 = \frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1);$$
$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{V \cdot \xi_1 - k \cdot \eta \cdot \psi + \xi_2}{V \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \psi + \xi_4}\right]$$

$$\mathrm{e}^{\left[\frac{V}{Vt}\right]} = \frac{V \cdot \xi_1 - k \cdot \eta \cdot \psi + \xi_2}{V \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \psi + \xi_4};\ \mathrm{e}^{\left[\frac{V}{Vt}\right]} \approx \frac{V}{V_t} + 1 \text{ (Taylor series approximation).}$$

$$\left(\frac{V}{V_t} + 1\right) \cdot (V \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \psi + \xi_4) = V \cdot \xi_1 - k \cdot \eta \cdot \psi + \xi_2$$

$$\frac{V^2}{V_t} \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \psi \cdot \frac{V}{V_t} + \frac{V}{V_t} \cdot \xi_4 + V \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \psi + \xi_4 = V \cdot \xi_1 - k \cdot \eta \cdot \psi + \xi_2$$

$$\frac{V^2}{V_t} \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \left[-\frac{1}{R_1} \cdot V^2 - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}\right] \cdot \frac{V}{V_t} + \frac{V}{V_t} \cdot \xi_4 + V \cdot \xi_3 - k \cdot \eta \cdot \alpha f \cdot \left[-\frac{1}{R_1} \cdot V^2 - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}\right] + \xi_4 = V \cdot \xi_1 - k \cdot \eta \cdot \left[-\frac{1}{R_1} \cdot V^2 - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}\right] + \xi_2$$

$$\frac{V^2}{V_t} \cdot \xi_3 + \frac{k \cdot \eta \cdot \alpha f}{R_1 \cdot V_t} \cdot V^3 + \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2} \cdot \frac{V}{V_t} - \frac{k \cdot \eta \cdot \alpha f}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot \frac{V}{V_t} + \frac{V}{V_t} \cdot \xi_4 + V \cdot \xi_3 + \frac{k \cdot \eta \cdot \alpha f}{R_1} \cdot V^2 + \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2} - \frac{k \cdot \eta \cdot \alpha f}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} + \xi_4 = V \cdot \xi_1 + \frac{k \cdot \eta}{R_1} \cdot V^2 + \frac{k \cdot \eta \cdot \Gamma_1}{R_2} - \frac{k \cdot \eta}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} + \xi_2$$

$$V^2 \cdot \left[\frac{\xi_3}{V_t} + \frac{k \cdot \eta \cdot \alpha f}{R_1} - \frac{k \cdot \eta}{R_1}\right] + V \cdot \left[\frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2 \cdot V_t} + \frac{\xi_4}{V_t} + \xi_3 - \xi_1\right] + \frac{k \cdot \eta \cdot \alpha f}{R_1 \cdot V_t} \cdot V^3 - \frac{k \cdot \eta \cdot \alpha f}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot \frac{V}{V_t} + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot \left[\frac{k \cdot \eta}{R_3} - \frac{k \cdot \eta \cdot \alpha f}{R_3}\right] + \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2} + \xi_4 - \frac{k \cdot \eta \cdot \Gamma_1}{R_2} - \xi_2 = 0$$

$$V^2 \cdot \left[\frac{\xi_3}{V_t} + \frac{k \cdot \eta \cdot (\alpha f - 1)}{R_1}\right] + V \cdot \left[\frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2 \cdot V_t} + \frac{\xi_4}{V_t} + \xi_3 - \xi_1\right]$$
$$+ \frac{k \cdot \eta \cdot \alpha f}{R_1 \cdot V_t} \cdot V^3 - \frac{k \cdot \eta \cdot \alpha f}{R_3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot \frac{V}{V_t} + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot \frac{k \cdot \eta \cdot (1 - \alpha f)}{R_3}$$
$$+ \frac{k \cdot \eta \cdot (\alpha f - 1) \cdot \Gamma_1}{R_2} + \xi_4 - \xi_2 = 0$$

We define the following new system parameters: m_1, m_2, m_3, m_4, m_5, m_6.

$$m_1 = \frac{\xi_3}{V_t} + \frac{k \cdot \eta \cdot (\alpha f - 1)}{R_1}; \quad m_2 = \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2 \cdot V_t} + \frac{\xi_4}{V_t} + \xi_3 - \xi_1;$$
$$m_3 = \frac{k \cdot \eta \cdot \alpha f}{R_1 \cdot V_t}; \quad m_4 = -\frac{k \cdot \eta \cdot \alpha f}{R_3 \cdot V_t}; \quad m_5 = \frac{k \cdot \eta \cdot (1 - \alpha f)}{R_3};$$
$$m_6 = \frac{k \cdot \eta \cdot (\alpha f - 1) \cdot \Gamma_1}{R_2} + \xi_4 - \xi_2$$

$$V^2 \cdot m_1 + V \cdot m_2 + m_3 \cdot V^3 + m_4 \cdot V \cdot \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot m_5 + m_6 = 0$$

$$V^2 \cdot m_1 + V \cdot m_2 + m_3 \cdot V^3 + \frac{\mathrm{d}V}{\mathrm{d}t} \cdot [m_4 \cdot V + m_5] + m_6 = 0$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} \cdot [m_4 \cdot V + m_5] = -V^2 \cdot m_1 - V \cdot m_2 - m_3 \cdot V^3 - m_6$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V^2 \cdot \frac{m_1}{(m_4 \cdot V + m_5)} - V \cdot \frac{m_2}{(m_4 \cdot V + m_5)}$$
$$- V^3 \cdot \frac{m_3}{(m_4 \cdot V + m_5)} - \frac{m_6}{(m_4 \cdot V + m_5)}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{(m_4 \cdot V + m_5)} \cdot \{V^2 \cdot m_1 + V \cdot m_2 + V^3 \cdot m_3 + m_6\}; \; \frac{1}{(m_4 \cdot V + m_5)} \neq 0$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{(m_4 \cdot V + m_5)} \cdot \{V^3 \cdot m_3 + V^2 \cdot m_1 + V \cdot m_2 + m_6\}; \; \frac{1}{(m_4 \cdot V + m_5)} \neq 0$$

$$m_1 = \frac{\xi_3}{V_t} + \frac{k \cdot \eta \cdot (\alpha f - 1)}{R_1} = -\frac{(1 - \alpha f)}{R_{CQ1} \cdot V_t}$$
$$+ \frac{k \cdot (\alpha f - 1)}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2}$$

$$m_2 = \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2 \cdot V_t} + \frac{\xi_4}{V_t} + \xi_3 - \xi_1$$
$$= k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2 \cdot V_t} \cdot \alpha f \cdot \Gamma_1$$
$$+ \frac{1}{V_t} \cdot \left\{\frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)\right\} - \frac{(1 - \alpha f)}{R_{CQ1}} + \frac{(\alpha r - 1)}{R_{CQ1}}$$

$$m_3 = \frac{k \cdot \eta \cdot \alpha f}{R_1 \cdot V_t} = k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_1 \cdot V_t} \cdot \alpha f$$

$$m_4 = -\frac{k \cdot \eta \cdot \alpha f}{R_3 \cdot V_t} = -k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3 \cdot V_t} \cdot \alpha f$$

$$m_5 = \frac{k \cdot \eta \cdot (1 - \alpha f)}{R_3} = k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3} \cdot (1 - \alpha f)$$

$$m_6 = \frac{k \cdot \eta \cdot (\alpha f - 1) \cdot \Gamma_1}{R_2} + \xi_4 - \xi_2$$
$$= k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2} \cdot (\alpha f - 1) \cdot \Gamma_1$$
$$+ \frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)$$

We define a new function:

$$\phi(V) = V^3 \cdot m_3 + V^2 \cdot m_1 + V \cdot m_2 + m_6 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t}$$
$$= -\frac{\phi(V)}{(m_4 \cdot V + m_5)}; \quad \frac{1}{(m_4 \cdot V + m_5)} \neq 0$$

We find our system fixed point by setting $\frac{\mathrm{d}V}{\mathrm{d}t} = 0 \Rightarrow \phi(V) = 0$. $\Phi(V)$ is a cubic function of V (main system variable).

Before checking system bifurcation, we need to check possible signs for equation $\Phi(V)$ parameters m_3, m_1, m_2, m_6.

$$\frac{1}{(m_4 \cdot V + m_5)} \neq 0 \;\; \& \;\; m_4 \cdot V + m_5 \neq 0 \Rightarrow V \neq -\frac{m_5}{m_4}$$
$$= -\frac{k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3} \cdot (1 - \alpha f)}{-k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3 \cdot V_t} \cdot \alpha f}$$

$$V \neq -\frac{m_5}{m_4} = \frac{V_t \cdot (1 - \alpha f)}{\alpha f} > 0; \quad m_3 > 0$$
$$\text{if } \Gamma_2 > 0; \quad m_3 < 0 \quad \text{if } \Gamma_2 < 0; \quad \Gamma_2 \neq 0$$

$$m_1 = (\alpha f - 1) \cdot \left\{ \frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2} \right\}; \quad (\alpha f - 1) < 0$$

Then $m_1 > 0$ if

$$\frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2} < 0; \quad \Gamma_2 \neq 0$$

$$\frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2} < 0 \Rightarrow \Gamma_2 < 0$$
$$\Rightarrow \frac{1}{\Gamma_2} < -\frac{R_1 \left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right]}{R_{CQ1} \cdot V_t \cdot k}$$

$$\frac{1}{\Gamma_2} < -\frac{R_1 \cdot \left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right]}{R_{CQ1} \cdot V_t \cdot k} \Rightarrow 0$$
$$> \Gamma_2 > -\frac{R_{CQ1} \cdot V_t \cdot k}{R_1 \cdot \left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right]}$$

$$m_1 < 0 \text{ if } \frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2} > 0$$

$$\frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2} > 0$$
$$\Rightarrow \Gamma_2 < -\frac{R_{CQ1} \cdot V_t \cdot k}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot R_1} \quad \text{OR} \quad \Gamma_2 > 0$$

$$m_2 > 0 \text{ if } k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2 \cdot V_t} \cdot \alpha f \cdot \Gamma_1$$
$$+ \frac{1}{V_t} \cdot \left\{ \frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1) \right\} - \frac{(1 - \alpha f)}{R_{CQ1}} + \frac{(\alpha r - 1)}{R_{CQ1}} > 0$$

$$\frac{\Gamma_1}{\Gamma_2} > \frac{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot R_2\cdot V_t}{k\cdot \alpha f} \cdot\left\{-\frac{1}{V_t}\cdot\left\{\frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}}+I_{sc}\cdot(\alpha r\cdot\alpha f-1)\right\}+\frac{(1-\alpha f)}{R_{CQ1}}+\frac{(1-\alpha r)}{R_{CQ1}}\right\}$$

$$m_2<0 \text{ if } k\cdot\frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_2\cdot V_t}\cdot\alpha f\cdot\Gamma_1 + \frac{1}{V_t}\cdot\left\{\frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}}+I_{sc}\cdot(\alpha r\cdot\alpha f-1)\right\}-\frac{(1-\alpha f)}{R_{CQ1}}+\frac{(\alpha r-1)}{R_{CQ1}}<0$$

$$\frac{\Gamma_1}{\Gamma_2} < \frac{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot R_2\cdot V_t}{k\cdot \alpha f} \cdot\left\{-\frac{1}{V_t}\cdot\left\{\frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}}+I_{sc}\cdot(\alpha r\cdot\alpha f-1)\right\}+\frac{(1-\alpha f)}{R_{CQ1}}+\frac{(1-\alpha r)}{R_{CQ1}}\right\}$$

$$m_3>0 \text{ if } \frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_1\cdot V_t}\cdot\alpha f>0\Rightarrow\Gamma_2>0$$

$$m_3<0 \text{ if } \frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_1\cdot V_t}\cdot\alpha f<0\Rightarrow\Gamma_2<0$$

$$m_4>0 \text{ if } -k\cdot\frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_3\cdot V_t}\cdot\alpha f>0\Rightarrow\Gamma_2<0$$

$$m_4<0 \text{ if } -k\cdot\frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_3\cdot V_t}\cdot\alpha f<0\Rightarrow\Gamma_2>0$$

$$m_5>0 \text{ if } \frac{k\cdot(1-\alpha f)}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_3}>0\Rightarrow\Gamma_2>0$$

$$m_5<0 \text{ if } \frac{k\cdot(1-\alpha f)}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_3}<0\Rightarrow\Gamma_2<0$$

$$m_6>0 \text{ if } k\cdot\frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_2}\cdot(\alpha f-1)\cdot\Gamma_1 + \frac{V_{DD}\cdot[(1-\alpha f)-(\alpha r-1)]}{R_{CQ1}}+(I_{sc}-I_{se})\cdot(\alpha r\cdot\alpha f-1)>0$$

$$\frac{\Gamma_1}{\Gamma_2} < \frac{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot R_2}{k \cdot (1 - \alpha f)} \cdot \left\{\frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)\right\}$$

$$m_6 < 0 \text{ if } k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2} \cdot (\alpha f - 1) \cdot \Gamma_1 + \frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1) < 0$$

$$\frac{\Gamma_1}{\Gamma_2} > \frac{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot R_2}{k \cdot (1 - \alpha f)} \cdot \left\{\frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)\right\}$$

We already defined:

$$\phi(V) = V^3 \cdot m_3 + V^2 \cdot m_1 + V \cdot m_2 + m_6 \Rightarrow \frac{dV}{dt} = g(\Gamma_1, \Gamma_2, \alpha_r, \ldots) = -\frac{\phi(V)}{(m_4 \cdot V + m_5)}; \quad \frac{1}{(m_4 \cdot V + m_5)} \neq 0$$

We wish to find roots of $g(V, \Gamma_1, \Gamma_2, \alpha_r, \ldots)$ which correspond to stationary solutions of the differential equation. Turning points of $g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)$ are values of $\frac{dg(V, \Gamma_1, \Gamma_2, \alpha_r, \ldots)}{dV}$ satisfying $\frac{dg(\Gamma_1, \Gamma_2, \alpha_r, \ldots)}{dV} = -\frac{d}{dV}\left\{\frac{\phi(V)}{(m_4 \cdot V + m_5)}\right\} = 0$

$$\frac{dg(\Gamma_1, \Gamma_2, \alpha_r, \ldots)}{dV} = -\frac{\frac{d\phi(V)}{dV} \cdot (m_4 \cdot V + m_5) - \phi(V) \cdot m_4}{(m_4 \cdot V + m_5)^2} = 0; \quad V \neq -\frac{m_5}{m_4}$$

$$\phi(V) = V^3 \cdot m_3 + V^2 \cdot m_1 + V \cdot m_2 + m_6$$

$$\frac{d\phi(V)}{dV} \cdot (m_4 \cdot V + m_5) - \phi(V) \cdot m_4 = 0; \quad \frac{d\phi(V)}{dV} = 3 \cdot V^2 \cdot m_3 + 2 \cdot V \cdot m_1 + m_2$$

$$(3 \cdot V^2 \cdot m_3 + 2 \cdot V \cdot m_1 + m_2) \cdot (m_4 \cdot V + m_5) - (V^3 \cdot m_3 + V^2 \cdot m_1 + V \cdot m_2 + m_6) \cdot m_4 = 0$$

$$3 \cdot V^3 \cdot m_3 \cdot m_4 + 3 \cdot V^2 \cdot m_3 \cdot m_5 + 2 \cdot V^2 \cdot m_1 \cdot m_4 + 2 \cdot V \cdot m_1 \cdot m_5 + m_2 \cdot m_4 \cdot V + m_2 \cdot m_5 - V^3 \cdot m_3 \cdot m_4 - V^2 \cdot m_1 \cdot m_4 - V \cdot m_2 \cdot m_4 - m_6 \cdot m_4 = 0$$

$$2 \cdot V^3 \cdot m_3 \cdot m_4 + V^2 \cdot (3 \cdot m_3 \cdot m_5 + m_1 \cdot m_4) + V \cdot 2 \cdot m_1 \cdot m_5 + [m_2 \cdot m_5 - m_6 \cdot m_4] = 0$$

The number of real roots in the above cubic equation is the number of turning points in our optoisolation system. We define the following parameters:

$$a = 2 \cdot m_3 \cdot m_4 = \frac{-2 \cdot k^2 \cdot [\alpha f]^2}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right]^2 \cdot \Gamma_2^2 \cdot V_t^2 \cdot R_1 \cdot R_3} < 0$$

$$b = 3 \cdot m_3 \cdot m_5 + m_1 \cdot m_4 = \frac{3 \cdot k^2 \cdot \alpha f \cdot (1 - \alpha f)}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right]^2 \cdot \Gamma_2^2 \cdot R_1 \cdot R_3 \cdot V_t} + (1 - \alpha f) \cdot \left\{\frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2}\right\} \cdot \left\{\frac{k \cdot \alpha f}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3 \cdot V_t}\right\}$$

$$\mathrm{c} = 2 \cdot m_1 \cdot m_5 = -2 \cdot (1 - \alpha f)^2 \cdot \left\{\frac{1}{R_{CQ1} \cdot V_t} + \frac{k}{R_1} \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2}\right\} \cdot \left\{\frac{k}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}\right\}$$

$$d = m_2 \cdot m_5 - m_6 \cdot m_4 = \left\{k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2 \cdot V_t} \cdot \alpha f \cdot \Gamma_1 + \frac{1}{V_t} \cdot \left\{\frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)\right\} - \frac{(1 - \alpha f)}{R_{CQ1}} + \frac{(\alpha r - 1)}{R_{CQ1}}\right\} \cdot \left\{\frac{k \cdot (1 - \alpha f)}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}\right\} + \left\{\frac{k \cdot (\alpha f - 1) \cdot \Gamma_1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2} + \frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)\right\} \cdot \left\{\frac{k \cdot \alpha f}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3 \cdot V_t}\right\}$$

The cubic equation with real coefficients (a, b, c, d) has at least one solution V among the real numbers. We can distinguish several possible cases using discriminant (Δ).

$$\Delta = -4 \cdot b^3 \cdot d + b^2 \cdot c^2 - 4 \cdot a \cdot c^3 + 18 \cdot a \cdot b \cdot c \cdot d - 27 \cdot a^2 \cdot d^2$$

If $\Delta > 0$, then the equation has three distinct real roots (three turning points).

If $\Delta < 0$, then the equation has one real root and a pair of complex conjugate roots (one turning point). If $\Delta = 0$, then at least two roots coincide. It may be that the equation has a double real root (one turning point) and another distinct singe real root (another turning point). Alternatively, all three roots coincide yielding a triple real root (triple turning point). The triple root exists if and only if this resultant vanishes.

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots) = -\frac{\phi(V)}{(m_4 \cdot V + m_5)};$$
$$V = 0 \Rightarrow \phi(V = 0) = m_6; \quad g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = -\frac{m_6}{m_5}$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = -\frac{m_6}{m_5}\bigg|_{V=0}$$
$$= -\frac{k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2} \cdot (\alpha f - 1) \cdot \Gamma_1 + \frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)}{k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3} \cdot (1 - \alpha f)}$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = \frac{m_6}{m_5}\bigg|_{V=0} = \frac{R_3}{R_2} \cdot \Gamma_1$$
$$- \frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)] \cdot \left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}{R_{CQ1} \cdot k \cdot (1 - \alpha f)}$$
$$- \frac{(I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1) \cdot \left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}{k \cdot (1 - \alpha f)}$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = -\frac{m_6}{m_5}\bigg|_{V=0} = \frac{R_3}{R_2} \cdot \Gamma_1$$
$$- \frac{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}{k \cdot (1 - \alpha f)}$$
$$\cdot \left\{\frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)\right\}$$

Next Step We investigate our system cusp catastrophe behavior [7, 8]. All parameters are fix except Γ_1, Γ_2, $k = 0.02 \ldots 0.1$. $k \neq 0$; $k > 0$ (Table 2.2).

Table 2.2 Cusp optoisolation circuit components and parameters values

V_t	0.026	α_f	0.98
I_0	1E−6	β_f	49
k	0.02–0.1	α_r	0.5
R_{CQ1}	2000 Ω	β_r	1
V_{DD}	15v	I_{sc}	$2uA$
R_1, R_2, R_3	1000 Ω	I_{se}	$1uA$

$$\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]=79;\quad \frac{1}{R_{CQ1}\cdot V_t}=\frac{1}{52}=0.019$$

$$m_1=-\frac{(1-\alpha f)}{R_{CQ1}\cdot V_t}+\frac{k\cdot(\alpha f-1)}{R_1}\cdot\frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2}$$
$$=-10^{-6}\cdot\left[380+\frac{k}{\Gamma_2}\cdot 0.253\right]$$

$$m_1>0\ \text{ if } 0>\frac{\Gamma_2}{k}>-0.665\cdot 10^{-3}\Rightarrow 0>\Gamma_2>-k\cdot 0.665\cdot 10^{-3};\quad \Gamma_2\neq 0$$

$$m_1<0\ \text{ if } \frac{\Gamma_2}{k}<-0.665\cdot 10^{-3}\quad \text{OR}\quad \frac{\Gamma_2}{k}>0$$
$$\Rightarrow \Gamma_2<-k\cdot 0.665\cdot 10^{-3}\quad \text{OR}\quad \Gamma_2>0$$

$$m_2=k\cdot\frac{1}{\left[1+\frac{V_t}{I_0}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\right]\cdot\Gamma_2\cdot R_2\cdot V_t}\cdot\alpha f\cdot\Gamma_1$$
$$+\frac{1}{V_t}\cdot\left\{\frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}}+I_{sc}\cdot(\alpha r\cdot\alpha f-1)\right\}$$
$$-\frac{(1-\alpha f)}{R_{CQ1}}+\frac{(\alpha r-1)}{R_{CQ1}}=k\cdot\frac{\Gamma_1}{\Gamma_2}\cdot 0.477\cdot 10^{-3}+5.47\cdot 10^{-3}$$

$$m_2=\left\{k\cdot\frac{\Gamma_1}{\Gamma_2}\cdot 0.477+5.47\right\}\cdot 10^{-3};\quad \Gamma_2\neq 0;$$
$$m_2>0\ \text{ if } k\cdot\frac{\Gamma_1}{\Gamma_2}>-11.46\Rightarrow\frac{\Gamma_1}{\Gamma_2}>-11.46\cdot\frac{1}{k}$$

$$m_2<0\ \text{ if } k\cdot\frac{\Gamma_1}{\Gamma_2}<-11.46\Rightarrow\frac{\Gamma_1}{\Gamma_2}<-11.46\cdot\frac{1}{k}$$

$$m_2=0\ \text{ if } k\cdot\frac{\Gamma_1}{\Gamma_2}\cdot 0.477+5.47=0\Rightarrow\frac{\Gamma_1}{\Gamma_2}=-11.46\cdot\frac{1}{k};\quad \Gamma_2\neq 0$$

$$m_3 = k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_1 \cdot V_t} \cdot \alpha f = \frac{k}{2054 \cdot \Gamma_2}$$
$$= 0.486 \cdot 10^{-3} \cdot \frac{k}{\Gamma_2}$$

$$m_3 < 0 \ \text{if} \ \Gamma_2 < 0; \quad m_3 > 0 \ \text{if} \ \Gamma_2 > 0;$$

$$m_6 = \frac{k \cdot \eta \cdot (\alpha f - 1) \cdot \Gamma_1}{R_2} + \xi_4 - \xi_2$$
$$= k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_2} \cdot (\alpha f - 1) \cdot \Gamma_1$$
$$+ \frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se})$$
$$\cdot (\alpha r \cdot \alpha f - 1) = -0.253 \cdot 10^{-6} \cdot k \cdot \frac{\Gamma_1}{\Gamma_2} + 3.89 \cdot 10^{-3}$$

$$m_6 > 0 \ \text{if} \ \frac{\Gamma_1}{\Gamma_2} < \frac{15375.49}{k}; \quad m_6 < 0 \ \text{if} \ \frac{\Gamma_1}{\Gamma_2} > \frac{15375.49}{k};$$
$$m_6 = 0 \ \text{if} \ \frac{\Gamma_1}{\Gamma_2} = \frac{15375.49}{k}$$

$$m_4 = -\frac{k \cdot \eta \cdot \alpha f}{R_3 \cdot V_t} = -k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3 \cdot V_t} \cdot \alpha f$$
$$= -0.477 \cdot 10^{-3} \cdot \frac{k}{\Gamma_2}$$

$$m_4 > 0 \ \text{if} \ \Gamma_2 < 0; \quad m_4 < 0 \ \text{if} \ \Gamma_2 > 0$$

$$m_5 = \frac{k \cdot \eta \cdot (1 - \alpha f)}{R_3} = k \cdot \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}$$
$$\cdot (1 - \alpha f) = 0.253 \cdot 10^{-6} \cdot \frac{k}{\Gamma_2}$$

$$m_5 > 0 \ \text{if} \ \Gamma_2 > 0; \quad m_5 < 0 \ \text{if} \ \Gamma_2 < 0$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = -\frac{m_6}{m_5}\bigg|_{V=0} = \frac{R_3}{R_2} \cdot \Gamma_1 - \frac{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\right] \cdot \Gamma_2 \cdot R_3}{k \cdot (1 - \alpha f)}$$
$$\cdot \left\{\frac{V_{DD} \cdot [(1 - \alpha f) - (\alpha r - 1)]}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)\right\} = \Gamma_1 - \frac{\Gamma_2}{k} \cdot 15.4 \cdot 10^3$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = \left.\frac{m_6}{m_5}\right|_{V=0} > 0 \Rightarrow \Gamma_1 - \frac{\Gamma_2}{k} \cdot 15.4 \cdot 10^3 > 0$$
$$\Rightarrow \Gamma_1 > \frac{\Gamma_2}{k} \cdot 15.4 \cdot 10^3 \quad \Gamma_2 > 0 \Rightarrow \frac{\Gamma_1}{\Gamma_2} > \frac{1}{k} \cdot 15.4 \cdot 10^3;$$
$$\Gamma_2 < 0 \Rightarrow \frac{\Gamma_1}{\Gamma_2} < \frac{1}{k} \cdot 15.4 \cdot 10^3$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = \left.\frac{m_6}{m_5}\right|_{V=0} < 0 \Rightarrow \Gamma_1 - \frac{\Gamma_2}{k} \cdot 15.4 \cdot 10^3 < 0$$
$$\Rightarrow \Gamma_1 < \frac{\Gamma_2}{k} \cdot 15.4 \cdot 10^3 \quad \Gamma_2 > 0 \Rightarrow \frac{\Gamma_1}{\Gamma_2} < \frac{1}{k} \cdot 15.4 \cdot 10^3;$$
$$\Gamma_2 < 0 \Rightarrow \frac{\Gamma_1}{\Gamma_2} > \frac{1}{k} \cdot 15.4 \cdot 10^3$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots)|_{V=0} = -\left.\frac{m_6}{m_5}\right|_{V=0} = 0$$
$$\Rightarrow \Gamma_1 - \frac{\Gamma_2}{k} \cdot 15.4 \cdot 10^3 = 0$$
$$\Rightarrow \frac{\Gamma_1}{\Gamma_2} = \frac{1}{k} \cdot 15.4 \cdot 10^3$$

$$g(\Gamma_1, \Gamma_2, \alpha_r, \ldots) = -\frac{\phi(V)}{(m_4 \cdot V + m_5)} \Rightarrow V \neq -\frac{m_5}{m_4} = \frac{V_t \cdot (1 - \alpha f)}{\alpha f}$$
$$= 0.53 \cdot 10^{-3} (\text{Asymptotic}).$$

In Table 2.3 we summarize all m_i ($i = 1,\ldots, 6$) expressions as a function of k, Γ_1, Γ_2.

MATLAB (k → k, Γ_1 → A1, Γ_2 → A2).

EDU≫v=.0004:0.00001:0.0006;k=0.02;A1=20;A2=2;m3=0.486*0.001*k/A2;m1=−0.000001*(380+(k/A2)*0.253);m2=0.001*(k*(A1/A2)*0.477+5.47);m6=−0.253*0.000001*k*(A1/A2)+3.89*0.001;

EDU≫m4=0.477*0.001*(k/A2);m5=0.253*0.000001*(k/A2);u1=v.^3*m3+v.^2*m1+v*m2+m6;u2=m4*v+m5;u=−u1./u2;plot(v,u,‘r’),grid

Results We see different behaviors for different k, Γ_1, Γ_2 values. The number of our optoisolation system fixed points change and bifurcation occurs (Figs. 2.4, 2.5, 2.6 and 2.7).

Table 2.3 Summary of all m_i ($i = 1,\ldots, 6$) expressions as a function of k, Γ_1, Γ_2

m_i	$m_i = \zeta(k, \Gamma_1, \Gamma_2)$
m_1	$-10^{-6} \cdot [380 + k/\Gamma_2 \cdot 0.253]$
m_2	$[k \cdot (\Gamma_1/\Gamma_2) \cdot 0.477 + 5.47] \cdot 10^{-3}$
m_3	$0.486 \cdot 10^{-3} \cdot k/\Gamma_2$
m_4	$-0.477 \cdot 10^{-3} \cdot k/\Gamma_2$
m_5	$0.253 \cdot 10^{-6} \cdot k/\Gamma_2$
m_6	$-0.253 \cdot 10^{-6} \cdot k \cdot (\Gamma_1/\Gamma_2) + 3.89 \cdot 10^{-3}$

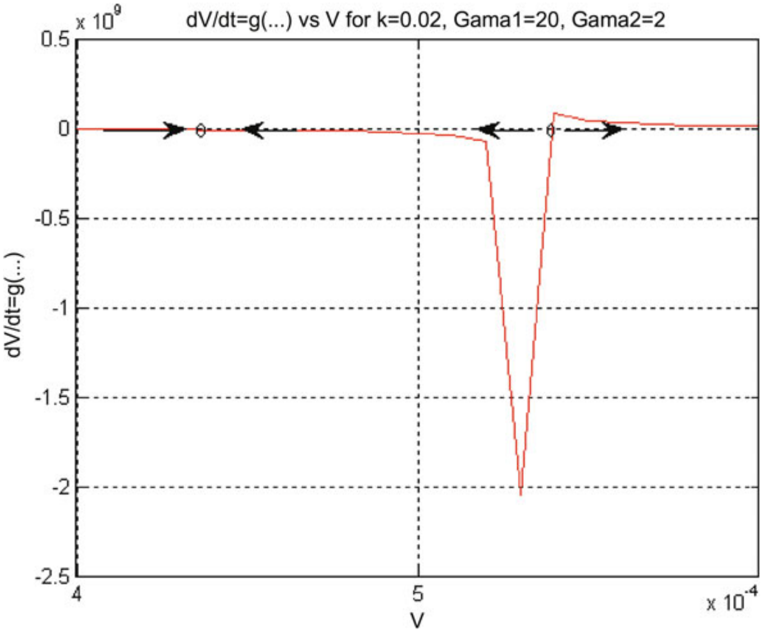

Fig. 2.4 Optoisolation system fixed points change and bifurcation occurs

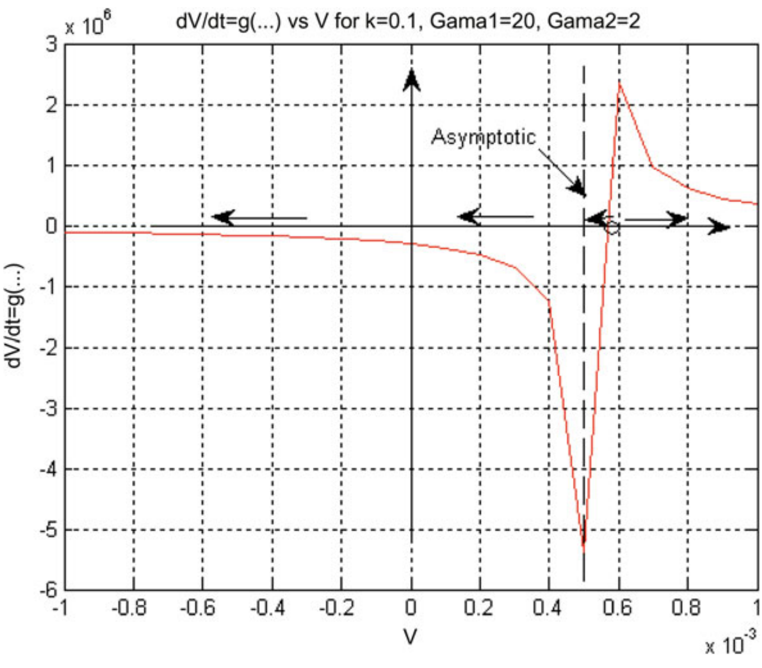

Fig. 2.5 Optoisolation system fixed points change and bifurcation occurs

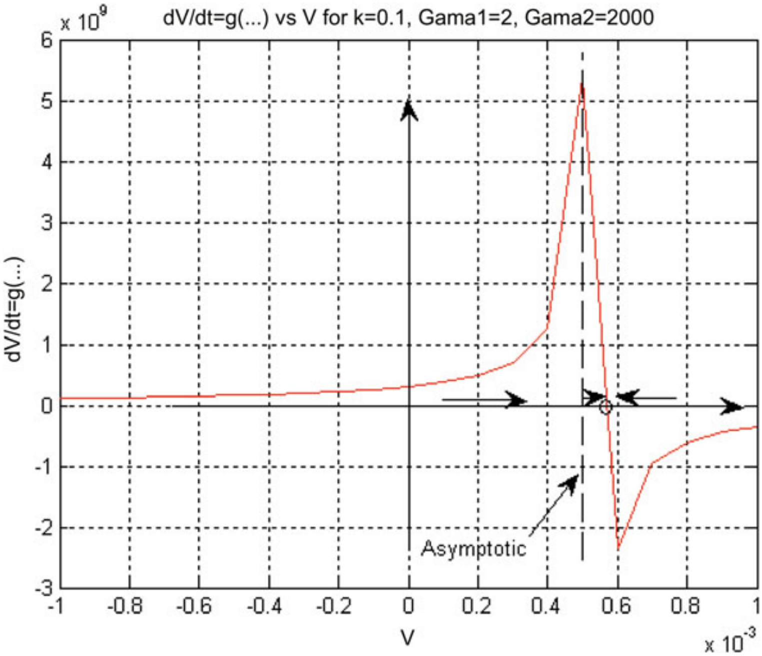

Fig. 2.6 Optoisolation system fixed points change and bifurcation occurs

2.3 Bautin Bifurcation Analysis System

The Bautin bifurcation is a bifurcation of an equilibrium in a two-parameter family of autonomous ODEs at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov–Hopf bifurcation vanishes (Generalized Hopf (GH) bifurcation). The bifurcation point separates branches of sub and supercritical Andronov–Hopf bifurcations in the parameter plain. For nearby parameter values, the system has two limit cycles which collide and disappear via a saddle node bifurcation of periodic orbits. Autonomous system of ODEs $\frac{dV}{dt} = f(V, \Gamma),\ \Gamma \in \mathbb{R}^n$ depending on two parameters $\Gamma \in \mathbb{R}^n$ where f is smooth. For all sufficiently small $\|\Gamma\|$, the system has an equilibrium at $V = 0$. Its Jacobian matrix $A(\Gamma) = f_V(0, \Gamma)$ has one pair of complex eigenvalues $\lambda_{1,2}(\Gamma) = \mu(\Gamma) \pm i \cdot \omega(\Gamma)$. Such that $\mu(\Gamma = 0) = 0$; $\omega(\Gamma = 0) = \omega_0 > 0 \Rightarrow \lambda_{1,2}(\Gamma = 0) = \pm i \cdot \omega_0$. The Generalized Hopf (Bautin) bifurcation in the planar system can be characterized by the following equations: $\frac{dr}{dt} = r \cdot (\Gamma_1 + \Gamma_2 \cdot r^2 + \sigma \cdot r^4)$; $\frac{d\varphi}{dt} = 1$. To describe Bautin bifurcation analytically, we consider system $\frac{dV}{dt} = f(V, \Gamma),\ \Gamma \in \mathbb{R}^n$ with $n = 2$.

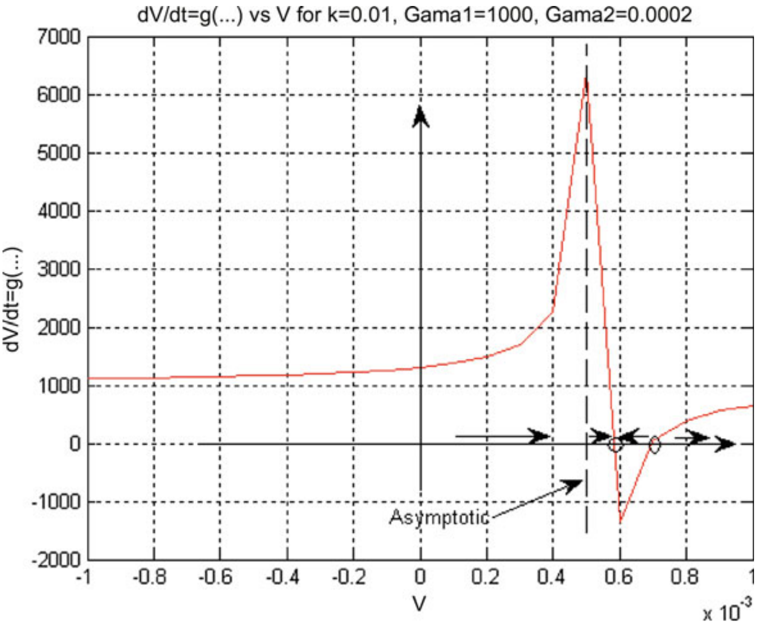

Fig. 2.7 Optoisolation system fixed points change and bifurcation occurs

We have a system with two main variables V_1, V_2. $\cos\varphi = \frac{V_1}{r} \Rightarrow r = \frac{V_1}{\cos\varphi}$ then

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V_1}{\mathrm{d}t}\cdot\cos\varphi + V_1\cdot\frac{\mathrm{d}\varphi}{\mathrm{d}t}\cdot\sin\varphi}{[\cos\varphi]^2}; \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V_1}{\mathrm{d}t}\cdot\cos\varphi + V_1\cdot\sin\varphi}{[\cos\varphi]^2}.$$

$\cos\varphi = \frac{V_1}{r} = \frac{V_1}{\sqrt{V_1^2+V_2^2}}$; $\quad \sin\varphi = \frac{V_2}{r} = \frac{V_2}{\sqrt{V_1^2+V_2^2}}$; Γ_1, Γ_2—control parameters.

$V = (V_1, V_2)^T \in \mathbb{R}^2, \Gamma \in \mathbb{R}^2$; $\sigma = \mathrm{sign}[l_2(0)] = \pm 1$. This normal form is simple in polar coordinates (r, φ). Non-degeneracy conditions hold: $l_2(0) \neq 0$, $l_2(0)$ is the second Lyapunov coefficient. The map $\alpha \to (\mu(\alpha), l_1(\alpha))$ is regular at $\alpha = 0$, where $l_1(\alpha)$ is parameter coefficient [71, 72].

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r\cdot(\Gamma_1 + \Gamma_2\cdot r^2 + \sigma\cdot r^4) \Rightarrow \frac{\frac{\mathrm{d}V_1}{\mathrm{d}t}\cdot\cos\varphi + V_1\cdot\sin\varphi}{[\cos\varphi]^2}$$

$$= \frac{V_1}{\cos\varphi}\cdot\left(\Gamma_1 + \Gamma_2\cdot\left[\frac{V_1}{\cos\varphi}\right]^2 + \sigma\cdot\left[\frac{V_1}{\cos\varphi}\right]^4\right)$$

$$\frac{dV_1}{dt} \cdot \cos\varphi + V_1 \cdot \sin\varphi = V_1 \cdot \cos\varphi \cdot \left(\Gamma_1 + \Gamma_2 \cdot \frac{V_1^2}{[\cos\varphi]^2} + \sigma \cdot \frac{V_1^4}{[\cos\varphi]^4}\right)$$

$$\frac{dV_1}{dt} \cdot \frac{V_1}{\sqrt{V_1^2 + V_2^2}} + V_1 \cdot \frac{V_2}{\sqrt{V_1^2 + V_2^2}} = V_1 \cdot \frac{V_1}{\sqrt{V_1^2 + V_2^2}} \cdot \left(\Gamma_1 + \Gamma_2 \cdot \frac{V_1^2 \cdot (V_1^2 + V_2^2)}{V_1^2} + \sigma \cdot \frac{V_1^4 \cdot (V_1^2 + V_2^2)^2}{V_1^4}\right)$$

$$\frac{dV_1}{dt} + V_2 = V_1 \cdot \left\{\Gamma_1 + \Gamma_2 \cdot (V_1^2 + V_2^2) + \sigma \cdot (V_1^2 + V_2^2)^2\right\}$$
$$\Rightarrow \frac{dV_1}{dt} = -V_2 + V_1 \cdot \Gamma_1 + V_1 \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + V_1 \cdot \sigma \cdot (V_1^2 + V_2^2)^2$$

In the same manner $\sin\varphi = \frac{V_2}{r} \Rightarrow r = \frac{V_2}{\sin\varphi} \Rightarrow \frac{dr}{dt} = \frac{\frac{dV_2}{dt} \cdot \sin\varphi - V_2 \cdot \frac{d\varphi}{dt} \cdot \cos\varphi}{[\sin\varphi]^2}$

$$\frac{dr}{dt} = r \cdot (\Gamma_1 + \Gamma_2 \cdot r^2 + \sigma \cdot r^4) \Rightarrow \frac{\frac{dV_2}{dt} \cdot \sin\varphi - V_2 \cdot \frac{d\varphi}{dt} \cdot \cos\varphi}{[\sin\varphi]^2}$$
$$= \frac{V_2}{\sin\varphi} \cdot \left(\Gamma_1 + \Gamma_2 \cdot \left[\frac{V_2}{\sin\varphi}\right]^2 + \sigma \cdot \left[\frac{V_2}{\sin\varphi}\right]^4\right)$$

$$\frac{d\varphi}{dt} = 1 \Rightarrow \frac{dV_2}{dt} \cdot \sin\varphi - V_2 \cdot \cos\varphi$$
$$= V_2 \cdot \sin\varphi \cdot \left(\Gamma_1 + \Gamma_2 \cdot \frac{V_2^2}{[\sin\varphi]^2} + \sigma \cdot \frac{V_2^4}{[\sin\varphi]^4}\right)$$

$$\frac{dV_2}{dt} \cdot \frac{V_2}{\sqrt{V_1^2 + V_2^2}} - V_2 \cdot \frac{V_1}{\sqrt{V_1^2 + V_2^2}} = V_2 \cdot \frac{V_2}{\sqrt{V_1^2 + V_2^2}} \cdot \left\{\Gamma_1 + \Gamma_2 \cdot (V_1^2 + V_2^2) + \sigma \cdot (V_1^2 + V_2^2)^2\right\}$$

$$\frac{dV_2}{dt} - V_1 = V_2 \cdot \left\{\Gamma_1 + \Gamma_2 \cdot (V_1^2 + V_2^2) + \sigma \cdot (V_1^2 + V_2^2)^2\right\} \Rightarrow \frac{dV_2}{dt}$$
$$= V_1 + V_2 \cdot \Gamma_1 + V_2 \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + V_2 \cdot \sigma \cdot (V_1^2 + V_2^2)^2$$

Finally, we get the form for our system, two differential equations:

$$\frac{dV_1}{dt} = -V_2 + V_1 \cdot \Gamma_1 + V_1 \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + V_1 \cdot \sigma \cdot (V_1^2 + V_2^2)^2$$

$$\frac{dV_2}{dt} = V_1 + V_2 \cdot \Gamma_1 + V_2 \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + V_2 \cdot \sigma \cdot (V_1^2 + V_2^2)^2$$

To find our system fixed points we set $\frac{dV_1}{dt} = 0$; $\frac{dV_2}{dt} = 0$ and get the following two equations:

$$(*) \quad V_1 + V_2 \cdot \Gamma_1 + V_2 \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + V_2 \cdot \sigma \cdot (V_1^2 + V_2^2)^2 = 0$$

$$(**) \quad -V_2 + V_1 \cdot \Gamma_1 + V_1 \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + V_1 \cdot \sigma \cdot (V_1^2 + V_2^2)^2 = 0$$

$$(*) + (**) \rightarrow \quad V_1 - V_2 + (V_1 + V_2) \cdot \Gamma_1 + (V_1 + V_2) \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + (V_1 + V_2) \cdot \sigma \cdot (V_1^2 + V_2^2)^2 = 0$$

$$V_1 - V_2 + (V_1 + V_2) \cdot \left\{ \Gamma_1 + \Gamma_2 \cdot (V_1^2 + V_2^2) + \sigma \cdot (V_1^2 + V_2^2)^2 \right\} = 0; \quad V_1 + V_2 \neq 0$$

$$\frac{V_1 - V_2}{V_1 + V_2} + \Gamma_1 + \Gamma_2 \cdot (V_1^2 + V_2^2) + \sigma \cdot (V_1^2 + V_2^2)^2 = 0; \quad V_1 + V_2 \neq 0$$

We define a new system variables: $X = \frac{V_1 - V_2}{V_1 + V_2}$; $Y = V_1^2 + V_2^2$

$$(***) \quad X + \Gamma_1 + \Gamma_2 \cdot Y + \sigma \cdot Y^2 = 0; \quad V_1 + V_2 \neq 0$$

$$(*) + (**) \rightarrow \quad V_1 + V_2 - (V_1 - V_2) \cdot \Gamma_1 - (V_1 - V_2) \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) - (V_1 - V_2) \cdot \sigma \cdot (V_1^2 + V_2^2)^2 = 0$$

$$V_1 + V_2 - (V_1 - V_2) \cdot \left\{ \Gamma_1 + \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + \cdot \sigma \cdot (V_1^2 + V_2^2)^2 \right\} = 0; \quad V_1 - V_2 \neq 0 \Rightarrow V_1 \neq V_2$$

$$\frac{V_1 + V_2}{V_1 - V_2} - \left\{ \Gamma_1 + \cdot \Gamma_2 \cdot (V_1^2 + V_2^2) + \cdot \sigma \cdot (V_1^2 + V_2^2)^2 \right\} = 0; \quad V_1 - V_2 \neq 0 \Rightarrow V_1 \neq V_2$$

$$X = \frac{V_1 - V_2}{V_1 + V_2} \Rightarrow \frac{1}{X} = \frac{V_1 + V_2}{V_1 - V_2}; \quad Y = V_1^2 + V_2^2; \quad V_1 - V_2 \neq 0 \Rightarrow V_1 \neq V_2$$

$$(****) \quad \frac{1}{X} - \left\{ \Gamma_1 + \cdot \Gamma_2 \cdot Y + \cdot \sigma \cdot Y^2 \right\} = 0; \quad V_1 - V_2 \neq 0 \Rightarrow V_1 \neq V_2$$

$$(***) + (****) \rightarrow \quad X + \frac{1}{X} = 0 \Rightarrow \frac{X^2+1}{X} = 0 \Rightarrow X \neq 0;$$
$$X^2 + 1 = 0 \Rightarrow X = \pm i \Rightarrow \frac{V_1 - V_2}{V_1 + V_2} = \pm i$$

Case (I)

$$X = i \Rightarrow \frac{V_1 - V_2}{V_1 + V_2} = i \Rightarrow V_1 = V_2 \cdot \frac{1+i}{1-i} = V_2 \cdot i \Rightarrow V_1 = V_2 \cdot i$$

$$X = i \Rightarrow i + \Gamma_1 + \Gamma_2 \cdot Y + \sigma \cdot Y^2 = 0 \Rightarrow Y^2 \cdot \sigma + Y \cdot \Gamma_2 + (\Gamma_1 + i) = 0$$

$$Y_{1,2} = \frac{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot (\Gamma_1 + i)}}{2 \cdot \sigma} = \frac{-\Gamma_2 \pm \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - 4 \cdot \sigma \cdot i}}{2 \cdot \sigma}$$

$$\sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - 4 \cdot \sigma \cdot i} = \gamma + i \cdot \delta \Rightarrow [\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - 4 \cdot \sigma \cdot i$$
$$= \gamma^2 - \delta^2 + i \cdot 2 \cdot \gamma \cdot \delta$$

$$\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 = \gamma^2 - \delta^2; \quad -2 \cdot \sigma = \gamma \cdot \delta \Rightarrow \delta = \frac{-2 \cdot \sigma}{\gamma};$$
$$\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 = \gamma^2 - \frac{4 \cdot \sigma^2}{\gamma^2}$$

$$\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 = \gamma^2 - \frac{4 \cdot \sigma^2}{\gamma^2} \Rightarrow \gamma^4 - [\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]$$
$$\cdot \gamma^2 - 4 \cdot \sigma^2 = 0; \quad \gamma^4 \rightarrow \phi^2; \quad \gamma^2 \rightarrow \phi$$

$$\phi^2 - [\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] \cdot \phi - 4 \cdot \sigma^2 = 0 \Rightarrow \phi_{1,2}$$
$$= \frac{1}{2} \cdot \left\{ [\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] \pm \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]^2 + 16 \cdot \sigma^2} \right\}$$

$\gamma^2 \rightarrow \phi \Rightarrow \gamma_{1,2} = \pm\sqrt{\phi}$; $\delta = \frac{-2 \cdot \sigma}{\gamma} \Rightarrow \delta_{1,2} = \frac{-2 \cdot \sigma}{\pm\sqrt{\phi}} = \mp \frac{2 \cdot \sigma}{\sqrt{\phi}}$; we have four possible solutions: $\sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - 4 \cdot \sigma \cdot i} \Rightarrow \gamma_1 + i \cdot \delta_1, \gamma_2 + i \cdot \delta_2, \ \gamma_3 + i \cdot \delta_3, \gamma_4 + i \cdot \delta_4$

$$\gamma_{1,2} = \pm \frac{1}{\sqrt{2}} \cdot \sqrt{\left\{ [\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] + \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]^2 + 16 \cdot \sigma^2} \right\}}$$

$$\gamma_{3,4} = \pm \frac{1}{\sqrt{2}} \cdot \sqrt{\left\{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]^2 + 16 \cdot \sigma^2}\right\}}$$

$$\delta_{1,2} = \frac{\mp 2 \cdot \sqrt{2} \cdot \sigma}{\sqrt{\left\{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] + \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]^2 + 16 \cdot \sigma^2}\right\}}};$$

$$\delta_{3,4} = \frac{\mp 2 \cdot \sqrt{2} \cdot \sigma}{\sqrt{\left\{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]^2 + 16 \cdot \sigma^2}\right\}}}$$

$$Y_{1,2} = \frac{1}{2 \cdot \sigma} \cdot \left\{-\Gamma_2 \pm \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] - 4 \cdot \sigma \cdot i}\right\}$$

$$\Rightarrow Y_{1,2} = \frac{1}{2 \cdot \sigma} \cdot \left\{-\Gamma_2 \pm (\gamma_j + i \cdot \delta_j)\right\} \quad \forall \, \mathrm{j} \in 1, \ldots, 4$$

Since there are four possible expressions to $\gamma_j + i \cdot \delta_j$, we get eight possible expressions for Y_j (j = 1,…, 8). $Y_{1,\ldots,8} = \frac{1}{2 \cdot \sigma} \cdot \left\{-\Gamma_2 \pm (\gamma_j + i \cdot \delta_j)\right\} \, \forall \, j \in 1, \ldots, 4.$

Finally, we get the following set of equations for all possible system complex fixed points.

$$V_1 = V_2 \cdot i; \quad V_1^2 + V_2^2 = \frac{1}{2 \cdot \sigma} \cdot \left\{-\Gamma_2 \pm (\gamma_j + i \cdot \delta_j)\right\} \quad \forall \, j \in 1, \ldots, 4$$

$$V_1 = V_2 \cdot i \Rightarrow V_1^2 + V_2^2 = 0 \Rightarrow \frac{1}{2 \cdot \sigma} \cdot \left\{-\Gamma_2 \pm (\gamma_j + i \cdot \delta_j)\right\} = 0;$$
$$\sigma \neq 0 \Rightarrow -\Gamma_2 \pm (\gamma_j + i \cdot \delta_j) = 0 \quad \forall \, j \in 1, \ldots, 4$$

Case (II)

$$X = -i \Rightarrow \frac{V_1 - V_2}{V_1 + V_2} = -i \Rightarrow V_1 = V_2 \cdot \frac{1 - i}{1 + i} = -V_2 \cdot i \Rightarrow V_1 = -V_2 \cdot i$$

$$X = -i \Rightarrow -i + \Gamma_1 + \Gamma_2 \cdot Y + \sigma \cdot Y^2 = 0 \Rightarrow Y^2 \cdot \sigma + Y \cdot \Gamma_2 + (\Gamma_1 - i) = 0$$

$$Y_{1,2} = \frac{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot (\Gamma_1 - i)}}{2 \cdot \sigma} = \frac{-\Gamma_2 \pm \sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] + 4 \cdot \sigma \cdot i}}{2 \cdot \sigma}$$

$$\sqrt{[\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1] + 4 \cdot \sigma \cdot i} = \gamma + i \cdot \delta \Rightarrow [\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1]$$
$$+ \; 4 \cdot \sigma \cdot i = \gamma^2 - \delta^2 + i \cdot 2 \cdot \gamma \cdot \delta$$

$$\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1 = \gamma^2 - \delta^2; \quad 2\cdot\sigma = \gamma\cdot\delta \Rightarrow \delta = \frac{2\cdot\sigma}{\gamma};$$
$$\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1 = \gamma^2 - \frac{4\cdot\sigma^2}{\gamma^2}$$

$$\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1 = \gamma^2 - \frac{4\cdot\sigma^2}{\gamma^2} \Rightarrow \gamma^4 - [\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]\cdot\gamma^2 - 4\cdot\sigma^2 = 0;$$
$$\gamma^4 \to \phi^2;\ \gamma^2 \to \phi$$

$$\phi^2 - [\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]\cdot\phi - 4\cdot\sigma^2 = 0 \Rightarrow \phi_{1,2} = \frac{1}{2}\cdot\left\{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] \pm \sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]^2 + 16\cdot\sigma^2}\right\}$$

$\gamma^2 \to \phi \Rightarrow \gamma_{1,2} = \pm\sqrt{\phi}$; $\delta = \frac{2\cdot\sigma}{\gamma} \Rightarrow \delta_{1,2} = \frac{2\cdot\sigma}{\pm\sqrt{\phi}} = \pm\frac{2\cdot\sigma}{\sqrt{\phi}}$; We have four possible solutions:

$$\sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] + 4\cdot\sigma\cdot i} \Rightarrow \gamma_1 + i\cdot\delta_1, \gamma_2 + i\cdot\delta_2, \gamma_3 + i\cdot\delta_3, \gamma_4 + i\cdot\delta_4$$

$$\gamma_{1,2} = \pm\frac{1}{\sqrt{2}}\cdot\sqrt{\left\{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] + \sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]^2 + 16\cdot\sigma^2}\right\}}$$

$$\gamma_{3,4} = \pm\frac{1}{\sqrt{2}}\cdot\sqrt{\left\{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] - \sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]^2 + 16\cdot\sigma^2}\right\}}$$

$$\delta_{1,2} = \frac{\pm 2\cdot\sqrt{2}\cdot\sigma}{\sqrt{\left\{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] + \sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]^2 + 16\cdot\sigma^2}\right\}}};$$

$$\delta_{3,4} = \frac{\pm 2\cdot\sqrt{2}\cdot\sigma}{\sqrt{\left\{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] - \sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1]^2 + 16\cdot\sigma^2}\right\}}}$$

$$Y_{1,2} = \frac{1}{2\cdot\sigma}\cdot\left\{-\Gamma_2 \pm \sqrt{[\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1] + 4\cdot\sigma\cdot i}\right\}$$
$$\Rightarrow Y_{1,2} = \frac{1}{2\cdot\sigma}\cdot\left\{-\Gamma_2 \pm (\gamma_j + i\cdot\delta_j)\right\} \quad \forall\, j \in 1,\ldots,4$$

Since there are four possible expressions to $\gamma_j + i\cdot\delta_j$, we get eight possible expressions for Y_j (j = 1,…, 8). $Y_{1,\ldots,8} = \frac{1}{2\cdot\sigma}\cdot\left\{-\Gamma_2 \pm (\gamma_j + i\cdot\delta_j)\right\}\ \forall\, j \in 1,\ldots,4$.

Finally, we get the following set of equations for all possible system complex fixed points.

$$V_1 = -V_2 \cdot i; \quad V_1^2 + V_2^2 = \frac{1}{2 \cdot \sigma} \cdot \{-\Gamma_2 \pm (\gamma_j + i \cdot \delta_j)\} \quad \forall j \in 1, \ldots, 4$$

$$V_1 = -V_2 \cdot i \Rightarrow V_1^2 + V_2^2 = 0 \Rightarrow \frac{1}{2 \cdot \sigma} \cdot \{-\Gamma_2 \pm (\gamma_j + i \cdot \delta_j)\} = 0;$$
$$\sigma \neq 0 \Rightarrow -\Gamma_2 \pm (\gamma_j + i \cdot \delta_j) = 0 \quad \forall j \in 1, \ldots, 4$$

We plot our system for different Γ_1, Γ_2, σ values and initial values V_{10}, V_{20}.

MATLAB Software $\Gamma_1 \to a$, $\Gamma_2 \to b$, $\sigma \to c$, $V_1 \to x(1)$, $V_2 \to x(2)$

```
function g=bautin(t,x,a,b,c)
g=zeros(2,1);
g(1)=-
x(2)+x(1)*a+x(1)*b*(x(1)*x(1)+x(2)*x(2))+x(1)*c*(x(1)*x(1)+x(2)*x(2))
*(x(1)*x(1)+x(2)*x(2));
g(2)=x(1)+x(2)*a+x(2)*b*(x(1)*x(1)+x(2)*x(2))+x(2)*c*(x(1)*x(1)+x(2)*
x(2))*(x(1)*x(1)+x(2)*x(2));

function h=bautin1 (a,b,c, V10, V20)
[t,x]=ODE45 (@bautin, [0,10] , [V10,V20] , [] , a,b,c);
%plot(t,x);
plot(x(:,1),x(:,2)) % V1 against V2 at time increase phase plan plot
```

Results There is different phase space behavior for different Γ_1, Γ_2, σ values. All possibilities of limit cycles exist and different spiral behavior (stable/unstable) (Fig. 2.8).

Our system Bautin bifurcation planar equations are $\frac{dr}{dt} = r \cdot (\Gamma_1 + \Gamma_2 \cdot r^2 + \sigma \cdot r^4)$ and $\frac{d\varphi}{dt} = 1$ where $r \geq 0$. Treating our planar equation $\frac{dr}{dt} = \ldots$ as vector field on the line, we see that there are some fixed points. $\frac{dr}{dt} = 0 \Rightarrow r^{(0)} = 0$ (first fixed point) and other fixed points fulfill $\Gamma_1 + \Gamma_2 \cdot [r^{(j)}]^2 + \sigma \cdot [r^{(j)}]^4 = 0$, $j = 1, 2, \ldots$

$$[r^{(j)}]^4 \to [s^{(j)}]^2; \quad [r^{(j)}]^2 \to s^{(j)} \Rightarrow r^{(j)} = \pm\sqrt{s^{(j)}}; \quad \sigma \cdot [s^{(j)}]^2 + \Gamma_2 \cdot s^{(j)} + \Gamma_1 = 0$$

$$\sigma \cdot [s^{(j)}]^2 + \Gamma_2 \cdot s^{(j)} + \Gamma_1 = 0 \Rightarrow s_{1,2}^{(j)} = \frac{1}{2 \cdot \sigma} \cdot \left(-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right)$$

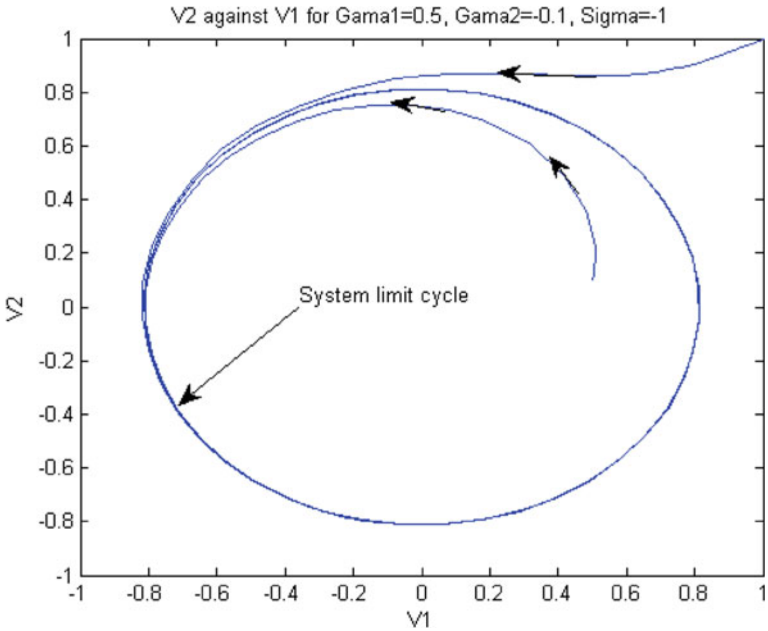

Fig. 2.8 Sorts of limit cycles exist and different spiral behaviors

$$r^{(j)}\big|_{r^{(j)}\geq 0} = \sqrt{\frac{1}{2\cdot\sigma}\cdot(-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1})} = \frac{1}{\sqrt{2\cdot\sigma}}\cdot\sqrt{(-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1})}$$

$$r_{(1)}^{(j)}\Big|_{r^{(j)}\geq 0} = \frac{1}{\sqrt{2\cdot\sigma}}\cdot\sqrt{(-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1})};$$

$$r_{(2)}^{(j)}\Big|_{r^{(j)}\geq 0} = \frac{1}{\sqrt{2\cdot\sigma}}\cdot\sqrt{(-\Gamma_2 - \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1})}$$

Case (I)

$$\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1 = 0 \Rightarrow \Gamma_2^2 = 4\cdot\sigma\cdot\Gamma_1; \quad s^{(j)} = -\frac{\Gamma_2}{2\cdot\sigma}$$

$$\Rightarrow r^{(j)}\big|_{r^{(j)}\geq 0} = \sqrt{-\frac{\Gamma_2}{2\cdot\sigma}} = \sqrt{\frac{|\Gamma_2|}{2\cdot|\sigma|}}$$

$$\Gamma_2^2 = 4 \cdot \sigma \cdot \Gamma_1 \Rightarrow \sigma \cdot \Gamma_1 > 0; \quad -\frac{\Gamma_2}{2 \cdot \sigma} > 0 \Rightarrow \frac{\Gamma_2}{\sigma} < 0$$

If $\sigma > 0$ then $\Gamma_1 > 0$ & $\Gamma_2 < 0$; if $\sigma < 0$ then $\Gamma_1 < 0$ & $\Gamma_2 > 0$.

Case (II)

$$\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 > 0 \Rightarrow (\Gamma_2 - 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) \cdot (\Gamma_2 + 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) > 0$$

II.1 $(\Gamma_2 - 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) > 0$ & $(\Gamma_2 + 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) > 0 \Rightarrow \Gamma_2 > 2 \cdot \sqrt{\sigma \cdot \Gamma_1}$

II.2 $(\Gamma_2 - 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) < 0$ & $(\Gamma_2 + 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) < 0 \Rightarrow \Gamma_2 < -2 \cdot \sqrt{\sigma \cdot \Gamma_1}$

$$r^{(j)}\big|_{r^{(j)} \geq 0} = \sqrt{\frac{1}{2 \cdot \sigma} \cdot (-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1})}$$

$$\sigma > 0 \Rightarrow \left\{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right\} > 0$$
$$\Rightarrow \Gamma_2 < \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1} \Rightarrow \Gamma_2 < -\sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}$$

$$\sigma < 0 \Rightarrow \left\{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right\} < 0$$
$$\Rightarrow \Gamma_2 > \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1} \Rightarrow \Gamma_2 > \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}$$

Case (III) $\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 < 0$ (r is a complex number which is impossible), no limit cycle.

$$\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 < 0 \Rightarrow (\Gamma_2 - 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) \cdot (\Gamma_2 + 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) < 0$$

III.1 $(\Gamma_2 - 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) > 0 \quad \& \quad (\Gamma_2 + 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) < 0$
$\Rightarrow \Gamma_2 > 2 \cdot \sqrt{\sigma \cdot \Gamma_1} \quad \& \quad \Gamma_2 < -2 \cdot \sqrt{\sigma \cdot \Gamma_1}$

Cannot exist!!

III.2 $(\Gamma_2 - 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) < 0 \quad \& \quad (\Gamma_2 + 2 \cdot \sqrt{\sigma \cdot \Gamma_1}) > 0$
$\Rightarrow \Gamma_2 < 2 \cdot \sqrt{\sigma \cdot \Gamma_1} \quad \& \quad \Gamma_2 > -2 \cdot \sqrt{\sigma \cdot \Gamma_1}$

We need to verify for our Bautin bifurcation system, when there is a stable limit cycle at specific r values and when the close orbit still exists. If we start from initial radius variable value r_0 then the equation for $r(t)$ is $r(t) = r_0 + [dr/dt]dt$. We define dr/dt as the change rate respect to time of radius vector $r(t)$. We define two main cases: in the inner circle and outer circle. We have three kinds of possible limit cycle in our system: Stable Limit Cycle (SLC), Half-Stable Limit Cycle (HLC), and Unstable Limit Cycle (USL). For each limit cycle we need to establish the $r_{\min}$ and $r_{\max}$ values for d*r*/d*t* conditions in the inner and outer circle. We need to find $r_{\min}$ and $r_{\max}$ values ($r > 0$) then Bautin bifurcation system implies existence of a closed orbit. Table 2.4 shows that for each limit cycle type, the conditions for d*r*/d*t* in the inner and outer circle [5–8] (Fig. 2.9).

Option No.1 $\mathrm{d}r/\mathrm{d}t > 0$; $\left(\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1 > 0; \left\{\frac{1}{2\cdot\sigma}\cdot\left[-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1}\right]\right\} > 0\right)$

$$\frac{\mathrm{d}r}{\mathrm{d}t} > 0 \Rightarrow r > 0 \quad \& \quad \Gamma_1 + \Gamma_2\cdot r^2 + \sigma\cdot r^4 > 0 \;\; r^2 \to s; \quad r^4 \to s^2$$

Table 2.4 Summary of each limit cycle type and the conditions for d*r*/d*t* in the inner and outer circle

Limit cycle type	d*r*/d*t* in the inner circle	d*r*/d*t* in the outer circle
Stable Limit Cycle (SLC)	d*r*/d*t* > 0	d*r*/d*t* < 0
Half-Stable Limit Cycle (HLC)	d*r*/d*t* < 0	d*r*/d*t* < 0
Unstable Limit Cycle (ULC)	d*r*/d*t* < 0	d*r*/d*t* > 0

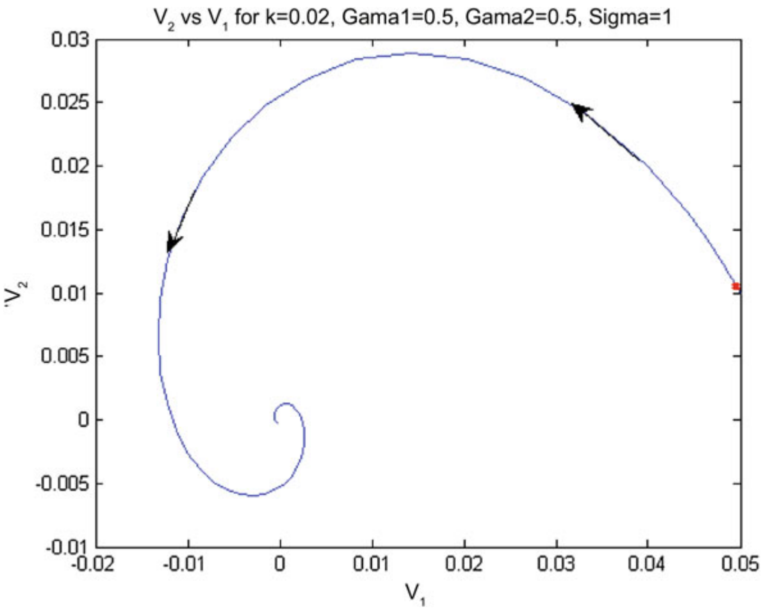

Fig. 2.9 Bautin bifurcation system

$$\sigma \cdot s^2 + \Gamma_2 \cdot s + \Gamma_1 > 0 \Rightarrow \text{ if } \sigma > 0 \Rightarrow s > \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right] \text{ OR } 0 < s < \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]$$

$$r = \pm\sqrt{s}; \quad r|_{r \geq 0} = \sqrt{s} \Rightarrow \text{ if } \sigma > 0 \Rightarrow r > \sqrt{\frac{1}{2 \cdot \sigma} \cdot [-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}]} \quad \text{OR} \quad 0 < r < \sqrt{\frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]}$$

$$\sigma \cdot s^2 + \Gamma_2 \cdot s + \Gamma_1 > 0 \Rightarrow \text{ if } \sigma < 0 \Rightarrow \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right] < s < \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]$$

$$r = \pm\sqrt{s}; \quad r|_{r \geq 0} = \sqrt{s} \Rightarrow \text{ if } \sigma < 0 \Rightarrow r < \sqrt{\frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]} \;\&\; r > \sqrt{\frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]}$$

Option No. 2 $\mathrm{d}r/\mathrm{d}t < 0$; $\left(\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1 > 0;\ \left\{\frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]\right\} > 0\right)$

$$\frac{\mathrm{d}r}{\mathrm{d}t} < 0 \Rightarrow r > 0 \quad \& \quad \Gamma_1 + \Gamma_2 \cdot r^2 + \sigma \cdot r^4 < 0; \quad r^2 \to s; \;\; r^4 \to s^2$$

$$\sigma \cdot s^2 + \Gamma_2 \cdot s + \Gamma_1 < 0 \Rightarrow \text{ if } \sigma > 0 \Rightarrow \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right] < s < \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}\right]$$

$$r = \pm\sqrt{s}; \; r|_{r \geq 0} = \sqrt{s} \Rightarrow \text{ if } \sigma < 0 \Rightarrow r < \sqrt{\frac{1}{2 \cdot \sigma} \cdot [-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}]} \;\&\; r > \sqrt{\frac{1}{2 \cdot \sigma} \cdot [-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1}]}$$

$$\sigma \cdot s^2 + \Gamma_2 \cdot s + \Gamma_1 < 0 \Rightarrow \text{ if } \sigma < 0 \Rightarrow 0 < s < \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1} \right] \text{ OR s} > \frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1} \right]$$

$$r = \pm\sqrt{s};\ r|_{r \geq 0} = \sqrt{s} \Rightarrow \text{ if } \sigma < 0 \Rightarrow 0 < r < \sqrt{\frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1} \right]} \text{ OR } r > \sqrt{\frac{1}{2 \cdot \sigma} \cdot \left[-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1} \right]}$$

Our Bautin bifurcation system can be represented by the equivalent set of differential equations:

$$V_1 = \frac{1}{\Gamma_1} \left\{ \frac{dV_1}{dt} + V_2 - \Gamma_2 \cdot V_1 \cdot (V_1^2 + V_2^2) - \sigma \cdot V_1 \cdot (V_1^2 + V_2^2)^2 \right\}$$

$$V_2 = \frac{1}{\Gamma_1} \left\{ \frac{dV_2}{dt} - V_1 - \Gamma_2 \cdot V_2 \cdot (V_1^2 + V_2^2) - \sigma \cdot V_2 \cdot (V_1^2 + V_2^2)^2 \right\}$$

The next step is to implement our Bautin bifurcation system by optoisolation circuits, discrete parts, op-amp, etc. $\sigma = \pm 1$. See Fig. 2.10 system functional block diagram.

2.4 Optoisolation Circuits Bautin Bifurcation Analysis

Figure 2.11 shows the implementation of our Bautin bifurcation system. We define four functional blocks $g_1(V_1, V_2)$, $g_2(V_1, V_2)$, $g_3(V_1, V_2)$, $g_4(V_1, V_2)$. Additionally, we have inverter functional block ("-[]") and derivative functional blocks ("d/dt").

They are not implemented in our circuit [15, 16].

$$g_1'(V_1, V_2) = -(V_1^2 + V_2^2) \cdot V_1; \quad g_2'(V_1, V_2) = -(V_1^2 + V_2^2)^2 \cdot V_1$$
$$g_3'(V_1, V_2) = -(V_1^2 + V_2^2) \cdot V_2; \quad g_4'(V_1, V_2) = -(V_1^2 + V_2^2)^2 \cdot V_2$$
$$g_1'(V_1, V_2) = -g_1(V_1, V_2) \Rightarrow g_1(V_1, V_2) = (V_1^2 + V_2^2) \cdot V_1$$
$$g_2'(V_1, V_2) = -g_2(V_1, V_2) \Rightarrow g_2(V_1, V_2) = (V_1^2 + V_2^2)^2 \cdot V_1$$
$$g_3'(V_1, V_2) = -g_3(V_1, V_2) \Rightarrow g_3(V_1, V_2) = (V_1^2 + V_2^2) \cdot V_2$$
$$g_4'(V_1, V_2) = -g_4(V_1, V_2) \Rightarrow g_4(V_1, V_2) = (V_1^2 + V_2^2)^2 \cdot V_2$$

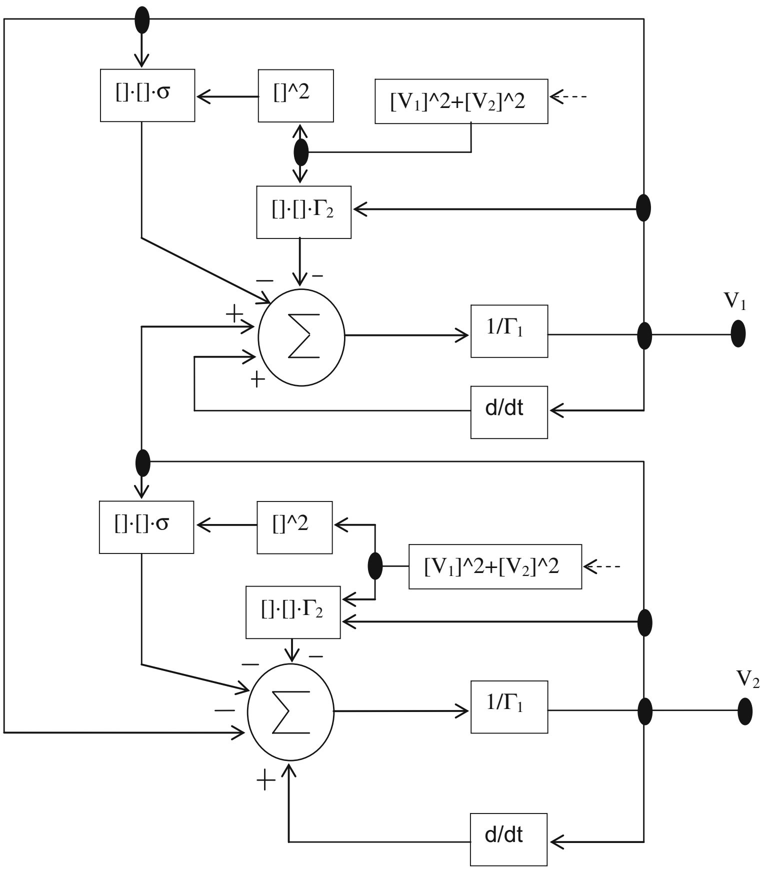

Fig. 2.10 System functional block diagram

Remark It is a reader exercise to implement those functional blocks using Op amps, resistors, capacitors, diodes, and other discrete components. All feedback loops are isolated (voltage follower op-amps) with infinite input impedance.

We define $g_1 = g_1(V_1, V_2);\ g_2 = g_2(V_1, V_2);\ g_3 = g_3(V_1, V_2);\ g_4 = g_4(V_1, V_2)$

$$I_{D1} = I_{R11} + I_{R12} + I_{R13} + I_{R14};$$

$$I_{D1} = \frac{-g_2 - V_{D1}}{R_{14}} + \frac{\frac{\mathrm{d}V_1}{\mathrm{d}t} - V_{D1}}{R_{11}} + \frac{-g_1 - V_{D1}}{R_{12}} + \frac{V_2 - V_{D1}}{R_{13}}$$

$$I_{R11} = \frac{\frac{\mathrm{d}V_1}{\mathrm{d}t} - V_{D1}}{R_{11}};\quad I_{R12} = \frac{-g_1 - V_{D1}}{R_{12}};\quad I_{R13} = \frac{V_2 - V_{D1}}{R_{13}};\quad I_{R14} = \frac{-g_2 - V_{D1}}{R_{14}}$$

We consider Taylor series approximation: $V_{D1} = V_t \cdot \ln(\frac{I_{D1}}{I_0} + 1) \approx V_t \cdot \frac{I_{D1}}{I_0}$

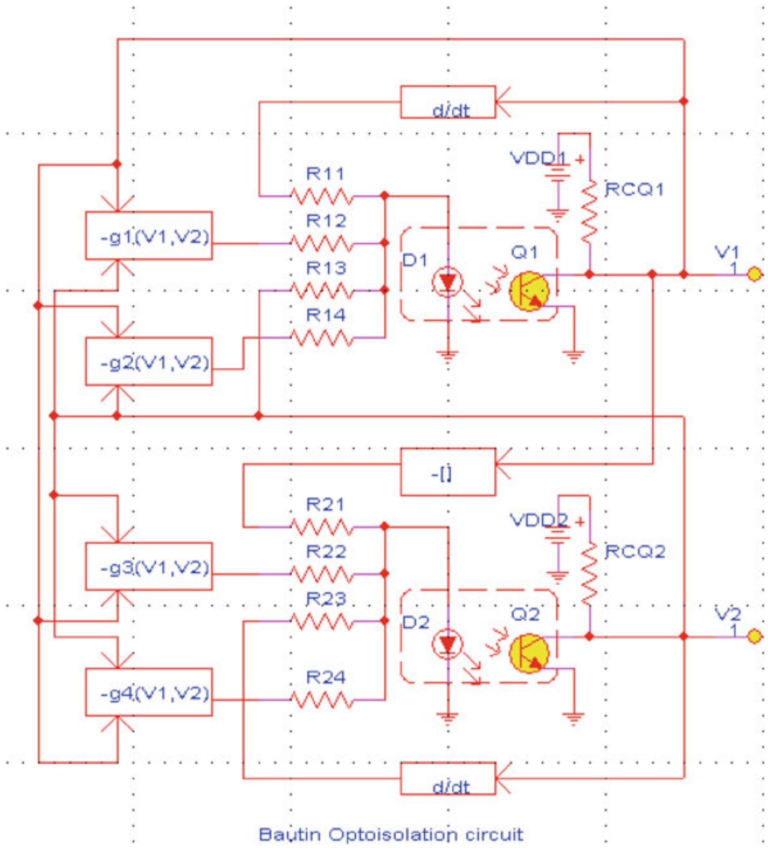

Fig. 2.11 Implementation of Bautin bifurcation system

$$I_{D1} = \frac{-g_2 - V_t \cdot \frac{I_{D1}}{I_0}}{R_{14}} + \frac{\frac{dV_1}{dt} - V_t \cdot \frac{I_{D1}}{I_0}}{R_{11}} + \frac{-g_1 - V_t \cdot \frac{I_{D1}}{I_0}}{R_{12}} + \frac{V_2 - V_t \cdot \frac{I_{D1}}{I_0}}{R_{13}}$$

$$I_{D1} = \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_{11}} - \frac{g_1}{R_{12}} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_{12}} + \frac{V_2}{R_{13}} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_{13}} - \frac{g_2}{R_{14}} - V_t \cdot \frac{I_{D1}}{I_0 \cdot R_{14}}$$

$$I_{D1} = -V_t \cdot \frac{I_{D1}}{I_0} \cdot \left\{ \frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}} \right\} + \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}$$

$$I_{D1} \cdot \left[1 + V_t \cdot \frac{1}{I_0} \cdot \left\{ \frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}} \right\} \right] = \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}$$

$$I_{D1} = \frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{ \frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}} \right\}} \cdot \left\{ \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}} \right\}$$

For simplicity, we define the following functions: $I_{D1} = \eta_1 \cdot \psi_1\left(\frac{dV_1}{dt}, V_1, V_2\right)$

$$\eta_1 = \frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{ \frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}} \right\}};$$

$$\psi_1\left(\frac{dV_1}{dt}, V_1, V_2\right) = \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; \quad I_{BQ1} = k_1 \cdot I_{D1} = k_1 \cdot \eta_1 \cdot \psi_1\left(\frac{dV_1}{dt}, V_1, V_2\right);$$

$$I_{EQ1} = k_1 \cdot \eta_1 \cdot \psi_1\left(\frac{dV_1}{dt}, V_1, V_2\right) + I_{CQ1}$$

The mathematical analysis is based on the basic transistor Ebers–Moll equations. We need to implement the regular Ebers–Moll Model to the above optocoupler circuit.

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r_1 \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right];$$

$$V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f_1}{I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]$$

$V_{CEQ1} = V_{CBQ1} + V_{BEQ1}$, but $V_{CBQ1} = -V_{BCQ1}$, then $\mathbf{V_{CEQ1}} = \mathbf{V_{BEQ1}} - \mathbf{V_{BCQ1}}$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r_1 \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f_1}{I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r_1 \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f_1) + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right] + Vt \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right)$$

Since $I_{sc} - I_{se} \to \varepsilon \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.

$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r_1 \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f_1) + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right];$$

$$\psi_1 = \psi_1\left(\frac{dV_1}{dt}, V_1, V_2\right)$$

$$\alpha r_1 \cdot I_{CQ1} - I_{EQ1} = \alpha r_1 \cdot I_{CQ1} - \left\{ k_1 \cdot \eta_1 \cdot \psi_1 \left(\frac{\mathrm{d}V_1}{\mathrm{d}t}, V_1, V_2 \right) + I_{CQ1} \right\}$$
$$= I_{CQ1} \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 \left(\frac{\mathrm{d}V_1}{\mathrm{d}t}, V_1, V_2 \right)$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f_1 = I_{CQ1} - \left\{ k_1 \cdot \eta_1 \cdot \psi_1 \left(\frac{\mathrm{d}V_1}{\mathrm{d}t}, V_1, V_2 \right) + I_{CQ1} \right\} \cdot \alpha f_1$$
$$= I_{CQ1} \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 \left(\frac{\mathrm{d}V_1}{\mathrm{d}t}, V_1, V_2 \right)$$

$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln \left[\frac{I_{CQ1} \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{I_{CQ1} \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)} \right];$$
$$V_{DD1} = I_{CQ1} \cdot R_{CQ1} + V_1 \Rightarrow I_{CQ1} = \frac{V_{DD1} - V_1}{R_{CQ1}}$$

$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln \left[\frac{\frac{(V_{DD1} - V_1)}{R_{CQ1}} \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{\frac{(V_{DD1} - V_1)}{R_{CQ1}} \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)} \right]$$

$$V_1 = V_{CEQ1}$$
$$\simeq Vt \cdot \ln \left[\frac{\frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - \frac{V_1 \cdot (\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{\frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - \frac{V_1 \cdot (1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)} \right]$$

$$V_1 = V_{CEQ1}$$
$$\simeq Vt \cdot \ln \left[\frac{-\frac{V_1 \cdot (\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + \left[\frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1) \right]}{-\frac{V_1 \cdot (1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + \left[\frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1) \right]} \right]$$

For simplicity, we define new system global parameters: ξ_{11}, ξ_{12}, ξ_{13}, ξ_{14}.

$$\xi_{11} = -\frac{(\alpha r_1 - 1)}{R_{CQ1}}; \quad \xi_{12} = \frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} + I_{se} \cdot (\alpha r_1 \cdot \alpha f_1 - 1);$$
$$\xi_{13} = -\frac{(1 - \alpha f_1)}{R_{CQ1}}$$

$$\xi_{14} = \frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1);$$
$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln \left[\frac{V_1 \cdot \xi_{11} - k_1 \cdot \eta_1 \cdot \psi_1 + \xi_{12}}{V_1 \cdot \xi_{13} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + \xi_{14}} \right]$$

$e^{\left[\frac{V_1}{V_t}\right]} = \frac{V_1 \cdot \xi_{11} - k_1 \cdot \eta_1 \cdot \psi_1 + \xi_{12}}{V_1 \cdot \xi_{13} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + \xi_{14}}$; $e^{\left[\frac{V_1}{V_t}\right]} \approx \frac{V_1}{V_t} + 1$ (Taylor series approximation).

$$\left(\frac{V_1}{V_t} + 1\right) \cdot (V_1 \cdot \xi_{13} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + \xi_{14}) = (V_1 \cdot \xi_{11} - k_1 \cdot \eta_1 \cdot \psi_1 + \xi_{12});$$

$$\psi_1 = \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}$$

$$\frac{V_1^2}{V_t} \cdot \xi_{13} - \frac{V_1}{V_t} \cdot k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + \frac{V_1}{V_t} \cdot \xi_{14} + V_1 \cdot \xi_{13} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \psi_1 + \xi_{14}$$
$$= V_1 \cdot \xi_{11} - k_1 \cdot \eta_1 \cdot \psi_1 + \xi_{12}; \quad \psi_1 = \frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}$$

$$\frac{V_1^2}{V_t} \cdot \xi_{13} - \frac{V_1}{V_t} \cdot k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \left\{\frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}\right\} + \frac{V_1}{V_t} \cdot \xi_{14}$$
$$+ V_1 \cdot \xi_{13} - k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot \left\{\frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}\right\} + \xi_{14}$$
$$= V_1 \cdot \xi_{11} - k_1 \cdot \eta_1 \cdot \left\{\frac{1}{R_{11}} \cdot \frac{dV_1}{dt} - \frac{g_1}{R_{12}} + \frac{V_2}{R_{13}} - \frac{g_2}{R_{14}}\right\} + \xi_{12}$$

$$\frac{V_1^2}{V_t} \cdot \xi_{13} - \frac{V_1 \cdot k_1 \cdot \eta_1 \cdot \alpha f_1}{R_{11} \cdot V_t} \cdot \frac{dV_1}{dt} + \frac{V_1 \cdot g_1 \cdot k_1 \cdot \eta_1 \cdot \alpha f_1}{R_{12} \cdot V_t}$$
$$- \frac{V_1 \cdot V_2 \cdot k_1 \cdot \eta_1 \cdot \alpha f_1}{R_{13} \cdot V_t} + \frac{V_1 \cdot g_2 \cdot k_1 \cdot \eta_1 \cdot \alpha f_1}{R_{14} \cdot V_t}$$
$$+ \frac{V_1}{V_t} \cdot \xi_{14} + V_1 \cdot \xi_{13} - \frac{k_1 \cdot \eta_1 \cdot \alpha f_1}{R_{11}} \cdot \frac{dV_1}{dt} + \frac{k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot g_1}{R_{12}}$$
$$- \frac{k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot V_2}{R_{13}} + \frac{k_1 \cdot \eta_1 \cdot \alpha f_1 \cdot g_2}{R_{14}} + \xi_{14}$$
$$= V_1 \cdot \xi_{11} - \frac{k_1 \cdot \eta_1}{R_{11}} \cdot \frac{dV_1}{dt} + \frac{k_1 \cdot \eta_1 \cdot g_1}{R_{12}}$$
$$- \frac{k_1 \cdot \eta_1 \cdot V_2}{R_{13}} + \frac{k_1 \cdot \eta_1 \cdot g_2}{R_{14}} + \xi_{12}$$

$$\frac{dV_1}{dt} \cdot \frac{k_1 \cdot \eta_1}{R_{11}} \cdot \left(-\frac{V_1 \cdot \alpha f_1}{V_t} - \alpha f_1 + 1\right) + V_1 \cdot \left(\frac{V_1}{V_t} \cdot \xi_{13} + \frac{\xi_{14}}{V_t} + \xi_{13} - \xi_{11}\right)$$
$$+ V_2 \cdot \frac{k_1 \cdot \eta_1}{R_{13}} \cdot \left(-\frac{V_1 \cdot \alpha f_1}{V_t} - \alpha f_1 + 1\right) + g_1 \cdot \frac{k_1 \cdot \eta_1}{R_{12}} \cdot \left(\frac{V_1 \cdot \alpha f_1}{V_t} + \alpha f_1 - 1\right)$$
$$+ g_2 \cdot \frac{k_1 \cdot \eta_1}{R_{14}} \cdot \left(\frac{V_1 \cdot \alpha f_1}{V_t} + \alpha f_1 - 1\right) + \xi_{14} - \xi_{12} = 0$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}\cdot\frac{k_1\cdot\eta_1}{R_{11}}\cdot\left(1-\alpha f\cdot\left\{\frac{V_1}{V_t}+1\right\}\right)+V_1\cdot\left(\frac{V_1}{V_t}\cdot\xi_{13}+\frac{\xi_{14}}{V_t}+\xi_{13}-\xi_{11}\right)$$
$$+V_2\cdot\frac{k_1\cdot\eta_1}{R_{13}}\cdot\left(1-\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}\right)+g_1\cdot\frac{k_1\cdot\eta_1}{R_{12}}\cdot\left(\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}-1\right)$$
$$+g_2\cdot\frac{k_1\cdot\eta_1}{R_{14}}\cdot\left(\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}-1\right)+\xi_{14}-\xi_{12}=0$$

We define $\Delta_1=\xi_{14}-\xi_{12};\ \Delta_1\to\varepsilon\Rightarrow\xi_{14}-\xi_{12}\to\varepsilon\Rightarrow\xi_{14}=\xi_{12}+\varepsilon;\ \varepsilon\to 0$

$$\Delta_1=\xi_{14}-\xi_{12}=\frac{V_{DD1}\cdot(1-\alpha f_1)}{R_{CQ1}}+I_{sc}\cdot(\alpha r_1\cdot\alpha f_1-1)$$
$$-\frac{V_{DD1}\cdot(\alpha r_1-1)}{R_{CQ1}}-I_{se}\cdot(\alpha r_1\cdot\alpha f_1-1)$$

$$\Delta_1=\xi_{14}-\xi_{12}=\frac{V_{DD1}\cdot(2-\alpha f_1-\alpha r_1)}{R_{CQ1}}$$
$$+(I_{sc}-I_{se})\cdot(\alpha r_1\cdot\alpha f_1-1);\quad\Delta_1\to\varepsilon\to 0$$

$$\frac{V_{DD1}}{R_{CQ1}}=\frac{(I_{se}-I_{sc})\cdot(\alpha r_1\cdot\alpha f_1-1)}{(2-\alpha f_1-\alpha r_1)};\quad I_{se}-I_{sc}<0$$
$$\Rightarrow I_{se}<I_{sc};\quad\alpha r_1\cdot\alpha f_1-1<0\Rightarrow\alpha r_1\cdot\alpha f_1<1$$

$$2-\alpha f_1-\alpha r_1>0;\ \frac{(I_{se}-I_{sc})\cdot(\alpha r_1\cdot\alpha f_1-1)}{(2-\alpha f_1-\alpha r_1)}>0\Rightarrow\frac{V_{DD1}}{R_{CQ1}}>0$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}\cdot\frac{k_1\cdot\eta_1}{R_{11}}\cdot\left(1-\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}\right)=V_1\cdot\left(-\frac{V_1}{V_t}\cdot\xi_{13}-\frac{\xi_{14}}{V_t}-\xi_{13}+\xi_{11}\right)$$
$$-V_2\cdot\frac{k_1\cdot\eta_1}{R_{13}}\cdot\left(1-\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}\right)+g_1\cdot\frac{k_1\cdot\eta_1}{R_{12}}\cdot\left(-\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}+1\right)$$
$$+g_2\cdot\frac{k_1\cdot\eta_1}{R_{14}}\cdot\left(-\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}+1\right)$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}=V_1\cdot\frac{\left(-\frac{V_1}{V_t}\cdot\xi_{13}-\frac{\xi_{14}}{V_t}-\xi_{13}+\xi_{11}\right)}{\frac{k_1\cdot\eta_1}{R_{11}}\cdot\left(1-\alpha f_1\cdot\left\{\frac{V_1}{V_t}+1\right\}\right)}-V_2\cdot\frac{R_{11}}{R_{13}}+g_1\cdot\frac{R_{11}}{R_{12}}+g_2\cdot\frac{R_{11}}{R_{14}}$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = V_1 \cdot \frac{R_{11}}{k_1 \cdot \eta_1} \cdot \frac{\left(-\frac{V_1}{V_t} \cdot \xi_{13} - \frac{\xi_{14}}{V_t} - \xi_{13} + \xi_{11}\right)}{\left(-\alpha f_1 \cdot \frac{V_1}{V_t} + 1 - \alpha f_1\right)} - V_2 \cdot \frac{R_{11}}{R_{13}} + g_1 \cdot \frac{R_{11}}{R_{12}} + g_2 \cdot \frac{R_{11}}{R_{14}}$$

$$\frac{\left(-\frac{V_1}{V_t} \cdot \xi_{13} - \frac{\xi_{14}}{V_t} - \xi_{13} + \xi_{11}\right)}{\left(-\alpha f_1 \cdot \frac{V_1}{V_t} + 1 - \alpha f_1\right)} = 1 \pm \varepsilon \Rightarrow V_1$$
$$= \frac{\left\{(1 - \alpha f_1) \cdot (1 \pm \varepsilon) + \frac{\xi_{14}}{V_t} + \xi_{13} - \xi_{11}\right\} \cdot V_t}{\alpha f_1 \cdot (1 \pm \varepsilon) - \xi_{13}}; \quad 0 < \varepsilon \ll 1$$

$$V_1 = \frac{\left\{(1 - \alpha f_1) \cdot (1 \pm \varepsilon) + \frac{1}{V_t} \cdot \left\{\frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} + I_{sc} \cdot (\alpha r_1 \cdot \alpha f_1 - 1)\right\} - \frac{(1 - \alpha f_1)}{R_{CQ1}} + \frac{(\alpha r_1 - 1)}{R_{CQ1}}\right\} \cdot V_t}{\alpha f_1 \cdot (1 \pm \varepsilon) + \frac{(1 - \alpha f_1)}{R_{CQ1}}}$$

Remark Our system $V_1(t = t_0)$ initial voltage and voltage in time $V_1(t)$ must be within the above V_1 values band to fulfill Bautin bifurcation differential equations.

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = V_1 \cdot \frac{R_{11}}{k_1 \cdot \eta_1} \cdot (1 \pm \varepsilon) - V_2 \cdot \frac{R_{11}}{R_{13}} + g_1 \cdot \frac{R_{11}}{R_{12}} + g_2 \cdot \frac{R_{11}}{R_{14}};$$
$$\Gamma_1 = \frac{R_{11}}{k_1 \cdot \eta_1} \cdot (1 \pm \varepsilon); \quad \frac{R_{11}}{R_{13}} = 1 \Rightarrow R_{11} = R_{13}$$

$$\Gamma_2 = \frac{R_{11}}{R_{12}}; \quad \sigma = \frac{R_{11}}{R_{14}}; \quad \Gamma_1 = \frac{R_{11}}{k_1 \cdot \eta_1} \cdot (1 \pm \varepsilon)$$
$$= \frac{R_{11}}{k_1 \cdot \left[\frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}}\right\}}\right]} \cdot (1 \pm \varepsilon)$$

Result Our Bautin bifurcation system control parameters depend on circuit resistors values and mainly optocoupler coupling coefficient k_1.

We find for our Optoisolation Bautin bifurcation system the radius planar fixed points $r^{(j)}|_{r^{(j)} \geq 0}$ as a function of system overall parameters.

$$r^{(j)}|_{r^{(j)} \geq 0} = \sqrt{\frac{1}{2 \cdot \sigma} \cdot (-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1})}$$
$$= \frac{1}{\sqrt{2 \cdot \sigma}} \cdot \sqrt{(-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \sigma \cdot \Gamma_1})}$$

$$r^{(j)}\Big|_{r^{(j)}\geq 0}=\sqrt{\frac{R_{14}}{2\cdot R_{11}}\cdot\left(-\frac{R_{11}}{R_{12}}\pm\sqrt{\left(\frac{R_{11}}{R_{12}}\right)^2-4\cdot\frac{R_{11}}{R_{14}}\cdot\left\{\frac{R_{11}}{k_1\cdot\left[\frac{1}{1+V_t\cdot\frac{1}{I_0}\cdot\left\{\frac{1}{R_{11}}+\frac{1}{R_{12}}+\frac{1}{R_{13}}+\frac{1}{R_{14}}\right\}}\right]}\cdot(1\pm\varepsilon)\right\}}\right)}$$

$$r^{(j)}\Big|_{r^{(j)}\geq 0}=\frac{\sqrt{R_{14}}}{\sqrt{2\cdot R_{11}}}\cdot\sqrt{\left(-\frac{R_{11}}{R_{12}}\pm\sqrt{\left(\frac{R_{11}}{R_{12}}\right)^2-4\cdot\frac{R_{11}}{R_{14}}\cdot\left\{\frac{R_{11}}{k_1\cdot\left[\frac{1}{1+V_t\cdot\frac{1}{I_0}\cdot\left\{\frac{1}{R_{11}}+\frac{1}{R_{12}}+\frac{1}{R_{13}}+\frac{1}{R_{14}}\right\}}\right]}\cdot(1\pm\varepsilon)\right\}}\right)}$$

$$I_{D2}=I_{R21}+I_{R22}+I_{R23}+I_{R24};$$

$$I_{D2}=\frac{-g_4-V_{D2}}{R_{24}}+\frac{\frac{dV_2}{dt}-V_{D2}}{R_{23}}+\frac{-g_3-V_{D2}}{R_{22}}+\frac{-V_1-V_{D2}}{R_{21}}$$

$$I_{R23}=\frac{\frac{dV_2}{dt}-V_{D2}}{R_{23}};\ I_{R22}=\frac{-g_3-V_{D2}}{R_{22}};\ I_{R21}=\frac{-V_1-V_{D2}}{R_{21}};\ I_{R24}=\frac{-g_4-V_{D2}}{R_{24}}$$

We consider Taylor series approximation: $V_{D2}=V_t\cdot\ln(\frac{I_{D2}}{I_0}+1)\approx V_t\cdot\frac{I_{D2}}{I_0}$

$$I_{D2}=\frac{-g_4-V_t\cdot\frac{I_{D2}}{I_0}}{R_{24}}+\frac{\frac{dV_2}{dt}-V_t\cdot\frac{I_{D2}}{I_0}}{R_{23}}+\frac{-g_3-V_t\cdot\frac{I_{D2}}{I_0}}{R_{22}}+\frac{-V_1-V_t\cdot\frac{I_{D2}}{I_0}}{R_{21}}$$

$$\begin{aligned}I_{D2}=&\frac{1}{R_{23}}\cdot\frac{dV_2}{dt}-V_t\cdot\frac{I_{D2}}{I_0\cdot R_{21}}-\frac{g_3}{R_{22}}-V_t\cdot\frac{I_{D2}}{I_0\cdot R_{22}}-\frac{V_1}{R_{21}}\\&-V_t\cdot\frac{I_{D2}}{I_0\cdot R_{23}}-\frac{g_4}{R_{24}}-V_t\cdot\frac{I_{D2}}{I_0\cdot R_{24}}\end{aligned}$$

$$I_{D2}=-V_t\cdot\frac{I_{D2}}{I_0}\cdot\left\{\frac{1}{R_{21}}+\frac{1}{R_{22}}+\frac{1}{R_{23}}+\frac{1}{R_{24}}\right\}+\frac{1}{R_{23}}\cdot\frac{dV_2}{dt}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}$$

$$I_{D2}\cdot\left[1+V_t\cdot\frac{1}{I_0}\cdot\left\{\frac{1}{R_{21}}+\frac{1}{R_{22}}+\frac{1}{R_{23}}+\frac{1}{R_{24}}\right\}\right]=\frac{1}{R_{23}}\cdot\frac{dV_2}{dt}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}$$

$$I_{D2}=\frac{1}{1+V_t\cdot\frac{1}{I_0}\cdot\left\{\frac{1}{R_{21}}+\frac{1}{R_{22}}+\frac{1}{R_{23}}+\frac{1}{R_{24}}\right\}}\cdot\left\{\frac{1}{R_{23}}\cdot\frac{dV_2}{dt}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}\right\}$$

For simplicity, we define the following functions: $I_{D2}=\eta_2\cdot\psi_2\left(\frac{dV_2}{dt},V_1,V_2\right)$

$$\eta_2 = \frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{21}} + \frac{1}{R_{22}} + \frac{1}{R_{23}} + \frac{1}{R_{24}}\right\}};$$

$$\psi_2\left(\frac{dV_2}{dt}, V_1, V_2\right) = \frac{1}{R_{23}} \cdot \frac{dV_2}{dt} - \frac{g_3}{R_{22}} - \frac{V_1}{R_{21}} - \frac{g_4}{R_{24}}$$

$$I_{EQ2} = I_{BQ2} + I_{CQ2}; \quad I_{BQ2} = k_2 \cdot I_{D2} = k_2 \cdot \eta_2 \cdot \psi_2\left(\frac{dV_2}{dt}, V_1, V_2\right);$$

$$I_{EQ2} = k_2 \cdot \eta_2 \cdot \psi_2\left(\frac{dV_2}{dt}, V_1, V_2\right) + I_{CQ2}$$

The mathematical analysis is based on the basic transistor Ebers–Moll equations. We need to implement the Regular Ebers–Moll Model to the above optocoupler circuit.

$$V_{BEQ2} = Vt \cdot \ln\left[\left(\frac{\alpha r_2 \cdot I_{CQ2} - I_{EQ2}}{I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right) + 1\right];$$

$$V_{BCQ2} = Vt \cdot \ln\left[\left(\frac{I_{CQ2} - I_{EQ2} \cdot \alpha f_2}{I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right) + 1\right]$$

$V_{CEQ2} = V_{CBQ2} + V_{BEQ2}$, but $V_{CBQ2} = -V_{BCQ2}$, then $\mathbf{V_{CEQ2}} = \mathbf{V_{BEQ2}} - \mathbf{V_{BCQ2}}$

$$V_{CEQ2} = Vt \cdot \ln\left[\left(\frac{\alpha r_2 \cdot I_{CQ2} - I_{EQ2}}{I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ2} - I_{EQ2} \cdot \alpha f_2}{I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right) + 1\right]$$

$$V_{CEQ2} = Vt \cdot \ln\left[\frac{(\alpha r_2 \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha f_2) + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right] + Vt \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right)$$

Since $I_{sc} - I_{se} \to \varepsilon \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$ then we can write V_{CEQ2} expression.

$$V_2 = V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{(\alpha r_2 \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha f_2) + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right];$$

$$\psi_2 = \psi_2(\frac{dV_2}{dt}, V_1, V_2)$$

$$\begin{aligned}\alpha r_2 \cdot I_{CQ2} - I_{EQ2} &= \alpha r_2 \cdot I_{CQ2} - \left\{k_2 \cdot \eta_2 \cdot \psi_2\left(\frac{dV_2}{dt}, V_1, V_2\right) + I_{CQ2}\right\} \\ &= I_{CQ2} \cdot (\alpha r_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2\left(\frac{dV_2}{dt}, V_1, V_2\right)\end{aligned}$$

$$I_{CQ2} - I_{EQ2} \cdot \alpha f_2 = I_{CQ2} - \left\{ k_2 \cdot \eta_2 \cdot \psi_2 \left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2 \right) + I_{CQ2} \right\} \cdot \alpha f_2$$
$$= I_{CQ2} \cdot (1 - \alpha f_2) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 \left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2 \right)$$

$$V_2 = V_{CEQ2} \simeq Vt \cdot \ln \left[\frac{I_{CQ2} \cdot (\alpha r_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2 + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{I_{CQ2} \cdot (1 - \alpha f_2) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)} \right];$$

$$V_{DD2} = I_{CQ2} \cdot R_{CQ2} + V_2 \Rightarrow I_{CQ1} = \frac{V_{DD2} - V_2}{R_{CQ2}}$$

$$V_2 = V_{CEQ2} \simeq Vt \cdot \ln \left[\frac{\frac{(V_{DD2} - V_2)}{R_{CQ2}} \cdot (\alpha r_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2 + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{\frac{(V_{DD2} - V_2)}{R_{CQ2}} \cdot (1 - \alpha f_2) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)} \right]$$

$$V_2 = V_{CEQ2}$$
$$\simeq Vt \cdot \ln \left[\frac{\frac{V_{DD2} \cdot (\alpha r_2 - 1)}{R_{CQ2}} - \frac{V_2 \cdot (\alpha r_2 - 1)}{R_{CQ2}} - k_2 \cdot \eta_2 \cdot \psi_2 + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{\frac{V_{DD2} \cdot (1 - \alpha f_2)}{R_{CQ2}} - \frac{V_2 \cdot (1 - \alpha f_2)}{R_{CQ2}} - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)} \right]$$

$$V_2 = V_{CEQ2}$$
$$\simeq Vt \cdot \ln \left[\frac{-\frac{V_2 \cdot (\alpha r_2 - 1)}{R_{CQ2}} - k_2 \cdot \eta_2 \cdot \psi_2 + \left[\frac{V_{DD2} \cdot (\alpha r_2 - 1)}{R_{CQ2}} + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1) \right]}{-\frac{V_2 \cdot (1 - \alpha f_2)}{R_{CQ2}} - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + \left[\frac{V_{DD2} \cdot (1 - \alpha f_2)}{R_{CQ2}} + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1) \right]} \right]$$

For simplicity, we define new system global parameters: ξ_{21}, ξ_{22}, ξ_{23}, ξ_{24}.

$$\xi_{21} = -\frac{(\alpha r_2 - 1)}{R_{CQ2}}; \quad \xi_{22} = \frac{V_{DD2} \cdot (\alpha r_2 - 1)}{R_{CQ2}} + I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1);$$
$$\xi_{23} = -\frac{(1 - \alpha f_2)}{R_{CQ2}}$$

$$\xi_{24} = \frac{V_{DD2} \cdot (1 - \alpha f_2)}{R_{CQ2}} + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1);$$
$$V_2 = V_{CEQ2} \simeq Vt \cdot \ln \left[\frac{V_2 \cdot \xi_{21} - k_2 \cdot \eta_2 \cdot \psi_2 + \xi_{22}}{V_2 \cdot \xi_{23} - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + \xi_{24}} \right]$$

$e^{\left[\frac{V_2}{Vt}\right]} = \frac{V_2 \cdot \xi_{21} - k_2 \cdot \eta_2 \cdot \psi_2 + \xi_{22}}{V_2 \cdot \xi_{23} - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + \xi_{24}}$; $e^{\left[\frac{V_2}{Vt}\right]} \approx \frac{V_2}{V_t} + 1$ (Taylor series approximation).

$$\left(\frac{V_2}{V_t}+1\right)\cdot(V_2\cdot\xi_{23}-k_2\cdot\eta_2\cdot\alpha f_2\cdot\psi_2+\xi_{24})=(V_2\cdot\xi_{21}-k_2\cdot\eta_2\cdot\psi_2+\xi_{22});$$
$$\psi_2=\frac{1}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}$$

$$\frac{V_2^2}{V_t}\cdot\xi_{23}-\frac{V_2}{V_t}\cdot k_2\cdot\eta_2\cdot\alpha f_2\cdot\psi_2+\frac{V_2}{V_t}\cdot\xi_{24}+V_2\cdot\xi_{23}-k_2\cdot\eta_2\cdot\alpha f_2\cdot\psi_2+\xi_{24}$$
$$=V_2\cdot\xi_{21}-k_2\cdot\eta_2\cdot\psi_2+\xi_{22};\quad \psi_2=\frac{1}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}$$

$$\frac{V_2^2}{V_t}\cdot\xi_{23}-\frac{V_2}{V_t}\cdot k_2\cdot\eta_2\cdot\alpha f_2\cdot\left\{\frac{1}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}\right\}+\frac{V_2}{V_t}\cdot\xi_{24}$$
$$+V_2\cdot\xi_{23}-k_2\cdot\eta_2\cdot\alpha f_2\cdot\left\{\frac{1}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}\right\}+\xi_{24}$$
$$=V_2\cdot\xi_{21}-k_2\cdot\eta_2\cdot\left\{\frac{1}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}-\frac{g_3}{R_{22}}-\frac{V_1}{R_{21}}-\frac{g_4}{R_{24}}\right\}+\xi_{22}$$

$$\frac{V_2^2}{V_t}\cdot\xi_{23}-\frac{V_2\cdot k_2\cdot\eta_2\cdot\alpha f_2}{R_{23}\cdot V_t}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}+\frac{V_2\cdot g_3\cdot k_2\cdot\eta_2\cdot\alpha f_2}{R_{22}\cdot V_t}$$
$$+\frac{V_1\cdot V_2\cdot k_2\cdot\eta_2\cdot\alpha f_2}{R_{21}\cdot V_t}+\frac{V_2\cdot g_4\cdot k_2\cdot\eta_2\cdot\alpha f_2}{R_{24}\cdot V_t}$$
$$+\frac{V_2}{V_t}\cdot\xi_{24}+V_2\cdot\xi_{23}-\frac{k_2\cdot\eta_2\cdot\alpha f_2}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}$$
$$+\frac{k_2\cdot\eta_2\cdot\alpha f_2\cdot g_3}{R_{22}}+\frac{k_2\cdot\eta_2\cdot\alpha f_2\cdot V_1}{R_{21}}+\frac{k_2\cdot\eta_2\cdot\alpha f_2\cdot g_4}{R_{24}}+\xi_{24}$$
$$=V_2\cdot\xi_{21}-\frac{k_2\cdot\eta_2}{R_{23}}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}+\frac{k_2\cdot\eta_2\cdot g_3}{R_{22}}+\frac{k_2\cdot\eta_2\cdot V_1}{R_{21}}$$
$$+\frac{k_2\cdot\eta_2\cdot g_4}{R_{24}}+\xi_{22}$$

$$\frac{\mathrm{d}V_2}{\mathrm{d}t}\cdot\frac{k_2\cdot\eta_2}{R_{23}}\cdot\left(-\frac{V_2\cdot\alpha f_2}{V_t}-\alpha f_2+1\right)+V_2\cdot\left(\frac{V_2}{V_t}\cdot\xi_{23}+\frac{\xi_{24}}{V_t}+\xi_{23}-\xi_{21}\right)$$
$$+V_1\cdot\frac{k_2\cdot\eta_2}{R_{21}}\cdot\left(\frac{V_2\cdot\alpha f_2}{V_t}+\alpha f_2-1\right)+g_3\cdot\frac{k_2\cdot\eta_2}{R_{22}}\cdot\left(\frac{V_2\cdot\alpha f_2}{V_t}+\alpha f_2-1\right)$$
$$+g_4\cdot\frac{k_2\cdot\eta_2}{R_{24}}\cdot\left(\frac{V_2\cdot\alpha f_2}{V_t}+\alpha f_2-1\right)+\xi_{24}-\xi_{22}=0$$

We define $\Delta_2=\xi_{24}-\xi_{22}$; $\Delta_2\to\varepsilon\Rightarrow\xi_{24}-\xi_{22}\to\varepsilon\Rightarrow\xi_{24}=\xi_{22}+\varepsilon$; $\varepsilon\to0$

$$\Delta_2 = \xi_{24} - \xi_{22} = \frac{V_{DD2} \cdot (1 - \alpha f_2)}{R_{CQ2}} + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - \frac{V_{DD2} \cdot (\alpha r_2 - 1)}{R_{CQ2}} - I_{se} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)$$

$$\Delta_2 = \xi_{24} - \xi_{22} = \frac{V_{DD2} \cdot (2 - \alpha f_2 - \alpha r_2)}{R_{CQ2}} + (I_{sc} - I_{se}) \cdot (\alpha r_2 \cdot \alpha f_2 - 1); \quad \Delta_2 \to \varepsilon \to 0$$

$$\frac{V_{DD2}}{R_{CQ2}} = \frac{(I_{se} - I_{sc}) \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(2 - \alpha f_2 - \alpha r_2)}; \quad I_{se} - I_{sc} < 0$$
$$\Rightarrow I_{se} < I_{sc};\ \alpha r_2 \cdot \alpha f_2 - 1 < 0 \Rightarrow \alpha r_2 \cdot \alpha f_2 < 1$$

$$2 - \alpha f_2 - \alpha r_2 > 0; \quad \frac{(I_{se} - I_{sc}) \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(2 - \alpha f_2 - \alpha r_2)} > 0 \Rightarrow \frac{V_{DD2}}{R_{CQ2}} > 0$$

$$\frac{dV_2}{dt} \cdot \frac{k_2 \cdot \eta_2}{R_{23}} \cdot \left(1 - \alpha f_2 \cdot \frac{V_2}{V_t} - \alpha f_2\right) = V_2 \cdot \left(-\frac{V_2}{V_t} \cdot \xi_{23} - \frac{\xi_{24}}{V_t} - \xi_{23} + \xi_{21}\right) + V_1 \cdot \frac{k_2 \cdot \eta_2}{R_{21}} \cdot \left(1 - \alpha f_2 \cdot \frac{V_2}{V_t} - \alpha f_2\right) + g_3 \cdot \frac{k_2 \cdot \eta_2}{R_{22}} \cdot \left(-\alpha f_2 \cdot \frac{V_2}{V_t} - \alpha f_2 + 1\right) + g_4 \cdot \frac{k_2 \cdot \eta_2}{R_{24}} \cdot \left(-\alpha f_2 \cdot \frac{V_2}{V_t} - \alpha f_2 + 1\right)$$

$$\frac{dV_2}{dt} = V_2 \cdot \frac{\left(-\frac{V_2}{V_t} \cdot \xi_{23} - \frac{\xi_{24}}{V_t} - \xi_{23} + \xi_{21}\right)}{\frac{k_2 \cdot \eta_2}{R_{21}} \cdot \left(1 - \alpha f_2 \cdot \frac{V_2}{V_t} - \alpha f_2\right)} + V_1 \cdot \frac{R_{23}}{R_{21}} + g_3 \cdot \frac{R_{23}}{R_{22}} + g_4 \cdot \frac{R_{23}}{R_{24}}$$

$$\frac{\left(-\frac{V_2}{V_t} \cdot \xi_{23} - \frac{\xi_{24}}{V_t} - \xi_{23} + \xi_{21}\right)}{\left(1 - \alpha f_2 \cdot \frac{V_2}{V_t} - \alpha f_2\right)} = 1 \pm \varepsilon$$
$$\Rightarrow V_2 = \frac{\left\{(1 - \alpha f_2) \cdot (1 \pm \varepsilon) + \frac{\xi_{24}}{V_t} + \xi_{23} - \xi_{21}\right\} \cdot V_t}{\alpha f_2 \cdot (1 \pm \varepsilon) - \xi_{23}}; \quad 0 < \varepsilon \ll 1$$

$$V_2 = \frac{\left\{(1 - \alpha f_2) \cdot (1 \pm \varepsilon) + \frac{1}{V_t} \cdot \left\{\frac{V_{DD2} \cdot (1 - \alpha f_2)}{R_{CQ2}} + I_{sc} \cdot (\alpha r_2 \cdot \alpha f_2 - 1)\right\} - \frac{(1 - \alpha f_2)}{R_{CQ2}} + \frac{(\alpha r_2 - 1)}{R_{CQ2}}\right\} \cdot V_t}{\alpha f_2 \cdot (1 \pm \varepsilon) + \frac{(1 - \alpha f_2)}{R_{CQ2}}}$$

Remark Our system $V_2(t = t_0)$ initial voltage and voltage in time $V_2(t)$ must be within the above V_2 values band to fulfill Bautin bifurcation differential equations.

$$\frac{dV_2}{dt} = V_2 \cdot \frac{R_{21}}{k_2 \cdot \eta_2} \cdot (1 \pm \varepsilon) + V_1 \cdot \frac{R_{23}}{R_{21}} + g_3 \cdot \frac{R_{23}}{R_{22}} + g_4 \cdot \frac{R_{23}}{R_{24}};$$

$$\Gamma_1 = \frac{R_{21}}{k_2 \cdot \eta_2} \cdot (1 \pm \varepsilon); \quad \frac{R_{23}}{R_{21}} = 1 \Rightarrow R_{23} = R_{21}$$

$$\Gamma_2 = \frac{R_{23}}{R_{22}}; \quad \sigma = \frac{R_{23}}{R_{24}}; \quad \Gamma_1 = \frac{R_{21}}{k_2 \cdot \eta_2} \cdot (1 \pm \varepsilon)$$
$$= \frac{R_{21}}{k_2 \cdot \left[\frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{21}} + \frac{1}{R_{22}} + \frac{1}{R_{23}} + \frac{1}{R_{24}}\right\}}\right]} \cdot (1 \pm \varepsilon)$$

If we compare the two Bautin bifurcation optoisolation circuit's branches ($Q1$ ($V1$) and $Q2(V2)$ circuit branches) Γ_1, Γ_2, σ parameters, we get the following propositions:

$$\Gamma_1 = \frac{R_{11}}{k_1 \cdot \eta_1} \cdot (1 \pm \varepsilon) \quad \& \quad \Gamma_1 = \frac{R_{21}}{k_2 \cdot \eta_2} \cdot (1 \pm \varepsilon) \Rightarrow \frac{R_{11}}{k_1 \cdot \eta_1} = \frac{R_{21}}{k_2 \cdot \eta_2}$$

$$\Gamma_2 = \frac{R_{11}}{R_{12}} \ \& \ \Gamma_2 = \frac{R_{23}}{R_{22}} \Rightarrow \frac{R_{11}}{R_{12}} = \frac{R_{23}}{R_{22}}; \quad \sigma = \frac{R_{11}}{R_{14}} \ \& \ \sigma = \frac{R_{23}}{R_{24}} \Rightarrow \frac{R_{11}}{R_{14}} = \frac{R_{23}}{R_{24}}$$

$$\Gamma_1 = \frac{R_{11}}{k_1 \cdot \eta_1} \cdot (1 \pm \varepsilon) = \frac{R_{11}}{k_1 \cdot \left[\frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}}\right\}}\right]} \cdot (1 \pm \varepsilon) \quad \&$$

$$\Gamma_1 = \frac{R_{21}}{k_2 \cdot \eta_2} \cdot (1 \pm \varepsilon) = \frac{R_{21}}{k_2 \cdot \left[\frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{21}} + \frac{1}{R_{22}} + \frac{1}{R_{23}} + \frac{1}{R_{24}}\right\}}\right]} \cdot (1 \pm \varepsilon)$$

$$\Rightarrow \frac{R_{11}}{k_1 \cdot \left[\frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{11}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}}\right\}}\right]} = \frac{R_{21}}{k_2 \cdot \left[\frac{1}{1 + V_t \cdot \frac{1}{I_0} \cdot \left\{\frac{1}{R_{21}} + \frac{1}{R_{22}} + \frac{1}{R_{23}} + \frac{1}{R_{24}}\right\}}\right]}$$

We find for our Optoisolation Bautin bifurcation system the radius planar fixed points $r^{(j)}|_{r^{(j)} \geq 0}$ as a function of system overall parameters. It is equivalent to our previous results [71, 72].

$$r^{(j)}\big|_{r^{(j)}\geq 0} = \sqrt{\frac{1}{2\cdot\sigma}\cdot\left(-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1}\right)}$$
$$= \frac{1}{\sqrt{2\cdot\sigma}}\cdot\sqrt{\left(-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4\cdot\sigma\cdot\Gamma_1}\right)}$$

$$r^{(j)}\big|_{r^{(j)}\geq 0} = \frac{1}{\sqrt{2\cdot\frac{R_{23}}{R_{24}}}} \cdot \sqrt{\left(-\frac{R_{23}}{R_{22}} \pm \sqrt{\left(\frac{R_{23}}{R_{22}}\right)^2 - 4\cdot\frac{R_{23}}{R_{24}}\cdot\left\{\frac{R_{21}}{k_2\cdot\left[\frac{1}{1+V_t\cdot\frac{1}{I_0}\cdot\{\frac{1}{R_{21}}+\frac{1}{R_{22}}+\frac{1}{R_{23}}+\frac{1}{R_{24}}\}}\right]}\cdot(1\pm\varepsilon)\right\}}\right)}$$

2.5 Bogdanov–Takens (Double-Zero) Bifurcation System

The Bogdanov–Takens (BT) bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue of multiplicity two. The system has two equilibria (a saddle and a non-saddle) which collide and disappear via saddle node bifurcation. The non-saddle equilibrium undergoes an Andronov–Hopf bifurcation generating a limit cycle. This cycle degenerates into an orbit homoclinic to the saddle and disappear via a saddle homoclinic bifurcation [5, 6, 37]. We consider an autonomous system of $\frac{dx}{dt} = f(x, \alpha)$, $x \in \mathbb{R}^n$ depending on two parameters $\alpha \in \mathbb{R}^2$, where f is smooth. Suppose that at $\alpha = 0$ the system has an equilibrium $x^0 = 0$. Assume that its Jacobian matrix $A_0 = f_x(0, 0)$ has zero eigenvalue of multiplicity two $\lambda_{1,2} = 0$. This bifurcation is characterized by two bifurcation conditions $\lambda_1 = \lambda_2 = 0$ which has co-dimension two and appears generically in two-parameter families of smooth ODEs. The critical equilibrium x^0 is a double root of the equation $f(x, 0) = 0$ and $\alpha = 0$ is the origin in the parameter plane of two branches of the saddle–node bifurcation curve, an Andronov–Hopf bifurcation curve, and a saddle homoclinic bifurcation curve. These bifurcations are non-degenerate and no other bifurcation occur in a small fixed neighborhood of x^0 for parameter values sufficiently close to $\alpha = 0$. In this neighborhood, the system has at most two equilibria and one limit cycle. For the private case with $n = 2$, $\frac{dx}{dt} = f(x, \alpha)$, $x \in \mathbb{R}^2$ if the following non-degeneracy conditions hold (BT.1) $a(0)\cdot b(0) \neq 0$, where $a(0)$ and $b(0)$ are certain quadratic coefficients, (BT.2) the map $(x, \alpha) \rightarrow (f(x, \alpha), \mathrm{Tr}(f_x(x, \alpha)), \det(f_x(x, \alpha)))$ is regular at $(x, \alpha) = (0, 0)$, then this system is locally topologically equivalent near the origin to the following normal form:

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = V_2; \quad \frac{\mathrm{d}V_2}{\mathrm{d}t} = \Gamma_1 + \Gamma_2 \cdot V_1 + V_1^2 + \sigma \cdot V_1 \cdot V_2;$$
$$V = (V_1, V_2)^T \in \mathbb{R}^2, \quad \sigma = \operatorname{sign}[a(0) \cdot b(0)] = \pm 1$$

The first stage is to find a system fixed point and discuss stability issue.

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = 0; \quad \frac{\mathrm{d}V_2}{\mathrm{d}t} = 0 \Rightarrow V_2^{(i)} = 0 \quad \& \quad \Gamma_1 + \Gamma_2 \cdot V_1^{(i)}$$
$$+ [V_1^{(i)}]^2 = 0 \Rightarrow V_1^{(i)} = \frac{1}{2} \cdot \left\{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\}$$

Our fixed points coordinates are as follows: $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = \left(\frac{1}{2} \cdot \left\{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\}, 0\right)$.

We have some possible cases in our system stability analysis:

Case I $\Gamma_2^2 - 4 \cdot \Gamma_1 = 0 \Rightarrow \Gamma_2^2 = 4 \cdot \Gamma_1 \Rightarrow \Gamma_2 = \pm 2 \cdot \sqrt{\Gamma_1} \Rightarrow V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = \left(-\frac{\Gamma_2}{2}, 0\right)$ and we need to check the stability (only one fixed point), $V^{(0)} = (V_1^{(0)}, V_2^{(0)}) = \left(-\frac{\Gamma_2}{2}, 0\right)$. We define $f_1(V_1, V_2) = V_2$; $f_2(V_1, V_2) = \Gamma_1 + \Gamma_2 \cdot V_1 + V_1^2 + \sigma \cdot V_1 \cdot V_2$; $\frac{\mathrm{d}V_1}{\mathrm{d}t} = f_1(V_1, V_2)$; $\frac{\mathrm{d}V_2}{\mathrm{d}t} = f_2(V_1, V_2)$ and calculated the related partial derivatives of f_1 and f_2 respect to V_1 and V_2, $\frac{\partial f_1}{\partial V_1} = 0$; $\frac{\partial f_1}{\partial V_2} = 1$; $\frac{\partial f_2}{\partial V_1} = \Gamma_2 + 2 \cdot V_1 + \sigma \cdot V_2$; $\frac{\partial f_2}{\partial V_2} = \sigma \cdot V_1$.

The matrix A is called the Jacobian matrix at the fixed point $(V_1^{(0)}, V_2^{(0)}) = \left(-\frac{\Gamma_2}{2}, 0\right)$.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{pmatrix} \Bigg|_{(V_1^{(0)}, V_2^{(0)}) = \left(-\frac{\Gamma_2}{2}, 0\right)} = \begin{pmatrix} 0 & 1 \\ \Gamma_2 + 2 \cdot V_1 + \sigma \cdot V_2 & \sigma \cdot V_1 \end{pmatrix} \Bigg|_{(V_1^{(0)}, V_2^{(0)}) = \left(-\frac{\Gamma_2}{2}, 0\right)}$$
$$= \begin{pmatrix} 0 & 1 \\ \Gamma_2 - 2 \cdot \frac{\Gamma_2}{2} & -\frac{\sigma \cdot \Gamma_2}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ \Gamma_2 - 2 \cdot \frac{\Gamma_2}{2} & -\frac{\sigma \cdot \Gamma_2}{2} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \lambda$$
$$= \begin{pmatrix} -\lambda & 1 \\ \Gamma_2 - 2 \cdot \frac{\Gamma_2}{2} & -\frac{\sigma \cdot \Gamma_2}{2} - \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 0 & -\frac{\sigma \cdot \Gamma_2}{2} - \lambda \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$(-\lambda) \cdot \left(-\frac{\sigma \cdot \Gamma_2}{2} - \lambda\right) = 0 \Rightarrow \lambda \cdot \left(\frac{\sigma \cdot \Gamma_2}{2} + \lambda\right) = 0 \Rightarrow \lambda_1 = 0;$$
$$\lambda_2 = -\frac{\sigma \cdot \Gamma_2}{2}; \quad \Delta = \lambda_1 \cdot \lambda_2 = 0; \quad \tau = \lambda_1 \cdot \lambda_2 = -\frac{\sigma \cdot \Gamma_2}{2}$$

$\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, or a plane of fixed points, if $A = 0$.

Case II

$$\Gamma_2^2 - 4\cdot\Gamma_1 > 0 \Rightarrow V^{(i)} = \left(V_1^{(i)}, V_2^{(i)}\right) = \left(\frac{1}{2}\cdot\left\{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0\right)$$

and we need to check the stability (two fixed points), first and second fixed points:

$$(V_1^{(0)}, V_2^{(0)}) = \left(\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0\right);$$
$$(V_1^{(1)}, V_2^{(1)}) = \left(\frac{1}{2}\cdot\left\{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0\right)$$

First, we check the stability of the first fixed point:

$$(V_1^{(0)}, V_2^{(0)}) = \left(\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0\right).$$

We define $f_1(V_1, V_2) = V_2$; $f_2(V_1, V_2) = \Gamma_1 + \Gamma_2\cdot V_1 + V_1^2 + \sigma\cdot V_1\cdot V_2$; $\frac{dV_1}{dt} = f_1(V_1, V_2)$; $\frac{dV_2}{dt} = f_2(V_1, V_2)$ and calculated the related partial derivatives of f_1 and f_2 respect to V_1 and V_2, $\frac{\partial f_1}{\partial V_1} = 0$; $\frac{\partial f_1}{\partial V_2} = 1$; $\frac{\partial f_2}{\partial V_1} = \Gamma_2 + 2\cdot V_1 + \sigma\cdot V_2$; $\frac{\partial f_2}{\partial V_2} = \sigma\cdot V_1$.

The matrix A is called the Jacobian matrix at the first fixed point.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{pmatrix}\Bigg|_{(V_1^{(0)}, V_2^{(0)}) = (\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0)}$$
$$= \begin{pmatrix} 0 & 1 \\ \Gamma_2 + 2\cdot V_1 + \sigma\cdot V_2 & \sigma\cdot V_1 \end{pmatrix}\Bigg|_{(V_1^{(0)}, V_2^{(0)}) = (\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0)}$$
$$= \begin{pmatrix} 0 & 1 \\ \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1} & \sigma\cdot\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1} & \sigma\cdot\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\lambda$$
$$= \begin{pmatrix} -\lambda & 1 \\ \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1} & \sigma\cdot\frac{1}{2}\cdot\left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\} - \lambda \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$[-\lambda] \cdot \left[\sigma \cdot \frac{1}{2} \cdot \left\{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} - \lambda\right] - [\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}] = 0;$$
$$\psi(\Gamma_1, \Gamma_2) = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$$

$$\psi = \psi(\Gamma_1, \Gamma_2) \Rightarrow -\lambda \cdot \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 + \psi\} + \lambda^2 - \psi = 0$$
$$\Rightarrow \lambda^2 - \lambda \cdot \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 + \psi\} - \psi = 0$$

$$\xi(\Gamma_1, \Gamma_2) = \xi = \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 + \psi\} \Rightarrow \lambda^2 - \lambda \cdot \xi - \psi = 0$$
$$\Rightarrow \lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$$

Option I $\lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$; $\xi^2 + 4 \cdot \psi = 0 \Rightarrow$ $\xi^2 = -4 \cdot \psi \Rightarrow$ $\sigma^2 \cdot \frac{1}{4} \cdot \{-\Gamma_2 + \psi\}^2 = -4 \cdot \psi$

$\psi(\Gamma_1, \Gamma_2) = \psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$ then option I can't exist.

Option II $\lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$; $\xi^2 + 4 \cdot \psi > 0$ and we have two real eigenvalues (λ_1, λ_2) and three subcases. $\sigma^2 \cdot \frac{1}{4} \cdot \{-\Gamma_2 + \psi\}^2 + 4 \cdot \psi > 0 \Rightarrow$

$$\sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2^2 - 2 \cdot \Gamma_2 \cdot \psi + \psi^2\} + 4 \cdot \psi > 0 \Rightarrow \sigma^2 \cdot \frac{1}{4}$$
$$\cdot \left\{\Gamma_2^2 - 2 \cdot \Gamma_2 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} + \Gamma_2^2 - 4 \cdot \Gamma_1\right\} + 4 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$$

$$\sigma^2 \cdot \frac{1}{4} \cdot \left\{2 \cdot \Gamma_2^2 - 2 \cdot \Gamma_2 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} - 4 \cdot \Gamma_1\right\} + 4 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$$

$$\frac{1}{2} \cdot \Gamma_2^2 \cdot \sigma^2 - \frac{1}{2} \cdot \sigma^2 \cdot \Gamma_2 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} - \Gamma_1 \cdot \sigma^2 + 4 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$$

$\sigma^2 \cdot \left(\frac{1}{2} \cdot \Gamma_2^2 - \Gamma_1\right) + \left\{4 - \frac{1}{2} \cdot \sigma^2 \cdot \Gamma_2\right\} \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$ and since $\psi(\Gamma_1, \Gamma_2) = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$.

We divide the above expression by ψ and get $\frac{\sigma^2}{\psi} \cdot \left(\frac{1}{2} \cdot \Gamma_2^2 - \Gamma_1\right) + 4 - \frac{1}{2} \cdot \sigma^2 \cdot \Gamma_2 > 0$.

Option II.1 $\xi \pm \sqrt{\xi^2 + 4 \cdot \psi} > 0$ then we have two positive eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ and the first fixed point is unstable node.

$$\xi + \sqrt{\xi^2 + 4 \cdot \psi} > 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 + \psi\} + \sqrt{\sigma^2 \cdot \frac{1}{4} \cdot \{-\Gamma_2 + \psi\}^2 + 4 \cdot \psi} > 0$$

$$\sigma \cdot [\psi - \Gamma_2] + \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} > 0; \quad \sigma > 0 \quad \& \quad \sigma < 0$$

$$\sigma > 0 \Rightarrow \psi - \Gamma_2 + \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} > 0$$

$$\Rightarrow \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} > \Gamma_2 - \psi$$

$$\sigma < 0 \Rightarrow \psi - \Gamma_2 + \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} < 0$$

$$\Rightarrow \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} < \Gamma_2 - \psi$$

$$\xi - \sqrt{\xi^2 + 4 \cdot \psi} > 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 + \psi\} - \sqrt{\sigma^2 \cdot \frac{1}{4} \cdot \{-\Gamma_2 + \psi\}^2 + 4 \cdot \psi} > 0$$

$$\sigma \cdot [\psi - \Gamma_2] - \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} > 0; \quad \sigma > 0 \quad \& \quad \sigma < 0$$

$$\sigma > 0 \Rightarrow \psi - \Gamma_2 - \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} > 0$$

$$\Rightarrow \psi - \Gamma_2 > \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi}$$

$$\sigma < 0 \Rightarrow \psi - \Gamma_2 - \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi} < 0$$

$$\Rightarrow \psi - \Gamma_2 < \frac{1}{\sigma} \cdot \sqrt{\sigma^2 \cdot \{-\Gamma_2 + \psi\}^2 + 16 \cdot \psi}$$

Option II.2 $\lambda_1 \cdot \lambda_2 < 0 \Rightarrow \left[\xi + \sqrt{\xi^2 + 4 \cdot \psi}\right] \cdot \left[\xi - \sqrt{\xi^2 + 4 \cdot \psi}\right] < 0$ then we have two opposite signs eigenvalues $\lambda_1 > 0$, $\lambda_2 < 0$ or $\lambda_1 < 0$, $\lambda_2 > 0$ and the first fixed point is hyperbolic fixed point (saddle point).

$$\lambda_1 > 0 \cdot \lambda_2 < 0 \Rightarrow \left[\xi + \sqrt{\xi^2 + 4\cdot\psi}\right] > 0 \cdot \left[\xi - \sqrt{\xi^2 + 4\cdot\psi}\right] < 0$$

$$\Rightarrow \left[\sigma\cdot\frac{1}{2}\cdot\{-\Gamma_2+\psi\} + \sqrt{\sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi} > 0\right.$$

$$\cdot\left[\sigma\cdot\frac{1}{2}\cdot\{-\Gamma_2+\psi\} - \sqrt{\sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi} < 0\right.$$

$$\lambda_1 < 0 \cdot \lambda_2 > 0 \Rightarrow \left[\xi + \sqrt{\xi^2 + 4\cdot\psi}\right] < 0 \cdot \left[\xi - \sqrt{\xi^2 + 4\cdot\psi}\right] > 0$$

$$\Rightarrow \left[\sigma\cdot\frac{1}{2}\cdot\{-\Gamma_2+\psi\} + \sqrt{\sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi} < 0\right.$$

$$\cdot\left[\sigma\cdot\frac{1}{2}\cdot\{-\Gamma_2+\psi\} - \sqrt{\sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi} > 0\right.$$

Option II.3 $\xi \pm \sqrt{\xi^2 + 4\cdot\psi} < 0$ then we have two negative eigenvalues $\lambda_1 < 0$ and $\lambda_2 < 0$ and the first fixed point is a stable node.

$$\lambda_1 < 0 \cdot \lambda_2 < 0 \Rightarrow [\xi + \sqrt{\xi^2 + 4\cdot\psi}] < 0 \cdot [\xi - \sqrt{\xi^2 + 4\cdot\psi}] < 0$$

$$\Rightarrow \left[\sigma\cdot\frac{1}{2}\cdot\{-\Gamma_2+\psi\} + \sqrt{\sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi} < 0\right.$$

$$\cdot\left[\sigma\cdot\frac{1}{2}\cdot\{-\Gamma_2+\psi\} - \sqrt{\sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi} < 0\right.$$

Option III $\lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4\cdot\psi}}{2}$; $\xi^2 + 4\cdot\psi < 0$ and we have two complex eigenvalues $(\lambda_1,\ \lambda_2)$, $\lambda_{1,2} = \frac{1}{2}\cdot(\xi \pm i\cdot\sqrt{|\xi^2 + 4\cdot\psi|})$ and three subcases $\xi = 0,\ \xi > 0,\ \xi < 0$. $\xi^2 + 4\cdot\psi < 0 \Rightarrow \sigma^2\cdot\frac{1}{4}\cdot\{-\Gamma_2+\psi\}^2 + 4\cdot\psi < 0$ cannot exist since $\xi^2 > 0\ \&\ 4\cdot\psi > 0$; $\psi(\Gamma_1, \Gamma_2) = \psi = \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1} > 0$.

Second we check the stability of the second fixed point:

$$\left(V_1^{(0)}, V_2^{(0)}\right) = \left(\frac{1}{2}\cdot\left\{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4\cdot\Gamma_1}\right\}, 0\right).$$

We define $f_1(V_1, V_2) = V_2$; $f_2(V_1, V_2) = \Gamma_1 + \Gamma_2 \cdot V_1 + V_1^2 + \sigma \cdot V_1 \cdot V_2$; $\frac{dV_1}{dt} = f_1(V_1, V_2)$; $\frac{dV_2}{dt} = f_2(V_1, V_2)$ and calculated the related partial derivatives of f_1 and f_2 respect to V_1 and V_2, $\frac{\partial f_1}{\partial V_1} = 0$; $\frac{\partial f_1}{\partial V_2} = 1$; $\frac{\partial f_2}{\partial V_1} = \Gamma_2 + 2 \cdot V_1 + \sigma \cdot V_2$; $\frac{\partial f_2}{\partial V_2} = \sigma \cdot V_1$.

The matrix A is called the jacobian matrix at the first fixed point.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{pmatrix} \Bigg|_{(V_1^{(0)}, V_2^{(0)}) = (\frac{1}{2} \cdot \{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}, 0)}$$

$$= \begin{pmatrix} 0 & 1 \\ \Gamma_2 + 2 \cdot V_1 + \sigma \cdot V_2 & \sigma \cdot V_1 \end{pmatrix} \Bigg|_{(V_1^{(0)}, V_2^{(0)}) = (\frac{1}{2} \cdot \{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}, 0)}$$

$$= \begin{pmatrix} 0 & 1 \\ -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} & \sigma \cdot \frac{1}{2} \cdot \left\{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} & \sigma \cdot \frac{1}{2} \cdot \left\{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \lambda$$

$$= \begin{pmatrix} -\lambda & 1 \\ -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} & \sigma \cdot \frac{1}{2} \cdot \left\{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} - \lambda \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$[-\lambda] \cdot \left[\sigma \cdot \frac{1}{2} \cdot \left\{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} - \lambda\right] + \left[\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right] = 0;$$

$$\psi(\Gamma_1, \Gamma_2) = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$$

$$\psi = \psi(\Gamma_1, \Gamma_2) \Rightarrow -\lambda \cdot \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 - \psi\} + \lambda^2 + \psi = 0$$

$$\Rightarrow \lambda^2 + \lambda \cdot \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} + \psi = 0$$

$$\zeta(\Gamma_1, \Gamma_2) = \zeta = \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} \Rightarrow \lambda^2 + \lambda \cdot \zeta + \psi = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$$

Option I $\lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$; $\zeta^2 - 4 \cdot \psi = 0 \Rightarrow$ $\zeta^2 = 4 \cdot \psi$ $\Rightarrow \sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 = 4 \cdot \psi$

$$\sigma^2 \cdot \{\Gamma_2 + \psi\}^2 = 16 \cdot \psi; \;\; \psi(\Gamma_1, \Gamma_2) = \psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0 \Rightarrow \lambda_{1,2} = \frac{-\zeta}{2}$$

We have three subcases: $\zeta > 0$; $\zeta < 0$; $\zeta = 0$.

Option I.1 $\zeta = 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} = 0 \Rightarrow \quad |_{\sigma=\pm 1}\Gamma_2 + \psi = 0 \Rightarrow \quad \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} = 0$ $\lambda_{1,2} = 0$; $\Delta = \lambda_1 \cdot \lambda_2 = 0$; $\tau = \lambda_1 + \lambda_2 = 0$, at least one of the eigenvalues is zero and the origin is not an isolated fixed point. There is either a whole line of fixed point, or a plane of fixed points, if $A = 0$.

Option I.2 $\zeta > 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} > 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \left\{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} > 0$

$$\sigma = +1 \Rightarrow \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0 \Rightarrow \Gamma_2 > -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$$
$$\sigma = -1 \Rightarrow \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} < 0 \Rightarrow \Gamma_2 < -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$$

$\lambda_1 = \lambda_2 = \frac{-\zeta}{2} \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0$; $\tau = \lambda_1 + \lambda_2 < 0$, we have a star node (degenerate node). We have two negative eigenvalues and the second fixed point is a stable node.

Option I.3 $\zeta < 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} < 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \left\{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\} < 0$

$$\sigma = +1 \Rightarrow \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} < 0 \Rightarrow \Gamma_2 < -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$$
$$\sigma = -1 \Rightarrow \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0 \Rightarrow \Gamma_2 > -\sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$$

$\lambda_1 = \lambda_2 = \frac{-\zeta}{2} \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0$; $\tau = \lambda_1 + \lambda_2 > 0$, we have star node (degenerate node). We have two positive eigenvalues and the second fixed point is an unstable node.

Option II $\lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$; $\quad \zeta^2 - 4 \cdot \psi > 0 \Rightarrow \quad \zeta^2 > 4 \cdot \psi \Rightarrow \sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 > 4 \cdot \psi \quad \sigma^2 \cdot \left\{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\}^2 > 16 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$. We have two real eigenvalues.

Option II.1 $-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi} > 0$ then we have two positive real eigenvalues and the second fixed point is an unstable node $\lambda_1 > 0$; $\lambda_2 > 0 \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0 \quad \tau = \lambda_1 + \lambda_2 > 0$; $\quad -\zeta + \sqrt{\zeta^2 - 4 \cdot \psi} > 0 \Rightarrow -\sigma \cdot \quad \frac{1}{2} \cdot \{\Gamma_2 + \psi\} +$

$\sqrt{\sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 - 4 \cdot \psi} > 0 \Rightarrow \sqrt{\sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 - 4 \cdot \psi} > \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\}$. Also must fulfill

$$-\zeta - \sqrt{\zeta^2 - 4 \cdot \psi} > 0 \Rightarrow -\sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} - \sqrt{\sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 - 4 \cdot \psi} > 0 \Rightarrow$$

$$-\sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} > \sqrt{\sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 - 4 \cdot \psi}$$

Option II.2 $[-\zeta + \sqrt{\zeta^2 - 4 \cdot \psi}] \cdot [-\zeta - \sqrt{\zeta^2 - 4 \cdot \psi}] < 0$ then we have two real eigenvalues with opposite signs. The second fixed point is hyperbolic fixed point (saddle point). $\Delta = \lambda_1 \cdot \lambda_2 < 0$ $\tau = \lambda_1 + \lambda_2 = -2 \cdot \zeta > 0$ and if $\zeta > 0 \Rightarrow \tau < 0$ and if $\zeta < 0 \Rightarrow \tau > 0$. We have two subcases: $\lambda_1 > 0$ & $\lambda_2 < 0$ and $\lambda_1 < 0$ & $\lambda_2 > 0$

$$\lambda_1 > 0 \quad \& \quad \lambda_2 < 0 \Rightarrow \left[-\zeta + \sqrt{\zeta^2 - 4 \cdot \psi}\right] > 0 \quad \& \quad \left[-\zeta - \sqrt{\zeta^2 - 4 \cdot \psi}\right] < 0$$

$$\lambda_1 < 0 \quad \& \quad \lambda_2 > 0 \Rightarrow \left[-\zeta + \sqrt{\zeta^2 - 4 \cdot \psi}\right] < 0 \quad \& \quad \left[-\zeta - \sqrt{\zeta^2 - 4 \cdot \psi}\right] > 0$$

Option II.3 $-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi} < 0$ then we have negative real eigenvalues and the second fixed point is stable node $\lambda_1 < 0$; $\lambda_2 < 0 \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0$, $\tau = \lambda_1 + \lambda_2 < 0$

$$[-\zeta + \sqrt{\zeta^2 - 4 \cdot \psi}] < 0 \quad \& \quad [-\zeta - \sqrt{\zeta^2 - 4 \cdot \psi}] < 0.$$

Option III $\lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$; $\zeta^2 - 4 \cdot \psi < 0 \Rightarrow \zeta^2 < 4 \cdot \psi \Rightarrow$ $\sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 < 4 \cdot \psi$ $\sigma^2 \cdot \{\Gamma_2 + \psi\}^2 < 16 \cdot \psi \Rightarrow$ $\sigma^2 \cdot \left\{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\}^2 < 16 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$. We have two complex eigenvalues $\lambda_{1,2} = \frac{-\zeta \pm i \cdot \sqrt{|\zeta^2 - 4 \cdot \psi|}}{2}$. We have three subcases: $\zeta = 0$, $\zeta > 0$, $\zeta < 0$. $\zeta = 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} = 0$ $\Rightarrow \Gamma_2 + \psi = 0 \Rightarrow \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} = 0$

$\lambda_{1,2} = \left.\frac{\pm i \cdot \sqrt{|\zeta^2 - 4 \cdot \psi|}}{2}\right|_{\zeta = 0} = \frac{\pm i \cdot \sqrt{|-4 \cdot \psi|}}{2} = \pm i \cdot \sqrt{|-\psi|}$ the second fixed point is elliptic and the eigenvalues are pure imaginary, then all solutions are periodic

Table 2.5 Summary results

Bogdanov-Takens system fixed point $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \left\{ -\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} \right\}, 0)$		
Case I $\Gamma_2^2 - 4 \cdot \Gamma_1 = 0 \Rightarrow \Gamma_2^2 = 4 \cdot \Gamma_1 \Rightarrow \Gamma_2 = \pm 2 \cdot \sqrt{\Gamma_1} \Rightarrow V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = \left(-\frac{\Gamma_2}{2}, 0\right)$ $\lambda_1 = 0;\ \lambda_2 = -\frac{\sigma \cdot \Gamma_2}{2};\ \Delta = \lambda_1 \cdot \lambda_2 = 0\ ;\ \tau = \lambda_1 \cdot \lambda_2 = -\frac{\sigma \cdot \Gamma_2}{2}$ $\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, or a plane of fixed points, if $A = 0$		
Bogdanov-Takens system fixed point $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \left\{ -\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} \right\}, 0)$		
Case II (first fixed point): $\Gamma_2^2 - 4 \cdot \Gamma_1 > 0 \Rightarrow V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \{-\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}, 0);$ $\psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0;\ \xi(\Gamma_1, \Gamma_2) = \xi = \sigma \cdot \frac{1}{2} \cdot \{-\Gamma_2 + \psi\} \Rightarrow \lambda^2 - \lambda \cdot \xi - \psi = 0$ $\Rightarrow \lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$		
Option I $\lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$ $\xi^2 + 4 \cdot \psi = 0 \Rightarrow \xi^2 = -4 \cdot \psi$ $\Rightarrow \sigma^2 \cdot \frac{1}{4} \cdot \{-\Gamma_2 + \psi\}^2 = -4 \cdot \psi$ $\psi(\Gamma_1, \Gamma_2) = \psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$ then option I can't exist	*Option II* $\lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$ $\xi^2 + 4 \cdot \psi > 0$ and we have two real eigenvalues (λ_1, λ_2) and three subcases *Option II.1* $\xi \pm \sqrt{\xi^2 + 4 \cdot \psi} > 0$ then we have two positive eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ and the first fixed point is unstable node *Option II.2* $\lambda_1 \cdot \lambda_2 < 0 \Rightarrow [\xi + \sqrt{\xi^2 + 4 \cdot \psi}] \cdot [\xi - \sqrt{\xi^2 + 4 \cdot \psi}] < 0$ then we have two opposite sign eigenvalues $\lambda_1 > 0$, $\lambda_2 < 0$ or $\lambda_1 < 0$, $\lambda_2 > 0$ and the first fixed point is hyperbolic fixed point (saddle point) *Option II.3* $\xi \pm \sqrt{\xi^2 + 4 \cdot \psi} < 0$ then we have two negative eigenvalues $\lambda_1 < 0$ and $\lambda_2 < 0$ and the first fixed point is stable node	*Option III* $\lambda_{1,2} = \frac{\xi \pm \sqrt{\xi^2 + 4 \cdot \psi}}{2}$ and we have two complex eigenvalues $\xi^2 + 4 \cdot \psi < 0$ (λ_1, λ_2), $\lambda_{1,2} = \frac{1}{2} \cdot (\xi \pm i \cdot \sqrt{\lvert \xi^2 + 4 \cdot \psi \rvert})$ can't exist since $\xi^2 > 0 \quad \& \quad 4 \cdot \psi > 0$ $\psi(\Gamma_1, \Gamma_2) = \psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$

(continued)

Table 2.5 (continued)

Bogdanov-Takens system fixed point $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \left\{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\right\}, 0)$		
Case III $\Gamma_2^2 - 4 \cdot \Gamma_1 < 0 \Rightarrow V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \{-\Gamma_2 \pm i \cdot \sqrt{\lvert\Gamma_2^2 - 4 \cdot \Gamma_1\rvert}\}, 0)$ Complex fixed points which are not discussed in our analysis, since we have only real fixed point coordinates in our system		
Bogdanov-Takens system fixed point $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}, 0)$		
Case II (second fixed point): $\Gamma_2^2 - 4 \cdot \Gamma_1 > 0 \Rightarrow V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \{-\Gamma_2 - \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}, 0)$ $\psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0;\ \zeta(\Gamma_1, \Gamma_2) = \zeta = \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} \Rightarrow \lambda^2 + \lambda \cdot \zeta + \psi = 0$ $\Rightarrow \lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$		
Option I $\lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$ $\zeta^2 - 4 \cdot \psi = 0 \Rightarrow \zeta^2 = 4 \cdot \psi$ $\Rightarrow \sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 = 4 \cdot \psi$ $\sigma^2 \cdot \{\Gamma_2 + \psi\}^2 = 16 \cdot \psi$ $\psi(\Gamma_1, \Gamma_2) = \psi = \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} > 0$ $\Rightarrow \lambda_{1,2} = \frac{-\zeta}{2}$ We have three subcases: $\zeta > 0;\ \zeta < 0;\ \zeta = 0$ *Option I.1* $\zeta = 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} = 0$ $\Rightarrow \rvert_{\sigma = \pm 1} \Gamma_2 + \psi = 0 \Rightarrow$ $\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} = 0$ $\lambda_{1,2} = 0;\ \Delta = \lambda_1 \cdot \lambda_2 = 0$ $\tau = \lambda_1 + \lambda_2 = 0$	*Option II* $\lambda_{1,2} = \dfrac{-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi}}{2}$ $\zeta^2 - 4 \cdot \psi > 0 \Rightarrow \zeta^2 > 4 \cdot \psi$ $\Rightarrow \sigma^2 \cdot \frac{1}{4} \cdot \{\Gamma_2 + \psi\}^2 > 4 \cdot \psi$ We have two real eigenvalues *Option II.1* $-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi} > 0$ then we have two positive real eigenvalues and the second fixed point is unstable node $\lambda_1 > 0;\ \lambda_2 > 0 \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0$ *Option II.2* $[-\zeta + \sqrt{\zeta^2 - 4 \cdot \psi}] \cdot [-\zeta - \sqrt{\zeta^2 - 4 \cdot \psi}] < 0$ then we have two real eigenvalues with opposite signs. The second fixed point is hyperbolic fixed point (saddle point)	*Option III* $\sigma^2 \cdot \{\Gamma_2 + \psi\}^2 < 16 \cdot \psi \Rightarrow$ $\sigma^2 \cdot \{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}^2$ $< 16 \cdot \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}$ We have two complex eigenvalues $\lambda_{1,2} = \frac{-\zeta \pm i \cdot \sqrt{\lvert\zeta^2 - 4 \cdot \psi\rvert}}{2}$ We have three subcases: $\zeta = 0,\ \zeta > 0,\ \zeta < 0$ $\zeta = 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} = 0$ $\Rightarrow \Gamma_2 + \psi = 0$ $\Rightarrow \Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1} = 0$ $\lambda_{1,2} = \frac{\pm i \cdot \sqrt{\lvert\zeta^2 - 4 \cdot \psi\rvert}}{2}$ $\lambda_{1,2}\rvert_{\zeta = 0} = \pm i \cdot \sqrt{\lvert - \cdot \psi\rvert}$

(continued)

Table 2.5 (continued)

Bogdanov-Takens system fixed point $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = (\frac{1}{2} \cdot \{-\Gamma_2 \pm \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\}, 0)$		
There is either a whole line of fixed point, or a plane of fixed points, if $A = 0$ *Option I.2* $\zeta > 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} > 0$ $\Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\} > 0$ $\lambda_1 = \lambda_2 = \frac{-\zeta}{2}$ $\Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0$ $\tau = \lambda_1 + \lambda_2 < 0$ We have star node (degenerate node). We have two negative eigenvalues and the second fixed point is stable node *Option I.3* $\zeta < 0 \Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \psi\} < 0$ $\Rightarrow \sigma \cdot \frac{1}{2} \cdot \{\Gamma_2 + \sqrt{\Gamma_2^2 - 4 \cdot \Gamma_1}\} < 0$ $\lambda_1 = \lambda_2 = \frac{-\zeta}{2} \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0$ $\tau = \lambda_1 + \lambda_2 > 0$ We have star node (degenerate node). We have two positive eigenvalues and the second fixed point is unstable node	*Option II.3* $-\zeta \pm \sqrt{\zeta^2 - 4 \cdot \psi} < 0$ then we have negative real eigenvalues and the second fixed point is stable node $\lambda_1 < 0;\ \lambda_2 < 0 \Rightarrow \Delta = \lambda_1 \cdot \lambda_2 > 0,\ \tau = \lambda_1 + \lambda_2 < 0$	the second fixed point is elliptic and the eigenvalues are pure imaginary, then all solutions are periodic with period T=2 · π/ω $\omega = \frac{\sqrt{\lvert\zeta^2 - 4 \cdot \psi\rvert}}{2}$ $\omega\rvert_{\zeta=0} = \sqrt{\lvert -\psi \rvert}$ $T = \frac{4\pi}{\sqrt{\lvert\zeta^2 - 4 \cdot \psi\rvert}}$ $T\rvert_{\zeta=0} = \frac{2 \cdot \pi}{\sqrt{\lvert -\psi \rvert}}$ $\zeta > 0$; We have complex eigenvalues with negative real part and the second fixed point is stable spiral point. $\zeta < 0$; We have complex eigenvalues with positive real part and the second fixed point is unstable spiral point $\lambda_{1,2} = \frac{-\zeta \pm i \cdot \sqrt{\lvert\zeta^2 - 4 \cdot \psi\rvert}}{2}$

with period $T = 2 \cdot \pi/\omega$. $\omega = \left.\frac{\sqrt{|\zeta^2-4\cdot\psi|}}{2}\right|_{\zeta=0} = \frac{\sqrt{|-4\cdot\psi|}}{2} = \sqrt{|-\psi|}$ $T = \left.\frac{4\pi}{\sqrt{|\zeta^2-4\cdot\psi|}}\right|_{\zeta=0} = \frac{4\cdot\pi}{\sqrt{|-4\cdot\psi|}} = \frac{2\cdot\pi}{\sqrt{|-\psi|}}$.

$\zeta > 0$ then we have complex eigenvalues with negative real part and the second fixed point is stable spiral point. $\zeta > 0$ then we have complex eigenvalues with positive real part and the second fixed point is unstable spiral point. $\lambda_{1,2} = \frac{-\zeta\pm i\cdot\sqrt{|\zeta^2-4\cdot\psi|}}{2}$.

Case III $\Gamma_2^2 - 4\cdot\Gamma_1 < 0 \Rightarrow$ $V^{(i)} = (V_1^{(i)}, V_2^{(i)}) = \left(\frac{1}{2}\cdot\{-\Gamma_2 \pm i\cdot \sqrt{|\Gamma_2^2 - 4\cdot\Gamma_1|}\}, 0\right)$.

Complex fixed points which not discuss in our analysis, since we have only real fixed point coordinates in our system. We can summarize our result in Table 2.5.

The next step is to move to system planar representation (r, φ) space and analyze limit cycle behavior at specific r values and when the close orbit still exist. We need to move from (V_1, V_2) space to (r, φ) space $((V_1, V_2) \rightarrow (r, \varphi))$.

Our system is locally topologically equivalent near the origin to the following normal form: $r = r(t)$; $\varphi = \varphi(t)$ [73, 74].

$$\frac{dV_1}{dt} = V_2; \quad \frac{dV_2}{dt} = \Gamma_1 + \Gamma_2\cdot V_1 + V_1^2 + \sigma\cdot V_1\cdot V_2;$$
$$V = (V_1, V_2)^T \in \mathbb{R}^2, \quad \sigma = \text{sign}[a(0)\cdot b(0)] = \pm 1$$

For planar representation, we have the following transformations: $V_1 = r\cdot\cos\varphi$; $V_2 = r\cdot\sin\varphi$; $\frac{dV_1}{dt} = -r\cdot\frac{d\varphi}{dt}\cdot\sin\varphi + \frac{dr}{dt}\cdot\cos\varphi$; $\frac{dV_2}{dt} = r\cdot\frac{d\varphi}{dt}\cdot\cos\varphi + \frac{dr}{dt}\cdot\sin\varphi$

$$\frac{dV_1}{dt} = V_2 \Rightarrow -r\cdot\frac{d\varphi}{dt}\cdot\sin\varphi + \frac{dr}{dt}\cdot\cos\varphi = r\cdot\sin\varphi$$
$$\Rightarrow -r\cdot\frac{d\varphi}{dt} + \frac{dr}{dt}\cdot\frac{1}{tg\varphi} = r \Rightarrow \frac{dr}{dt} = r\cdot\left(1+\frac{d\varphi}{dt}\right)\cdot tg\varphi$$

$$\frac{dV_2}{dt} = \Gamma_1 + \Gamma_2\cdot V_1 + V_1^2 + \sigma\cdot V_1\cdot V_2 \Rightarrow r\cdot\frac{d\varphi}{dt}\cdot\cos\varphi + \frac{dr}{dt}\cdot\sin\varphi$$
$$= \Gamma_1 + \Gamma_2\cdot r\cdot\cos\varphi + r^2\cdot\cos^2\varphi + \sigma\cdot r^2\cdot\sin\varphi\cdot\cos\varphi$$

$$\frac{dr}{dt} = r\cdot\left(1+\frac{d\varphi}{dt}\right)\cdot tg\varphi \Rightarrow r\cdot\frac{d\varphi}{dt}\cdot\cos\varphi + r\cdot\left(1+\frac{d\varphi}{dt}\right)\cdot tg\varphi\cdot\sin\varphi$$
$$= \Gamma_1 + \Gamma_2\cdot r\cdot\cos\varphi + r^2\cdot\cos^2\varphi + \sigma\cdot r^2\cdot\sin\varphi\cdot\cos\varphi$$

$$r\cdot\frac{d\varphi}{dt}\cdot[\cos\varphi + \ \cdot tg\varphi\cdot\sin\varphi] = \Gamma_1 + \Gamma_2\cdot r\cdot\cos\varphi + r^2\cdot\cos^2\varphi$$
$$+\sigma\cdot r^2\cdot\sin\varphi\cdot\cos\varphi_1 - r\cdot tg\varphi\cdot\sin\varphi$$

$$\frac{d\varphi}{dt} = \frac{\Gamma_1 + \Gamma_2 \cdot r \cdot \cos\varphi + r^2 \cdot \cos^2\varphi + \sigma \cdot r^2 \cdot \sin\varphi \cdot \cos\varphi - r \cdot tg\varphi \cdot \sin\varphi}{r \cdot [\cos\varphi + \ \cdot tg\varphi \cdot \sin\varphi]}$$

$$\cos\varphi + \ \cdot tg\varphi \cdot \sin\varphi = \frac{1}{\cos\varphi} \Rightarrow \frac{d\varphi}{dt} = \frac{1}{r} \cdot \Gamma_1 \cdot \cos\varphi + \Gamma_2 \cdot \cos^2\varphi + r \cdot \cos^3\varphi + \sigma \cdot r \cdot \sin\varphi \cdot \cos^2\varphi - \sin^2\varphi$$

$$\frac{dV_1}{dt} = V_2 \Rightarrow -r \cdot \frac{d\varphi}{dt} \cdot \sin\varphi + \frac{dr}{dt} \cdot \cos\varphi = r \cdot \sin\varphi$$
$$\Rightarrow -r \cdot \frac{d\varphi}{dt} + \frac{dr}{dt} \cdot \frac{1}{tg\varphi} = r \Rightarrow \frac{d\varphi}{dt} = \frac{1}{r} \cdot \frac{dr}{dt} \cdot \frac{1}{tg\varphi} - 1$$

$$r \cdot \left[\frac{1}{r} \cdot \frac{dr}{dt} \cdot \frac{1}{tg\varphi} - 1\right] \cdot \cos\varphi + \frac{dr}{dt} \cdot \sin\varphi = \Gamma_1 + \Gamma_2 \cdot r \cdot \cos\varphi + r^2 \cdot \cos^2\varphi + \sigma \cdot r^2 \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dr}{dt} \cdot \frac{\cos\varphi}{tg\varphi} - r \cdot \cos\varphi + \frac{dr}{dt} \cdot \sin\varphi = \Gamma_1 + \Gamma_2 \cdot r \cdot \cos\varphi + r^2 \cdot \cos^2\varphi + \sigma \cdot r^2 \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dr}{dt} \cdot \left[\frac{\cos\varphi}{tg\varphi} + \sin\varphi\right] = \Gamma_1 + \Gamma_2 \cdot r \cdot \cos\varphi + r^2 \cdot \cos^2\varphi + \sigma \cdot r^2 \cdot \sin\varphi \cdot \cos\varphi + r \cdot \cos\varphi$$

$$\frac{dr}{dt} = \frac{\Gamma_1 + \Gamma_2 \cdot r \cdot \cos\varphi + r^2 \cdot \cos^2\varphi + \sigma \cdot r^2 \cdot \sin\varphi \cdot \cos\varphi_1 + r \cdot \cos\varphi}{\frac{\cos\varphi}{tg\varphi} + \sin\varphi}$$

$$\frac{\cos\varphi}{tg\varphi} + \sin\varphi = \frac{1}{\sin\varphi} \Rightarrow \frac{dr}{dt} = \Gamma_1 \cdot \sin\varphi + \Gamma_2 \cdot r \cdot \cos\varphi \cdot \sin\varphi + r^2 \cdot \cos^2\varphi \cdot \sin\varphi + \sigma \cdot r^2 \cdot \sin^2\varphi \cdot \cos\varphi + r \cdot \cos\varphi \cdot \sin\varphi$$

$$\frac{dr}{dt} = \Gamma_1 \cdot \sin\varphi + \frac{1}{2} \cdot \Gamma_2 \cdot r \cdot \sin[2 \cdot \varphi] + \frac{1}{2} \cdot r^2 \cdot \cos\varphi \cdot \sin[2 \cdot \varphi] + \frac{1}{2} \cdot \sigma \cdot r^2 \cdot \sin[2 \cdot \varphi] \cdot \sin\varphi + \frac{1}{2} \cdot r \cdot \sin[2 \cdot \varphi]$$

$$\frac{dr}{dt} = \Gamma_1 \cdot \sin\varphi + \frac{1}{2} \cdot r^2 \cdot \sin[2 \cdot \varphi] \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} + \frac{1}{2} \cdot r \cdot \sin[2 \cdot \varphi] \cdot \{1 + \Gamma_2\}$$

There is a different phase space behavior for different Γ_1, Γ_2, σ values. All sorts of limit cycles exist and different spiral behavior (stable/Unstable).

To find our limit cycle radius we set $\frac{dr}{dt} = 0$ (no planar radius change in time).

$$\frac{dr}{dt} = 0 \Rightarrow \Gamma_1 \cdot \sin\varphi + \frac{1}{2} \cdot r^2 \cdot \sin[2 \cdot \varphi] \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} + \frac{1}{2} \cdot r \cdot \sin[2 \cdot \varphi] \cdot \{1 + \Gamma_2\} = 0$$

$$\phi_1(\varphi) = \frac{1}{2} \cdot \sin[2 \cdot \varphi] \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\}; \; \phi_2(\varphi) = \frac{1}{2} \cdot \sin[2 \cdot \varphi] \cdot \{1 + \Gamma_2\}$$

$$r^2 \cdot \phi_1(\varphi) + r \cdot \phi_2(\varphi) + \Gamma_1 \cdot \sin\varphi = 0$$
$$\Rightarrow r^{(j)} = \frac{-\phi_2(\varphi) \pm \sqrt{\phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi}}{2 \cdot \phi_1(\varphi)}$$

$r^{(j)} \geq 0$ then we have some subcases.

We define

$$\chi(\varphi) = \phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi \Rightarrow r^{(j)} = \frac{-\phi_2(\varphi) \pm \sqrt{\chi(\varphi)}}{2 \cdot \phi_1(\varphi)};$$
$$\chi(\varphi) = 0; \quad \chi(\varphi) > 0$$

Case I $\chi(\varphi) = 0 \Rightarrow \phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi = 0 \Rightarrow \phi_2^2(\varphi) = 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi$

$$\sin^2[2 \cdot \varphi] \cdot \{1 + \Gamma_2\}^2 = 8 \cdot \sin[2 \cdot \varphi] \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 \cdot \sin\varphi$$

$$\sin[2 \cdot \varphi] \cdot \left\{\sin[2 \cdot \varphi] \cdot \{1 + \Gamma_2\}^2 - 8 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 \cdot \sin\varphi\right\} = 0$$

$$\sin[2 \cdot \varphi] = 0 \Rightarrow 2 \cdot \varphi = n \cdot \pi \Rightarrow \varphi = \frac{n \cdot \pi}{2} \quad \forall\, n = 0, 1, 2, \ldots$$

or other option:

$$\sin[2 \cdot \varphi] \cdot \{1 + \Gamma_2\}^2 - 8 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 \cdot \sin\varphi = 0$$

$$2 \cdot \sin\varphi \cdot \cos\varphi \cdot \{1 + \Gamma_2\}^2 - 8 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 \cdot \sin\varphi = 0$$

$$2 \cdot \sin\varphi \cdot \left[\cos\varphi \cdot \{1+\Gamma_2\}^2 - 4 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1\right] = 0$$

$$\sin\varphi = 0 \Rightarrow \varphi = n \cdot \pi \quad \forall\, n = 0,1,2,\ldots \quad \text{or}$$
$$\cos\varphi \cdot \{1+\Gamma_2\}^2 - 4 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 = 0$$

$$\cos\varphi \cdot \{1+\Gamma_2\}^2 - 4 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 = 0$$
$$\Rightarrow \{1+\Gamma_2\}^2 - 4 \cdot \{1 + \sigma \cdot tg\varphi\} \cdot \Gamma_1 = 0$$

$$\{1+\Gamma_2\}^2 - 4 \cdot \{1 + \sigma \cdot tg\varphi\} \cdot \Gamma_1 = 0 \Rightarrow tg\varphi$$
$$= \frac{1}{\sigma} \cdot \left[\frac{\{1+\Gamma_2\}^2}{4 \cdot \Gamma_1} - 1\right] \Rightarrow \varphi = \operatorname{arc}tg\left\{\frac{1}{\sigma} \cdot \left[\frac{\{1+\Gamma_2\}^2}{4 \cdot \Gamma_1} - 1\right]\right\}$$

For the first case we have three possible $\underline{\varphi \text{ values}}$:

$$\varphi = \frac{n \cdot \pi}{2}; \quad \varphi = n \cdot \pi; \quad \varphi = \operatorname{arc}tg\left\{\frac{1}{\sigma} \cdot \left[\frac{\{1+\Gamma_2\}^2}{4 \cdot \Gamma_1} - 1\right]\right\} \quad \forall\, n = 0,1,2,\ldots$$

$$r^{(j)} = \left.\frac{-\phi_2(\varphi) \pm \sqrt{\chi(\varphi)}}{2 \cdot \phi_1(\varphi)}\right|_{\chi(\varphi)=0} = \frac{-\phi_2(\varphi)}{2 \cdot \phi_1(\varphi)} = \frac{-\{1+\Gamma_2\}}{2 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\}}$$

$r^{(j)} = \frac{-\{1+\Gamma_2\}}{2 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\}} > 0$ then we have two subcases:

(1) $-\{1+\Gamma_2\} > 0 \Rightarrow 1+\Gamma_2 < 0 \Rightarrow \Gamma_2 < -1 \;\&\; \cos\varphi + \sigma \cdot \sin\varphi > 0$

(2) $-\{1+\Gamma_2\} < 0 \Rightarrow 1+\Gamma_2 > 0 \Rightarrow \Gamma_2 > -1 \;\&\; \cos\varphi + \sigma \cdot \sin\varphi < 0$

Case II $\chi(\varphi) > 0 \Rightarrow \phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi > 0 \Rightarrow \phi_2^2(\varphi) > 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi$

$$\sin[2 \cdot \varphi] \cdot \left\{\sin[2 \cdot \varphi] \cdot \{1+\Gamma_2\}^2 - 8 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 \cdot \sin\varphi\right\} > 0$$

Case II.1 $\sin[2 \cdot \varphi] > 0 \;\&\; \{\sin[2 \cdot \varphi] \cdot \{1+\Gamma_2\}^2 - 8 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\} \cdot \Gamma_1 \cdot \sin\varphi\} > 0$

$$\sin[2 \cdot \varphi] > 0 \Rightarrow (2 \cdot n + 1) \cdot \pi > 2\varphi > 2 \cdot n \cdot \pi \Rightarrow \left(n + \frac{1}{2}\right) \cdot \pi > \varphi > n \cdot \pi$$

$$\sin[2\cdot\varphi]\cdot\{1+\Gamma_2\}^2 - 8\cdot\{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1\cdot\sin\varphi > 0$$
$$\Rightarrow 2\cdot\sin\varphi\cdot[\cos\varphi\cdot\{1+\Gamma_2\}^2 - 4\cdot\{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1] > 0$$

Case II.1-1 $\sin\varphi > 0$ & $[\cos\varphi\cdot\{1+\Gamma_2\}^2 - 4\cdot\ \{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1] > 0$

$$\sin\varphi > 0 \Rightarrow (2\cdot n+1)\cdot\pi > \varphi > 2\cdot n\cdot\pi \quad \&$$
$$[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1]\cdot\cos\varphi - 4\cdot\Gamma_1\cdot\sigma\cdot\sin\varphi > 0$$

We divide two sides by cos φ:

$$\cos\varphi > 0 \Rightarrow \left(\frac{3}{2}+2\cdot n\right)\cdot\pi<\varphi<\left(\frac{1}{2}+2\cdot n\right)\cdot\pi$$
$$\Rightarrow [\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1] > 4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi;$$
$$\Gamma_1\cdot\sigma > 0 \Rightarrow tg\varphi < \frac{[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1]}{4\cdot\Gamma_1\cdot\sigma}$$
$$\Gamma_1\cdot\sigma < 0 \Rightarrow tg\varphi > \frac{[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1]}{4\cdot\Gamma_1\cdot\sigma}$$

If $\cos\varphi<0 \Rightarrow \left(\frac{1}{2}+2\cdot n\right)\cdot\ \pi<\varphi<\left(\frac{3}{2}+2\cdot n\right)\cdot\ \pi \Rightarrow [\{1+\Gamma_2\}^2 - 4\cdot\ \Gamma_1]\cdot \cos\varphi - 4\cdot\ \Gamma_1\cdot\sigma\cdot\sin\varphi > 0$

We divide two sides by cos φ:

$$[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1] < 4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi; \quad \Gamma_1\cdot\sigma > 0$$

$$\Rightarrow tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}; \quad \Gamma_1\cdot\sigma<0 \Rightarrow tg\varphi < \frac{[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1]}{4\cdot\Gamma_1\cdot\sigma}$$

Case II.1-2 $\sin\varphi<0$ & $[\cos\varphi\cdot\{1+\Gamma_2\}^2 - 4\cdot\ \{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1]<0$

$$\sin\varphi<0 \Rightarrow 2\cdot\pi\cdot(n+1) > \varphi > \pi\cdot(1+2\cdot n) \quad \&$$
$$\cos\varphi\cdot[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1] - 4\cdot\Gamma_1\cdot\sigma\cdot\sin\varphi<0$$

We divide two sides by cos φ:

$$\cos\varphi > 0 \Rightarrow \left(\frac{3}{2}+2\cdot n\right)\cdot\pi<\varphi<\left(\frac{1}{2}+2\cdot n\right)\cdot\pi$$
$$\Rightarrow [\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1] - 4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi<0$$
$$\Rightarrow \{1+\Gamma_2\}^2 - 4\cdot\Gamma_1<4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi; \quad \Gamma_1\cdot\sigma > 0$$

$$tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}; \quad \Gamma_1\cdot\sigma < 0 \Rightarrow tg\varphi < \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}.$$

$$\cos\varphi < 0 \Rightarrow \left(\frac{1}{2}+2\cdot n\right)\cdot\pi < \varphi < \left(\frac{3}{2}+2\cdot n\right)\cdot\pi;$$

$$\Gamma_1\cdot\sigma > 0 \Rightarrow tg\varphi < \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}$$

$$\Gamma_1\cdot\sigma < 0 \Rightarrow tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}$$

Case II.2 $\sin[2\cdot\varphi] < 0$ & $\{\sin[2\cdot\varphi]\cdot\{1+\Gamma_2\}^2 - 8\cdot \{\cos\varphi + \sigma\cdot\sin\varphi\}\cdot\Gamma_1\cdot\sin\varphi\} < 0$

$$\sin[2\cdot\varphi] < 0 \Rightarrow 2\cdot\pi\cdot[n+1] > 2\varphi > \pi\cdot[1+2\cdot n]$$
$$\Rightarrow \pi\cdot[n+1] > \varphi > \pi\cdot\left[\frac{1}{2}+n\right]$$

$$2\cdot\sin\varphi\cdot\left[\cos\varphi\cdot\{1+\Gamma_2\}^2 - 4\cdot\{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1\right] < 0$$

Case II.2-1

$$\sin\varphi > 0 \quad \& \quad [\cos\varphi\cdot\{1+\Gamma_2\}^2 - 4\cdot\{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1] < 0$$

$$\sin\varphi > 0 \Rightarrow (2\cdot n+1)\cdot\pi > \varphi > 2\cdot n\cdot\pi;$$
$$\cos\varphi\cdot\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\cdot\cos\varphi - \sigma\cdot 4\cdot\Gamma_1\cdot\sin\varphi < 0$$

$\cos\varphi\cdot[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1] - \sigma\cdot 4\cdot\Gamma_1\cdot\sin\varphi < 0$; We divide two sides by cos φ:

$$\cos\varphi > 0 \Rightarrow \left(\frac{3}{2}+2\cdot n\right)\cdot\pi < \varphi < \left(\frac{1}{2}+2\cdot n\right)\cdot\pi$$
$$\Rightarrow [\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1] - 4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi < 0$$
$$\Rightarrow \{1+\Gamma_2\}^2 - 4\cdot\Gamma_1 < 4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi;$$
$$\Gamma_1\cdot\sigma > 0; \quad tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}$$
$$\Gamma_1\cdot\sigma < 0 \Rightarrow tg\varphi < \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}\cdot\cos\varphi < 0$$
$$\Rightarrow \left(\frac{1}{2}+2\cdot n\right)\cdot\pi < \varphi < \left(\frac{3}{2}+2\cdot n\right)\cdot\pi$$

$$\cos\varphi\cdot[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]-\sigma\cdot 4\cdot\Gamma_1\cdot\sin\varphi<0$$
$$\Rightarrow \cos\varphi\cdot[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]<\sigma\cdot 4\cdot\Gamma_1\cdot\sin\varphi$$

$$\left[\{1+\Gamma_2\}^2-4\cdot\Gamma_1\right]>\sigma\cdot 4\cdot\Gamma_1\cdot tg\varphi$$
$$\Rightarrow \sigma\cdot\Gamma_1>0\Rightarrow tg\varphi<\frac{\left[\{1+\Gamma_2\}^2-4\cdot\Gamma_1\right]}{\sigma\cdot 4\cdot\Gamma_1}$$
$$\sigma\cdot\Gamma_1<0\Rightarrow tg\varphi>\frac{\left[\{1+\Gamma_2\}^2-4\cdot\Gamma_1\right]}{\sigma\cdot 4\cdot\Gamma_1}$$

Case II.2-2 $\sin\varphi<0$ & $[\cos\varphi\cdot\{1+\Gamma_2\}^2-4\cdot\ \{\cos\varphi+\sigma\cdot\sin\varphi\}\cdot\Gamma_1]>0$

$$\sin\varphi<0\Rightarrow 2\cdot\pi\cdot(n+1)>\varphi>\pi\cdot(1+2\cdot n)\quad \&$$
$$\cos\varphi\cdot\left[\{1+\Gamma_2\}^2-4\cdot\Gamma_1\right]-4\cdot\Gamma_1\cdot\sigma\cdot\sin\varphi>0$$

We divide two sides by cos φ:

$$\cos\varphi>0\Rightarrow\left(\frac{3}{2}+2\cdot n\right)\cdot\pi<\varphi<\left(\frac{1}{2}+2\cdot n\right)\cdot\pi$$
$$\Rightarrow[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]>4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi;$$
$$\Gamma_1\cdot\sigma>0\Rightarrow tg\varphi<\frac{[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]}{4\cdot\Gamma_1\cdot\sigma}$$
$$\Gamma_1\cdot\sigma<0\Rightarrow tg\varphi>\frac{[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]}{4\cdot\Gamma_1\cdot\sigma}$$

If $\cos\varphi<0\Rightarrow\ \left(\frac{1}{2}+2\cdot n\right)\cdot\ \ \pi<\varphi<\left(\frac{3}{2}+2\cdot n\right)\cdot\ \ \pi\Rightarrow[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]$ $\cdot\cos\varphi-4\cdot\Gamma_1\cdot\sigma\cdot\sin\varphi>0$

We divide two sides by cos φ: $[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]<4\cdot\Gamma_1\cdot\sigma\cdot tg\varphi;\ \Gamma_1\cdot\sigma>0$

$$\Rightarrow tg\varphi>\frac{\{1+\Gamma_2\}^2-4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma};\ \Gamma_1\cdot\sigma<0\Rightarrow tg\varphi<\frac{[\{1+\Gamma_2\}^2-4\cdot\Gamma_1]}{4\cdot\Gamma_1\cdot\sigma}$$

Table 2.6 summaries for each Bogdanov–Takens bifurcation system's limit cycle type, the conditions for d*r*/d*t* in the inner and outer circle.

Table 2.6 Summary of Bogdanov–Takens bifurcation system

Limit cycle type	d*r*/d*t* in the inner circle	d*r*/d*t* in the outer circle
Stable Limit Cycle (SLC)	d*r*/d*t* > 0	d*r*/d*t* < 0
Half-Stable Limit Cycle (HLC)	d*r*/d*t* < 0	d*r*/d*t* < 0
Unstable Limit Cycle (ULC)	d*r*/d*t* < 0	d*r*/d*t* > 0

In Table 2.7 we summarize our last result for Bogdanov–Takens bifurcation system's limit cycle planar radius $\left(\frac{dr}{dt}=0\right)$ and all possible options for control parameters (Γ_1, Γ_2, σ) and planar angular angle φ [6–8].

Our Bogdanov–Takens bifurcation system can be represented by the following equivalent differential equations: $V_2=\frac{dV_1}{dt}$; $V_1=\frac{1}{\Gamma_2}\cdot\left\{\frac{dV_2}{dt}-\Gamma_1-V_1^2-\sigma\cdot V_1\cdot V_2\right\}$.

Figure 2.12 describes our Bogdanov–Takens bifurcation system.

2.6 Optoisolation Circuits Bogdanov–Takens (Double-Zero) Bifurcation

We need to implement our Bogdanov–Takens bifurcation system block diagram by using optoisolation elements, op-amps, resistors, capacitors, etc. Figure 2.13 implements our system [10, 15, 16].

Remark There is no match between system block diagram feedback loop $[]^2$ operator for V_1 output voltage variable and circuit feedback term []. We will get later from optoisolation circuit implementation our feedback loop $[]^2$ operator for V_1 output voltage variable. Gama1 = Γ_1, Gama2 = Γ_2, Sigma = σ.

$$\frac{R_{f1}}{R_{\text{in}1}}=1;\quad \frac{R_{f4}}{R_{\text{in}4}}=1;\quad \frac{R_{f5}}{R_{\text{in}5}}=1;\quad R_{f3}\cdot C_{\text{in}3}=1;$$
$$R_{f2}\cdot C_{\text{in}2}=1;\quad V_{o1}=-V_1;\quad V_{o3}=-\sigma\cdot V_1\cdot V_2$$

$$V_{o4}=-R_{f2}\cdot C_{\text{in}2}\cdot\left.\frac{dV_1}{dt}\right|_{R_{f2}\cdot C_{\text{in}2}=1}=-\frac{dV_1}{dt};$$
$$V_2=-V_{o4}=R_{f2}\cdot C_{\text{in}2}\cdot\left.\frac{dV_1}{dt}\right|_{R_{f2}\cdot C_{in2}=1}=\frac{dV_1}{dt}$$

$$V_{o5}=-R_{f3}\cdot C_{\text{in}3}\cdot\left.\frac{dV_2}{dt}\right|_{R_3\cdot C_{in3}=1}=-\frac{dV_2}{dt};$$
$$V_{o2}=-V_{o5}=R_{f3}\cdot C_{\text{in}3}\cdot\left.\frac{dV_2}{dt}\right|_{R_3\cdot C_{\text{in}3}=1}=\frac{dV_2}{dt}$$

$$I_{D1}=I_{R1}+I_{R2}+I_{R3}+I_{R4}=\sum_{j=1}^{4}I_{Rj};\quad I_{R1}=\frac{-V_1-V_{D1}}{R_1};\quad I_{R2}=\frac{-\Gamma_1-V_{D1}}{R_2}$$

$$I_{R3}=\frac{V_{o2}-V_{D1}}{R_3}=\frac{\frac{dV_2}{dt}-V_{D1}}{R_3};\quad I_{R4}=\frac{V_{o3}-V_{D1}}{R_4}=\frac{-\sigma\cdot V_1\cdot V_2-V_{D1}}{R_4}$$

Table 2.7 Summary of Bogdanov–Takens bifurcation system

$\frac{dr}{dt} = 0 \Rightarrow r^2 \cdot \phi_1(\varphi) + r \cdot \phi_2(\varphi) + \Gamma_1 \cdot \sin\varphi = 0 \Rightarrow r^{(j)} = \frac{-\phi_2(\varphi) \pm \sqrt{\phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi}}{2 \cdot \phi_1(\varphi)}$ $\chi(\varphi) = \phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi \Rightarrow r^{(j)} = \frac{-\phi_2(\varphi) \pm \sqrt{\chi(\varphi)}}{2 \cdot \phi_1(\varphi)}$; $\chi(\varphi) = 0$; $\chi(\varphi) > 0$ $\phi_1(\varphi) = \frac{1}{2} \cdot \sin[2 \cdot \varphi] \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\}$; $\phi_2(\varphi) = \frac{1}{2} \cdot \sin[2 \cdot \varphi] \cdot \{1 + \Gamma_2\}$; $r^{(j)} \geq 0$			
Case I $\chi(\varphi) = 0 \Rightarrow r^{(j)} = \frac{-\phi_2(\varphi)}{2 \cdot \phi_1(\varphi)} = \frac{-\{1+\Gamma_2\}}{2 \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\}}$ $\varphi = \frac{n \cdot \pi}{2}$; $\varphi = n \cdot \pi$ $\varphi = \mathrm{arctg}\left\{\frac{1}{\sigma} \cdot \left[\frac{\{1+\Gamma_2\}^2}{4 \cdot \Gamma_1} - 1\right]\right\} \quad \forall\, n = 0, 1, 2, \ldots$		*Case II* $\chi(\varphi) > 0 \Rightarrow \phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot$ $\Gamma_1 \cdot \sin\varphi > 0 \Rightarrow \phi_2^2(\varphi) > 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi$	
		Case II.1 $\sin[2 \cdot \varphi] > 0 \Rightarrow$ $\left(n + \frac{1}{2}\right) \cdot \pi > \varphi > n \cdot \pi$ $2 \cdot \sin\varphi \cdot [\cos\varphi$ $\cdot \{1+\Gamma_2\}^2$ $- 4 \cdot \{\cos\varphi$ $+ \sigma \cdot \sin\varphi\} \cdot \Gamma_1] > 0$	*Case II.2* $\sin[2 \cdot \varphi] < 0 \Rightarrow$ $(n+1) \cdot \pi > \varphi > \left[\frac{1}{2} + n\right] \cdot \pi$ $2 \cdot \sin\varphi \cdot [\cos\varphi$ $\cdot \{1+\Gamma_2\}^2$ $- 4 \cdot \{\cos\varphi$ $+ \sigma \cdot \sin\varphi\} \cdot \Gamma_1] < 0$
Case I.1 $\Gamma_2 < -1$ $\cos\varphi + \sigma \cdot \sin\varphi > 0$	*Case I.2* $\Gamma_2 > -1$ $\cos\varphi + \sigma \cdot \sin\varphi < 0$	*Case II.1-1* $\sin\varphi > 0 \Rightarrow$ $(2 \cdot n+1) \cdot \pi > \varphi > 2 \cdot n \cdot \pi$ $\cos\varphi > 0 \Rightarrow$ $\Gamma_1 \cdot \sigma > 0 \Rightarrow$ $tg\varphi < \frac{\left[\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1\right]}{4 \cdot \Gamma_1 \cdot \sigma}$ $\Gamma_1 \cdot \sigma < 0 \Rightarrow$ $tg\varphi > \frac{\left[\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1\right]}{4 \cdot \Gamma_1 \cdot \sigma}$ $\cos\varphi < 0 \Rightarrow$ $\Gamma_1 \cdot \sigma > 0$ $tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1}{4 \cdot \Gamma_1 \cdot \sigma}$	*Case II.2-1* $\sin\varphi > 0 \Rightarrow$ $(2 \cdot n+1) \cdot \pi > \varphi > 2 \cdot n \cdot \pi$ $\Gamma_1 \cdot \sigma > 0 \Rightarrow$ $tg\varphi > \frac{\left[\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1\right]}{4 \cdot \Gamma_1 \cdot \sigma}$ $\Gamma_1 \cdot \sigma < 0 \Rightarrow$ $tg\varphi < \frac{\left[\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1\right]}{4 \cdot \Gamma_1 \cdot \sigma}$ $\cos\varphi < 0 \Rightarrow$ $\Gamma_1 \cdot \sigma > 0$ $tg\varphi < \frac{\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1}{4 \cdot \Gamma_1 \cdot \sigma}$ $\Gamma_1 \cdot \sigma < 0 \Rightarrow$ $tg\varphi > \frac{\left[\{1+\Gamma_2\}^2 - 4 \cdot \Gamma_1\right]}{4 \cdot \Gamma_1 \cdot \sigma}$

(continued)

Table 2.7 (continued)

		Case II.1-2 $\sin\varphi < 0 \Rightarrow$ $(n+1)\cdot 2\cdot\pi > \varphi > (1+2\cdot n)\cdot$ $\pi \cos\varphi > 0 \Rightarrow$ $\Gamma_1\cdot\sigma > 0 \Rightarrow$ $tg\varphi > \frac{\left[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\right]}{4\cdot\Gamma_1\cdot\sigma}$ $\Gamma_1\cdot\sigma < 0 \Rightarrow$ $tg\varphi < \frac{\left[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\right]}{4\cdot\Gamma_1\cdot\sigma}$ $\cos\varphi < 0 \Rightarrow$ $\sigma\cdot\Gamma_1 > 0 \Rightarrow$ $tg\varphi < \frac{\left[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\right]}{\sigma\cdot 4\cdot\Gamma_1}$ $\Gamma_1\cdot\sigma < 0 \Rightarrow$ $tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}$	*Case II.2-2* $\sin\varphi < 0 \Rightarrow$ $(n+1)\cdot 2\cdot\pi > \varphi > [1+2\cdot n]\cdot$ $\pi \cos\varphi > 0 \Rightarrow$ $\Gamma_1\cdot\sigma > 0 \Rightarrow$ $tg\varphi < \frac{\left[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\right]}{4\cdot\Gamma_1\cdot\sigma}$ $\Gamma_1\cdot\sigma < 0 \Rightarrow$ $tg\varphi > \frac{\left[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\right]}{4\cdot\Gamma_1\cdot\sigma}$ $\cos\varphi < 0 \Rightarrow$ $\Gamma_1\cdot\sigma > 0 \Rightarrow$ $tg\varphi > \frac{\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1}{4\cdot\Gamma_1\cdot\sigma}$ $\Gamma_1\cdot\sigma < 0 \Rightarrow$ $tg\varphi < \frac{\left[\{1+\Gamma_2\}^2 - 4\cdot\Gamma_1\right]}{4\cdot\Gamma_1\cdot\sigma}$

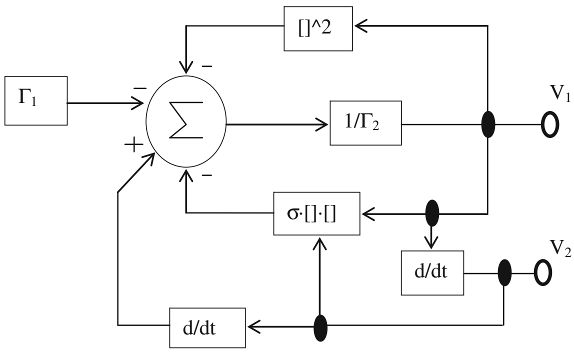

Fig. 2.12 Bogdanov–Takens bifurcation system

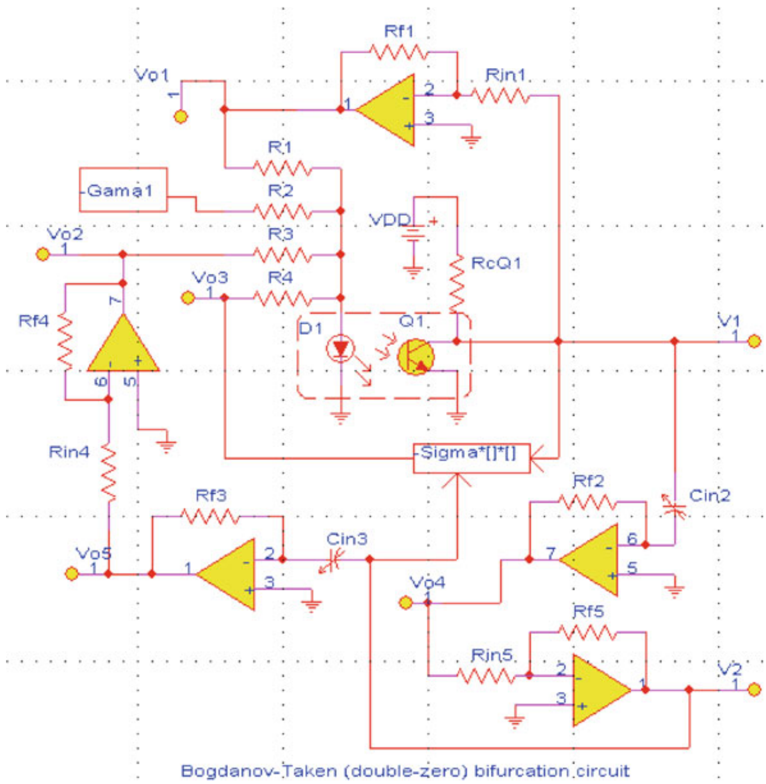

Fig. 2.13 Implementation of Bogdanov–Takens bifurcation system

$$I_{D1} = \sum_{j=1}^{4} I_{Rj} = \frac{-V_1 - V_{D1}}{R_1} + \frac{-\Gamma_1 - V_{D1}}{R_2} + \frac{\frac{dV_2}{dt} - V_{D1}}{R_3} + \frac{-\sigma \cdot V_1 \cdot V_2 - V_{D1}}{R_4}$$

We consider Taylor series approximation: $V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D1}}{I_0}$

$$\begin{aligned} I_{D1} &= \sum_{j=1}^{4} I_{Rj} \\ &= \frac{-V_1 - V_t \cdot \frac{I_{D1}}{I_0}}{R_1} + \frac{-\Gamma_1 - V_t \cdot \frac{I_{D1}}{I_0}}{R_2} + \frac{\frac{dV_2}{dt} - V_t \cdot \frac{I_{D1}}{I_0}}{R_3} + \frac{-\sigma \cdot V_1 \cdot V_2 - V_t \cdot \frac{I_{D1}}{I_0}}{R_4} \end{aligned}$$

$$I_{D1} = -\frac{V_1}{R_1} - \frac{V_t \cdot I_{D1}}{I_0 \cdot R_1} - \frac{\Gamma_1}{R_2} - \frac{V_t \cdot I_{D1}}{I_0 \cdot R_2} + \frac{1}{R_3} \cdot \frac{dV_2}{dt} - \frac{V_t \cdot I_{D1}}{I_0 \cdot R_3} - \frac{\sigma \cdot V_1 \cdot V_2}{R_4} - \frac{V_t \cdot I_{D1}}{I_0 \cdot R_4}$$

$$I_{D1} = -\frac{V_t \cdot I_{D1}}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right] - \frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{dV_2}{dt} - \frac{\sigma \cdot V_1 \cdot V_2}{R_4}$$

$$I_{D1} \cdot \left\{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]\right\} = -\frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{dV_2}{dt} - \frac{\sigma \cdot V_1 \cdot V_2}{R_4}$$

$$I_{D1} = \frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]} \cdot \left\{-\frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{dV_2}{dt} - \frac{\sigma \cdot V_1 \cdot V_2}{R_4}\right\}$$

$$\eta = \frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]};$$

$$\psi\left(\frac{dV_2}{dt}, V_1, V_2, \ldots\right) = -\frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3} \cdot \frac{dV_2}{dt} - \frac{\sigma \cdot V_1 \cdot V_2}{R_4}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; \quad I_{BQ1} = k \cdot I_{D1} = k \cdot \eta \cdot \psi\left(\frac{dV_2}{dt}, V_1, V_2, \ldots\right);$$

$$I_{EQ1} = k \cdot \eta \cdot \psi\left(\frac{dV_2}{dt}, V_1, V_2, \ldots\right) + I_{CQ1}$$

The Mathematical analysis is based on the basic transistor Ebers–Moll equations. We need to implement the regular Ebers–Moll Model to the above optocoupler circuit.

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right];$$
$$V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$V_{CEQ1} = V_{CBQ1} + V_{BEQ1}$, but $V_{CBQ1} = -V_{BCQ1}$, then $\mathbf{V_{CEQ1} = V_{BEQ1} - V_{BCQ1}}$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right)$$

Since $I_{sc} - I_{se} \to \varepsilon \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right];$$
$$\psi = \psi\left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2, \ldots\right)$$

$$\alpha r \cdot I_{CQ1} - I_{EQ1} = \alpha r \cdot I_{CQ1} - \left\{k \cdot \eta \cdot \psi\left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2, \ldots\right) + I_{CQ1}\right\}$$
$$= I_{CQ1} \cdot (\alpha r - 1) - k \cdot \eta \cdot \psi\left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2, \ldots\right)$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f = I_{CQ1} - \left\{k \cdot \eta \cdot \psi\left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2, \ldots\right) + I_{CQ1}\right\} \cdot \alpha f$$
$$= I_{CQ1} \cdot (1 - \alpha f) - k \cdot \eta \cdot \alpha f \cdot \psi\left(\frac{\mathrm{d}V_2}{\mathrm{d}t}, V_1, V_2, \ldots\right)$$

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{I_{CQ1} \cdot (\alpha r - 1) - k \cdot \eta \cdot \psi + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{I_{CQ1} \cdot (1 - \alpha f) - k \cdot \eta \cdot \alpha f \cdot \psi + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right];$$
$$V_{DD} = I_{CQ1} \cdot R_{CQ1} + V_1 \Rightarrow I_{CQ1} = \frac{V_{DD} - V_1}{R_{CQ1}}$$

$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\left(\frac{V_{DD} - V_1}{R_{CQ1}}\right) \cdot (\alpha r - 1) - k \cdot \eta \cdot \psi + I_{se} \cdot (\alpha r \cdot \alpha f - 1)}{\left(\frac{V_{DD} - V_1}{R_{CQ1}}\right) \cdot (1 - \alpha f) - k \cdot \eta \cdot \alpha f \cdot \psi + I_{sc} \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\frac{V_{DD}\cdot(\alpha r-1)}{R_{CQ1}} - \frac{V_1\cdot(\alpha r-1)}{R_{CQ1}} - k\cdot\eta\cdot\psi + I_{se}\cdot(\alpha r\cdot\alpha f-1)}{\frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}} - \frac{V_1\cdot(1-\alpha f)}{R_{CQ1}} - k\cdot\eta\cdot\alpha f\cdot\psi + I_{sc}\cdot(\alpha r\cdot\alpha f-1)}\right]$$

$$V_1 = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{-\frac{V_1\cdot(\alpha r-1)}{R_{CQ1}} - k\cdot\eta\cdot\psi + \left[\frac{V_{DD}\cdot(\alpha r-1)}{R_{CQ1}} + I_{se}\cdot(\alpha r\cdot\alpha f-1)\right]}{-\frac{V_1\cdot(1-\alpha f)}{R_{CQ1}} - k\cdot\eta\cdot\alpha f\cdot\psi + \left[\frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}} + I_{sc}\cdot(\alpha r\cdot\alpha f-1)\right]}\right]$$

For simplicity we define new system global parameters as follows: $\xi_1, \xi_2, \xi_3, \xi_4$.

$$\xi_1 = -\frac{(\alpha r-1)}{R_{CQ1}}; \quad \xi_2 = \frac{V_{DD}\cdot(\alpha r-1)}{R_{CQ1}} + I_{se}\cdot(\alpha r\cdot\alpha f-1);$$
$$\xi_3 = -\frac{(1-\alpha f)}{R_{CQ1}}; \quad \xi_4 = \frac{V_{DD}\cdot(1-\alpha f)}{R_{CQ1}} + I_{sc}\cdot(\alpha r\cdot\alpha f-1);$$
$$V = V_{CEQ1} \simeq Vt\cdot\ln\left[\frac{V\cdot\xi_1 - k\cdot\eta\cdot\psi+\xi_2}{V\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\psi+\xi_4}\right]$$

$e^{\left[\frac{V_1}{Vt}\right]} = \frac{V_1\cdot\xi_1 - k\cdot\eta\cdot\psi+\xi_2}{V_1\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\psi+\xi_4}$; $e^{\left[\frac{V_1}{Vt}\right]} \approx \frac{V_1}{V_t} + 1$ (Taylor series approximation).

$$\left(\frac{V_1}{V_t}+1\right)\cdot(V_1\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\psi + \xi_4) = V_1\cdot\xi_1 - k\cdot\eta\cdot\psi + \xi_2$$

$$\frac{V_1^2}{V_t}\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\psi\cdot\frac{V_1}{V_t} + \frac{V_1}{V_t}\cdot\xi_4 + V_1\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\psi + \xi_4$$
$$= V_1\cdot\xi_1 - k\cdot\eta\cdot\psi + \xi_2$$

$$\frac{V_1^2}{V_t}\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\left[-\frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t} - \frac{\sigma\cdot V_1\cdot V_2}{R_4}\right]\cdot\frac{V_1}{V_t} + \frac{V_1}{V_t}\cdot\xi_4$$
$$+ V_1\cdot\xi_3 - k\cdot\eta\cdot\alpha f\cdot\left[-\frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t} - \frac{\sigma\cdot V_1\cdot V_2}{R_4}\right] + \xi_4$$
$$= V_1\cdot\xi_1 - k\cdot\eta\cdot\left[-\frac{V_1}{R_1} - \frac{\Gamma_1}{R_2} + \frac{1}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t} - \frac{\sigma\cdot V_1\cdot V_2}{R_4}\right] + \xi_2$$

$$\frac{V_1^2}{V_t}\cdot\xi_3+\frac{k\cdot\eta\cdot\alpha f\cdot V_1^2}{R_1\cdot V_t}+\frac{k\cdot\eta\cdot\alpha f\cdot\Gamma_1\cdot V_1}{R_2\cdot V_t}-\frac{k\cdot\eta\cdot\alpha f\cdot V_1}{R_3\cdot V_t}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}$$
$$+\frac{k\cdot\eta\cdot\alpha f\cdot\sigma\cdot V_1^2\cdot V_2}{R_4\cdot V_t}+\frac{V_1}{V_t}\cdot\xi_4+V_1\cdot\xi_3+\frac{k\cdot\eta\cdot\alpha f\cdot V_1}{R_1}$$
$$+\frac{k\cdot\eta\cdot\alpha f\cdot\Gamma_1}{R_2}-\frac{k\cdot\eta\cdot\alpha f}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}+\frac{k\cdot\eta\cdot\alpha f\cdot\sigma\cdot V_1\cdot V_2}{R_4}+\xi_4$$
$$=V_1\cdot\xi_1+\frac{k\cdot\eta\cdot V_1}{R_1}+\frac{k\cdot\eta\cdot\Gamma_1}{R_2}-\frac{k\cdot\eta}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}+\frac{k\cdot\eta\cdot\sigma\cdot V_1\cdot V_2}{R_4}+\xi_2$$

$$\frac{V_1^2}{V_t}\cdot\left[\xi_3+\frac{k\cdot\eta\cdot\alpha f}{R_1}\right]+\frac{k\cdot\eta\cdot\alpha f\cdot\Gamma_1\cdot V_1}{R_2\cdot V_t}-\frac{k\cdot\eta\cdot\alpha f\cdot V_1}{R_3\cdot V_t}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}$$
$$+\frac{k\cdot\eta\cdot\alpha f\cdot\sigma\cdot V_1^2\cdot V_2}{R_4\cdot V_t}+\frac{V_1}{V_t}\cdot\xi_4+V_1\cdot\left[\xi_3+\frac{k\cdot\eta\cdot\alpha f}{R_1}\right]$$
$$+\frac{k\cdot\eta\cdot\alpha f\cdot\Gamma_1}{R_2}-\frac{k\cdot\eta\cdot\alpha f}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}+\frac{k\cdot\eta\cdot\alpha f\cdot\sigma\cdot V_1\cdot V_2}{R_4}+\xi_4$$
$$=V_1\cdot\left[\xi_1+\frac{k\cdot\eta}{R_1}\right]+\frac{k\cdot\eta\cdot\Gamma_1}{R_2}-\frac{k\cdot\eta}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}+\frac{k\cdot\eta\cdot\sigma\cdot V_1\cdot V_2}{R_4}+\xi_2$$

$$V_1\cdot\left\{\frac{k\cdot\eta\cdot\alpha f\cdot\Gamma_1}{R_2\cdot V_t}+\frac{k\cdot\eta\cdot(\alpha f-1)}{R_1}+\frac{\xi_4}{V_t}+\xi_3-\xi_1\right\}=-\frac{V_1^2}{V_t}\cdot\left[\xi_3+\frac{k\cdot\eta\cdot\alpha f}{R_1}\right]$$
$$+\frac{k\cdot\eta}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}\cdot\left\{-1+\alpha f+\frac{\alpha f\cdot V_1}{V_t}\right\}-\frac{k\cdot\eta\cdot\sigma\cdot V_1\cdot V_2}{R_4}$$
$$\cdot\left\{-1+\alpha f+\frac{\alpha f\cdot V_1}{V_t}\right\}+\left\{\xi_2+\frac{k\cdot\eta\cdot\Gamma_1\cdot(1-\alpha f)}{R_2}-\xi_4\right\}$$

We define the following global parameters functions for simplicity:

$$\Omega_1=\frac{k\cdot\eta\cdot\alpha f\cdot\Gamma_1}{R_2\cdot V_t}+\frac{k\cdot\eta\cdot(\alpha f-1)}{R_1}+\frac{\xi_4}{V_t}+\xi_3-\xi_1;\ \Omega_1=\Omega_1(\Gamma_1,\eta,k,\alpha f,\ldots)$$

$$\Omega_2=-\xi_2-\frac{k\cdot\eta\cdot\Gamma_1\cdot(1-\alpha f)}{R_2}+\xi_4;\quad \Omega_2=\Omega_2(\Gamma_1,\eta,k,\alpha f,\ldots)$$

$$V_1\cdot\Omega_1=-\frac{V_1^2}{V_t}\cdot\left[\xi_3+\frac{k\cdot\eta\cdot\alpha f}{R_1}\right]+\frac{k\cdot\eta}{R_3}\cdot\frac{\mathrm{d}V_2}{\mathrm{d}t}\cdot\left\{-1+\alpha f+\frac{\alpha f\cdot V_1}{V_t}\right\}$$
$$-\frac{k\cdot\eta\cdot\sigma\cdot V_1\cdot V_2}{R_4}\cdot\left\{-1+\alpha f+\frac{\alpha f\cdot V_1}{V_t}\right\}-\Omega_2$$

$$V_1 = -\frac{V_1^2}{V_t} \cdot \left[\xi_3 + \frac{k \cdot \eta \cdot \alpha f}{R_1}\right] \cdot \frac{1}{\Omega_1} + \frac{k \cdot \eta}{R_3} \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \left\{-1 + \alpha f + \frac{\alpha f \cdot V_1}{V_t}\right\} \cdot \frac{1}{\Omega_1}$$
$$- \frac{k \cdot \eta \cdot \sigma \cdot V_1 \cdot V_2}{R_4} \cdot \left\{-1 + \alpha f + \frac{\alpha f \cdot V_1}{V_t}\right\} \cdot \frac{1}{\Omega_1} - \frac{\Omega_2}{\Omega_1}$$

$$\Gamma_2 = \Omega_1; \quad \frac{k \cdot \eta}{R_3} = 1; \quad \Gamma_1 = \Omega_2; \quad \frac{k \cdot \eta}{R_4} = 1; \quad \frac{1}{V_t} \cdot \left[\xi_3 + \frac{k \cdot \eta \cdot \alpha f}{R_1}\right] = 1$$

$$\frac{1}{V_t} \cdot \left[-\frac{(1 - \alpha f)}{R_{CQ1}} + \frac{k \cdot \eta \cdot \alpha f}{R_1}\right] = 1 \Rightarrow \alpha f = \frac{\frac{1}{R_{CQ1}} + V_t}{\frac{1}{R_{CQ1}} + \frac{k \cdot \eta}{R_1}}$$

$$\alpha f = \frac{\frac{1}{R_{CQ1}} + V_t}{\frac{1}{R_{CQ1}} + \frac{k \cdot \eta}{R_1}} = \frac{\frac{1}{R_{CQ1}} + V_t}{\frac{1}{R_{CQ1}} + \frac{k \cdot \left\{\frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]}\right\}}{R_1}}$$

$$\frac{k \cdot \eta}{R_3} = 1 \Rightarrow \frac{k}{R_3} \cdot \left\{\frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]}\right\} = 1 \Rightarrow \frac{k}{R_3} = 1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]$$

$$\frac{k \cdot \eta}{R_4} = 1 \Rightarrow \frac{k}{R_4} \cdot \left\{\frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]}\right\} = 1 \Rightarrow \frac{k}{R_4} = 1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]$$

$$\Gamma_1 = \Omega_2 \Rightarrow \Gamma_1 = -\xi_2 - \frac{k \cdot \eta \cdot \Gamma_1 \cdot (1 - \alpha f)}{R_2} + \xi_4$$
$$\Rightarrow \Gamma_1 = \frac{\xi_4 - \xi_2}{1 + \frac{k \cdot \eta \cdot (1 - \alpha f)}{R_2}} = \frac{R_2 \cdot (\xi_4 - \xi_2)}{R_2 + k \cdot \eta \cdot (1 - \alpha f)}$$

$$\Gamma_1 = \frac{R_2 \cdot (\xi_4 - \xi_2)}{R_2 + k \cdot \eta \cdot (1 - \alpha f)}$$
$$= \frac{R_2 \cdot \left(\frac{V_{DD} \cdot (1 - \alpha f)}{R_{CQ1}} + I_{sc} \cdot (\alpha r \cdot \alpha f - 1) - \frac{V_{DD} \cdot (\alpha r - 1)}{R_{CQ1}} - I_{se} \cdot (\alpha r \cdot \alpha f - 1)\right)}{R_2 + k \cdot \left(\frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]}\right) \cdot (1 - \alpha f)}$$

$$\Gamma_1 = \frac{R_2 \cdot \left[\frac{V_{DD} \cdot (2 - \alpha f - \alpha r)}{R_{CQ1}} + (I_{sc} - I_{se}) \cdot (\alpha r \cdot \alpha f - 1)\right]}{R_2 + k \cdot \left(\frac{1}{1 + \frac{V_t}{I_0} \cdot \left[\sum_{j=1}^{4} \frac{1}{R_j}\right]}\right) \cdot (1 - \alpha f)}$$

$$\Gamma_2 = \Omega_1 \Rightarrow \Gamma_2 = \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2 \cdot V_t} + \frac{k \cdot \eta \cdot (\alpha f - 1)}{R_1} + \frac{\xi_4}{V_t} + \xi_3 - \xi_1$$

Bogdanov–Takens basic differential equation $\left(\frac{dV_2}{dt} = \Gamma_1 + \Gamma_2 \cdot V_1 + V_1^2 + \sigma \cdot V_1 \cdot V_2\right)$ must fulfil our optoisolation circuit implementation. The following expression must be within $1 \pm \varepsilon$; $\varepsilon > 0$; $\varepsilon \ll 1$.

$$\frac{dV_2}{dt} = \Gamma_1 + \Gamma_2 \cdot V_1 + V_1^2 + \sigma \cdot V_1 \cdot V_2 \Rightarrow V_1 \cdot \Gamma_2 = \frac{dV_2}{dt} - \Gamma_1 - V_1^2 - \sigma \cdot V_1 \cdot V_2$$

$\left\{-1 + \alpha f + \frac{\alpha f \cdot V_1}{V_t}\right\} = (1 \pm \varepsilon) \to 1;\ \varepsilon > 0;\ \varepsilon \ll 1$. We need to find the values gap for V_1(t) which fulfil Bogdanov–Takens basic differential equation.

$$\left\{-1 + \alpha f + \frac{\alpha f \cdot V_1}{V_t}\right\} = (1 \pm \varepsilon) \Rightarrow \frac{\alpha f \cdot V_1}{V_t} = 2 \pm \varepsilon - \alpha f \Rightarrow V_1(t) = \frac{[2 \pm \varepsilon - \alpha f] \cdot V_t}{\alpha f}$$

$$V_1(t) = \frac{[2 \pm \varepsilon - \alpha f] \cdot V_t}{\alpha f} = \left[\frac{1}{\alpha f} \cdot (2 \pm \varepsilon) - 1\right] \cdot V_t;\quad V_1(t)|_{\varepsilon=0} = \left[\frac{2}{\alpha f} - 1\right] \cdot V_t$$

$$\alpha f = 0.98;\ V_t = 0.026\,\mathrm{V} \Rightarrow V_1(t)|_{\varepsilon=0} = \left[\frac{2}{\alpha f} - 1\right] \cdot V_t = \left[\frac{2}{0.98} - 1\right] \cdot 0.026 \simeq 27.06\,\mathrm{mV}$$

$$V_1(t) = \frac{[2 - \alpha f \pm \varepsilon] \cdot V_t}{\alpha f} = \left[\frac{2}{\alpha f} - 1\right] \cdot V_t \pm \varepsilon \cdot \frac{V_t}{\alpha f}$$

Back to Bogdanov-Takens planar radius:

$$\phi_1(\varphi) = \frac{1}{2} \cdot \sin[2 \cdot \varphi] \cdot \{\cos\varphi + \sigma \cdot \sin\varphi\}$$

$$\phi_2(\varphi) = \frac{1}{2} \cdot \sin[2 \cdot \varphi] \cdot \left\{1 + \frac{k \cdot \eta \cdot \alpha f \cdot \Gamma_1}{R_2 \cdot V_t} + \frac{k \cdot \eta \cdot (\alpha f - 1)}{R_1} + \frac{\xi_4}{V_t} + \xi_3 - \xi_1\right\}$$

$$r^2 \cdot \phi_1(\varphi) + r \cdot \phi_2(\varphi) + \left\{\xi_4 - \xi_2 - \frac{k \cdot \eta \cdot \Gamma_1 \cdot (1 - \alpha f)}{R_2}\right\} \cdot \sin\varphi = 0$$

$$\Rightarrow r^{(j)} = \frac{-\phi_2(\varphi) \pm \sqrt{\phi_2^2(\varphi) - 4 \cdot \phi_1(\varphi) \cdot \Gamma_1 \cdot \sin\varphi}}{2 \cdot \phi_1(\varphi)}$$

2.7 Exercises

1. We have optoisolation cusp points in Bazykin system. The system is characterized by the following two differential equations:

$$\frac{dV_1}{dt} = V_1 - \frac{V_1 \cdot V_2}{1 + \Gamma_1 \cdot V_1} - \eta \cdot V_1^2; \quad \frac{dV_2}{dt} = -\Gamma_2 \cdot V_2 + \frac{V_1 \cdot V_2}{1 + \Gamma_1 \cdot V_1} - \xi \cdot V_2^2$$

   1.1 Compute the (Γ_1, ζ) coordinate of the cusp bifurcation points in the system.
   1.2 How many fixed points are there in the system? Analyze stability?
   1.3 Implement Bazykin system with optoisolation parts, op-amps, capacitors, resistors, etc.
   1.4 Try to find Global Bazykin system parameters (Γ_1, Γ_2, ζ, η) as a function of optoisolation circuit parameters (α_f, α_r, I_{se}, I_{sc}, etc.).
   1.5 Discuss cusp bifurcation as a function of optoisolation circuit parameters (α_f, α_r, I_{se}, I_{sc}, etc.) variation.

2. We have optoisolation system which are characterized by the differential equation: $\frac{dV}{dt} = V^4 + \Gamma_1 \cdot V^2 + \Gamma_2 \cdot V$ (quartic behavior).

   2.1 Shows the set of points (Γ_1, Γ_2, V^*) in space, where V^* is a critical point of $f(V) = V^4 + \Gamma_1 \cdot V^2 + \Gamma_2 \cdot V$.
   2.2 Find fixed points and discuss stability and cusp catastrophe behavior.
   2.3 Implement the system differential by optoisolation elements, op-amps, resistors, capacitors, etc. Find Γ_1, Γ_2 as a function of optoisolation circuit parameters.
   2.4 Discuss cusp catastrophe behavior changes when all optoisolation parameters are fixed except k (optoisolation coupling coefficient).

3. We have optoisolation system which is characterized by Bautin bifurcation in a predator–prey system differential equations.

$$\frac{dV_1}{dt} = \frac{V_1^2 \cdot (1 - V_1)}{\Gamma_1 + V_1} - V_1 \cdot V_2; \quad \frac{dV_2}{dt} = -\sigma \cdot V_2 \cdot (\Gamma_2 - V_1)$$

   3.1 Derive the equation for the Hopf bifurcation curve in the system and show that it is dependent of σ parameter.
   3.2 Find system first and second Lyapunov coefficients and sketch the bifurcation diagram (the parametric portrait on the (Γ_1,Γ_2)-plane and all possible phase portraits) of the system.

3.3 Try to sketch optoisolation circuit, which fulfills Bautin bifurcation in a predator–prey system. Find Γ_1, Γ_2, σ parameters as a function of circuit overall parameters (α_f, α_r, I_{se}, I_{sc}, k_1, k_2,…).

3.4 Discuss Bautin bifurcation as a function of optoisolation overall parameters (α_f, α_r, I_{se}, I_{sc}, k_1, k_2,…).

4. We have an optoisolation system which shows the model of a Bautin bifurcation controllable resonator. We have two main system variables V_1, V_2 and parameters Γ_1, Γ_2, σ_1, σ_2.

$$\frac{dV_1}{dt} = \sigma_1 \cdot V_1 \cdot (V_2 - \frac{\Gamma_2}{\sigma_2 \cdot V_1 + 1} - 1); \quad \frac{dV_2}{dt} = \Gamma_1 - (V_1 + 1) \cdot V_2$$

When $V_1 \geq 0$, $\sigma_2 > 0$, $\sigma_1 > 1$, and Γ_1, $\Gamma_2 > 0$, has a Bautin bifurcation point on the Hopf curve on the (Γ_1, Γ_2)-plane if $\sigma_2 < 1$ and $\sigma_2 - 1 + \sigma_1 \cdot \sigma_2 > 0$.

4.1 Find system fixed points and discuss Bautin bifurcation as a function of parameters Γ_1, Γ_2, σ_1, σ_2 values.

4.2 Implement our system differential equations by optoisolation circuit. Find system parameters (Γ_1, Γ_2, σ_1, σ_2) as a function of optoisolation circuit parameters (α_f, α_r, I_{se}, I_{sc}, k_1, k_2,…).

4.3 Discuss Bautin bifurcation as a function of optocouplers coupling coefficients k_1, k_2,… when all other parameters have fixed values (α_f, α_r, I_0, I_{se}, I_{sc},…).

4.4 We switch between V_1, V_2 system variables ($V_1 \rightleftharpoons V_2$) and restrict our system parameters space to only two (Γ, σ); $\Gamma_1 = \ln(\Gamma_2) = \Gamma$; $\sigma_1 = \sin(\sigma_2) = \sigma$. Discuss how the system dynamical behavior change. Find fixed points and bifurcation behavior. Try to implement it by using optoisolation circuit.

5. Consider the planar system $dV/dt = F(V)$; $\frac{dV}{dt} = F(V) + G(V) \cdot u$; where $F(V) = \begin{pmatrix} -V_2 - V_1 \cdot V_2^2 \\ V_1 \end{pmatrix}$. It possesses the non-hyperbolic equilibrium $V = 0$, and $J = dF(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the control $u(V, \mu, \beta) \cdot G(V)$, with $u(V, \mu, \beta)$ given by $u(V, \mu, \beta) = \beta_1 \cdot \mu_1 + (\beta_2 + \mu_2) \cdot (V_1^2 + V_2^2) + \beta_3 \cdot (V_1^2 + V_2^2)^2$, and $G(V) = s \cdot V$, $s \neq 0$. We have $G(0) = 0$. $M = dG(0) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$; $tr(M) = 2 \cdot s \neq 0$.
The bifurcation parameter $\mu = (\mu_1, \mu_2)$.

5.1 Find system fixed points and discuss stability.

5.2 Find first and second Lyapunov coefficients.

5.3 Discuss how the bifurcation parameters influence our system dynamic.

5.4 Implement the system by using optoisolation circuit which includes optocouplers, op-amps, resistors, capacitors, diodes, etc.
5.5 How the dynamic of the system change for different optocoupling coefficient values (k parameter).

6. We have the following system $\mathrm{d}V/\mathrm{d}t = F(V)$ where $F(V) = \begin{pmatrix} -V_2 + V_1^2 + \frac{1}{3} \cdot V_1^3 \\ V_1 \end{pmatrix}$ $\frac{\mathrm{d}V}{\mathrm{d}t} = F(V) + G(V) \cdot u$. Which possess the non-hyperbolic equilibrium $V = 0$ and $J = \mathrm{d}F(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the control $u(V, \mu, \beta) \cdot G(V)$, with $u(V, \mu, \beta)$ given by $u(V, \mu, \beta) = \beta_1 \cdot \mu_1 + (\beta_2 + \mu_2) \cdot (V_1^2 + V_2^2) + \beta_3 \cdot (V_1^2 + V_2^2)^2$, and $G(V) = \begin{pmatrix} s \cdot V_1 + V_1 \cdot V_2^2 \\ s \cdot V_2 + V_2^3 \end{pmatrix}$; $s \neq 0$. We have $G(0) = 0.M = \mathrm{d}gG(0) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ and $tr(M) = 2 \cdot s \neq 0$. We have nonlinear control system. The bifurcation parameter $\mu = (\mu_1, \mu_2)$.

6.1 Find system fixed points and discuss stability.
6.2 Find first and second Lyapunov coefficients.
6.3 Discuss how the bifurcation parameters influence our system dynamic.
6.4 Implement the system by using optoisolation circuit which includes, optocouplers, op-amps, resistors, capacitors, diodes, etc.
6.5 How the dynamic of the system change for different optocoupling coefficient values (k parameter), α_f, α_r, etc.

7. We have the below Bogdanov–Takens bifurcation circuit implementation. The circuit implementation includes two optoisolation elements: LED D1 which coupled with photo transistor Q1 (coupling coefficient is $k1$), LED D3 coupled with photo LED D2 coupling coefficient is $k2$). Additionally the circuit includes op-amps, resistors, capacitors, etc. We consider the parameters of two optoisolation elements are not the same. $V_{DD1} \neq V_{DD2}$ and the related resistors are not the same.

7.1 Find circuit differential equations which based on Ebers–Moll transistor formulas. Try to get the same shape of Bogdanov–Takens bifurcation set of differential equations.
7.2 Discuss stability and fixed points as a functions of circuit overall parameters ($k1$, $k2$, α_{f1}, α_{r1}, etc.).
7.3 What are the main circuit parameters which act as Bogdanov–Takens bifurcation control parameters.
7.4 Consider $k1 = \Delta \cdot k2$; $k2 = k$. How circuit equations change? Discuss stability and bifurcation as a function of k parameter and Δ coefficient.
7.5 Consider $k2 = \Delta \cdot k1$; $k1 = k$, How the results in (7.4) change (Fig. 2.14).

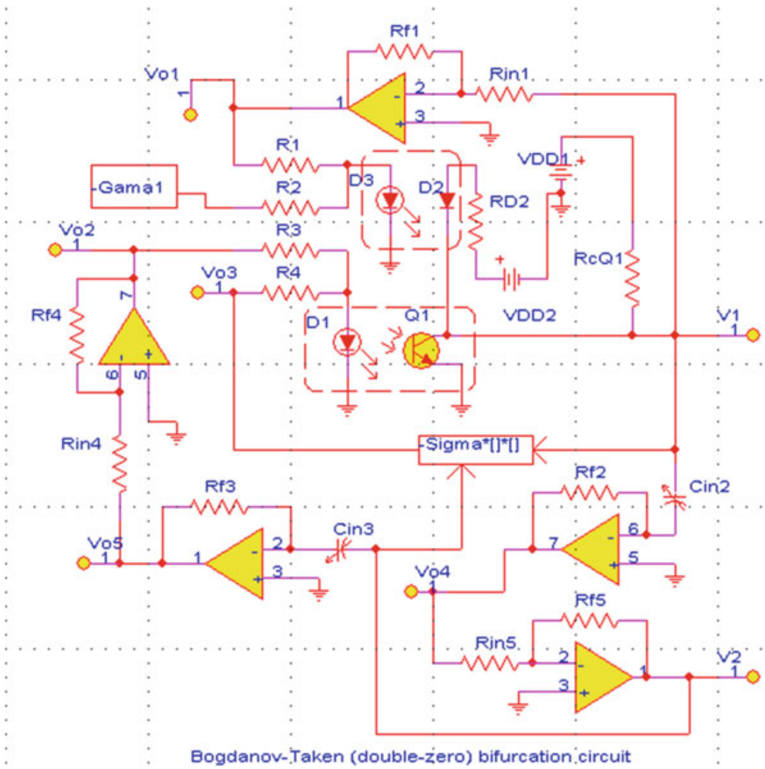

Fig. 2.14 Stability and bifurcation as a function of k parameter and Δ coefficient

8. We have a system which characterize by Bogdanov–Takens bifurcation. $\Gamma_2 \to e^{-[V_1 \cdot \Gamma_2]}$; and the system differential equations:

$$\frac{dV_1}{dt} = V_2; \quad \frac{dV_2}{dt} = \Gamma_1 + e^{-V_1 \cdot \Gamma_2} \cdot V_1 + V_1^2 + \sigma \cdot V_1 \cdot V_2; \quad f(V_1, \Gamma_2) = e^{-V_1 \cdot \Gamma_2}$$

   8.1 How the transformation $\Gamma_2 \to e^{-[V_1 \cdot \Gamma_2]}$ changes our system dynamic?
   8.2 Find fixed points, discuss stability. You can use Taylor series approximation but not less than second order.
   8.3 How the control parameters Γ_1, Γ_2 influence our system dynamics?
   8.4 Find implementation of our system by using optoisolation circuit.
   8.5 Discuss bifurcation for different circuit overall parameters.

9. We have a system which characterize by bifurcations of a triple equilibrium with elliptic sector. The truncated and scaled critical normal form: X, Y our main system variables. $\Gamma_1 = \pm 1$, $\Gamma_2 = \pm 2$, and $\beta > 0$.

$$\frac{dX}{dt} = Y; \quad \frac{dY}{dt} = \beta \cdot X \cdot Y + \Gamma_1 \cdot X^3 + \Gamma_2 \cdot X^2 \cdot Y$$

9.1 Find system fixed points and discuss stability.
9.2 Discuss system bifurcation and for which Γ_1, Γ_2, and β values we have saddle case, focus case, elliptic case.
9.3 Try to implement the system by using optoisolation circuit.
9.4 How the bifurcation behavior changes when optoisolation coupling coefficients spread among some discrete values.
9.5 Try to find our system parameters Γ_1, Γ_2, and β as a function of optoisolation circuit parameters.

10. We have system with bifurcation of a triple equilibrium with elliptic sector with the unfolding representation. Our system main variables X, Y and system parameters μ_1, μ_2, v, β. The system differential equations are as follows:

$$\frac{dX}{dt} = Y; \quad \frac{dY}{dt} = -\mu_1 - \mu_2 \cdot X + v \cdot Y + \beta \cdot X \cdot Y - X^3 - X^2 \cdot Y$$

10.1 Find system fixed points and discuss stability.
10.2 Draw 3D bifurcation graphs diagram, μ_2 parameter as a function of v, μ_1 parameters.
10.3 Draw μ_1 parameter as a function of v parameter when μ_2 and β have a fixed value.
10.4 Implement the system by using optoisolation circuit.
10.5 How phototransistor parameters, α_f, α_r,… values influence our system bifurcation behavior.

Chapter 3
Optoisolation Circuits Bifurcation Analysis (II)

The basic definition of bifurcation describes the qualitative alterations that occur in the orbit structure of a dynamical system as the parameters on which the system depends are varied. In this chapter, we discuss various bifurcations which exhibit by optoisolation circuits. The Fold-Hopf bifurcation is a bifurcation of an equilibrium point in a two parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues. The Hopf–Hopf bifurcation is when the critical equilibrium has two pairs of purely imaginary eigenvalues in system's two parameter family of autonomous differential equations. Torus bifurcations (Neimark–Sacker bifurcations) of the limit cycles generated by the Hopf bifurcations. This curve of torus bifurcations is transversal to the saddle–node and Andronov–Hopf bifurcation curves. The torus bifurcation generates an invariant two-dimensional torus. The invariant torus disappears via either a "heteroclinic destruction" or a "blow-up". Saddle-loop or Homoclinic bifurcation is when part of a limit cycle moves closer and closer to a saddle point. At the homoclinic bifurcation the cycle touches the saddle point and becomes a homoclinic orbit (infinite period bifurcation) [5–8].

3.1 Fold-Hopf Bifurcation System

The Fold-Hopf bifurcation is a bifurcation of an equilibrium point in a two parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues. Other names for Fold-Hopf bifurcation are Zero-Hopf (ZH) bifurcation, saddle-node Hopf bifurcation and Gavrilov–Guckenheimer bifurcation. We now consider the autonomous system of ODEs $\frac{\mathrm{d}V}{\mathrm{d}t} = f(V, \alpha)\,\forall\,\alpha \in \mathbb{R}^n$ depending on two parameters $\alpha \in \mathbb{R}^2$, where f is smooth function. Suppose that at $\alpha = 0$ the system has an equilibrium $V = 0$. The Jacobian matrix $A = f_V(0,0)$ has a zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary eigenvalues

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$\lambda_{2,3} = \pm i \cdot \omega$ with $\omega > 0$. The co-dimension two bifurcation is characterized by the conditions $\lambda_1 = 0$ and Re $\lambda_{2,3} = 0$ and we get appearance in open sets of two parameter families of smooth ODEs. In the three-dimensional case we can describe the fold-Hopf bifurcation analytically by system $\frac{dV}{dt} = f(V, \alpha) \quad \forall\, \alpha \in \mathbb{R}^n; n = 3 \Rightarrow V \in \mathbb{R}^3$. if the nondegeneracy conditions hold: (ZH.1) $B(0) \cdot C(0) \cdot E(0) \neq 0$ and (ZH.2) map $(V, \alpha) \to (f(V, \alpha), \mathrm{Tr}(f_V(V, \alpha)), \det(f_V(V, \alpha))$ is regular at $(V, \alpha) = (0,0)$. This system is locally orbit ally smoothly equivalent near the origin to the complex normal form: $\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot |\zeta|^2 + O\left(\|(\xi, \zeta)\|^4\right)$ and

$$\frac{d\zeta}{dt} = (\beta_2 + i \cdot \omega) \cdot \zeta + (\theta(\beta) + i \cdot \theta_1(\beta)) \cdot \xi \cdot \zeta + \xi^2 \cdot \zeta + O\left(\|(\xi, \zeta)\|^4\right)$$

where $\xi \in \mathbb{R}, \zeta \in \mathbb{C}, \beta \in \mathbb{R}^2, s = \mathrm{sign} B(0) \cdot C(0) = \pm 1, \theta(0) = \frac{\mathrm{Re} H_{110}}{B(0)}$ when $E(0) < 0$, the orbital equivalence includes reversal of time. The formulas $B(0)$, $C(0)$, $E(0)$, and H_{110} are given in Appendix A.4. This normal form is particularly simple in real cylindrical coordinates (r, φ, ξ) and take the form

$$\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot r^2 + O\left((\xi^2 + r^2)\right)^2;$$
$$\frac{dr}{dt} = r \cdot (\beta_2 + \theta(\beta) \cdot \xi + \xi^2) + O\left((\xi^2 + r^2)\right)^2$$

$\frac{d\varphi}{dt} = \omega + \theta_1(\beta) \cdot \xi + O\left((\xi^2 + r^2)\right)^2$, where the O-terms are 2π periodic in φ. The bifurcation diagram of the normal form depends on the O-terms, and its essential features are determined by the form:

$$\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot r^2; \frac{dr}{dt} = r \cdot \left(\beta_2 + \theta(\beta) \cdot \xi + \xi^2\right); \frac{d\varphi}{dt} = \omega + \theta_1(\beta) \cdot \xi.$$

The first two equations are independent of the third one, which describes a monotone rotation. The local bifurcation diagrams of the planar system:

$$\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot r^2; \frac{dr}{dt} = r \cdot \left(\beta_2 + \theta(\beta) \cdot \xi + \xi^2\right) \text{ with (ZH.3)} \theta(0) \neq 0.$$

We can distinguish four cases:

(1) $s = 1$, $\theta(0) > 0$ (subcritical Hopf bifurcations and no tori).
(2) $s = -1$, $\theta(0) < 0$ (subcritical Hopf bifurcations and no tori).
(3) $s = 1$, $\theta(0) < 0$ (sub and supercritical Hopf bifurcations and torus heteroclinic destruction).
(4) $s = -1$, $\theta(0) > 0$ (sub and supercritical Hopf bifurcations and torus blow up).

The next system we discuss and implement by using optoisolation circuit is Fold-Hopf bifurcation is Rossler's prototype chaotic system [5–9].

$$(*)\quad \frac{dX}{dt} = -Y - Z; \frac{dY}{dt} = X + a \cdot Y; \frac{dZ}{dt} = b \cdot X - c \cdot Z + X \cdot Z \quad \forall \quad a, b, c \in \mathbb{R}^3$$

The system possesses at most two equilibrium points O and P, and the coordinates can be calculated by setting $\frac{dX}{dt} = 0; \frac{dY}{dt} = 0; \frac{dZ}{dt} = 0$ (fixed points).

$$\frac{dX}{dt} = 0 \Rightarrow -Y - Z = 0; \frac{dY}{dt} = 0 \Rightarrow X + a \cdot Y = 0;$$
$$\frac{dZ}{dt} = 0 \Rightarrow b \cdot X - c \cdot Z + X \cdot Z = 0 \ \forall \ a, b, c \in \mathbb{R}^3$$
$$-Y - Z = 0 \Rightarrow -a \cdot Y - a \cdot Z = 0 \,\&\, X + a \cdot Y = 0 \Rightarrow X - a \cdot Z = 0 \Rightarrow X = a \cdot Z$$
$$X = a \cdot Z \Rightarrow b \cdot X - c \cdot Z + X \cdot Z = 0 \Rightarrow b \cdot a \cdot Z - c \cdot Z + a \cdot Z^2 = 0$$
$$b \cdot a \cdot Z - c \cdot Z + a \cdot Z^2 = 0 \Rightarrow Z \cdot \{b \cdot a - c + a \cdot Z\} = 0 \Rightarrow Z^{(0)} = 0; Z^{(1)} = \frac{c}{a} - b$$
$$Z^{(0)} = 0; Z^{(1)} = \frac{c}{a} - b \Rightarrow X^{(0)} = 0 \,\&\, X^{(1)} = c - ab \Rightarrow Y^{(0)} = 0 \,\&\, Y^{(1)} = b - \frac{c}{a}$$

First equilibria point O (first fixed point): $(X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$. Second equilibria point P (second fixed point): $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (c - a \cdot b,\ b - c/a,\ c/a - b)$. The equilibria O and P points collide at the surface $T = \{(a, b, c): c = a \cdot b\}$

If $c = a \cdot b$ then both $(X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$ and $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (0, 0, 0)$.

We need to represent our Fold-Hopf bifurcation Rossler's prototype chaotic system by one high-order and nonlinear differential equation. The target is to use it for optoisolation circuit implementation [45, 52].

$$\frac{dX}{dt} = -Y - Z; \frac{dY}{dt} = X + a \cdot Y; \frac{dZ}{dt} = b \cdot X - c \cdot Z + X \cdot Z \quad \forall \quad a, b, c \in \mathbb{R}^3$$
$$\frac{dY}{dt} = X + a \cdot Y \Rightarrow X = \frac{dY}{dt} - a \cdot Y \Rightarrow \frac{d}{dt}\left\{\frac{dY}{dt} - a \cdot Y\right\}$$
$$= -Y - Z \Rightarrow \frac{d^2Y}{dt^2} - a \cdot \frac{dY}{dt} = -Y - Z \Rightarrow Z = -Y + a \cdot \frac{dY}{dt} - \frac{d^2Y}{dt^2}$$
$$\frac{dY}{dt} = X + a \cdot Y \Rightarrow X = \frac{dY}{dt} - a \cdot Y \Rightarrow \frac{dZ}{dt} = b \cdot \left\{\frac{dY}{dt} - a \cdot Y\right\} - c \cdot Z + \left\{\frac{dY}{dt} - a \cdot Y\right\} \cdot Z$$
$$\frac{dZ}{dt} = b \cdot \left\{\frac{dY}{dt} - a \cdot Y\right\} - c \cdot Z + \left\{\frac{dY}{dt} - a \cdot Y\right\} \cdot Z \Rightarrow \frac{dZ}{dt}$$
$$= b \cdot \frac{dY}{dt} - b \cdot a \cdot Y - c \cdot Z + \frac{dY}{dt} \cdot Z - a \cdot Y \cdot Z$$

We get two intermediate equations:

$$(1) \quad Z = -Y + a \cdot \frac{dY}{dt} - \frac{d^2Y}{dt^2}; \quad (2)\, \frac{dZ}{dt} = b \cdot \frac{dY}{dt} - b \cdot a \cdot Y - c \cdot Z + \frac{dY}{dt} \cdot Z - a \cdot Y \cdot Z$$

$$(1) \rightarrow (2): \begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t}\left\{-Y + a \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2}\right\} = b \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - b \cdot a \cdot Y - c \cdot \left\{-Y + a \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2}\right\} \\ + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot \left\{-Y + a \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2}\right\} - a \cdot Y \cdot \left\{-Y + a \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2}\right\} \end{array}$$

$$\begin{array}{l} -\frac{\mathrm{d}Y}{\mathrm{d}t} + a \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} - \frac{\mathrm{d}^3 Y}{\mathrm{d}t^3} = b \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - b \cdot a \cdot Y + c \cdot Y - c \cdot a \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} + c \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} \\ -\frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y + a \cdot \left[\frac{\mathrm{d}Y}{\mathrm{d}t}\right]^2 - \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + a \cdot Y^2 - a^2 \cdot Y \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} + a \cdot Y \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} \end{array}$$

$$(**) \quad \begin{array}{l} \{-b \cdot a + c\} \cdot Y + \{b - c \cdot a + 1\} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} + (c - a) \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} \\ + a \cdot \left[\frac{\mathrm{d}Y}{\mathrm{d}t}\right]^2 - \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + a \cdot Y^2 - \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y \cdot \{a^2 + 1\} + a \cdot Y \cdot \frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + \frac{\mathrm{d}^3 Y}{\mathrm{d}t^3} = 0 \end{array}$$

The above nonlinear differential equation represent our system with one variable (Y) and parameters a, b, c. We will use it latter for our optoisolation circuit implementation. We need to investigate system fixed points stability. We define the following three functions:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = f_1(X, Y, Z); \; \frac{\mathrm{d}Y}{\mathrm{d}t} = f_2(X, Y, Z); \; \frac{\mathrm{d}Z}{\mathrm{d}t} = f_3(X, Y, Z) \; \forall \, a, b, c \in \mathbb{R}^3$$
$$f_1(X, Y, Z) = -Y - Z; \; f_2(X, Y, Z) = X + a \cdot Y; \; f_3(X, Y, Z) = b \cdot X - c \cdot Z + X \cdot Z$$
$$f_1 = f_1(X, Y, Z); \; f_2 = f_2(X, Y, Z); \; f_3 = f_3(X, Y, Z)$$

We calculated the related partial derivatives of f_1, f_2, f_3 respect to X, Y, Z.

$$\frac{\partial f_1}{\partial X} = 0; \; \frac{\partial f_1}{\partial Y} = -1; \; \frac{\partial f_1}{\partial Z} = -1; \; \frac{\partial f_2}{\partial X} = 1; \; \frac{\partial f_2}{\partial Y} = a; \; \frac{\partial f_2}{\partial Z} = 0; \; \frac{\partial f_3}{\partial X} = b + Z$$
$$\frac{\partial f_3}{\partial Y} = 0; \; \frac{\partial f_3}{\partial Z} = -c + X$$

The matrix A is called the jacobian matrix at the first fixed point ($X^{(0)}$, $Y^{(0)}$, $Z^{(0)}$) = (0, 0, 0).

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\right|_{(X^{(0)}, Y^{(0)}, Z^{(0)}) = (0,0,0)} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -c \end{pmatrix} - I \cdot \lambda = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -c \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} -\lambda & -1 & -1 \\ 1 & a - \lambda & 0 \\ b & 0 & -c - \lambda \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$\det\begin{pmatrix} -\lambda & -1 & -1 \\ 1 & a-\lambda & 0 \\ b & 0 & -(c+\lambda) \end{pmatrix} = -\lambda \cdot \begin{pmatrix} a-\lambda & 0 \\ 0 & -(c+\lambda) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ b & -(c+\lambda) \end{pmatrix} - \begin{pmatrix} 1 & a-\lambda \\ b & 0 \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = -\lambda \cdot \begin{pmatrix} a-\lambda & 0 \\ 0 & -(c+\lambda) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ b & -(c+\lambda) \end{pmatrix} - \begin{pmatrix} 1 & a-\lambda \\ b & 0 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda \cdot I) &= \lambda \cdot (a-\lambda) \cdot (c+\lambda) - (c+\lambda) + b \cdot (a-\lambda) \\ &= -\lambda^3 + \lambda^2 \cdot (a-c) + \lambda \cdot (a \cdot c - b - 1) + b \cdot a - c \\ \det(A - \lambda \cdot I) &= \lambda^3 + \lambda^2 \cdot (c-a) + \lambda \cdot (1 + b - a \cdot c) + c - b \cdot a \\ \det(A - \lambda \cdot I) &= 0 \Rightarrow \lambda^3 + \lambda^2 \cdot (c-a) + \lambda \cdot (1 + b - a \cdot c) + c - b \cdot a = 0 \end{aligned}$$

For simplicity we define three system parameters functions:

$$\psi_1(a,b,c) = c - a;\ \psi_2(a,b,c) = 1 + b - a \cdot c;\ \psi_3(a,b,c) = c - b \cdot a$$
$$\psi_1 = \psi_1(a,b,c);\ \psi_2 = \psi_2(a,b,c);\ \psi_3 = \psi_3(a,b,c)$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot \psi_1 + \lambda \cdot \psi_2 + \psi_3 = 0;\ P_1(\lambda) = \lambda^3 + \lambda^2 \cdot \psi_1 + \lambda \cdot \psi_2 + \psi_3$$

We got an eigenvalues cubic function (first fixed point) with real parameters functions ψ_1, ψ_2, ψ_3. The eigenvalues polynomial is of degree three. The derivative of an eigenvalues cubic function is a quadratic function and the integral of a cubic function is a quadratic function. Our eigenvalues cubic equation with real coefficients functions (ψ_1, ψ_2, ψ_3) has at least one solution λ among the real numbers which is a consequence of the intermediate value theorem. We can distinguish several possible cases for our eigenvalues cubic function using a discriminant (Δ_1).

$$\Delta_1 = -4 \cdot \psi_1^3 \cdot \psi_3 + \psi_1^2 \cdot \psi_2^2 - 4 \cdot \psi_2^3 + 18 \cdot \psi_1 \cdot \psi_2 \cdot \psi_3 - 27 \cdot \psi_3^2$$

The following cases need to be considered for all first fixed point eigenvalues options:

$\Delta_1 > 0 \rightarrow$ we have three distinct real eigenvalues for the first fixed point.
$\Delta_1 < 0 \rightarrow$ we have a real eigenvalue and a pair of complex conjugate roots.
$\Delta_1 = 0 \rightarrow$ at least two eigenvalues coincide. We get double real eigenvalue and another distinct single real eigenvalue. Alternatively all three roots coincide yielding a triple real eigenvalues.

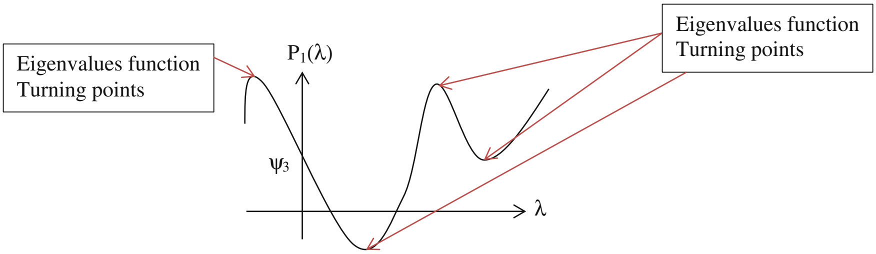

Fig. 3.1 Eigenvalues Turning Point (TP) for the first fixed point

We can decide between these subcases by computing the resultant of the cubic eigenvalue function and its second derivative. A triple root exists if and only if this resultant vanishes [5, 6].

Eigenvalues turning point (TP) for the first fixed point (Fig. 3.1):

Eigenvalues turning points (TPs):

$$\frac{dP_1(\lambda)}{d\lambda} = 3\cdot\lambda^2 + \lambda\cdot 2\cdot\psi_1 + \psi_2; \frac{dP_1(\lambda)}{d\lambda} = 0 \Rightarrow \lambda_{TP} = \frac{1}{3}\cdot(-\psi_1 \pm \sqrt{\psi_1^2 - 3\cdot\psi_2})$$

Case I One eigenvalue turning point $\psi_1^2 - 3\cdot\psi_2 = 0 \Rightarrow \lambda_{\text{TP}} = -\frac{\psi_1}{3} = -\frac{(c-a)}{3} = \frac{(a-c)}{3}$

Case II Two distinct complex eigenvalues turning points

$$\psi_1^2 - 3\cdot\psi_2 < 0 \Rightarrow \lambda_{\text{TP}} = \frac{1}{3}\cdot\left(-\psi_1 \pm i\cdot\sqrt{|\psi_1^2 - 3\cdot\psi_2|}\right)$$
$$= \frac{1}{3}\cdot\left\{-(c-a) \pm i\cdot\sqrt{|[c-a]^2 - 3\cdot(1+b-a\cdot c)|}\right\}$$
$$\psi_1^2 - 3\cdot\psi_2 < 0 \Rightarrow \lambda_{\text{TP}} = \frac{1}{3}\cdot\left\{a - c \pm i\cdot\sqrt{|c^2 + a\cdot c + a^2 - 3\cdot(1+b)|}\right\}$$

Case III Two distinct real eigenvalues turning points

$$\psi_1^2 - 3\cdot\psi_2 > 0 \Rightarrow \lambda_{\text{TP}} = \frac{1}{3}\cdot\left(-\psi_1 \pm \sqrt{\psi_1^2 - 3\cdot\psi_2}\right)$$
$$= \frac{1}{3}\cdot\left\{-(c-a) \pm \sqrt{[c-a]^2 - 3\cdot(1+b-a\cdot c)}\right\}$$
$$\psi_1^2 - 3\cdot\psi_2 > 0 \Rightarrow \lambda_{\text{TP}} = \frac{1}{3}\cdot\left\{a - c \pm \sqrt{c^2 + a\cdot c + a^2 - 3\cdot(1+b)}\right\}$$

The next step is to find matrix A is called the jacobian matrix at the second fixed point $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (c - a\cdot b, b - c/a, c/a - b)$.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\Bigg|_{(X^{(1)},Y^{(1)},Z^{(1)})=(c-a\cdot b,b-\frac{c}{a},\frac{c}{a}-b)} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c}{a} & 0 & -a\cdot b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c}{a} & 0 & -a\cdot b \end{pmatrix} - I\cdot\lambda = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c}{a} & 0 & -a\cdot b \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & -1 & -1 \\ 1 & a-\lambda & 0 \\ \frac{c}{a} & 0 & -a\cdot b-\lambda \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det ($A - \lambda \cdot I$), where I is the identity matrix.

$$\det\begin{pmatrix} -\lambda & -1 & -1 \\ 1 & a-\lambda & 0 \\ \frac{c}{a} & 0 & -a\cdot b-\lambda \end{pmatrix} = -\lambda \cdot \begin{pmatrix} a-\lambda & 0 \\ 0 & -a\cdot b-\lambda \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & -a\cdot b-\lambda \end{pmatrix} - \begin{pmatrix} 1 & a-\lambda \\ \frac{c}{a} & 0 \end{pmatrix}$$

$$\det(A-\lambda\cdot I) = \lambda\cdot(a-\lambda)\cdot(a\cdot b+\lambda) - a\cdot b - \lambda + c - \tfrac{c}{a}\cdot\lambda$$
$$\det(A-\lambda\cdot I) = a^2\cdot b\cdot\lambda + a\cdot\lambda^2 - \lambda^2\cdot a\cdot b - \lambda^3 - a\cdot b - \lambda + c - \tfrac{c}{a}\cdot\lambda$$
$$\det(A-\lambda\cdot I) = -\lambda^3 + \lambda^2\cdot a\cdot(1-b) + \lambda\cdot(a^2\cdot b - 1 - \tfrac{c}{a}) - a\cdot b + c$$
$$\det(A-\lambda\cdot I) = \lambda^3 + \lambda^2\cdot a\cdot(b-1) + \lambda\cdot(1 + \tfrac{c}{a} - a^2\cdot b) + a\cdot b - c$$

For simplicity we define three system parameters functions:

$$\psi_4(a,b,c) = a\cdot(b-1);\ \psi_5(a,b,c) = 1 + \tfrac{c}{a} - a^2\cdot b;\ \psi_6(a,b,c) = a\cdot b - c$$
$$\psi_4 = \psi_4(a,b,c);\ \psi_5 = \psi_5(a,b,c);\ \psi_6 = \psi_6(a,b,c)$$
$$\det(A-\lambda\cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2\cdot\psi_4 + \lambda\cdot\psi_5 + \psi_6 = 0;$$
$$P_2(\lambda) = \lambda^3 + \lambda^2\cdot\psi_4 + \lambda\cdot\psi_5 + \psi_6$$

We got an eigenvalues cubic function (second fixed point) with real parameters functions ψ_4, ψ_5, ψ_6. The eigenvalues polynomial is of degree three. The derivative of an eigenvalues cubic function is a quadratic function and the integral of a cubic function is a quadratic function. Our eigenvalues cubic equation with real coefficients functions (ψ_4, ψ_5, ψ_6) has at least one solution λ among the real numbers which is a consequence of the intermediate value theorem. We can distinguish several possible cases for our eigenvalues cubic function using a discriminant (Δ_2).

$$\Delta_2 = -4 \cdot \psi_4^3 \cdot \psi_6 + \psi_4^2 \cdot \psi_5^2 - 4 \cdot \psi_5^3 + 18 \cdot \psi_4 \cdot \psi_5 \cdot \psi_6 - 27 \cdot \psi_6^2$$

The following cases need to be considered for all first fixed point eigenvalues options:

$\Delta_2 > 0 \rightarrow$ we have three distinct real eigenvalues for the first fixed point.
$\Delta_2 < 0 \rightarrow$ we have a real eigenvalue and a pair of complex conjugate roots.
$\Delta_2 = 0 \rightarrow$ at least two eigenvalues coincide. We get double real eigenvalue and another distinct single real eigenvalue. Alternatively all three roots coincide yielding a triple real eigenvalues.

We can decide between these subcases by computing the resultant of the cubic eigenvalue function and its second derivative. A triple root exists if and only if this resultant vanishes.

Eigenvalues turning points (TPs) for the second fixed point (Fig. 3.2):

Eigenvalues turning points (TPs):

$$\frac{dP_2(\lambda)}{d\lambda} = 3 \cdot \lambda^2 + \lambda \cdot 2 \cdot \psi_4 + \psi_5; \frac{dP_2(\lambda)}{d\lambda} = 0 \Rightarrow \lambda_{TP} = \frac{1}{3} \cdot (-\psi_4 \pm \sqrt{\psi_4^2 - 3 \cdot \psi_5})$$

Case I One eigenvalue turning point $\psi_4^2 - 3 \cdot \psi_5 = 0 \Rightarrow \lambda_{\text{TP}} = -\frac{\psi_4}{3} = -\frac{a \cdot (b-1)}{3} = \frac{a \cdot (1-b)}{3}$

Case II Two distinct complex eigenvalues turning points

$$\psi_4^2 - 3 \cdot \psi_5 < 0 \Rightarrow \lambda_{\text{TP}} = \frac{1}{3} \cdot \left(-\psi_4 \pm i \cdot \sqrt{|\psi_4^2 - 3 \cdot \psi_5|} \right)$$
$$= \frac{1}{3} \cdot \left\{ -a \cdot (b-1) \pm i \cdot \sqrt{\left| a^2 \cdot (b-1)^2 - 3 \cdot \left(1 + \frac{c}{a} - a^2 \cdot b \right) \right|} \right\}$$
$$\psi_4^2 - 3 \cdot \psi_5 < 0 \Rightarrow \lambda_{\text{TP}} = \frac{1}{3} \cdot \left\{ a \cdot (1-b) \pm i \cdot \sqrt{\left| a^2 \cdot b^2 + a^2 - 3 \cdot \left(1 + \frac{c}{a} \right) + a^2 \cdot b \right|} \right\}$$

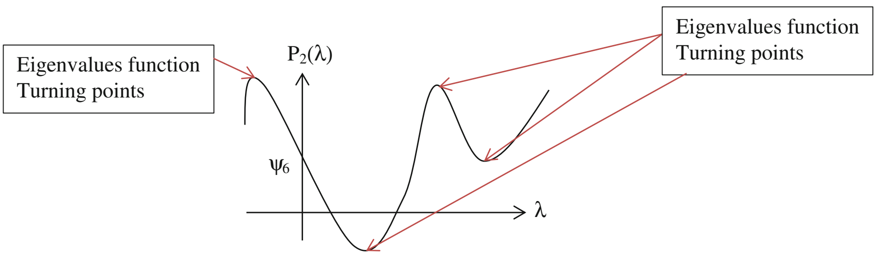

Fig. 3.2 Eigenvalues Turning Points (TPs) for the second fixed point

Case III Two distinct real eigenvalues turning points

$$\psi_4^2 - 3\cdot\psi_5 > 0 \Rightarrow \lambda_{\mathrm{TP}} = \frac{1}{3}\cdot\left(-\psi_4 \pm \sqrt{\psi_4^2 - 3\cdot\psi_5}\right)$$
$$= \frac{1}{3}\cdot\left\{-a\cdot(b-1) \pm \sqrt{a^2\cdot(b-1)^2 - 3\cdot\left(1+\frac{c}{a} - a^2\cdot b\right)}\right\}$$
$$\psi_4^2 - 3\cdot\psi_5 > 0 \Rightarrow \lambda_{\mathrm{TP}} = \frac{1}{3}\cdot\left\{a\cdot(1-b) \pm \sqrt{a^2\cdot b^2 + a^2 - 3\cdot(1+\frac{c}{a}) + a^2\cdot b}\right\}$$

Ψ_3 is the y-axis intersection value ($\lambda = 0$) of polynomial eigenvalue function $P_1(\lambda)$ for the first fixed point and we distinguish three sub cases:

$$\psi_3 = 0 \Rightarrow c - b\cdot a = 0 \Rightarrow c = b\cdot a;\ \psi_3 > 0 \Rightarrow c - b\cdot a > 0 \Rightarrow c > b\cdot a$$
$$\psi_3 < 0 \Rightarrow c - b\cdot a < 0 \Rightarrow c < b\cdot a$$

Ψ_6 is the y-axis intersection value ($\lambda = 0$) of polynomial eigenvalue function $P_2(\lambda)$ for the second fixed point and we distinguish three sub cases:

$$\psi_6 = 0 \Rightarrow a\cdot b - c = 0 \Rightarrow c = b\cdot a;\ \psi_6 > 0 \Rightarrow a\cdot b - c > 0 \Rightarrow c < b\cdot a$$
$$\psi_6 < 0 \Rightarrow a\cdot b - c < 0 \Rightarrow c > b\cdot a$$

We can summery our results in Table 3.1:

We need to check in which conditions our Rossler's prototype chaotic system has Fold-Hopf bifurcation. We define vector V as (X, Y, Z); $V = (X, Y, Z)$ and $\alpha = (a, b, c); \alpha = (a, b, c) \in \mathbb{R}^3, \frac{\mathrm{d}V}{\mathrm{d}t} = f(V, \alpha) \forall \alpha \in \mathbb{R}^{n=3}$. We check for the first fixed point $V^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$ the Jacobian matrix (A) and in which parameters conditions Jacobian matrix A has a zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\cdot\omega$ with $\omega > 0$. For the first fixed point $\lambda^3 + \lambda^2\cdot\psi_1 + \lambda\cdot\psi_2 + \psi_3 = 0; P_1(\lambda) = \lambda^3 + \lambda^2\cdot\psi_1 + \lambda\cdot\psi_2 + \psi_3$ and the condition for $P_1(\lambda = \lambda_1 = 0) = 0 \Rightarrow \psi_3 = 0 \Rightarrow c = b\cdot a$. We get the condition $\lambda_1 = 0 \Rightarrow \psi_3 = 0 \Rightarrow c = b\cdot a$. $P_1(\lambda)|_{\psi_3=0} = \lambda^3 + \lambda^2\cdot\psi_1 + \lambda\cdot\psi_2 = \lambda\cdot[\lambda^2 + \lambda\cdot\psi_1 + \psi_2]$.

The *first case* we choose $\lambda = \lambda_2 = i\cdot\omega$; $P_1(\lambda)|_{\psi_3=0,\lambda=i\cdot\omega} = i\cdot\omega\cdot[-\omega^2 + i\cdot\omega\cdot\psi_1 + \psi_2]$ $P_1(\lambda = \lambda_2)|_{\psi_3=0,\lambda=\lambda_2=i\cdot\omega} = 0 \Rightarrow i\cdot\omega\cdot[-\omega^2 + i\cdot\omega\cdot\psi_1 + \psi_2] = 0$;

Option I: $\omega = 0$

Option II: $-\omega^2 + i\cdot\omega\cdot\psi_1 + \psi_2 = 0 \Rightarrow [\psi_2 - \omega^2] + i\cdot\omega\cdot\psi_1 = 0 \Rightarrow \psi_2 - \omega^2 = 0 \,\&\, \omega\cdot\psi_1 = 0$.

$$\omega = \pm\sqrt{\psi_2} \,\&\, \omega\cdot\psi_1 = 0.$$

Option II.1: $\omega = 0$ & $\omega = \pm\sqrt{\psi_2} = \pm\sqrt{1 + b - a\cdot c} \Rightarrow 1 + b - a\cdot c = 0 \Rightarrow a\cdot c - b = 1$

Option II.2: $\omega = \pm\sqrt{\psi_2} \,\&\, \psi_1 = 0 \Rightarrow \omega = \pm\sqrt{\psi_2} = \pm\sqrt{1 + b - a^2} \,\&\, c = a$

Table 3.1 Fixed points

Fixed points	First fixed point $(X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$	Second fixed point $(X^{(1)}, Y^{(1)}, Z^{(1)}) = \left(c - a \cdot b, b - \frac{c}{a}, \frac{c}{a} - b\right)$
Eigenvalues characteristic equation	$\lambda^3 + \lambda^2 \cdot \psi_1 + \lambda \cdot \psi_2 + \psi_3 = 0$ $P_1(\lambda) = \lambda^3 + \lambda^2 \cdot \psi_1 + \lambda \cdot \psi_2 + \psi_3$ $\psi_1 = c - a; \psi_2 = 1 + b - a \cdot c$ $\psi_3 = c - b \cdot a$	$\lambda^3 + \lambda^2 \cdot \psi_4 + \lambda \cdot \psi_5 + \psi_6 = 0$ $P_2(\lambda) = \lambda^3 + \lambda^2 \cdot \psi_4 + \lambda \cdot \psi_5 + \psi_6$ $\psi_4 = a \cdot (b - 1); \psi_5 = 1 + \frac{c}{a} - a^2 \cdot b$ $\psi_6 = a \cdot b - c$
Eigenvalues cubic function using a discriminant	$\Delta_1 = -4 \cdot \psi_1^3 \cdot \psi_3 + \psi_1^2 \cdot \psi_2^2 - 4 \cdot \psi_2^3 + 18 \cdot \psi_1 \cdot \psi_2 \cdot \psi_3 - 27 \cdot \psi_3^2$	$\Delta_2 = -4 \cdot \psi_4^3 \cdot \psi_6 + \psi_4^2 \cdot \psi_5^2 - 4 \cdot \psi_5^3 + 18 \cdot \psi_4 \cdot \psi_5 \cdot \psi_6 - 27 \cdot \psi_6^2$
Eigenvalues Turning Points (TPs)	$\frac{dP_1(\lambda)}{d\lambda} = 3 \cdot \lambda^2 + \lambda \cdot 2 \cdot \psi_1 + \psi_2$ $\frac{dP_1(\lambda)}{d\lambda} = 0 \Rightarrow$ $\lambda_{TP} = \frac{1}{3} \cdot (-\psi_1 \pm \sqrt{\psi_1^2 - 3 \cdot \psi_2})$ *Case I* one eigenvalue turning point. $\psi_1^2 - 3 \cdot \psi_2 = 0 \Rightarrow$ $\lambda_{TP} = -\frac{\psi_1}{3} = -\frac{(c-a)}{3} = \frac{(a-c)}{3}$ *Case II* Two distinct complex eigenvalues turning points. $\psi_1^2 - 3 \cdot \psi_2 < 0 \Rightarrow$ $\lambda_{TP} = \frac{1}{3} \cdot \{a - c$ $\pm i \cdot \sqrt{\lvert c^2 + a \cdot c + a^2 - 3 \cdot (1 + b) \rvert}\}$ *Case III* Two distinct real eigenvalues turning points $\psi_1^2 - 3 \cdot \psi_2 > 0 \Rightarrow$ $\lambda_{TP} = \frac{1}{3} \cdot \{a - c$ $\pm \sqrt{c^2 + a \cdot c + a^2 - 3 \cdot (1 + b)}\}$	$\frac{dP_2(\lambda)}{d\lambda} = 3 \cdot \lambda^2 + \lambda \cdot 2 \cdot \psi_4 + \psi_5$ $\frac{dP_2(\lambda)}{d\lambda} = 0 \Rightarrow$ $\lambda_{TP} = \frac{1}{3} \cdot (-\psi_4 \pm \sqrt{\psi_4^2 - 3 \cdot \psi_5})$ *Case I* one eigenvalue turning point. $\psi_4^2 - 3 \cdot \psi_5 = 0 \Rightarrow$ $\lambda_{TP} = -\frac{\psi_4}{3} = -\frac{a \cdot (b-1)}{3} = \frac{a \cdot (1-b)}{3}$ *Case II* Two distinct complex eigenvalues turning points $\psi_4^2 - 3 \cdot \psi_5 < 0 \Rightarrow$ $\lambda_{TP} = \frac{1}{3} \cdot \{a \cdot (1 - b)$ $\pm i \cdot \sqrt{\lvert a^2 \cdot b^2 + a^2 - 3 \cdot (1 + \frac{c}{a}) + a^2 \cdot b \rvert}\}$ *Case III* Two distinct real eigenvalues turning points. $\psi_4^2 - 3 \cdot \psi_5 > 0 \Rightarrow$ $\lambda_{TP} = \frac{1}{3} \cdot \{a \cdot (1 - b)$ $\pm \sqrt{a^2 \cdot b^2 + a^2 - 3 \cdot (1 + \frac{c}{a}) + a^2 \cdot b}\}$

$\omega|_{\omega>0} = +\sqrt{\psi_2} = +\sqrt{1+b-a^2}\,\&\,1+b-a^2>0 \Rightarrow a^2-b<1;\ c=a\ \&$ $c = b \cdot a \rightarrow a = 0$ or $b = 1$. $\lambda = \lambda_2 = i\cdot\omega \Rightarrow \lambda_2' = 0; \lambda_2'' = i\cdot\sqrt{1+b-a^2}$. Since we ask $\lambda_2 = i\cdot\omega|_{\omega>0}$ then $\lambda_2 = i\cdot\omega|_{\omega>0} = i\cdot\sqrt{1+b-a^2}$.

The *second case* we choose $\lambda = \lambda_3 = -i\cdot\omega$; $P_1(\lambda)|_{\psi_3=0,\lambda=-i\cdot\omega} = -i\cdot\omega\cdot[-\omega^2 - i\cdot\omega\cdot\psi_1 + \psi_2]$;
$P_1(\lambda=\lambda_3)|_{\psi_3=0,\lambda=\lambda_3=-i\cdot\omega} = 0 \Rightarrow -i\cdot\omega\cdot[-\omega^2 - i\cdot\omega\cdot\psi_1+\psi_2] = 0$;

Option I: $\omega = 0$.

Option II: $-\omega^2 - i\cdot\omega\cdot\psi_1 + \psi_2 = 0 \Rightarrow [\psi_2 - \omega^2] - i\cdot\omega\cdot\psi_1 \Rightarrow \psi_2 - \omega^2 = 0$, $\&\ \omega\cdot\psi_1 = 0$

$$\omega = \pm\sqrt{\psi_2}\,\&\,\omega\cdot\psi_1 = 0.$$

Option II.1: $\omega = 0\ \&\ \omega = \pm\sqrt{\psi_2} = \pm\sqrt{1+b-a\cdot c} \Rightarrow 1+b-a\cdot c = 0 \Rightarrow a\cdot c - b = 1$.

Option II.2: $\omega = \pm\sqrt{\psi_2}\,\&\,\psi_1 = 0 \Rightarrow \omega = \pm\sqrt{\psi_2} = \pm\sqrt{1+b-a^2}\,\&\,c = a$

$\omega|_{\omega>0} = +\sqrt{\psi_2} = +\sqrt{1+b-a^2}\,\&\,1+b-a^2>0 \Rightarrow a^2-b<1;\ c=a\ \&$ $c = b\cdot a \rightarrow a = 0$ or $b = 1$. $\lambda = \lambda_3 = -i\cdot\omega \Rightarrow \lambda_3' = 0; \lambda_3'' = -i\cdot\sqrt{1+b-a^2}$ since we ask $\lambda_3 = -i\cdot\omega|_{\omega>0}$ then $\lambda_3 = -i\cdot\omega|_{\omega>0} = -i\cdot\sqrt{1+b-a^2}$.

Results We got the expressions for Rossler's prototype chaotic system Fold-Hopf bifurcation eigenvalues first fixed point. It is a reader exercise to check for our system second fixed point and in which conditions [9, 45, 52].

3.2 Optoisolation Circuits Fold-Hopf Bifurcation

We need to implement our Rossler's prototype chaotic system by using optoisolation circuits. In the previous Sect. 3.1 we represent our system by one high order nonlinear differential equation with one variable (Y). The optoisolation circuits must fulfill this differential equation (**) [5, 6].

$$(**)\quad \begin{aligned} &\{-b\cdot a + c\}\cdot Y + \{b - c\cdot a + 1\}\cdot \frac{dY}{dt} + (c-a)\cdot\frac{d^2Y}{dt^2} \\ &+ a\cdot\left[\frac{dY}{dt}\right]^2 - \frac{dY}{dt}\cdot\frac{d^2Y}{dt^2} + a\cdot Y^2 - \frac{dY}{dt}\cdot Y\cdot\{a^2+1\} + a\cdot Y\cdot\frac{d^2Y}{dt^2} + \frac{d^3Y}{dt^3} = 0 \end{aligned}$$

We define new parameters for our above differential equation m_1, … m_9.

$$(**)\quad \begin{aligned} &m_1\cdot Y + m_2\cdot\frac{dY}{dt} + m_3\cdot\frac{d^2Y}{dt^2} + m_4\cdot\left[\frac{dY}{dt}\right]^2 + m_5\cdot\frac{dY}{dt}\cdot\frac{d^2Y}{dt^2} + m_6\cdot Y^2 + m_7\frac{dY}{dt}\cdot Y \\ &+ m_8\cdot Y\cdot\frac{d^2Y}{dt^2} + m_9\cdot\frac{d^3Y}{dt^3} = 0 \\ &m_1 = -b\cdot a + c;\ m_2 = b - c\cdot a + 1;\ m_3 = c - a;\ m_4 = a;\ m_5 = -1 \\ &m_6 = a;\ m_7 = -\{a^2+1\};\ m_8 = a;\ m_9 = 1 \end{aligned}$$

The next block diagram demonstrates our system high order and nonlinear differential equation.

$$m_2 = \delta_1;\ m_3 = \delta_2;\ m_4 = \delta_1^2 \cdot \delta_8;\ m_5 = \delta_1 \cdot \delta_2 \cdot \delta_4;\ m_6 = \delta_5;$$
$$m_7 = \delta_1 \cdot \delta_6;\ m_8 = \delta_2 \cdot \delta_7;\ m_9 = \delta_3$$

$$\delta_1 = m_2 = b - c \cdot a + 1; \delta_2 = m_3 = c - a; \delta_3 = m_9 = 1; \delta_4 = \frac{m_5}{m_2 \cdot m_3} = \frac{-1}{(b - c \cdot a + 1) \cdot (c - a)}$$

$$\delta_5 = m_6 = a; \delta_6 = \frac{m_7}{\delta_1} = \frac{m_7}{m_2} = \frac{-\{a^2 + 1\}}{b - c \cdot a + 1}; \delta_7 = \frac{m_8}{\delta_2} = \frac{m_8}{m_3} = \frac{a}{c - a}; \delta_8 = \frac{m_4}{\delta_1^2} = \frac{m_4}{m_2^2} = \frac{a}{(b - c \cdot a + 1)^2}$$

We can summarize our last results in Table 3.2 (Fig. 3.3):

The below optoisolation circuit implements our Fold-Hopf bifurcation system.

We consider $R_{\text{in}1} \gg R_{\text{f}1}$ then $R_{\text{f}1}/R_{\text{in}1} \to \varepsilon$ and our Op-amp transfer function is one. Photo transistor is coupled with two LEDs D1 and D2.

Remark It is reader exercise to implement all derivative elements (d/d*t*), multiplication elements ([] · []) and square element ($[]^2$) by using Op-Amps, resistors, capacitors, diodes, etc. [16, 25, 26] (Fig. 3.4).

$$I_{D1} = I_{R_{11}} + I_{R_{12}} + I_{R_{13}} + I_{R_{14}} = \sum_{j=1}^{4} I_{R_{1j}};\ I_{D2} = I_{R_{21}} + I_{R_{22}} + I_{R_{23}} + I_{R_{24}} = \sum_{j=1}^{4} I_{R_{2j}}$$

We consider $V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D1}}{I_0}$; $V_{D2} = V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D2}}{I_0}$ (Taylor series approximation). $I_{BQ1} = k_1 \cdot I_{D1} + k_2 \cdot I_{D2}$; $I_{R_{11}} = \frac{V - V_{D1}}{R_{11}}$; $I_{R_{12}} = \frac{\dddot{V} - V_{D1}}{R_{12}}$

$$I_{R_{13}} = \frac{\ddot{V} - V_{D1}}{R_{13}};\ I_{R_{14}} = \frac{V \cdot \ddot{V} - V_{D1}}{R_{14}};\ I_{R_{21}} = \frac{V \cdot \dot{V} - V_{D2}}{R_{21}};\ I_{R_{22}} = \frac{\ddot{V} \cdot \dot{V} - V_{D2}}{R_{22}}$$

Table 3.2 Summary last results ($\delta_i \forall\, i = 1, 2, \cdots, 8$ expressions)

δ_1	$b - c \cdot a + 1$
δ_2	$c - a$
δ_3	1
δ_4	$\frac{-1}{(b - c \cdot a + 1) \cdot (c - a)}$
δ_5	a
δ_6	$\frac{-\{a^2 + 1\}}{b - c \cdot a + 1}$
δ_7	$\frac{a}{c - a}$
δ_8	$\frac{a}{(b - c \cdot a + 1)^2}$

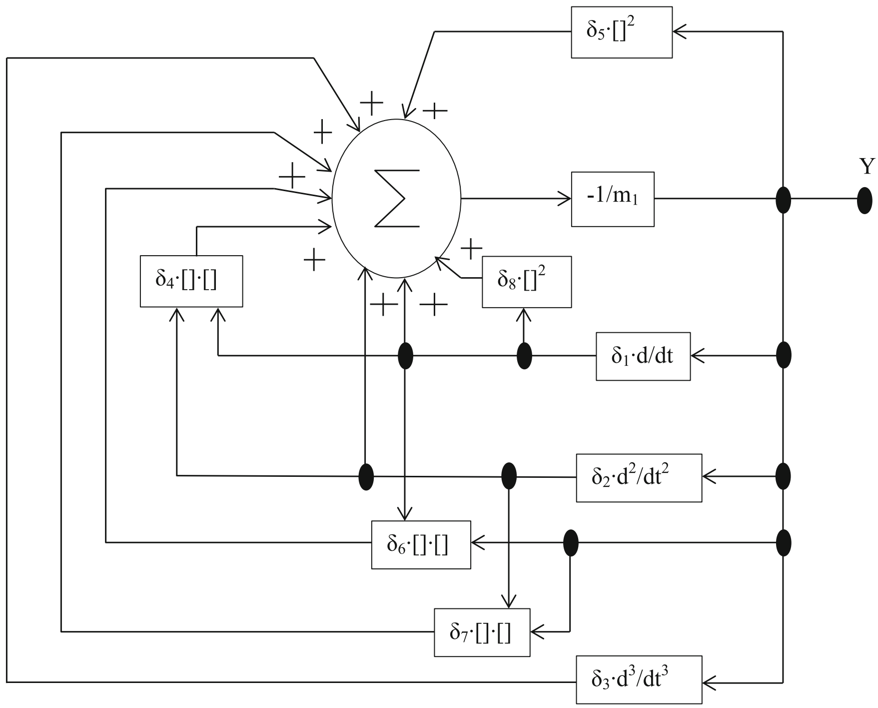

Fig. 3.3 Our system high order and nonlinear differential equation

$$I_{R_{23}} = \frac{\dot{V} - V_{D2}}{R_{23}};\ I_{R_{24}} = \frac{[\dot{V}]^2 - V_{D2}}{R_{24}};\ I_{D1} = \sum_{j=1}^{4} I_{R_{1j}}$$
$$= \frac{V - V_{D1}}{R_{11}} + \frac{\dddot{V} - V_{D1}}{R_{12}} + \frac{\ddot{V} - V_{D1}}{R_{13}} + \frac{V \cdot \ddot{V} - V_{D1}}{R_{14}}$$

$$I_{D2} = \sum_{j=1}^{4} I_{R_{2j}} = \frac{V \cdot \dot{V} - V_{D2}}{R_{21}} + \frac{\ddot{V} \cdot \dot{V} - V_{D2}}{R_{22}} + \frac{\dot{V} - V_{D2}}{R_{23}} + \frac{[\dot{V}]^2 - V_{D2}}{R_{24}}$$

$$I_{D1} = \sum_{j=1}^{4} I_{R_{1j}} = \frac{V}{R_{11}} + \frac{\dddot{V}}{R_{12}} + \frac{\ddot{V}}{R_{13}} + \frac{V \cdot \ddot{V}}{R_{14}} - V_{D1} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}} \right)$$
$$= \frac{V}{R_{11}} + \frac{\dddot{V}}{R_{12}} + \frac{\ddot{V}}{R_{13}} + \frac{V \cdot \ddot{V}}{R_{14}} - V_t \cdot \frac{I_{D1}}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}} \right)$$

$$I_{D2} = \sum_{j=1}^{4} I_{R_{2j}} = \frac{V \cdot \dot{V}}{R_{21}} + \frac{\ddot{V} \cdot \dot{V}}{R_{22}} + \frac{\dot{V}}{R_{23}} + \frac{[\dot{V}]^2}{R_{24}} - V_{D2} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}} \right)$$
$$= \frac{V \cdot \dot{V}}{R_{21}} + \frac{\ddot{V} \cdot \dot{V}}{R_{22}} + \frac{\dot{V}}{R_{23}} + \frac{[\dot{V}]^2}{R_{24}} - V_t \cdot \frac{I_{D2}}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}} \right)$$

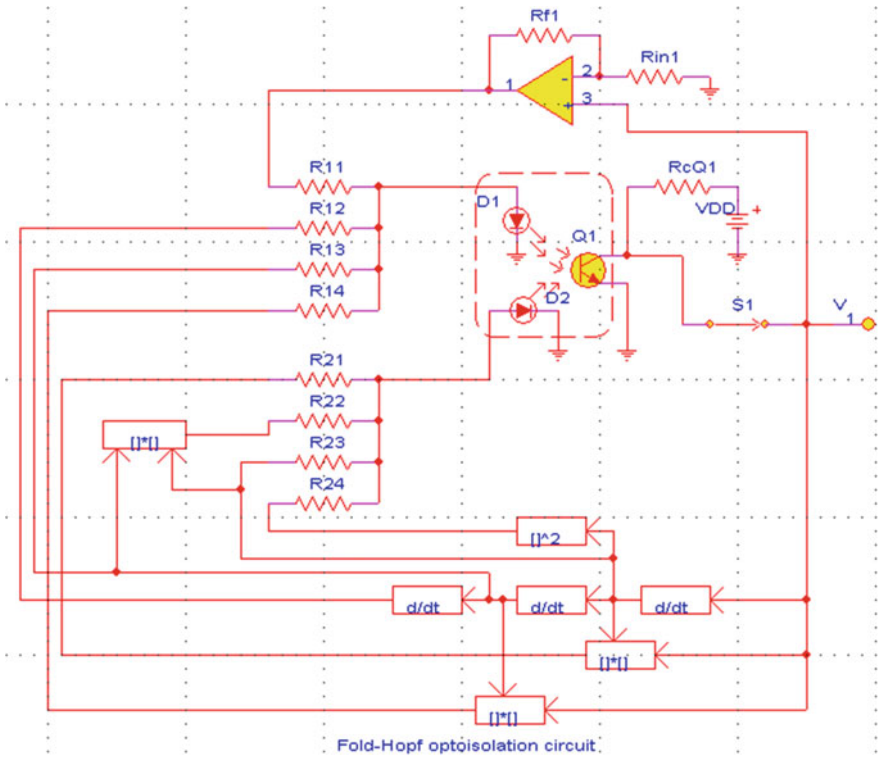

Fig. 3.4 Fold-Hopf optoisolation circuit

$$I_{D1}\cdot\left[1+V_t\cdot\frac{1}{I_0}\cdot\left(\sum_{j=1}^{4}\frac{1}{R_{1j}}\right)\right]=\frac{V}{R_{11}}+\frac{\dddot{V}}{R_{12}}+\frac{\ddot{V}}{R_{13}}+\frac{V\cdot\ddot{V}}{R_{14}};$$
$$I_{D2}\cdot\left[1+V_t\cdot\frac{1}{I_0}\cdot\left(\sum_{j=1}^{4}\frac{1}{R_{2j}}\right)\right]=\frac{V\cdot\dot{V}}{R_{21}}+\frac{\ddot{V}\cdot\dot{V}}{R_{22}}+\frac{\dot{V}}{R_{23}}+\frac{[\dot{V}]^2}{R_{24}}$$

$$I_{D1}=\frac{1}{\left[1+V_t\cdot\frac{1}{I_0}\cdot\left(\sum_{j=1}^{4}\frac{1}{R_{1j}}\right)\right]}\cdot\left\{\frac{V}{R_{11}}+\frac{\dddot{V}}{R_{12}}+\frac{\ddot{V}}{R_{13}}+\frac{V\cdot\ddot{V}}{R_{14}}\right\};$$
$$I_{D2}=\frac{1}{\left[1+V_t\cdot\frac{1}{I_0}\cdot\left(\sum_{j=1}^{4}\frac{1}{R_{2j}}\right)\right]}\cdot\left\{\frac{V\cdot\dot{V}}{R_{21}}+\frac{\ddot{V}\cdot\dot{V}}{R_{22}}+\frac{\dot{V}}{R_{23}}+\frac{[\dot{V}]^2}{R_{24}}\right\}$$

For simplicity we define

$$\eta_1 = \frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]}; \eta_2 = \frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right]}; \quad \eta_1 > 0; \eta_2 > 0$$

$$\psi_1(V, \ddot{V}, \dddot{V}) = \frac{V}{R_{11}} + \frac{\dddot{V}}{R_{12}} + \frac{\ddot{V}}{R_{13}} + \frac{V \cdot \ddot{V}}{R_{14}};\ \psi_2(V, \dot{V}, \ddot{V}) = \frac{V \cdot \dot{V}}{R_{21}} + \frac{\ddot{V} \cdot \dot{V}}{R_{22}} + \frac{\dot{V}}{R_{23}} + \frac{[\dot{V}]^2}{R_{24}}$$

$$\psi_1 = \psi_1(V, \ddot{V}, \dddot{V}); \psi_2 = \psi_2(V, \dot{V}, \ddot{V}); I_{D1} = \eta_1 \cdot \psi_1; I_{D2} = \eta_2 \cdot \psi_2$$

$$I_{BQ1} = k_1 \cdot I_{D1} + k_2 \cdot I_{D2} = k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2$$

$$I_{BQ1} = k_1 \cdot I_{D1} + k_2 \cdot I_{D2} = k_1 \cdot \left\{ \frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]} \right\} \cdot \left[\frac{V}{R_{11}} + \frac{\dddot{V}}{R_{12}} + \frac{\ddot{V}}{R_{13}} + \frac{V \cdot \ddot{V}}{R_{14}} \right] + k_1 \cdot \left\{ \frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right]} \right\} \cdot \left[\frac{V \cdot \dot{V}}{R_{21}} + \frac{\ddot{V} \cdot \dot{V}}{R_{22}} + \frac{\dot{V}}{R_{23}} + \frac{[\dot{V}]^2}{R_{24}} \right]$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1};\ I_{BQ1} = k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2;$$
$$I_{EQ1} = k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ1}$$

The Mathematical analysis is based on the basic Transistor Ebers–Moll equations. We need to implement the Regular Ebers–Moll Model to the above Opto Coupler circuit.

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right];$$
$$V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{Isc \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = V_{CBQ1} + V_{BEQ1},\ \text{but}\ V_{CBQ1} = -V_{BCQ1},\ \text{then}\ V_{CEQ1} = V_{BEQ1} - V_{BCQ1}$$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{Isc \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{Isc}{Ise}\right)$$

Since $Isc - Ise \to \varepsilon \Rightarrow \ln\left(\frac{Isc}{Ise}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$\alpha r \cdot I_{CQ1} - I_{EQ1} = \alpha r \cdot I_{CQ1} - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] - I_{CQ1}$$
$$= I_{CQ1} \cdot (\alpha r - 1) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2]$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f = I_{CQ1} - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f - I_{CQ1} \cdot \alpha f$$
$$= I_{CQ1} \cdot (1 - \alpha f) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f$$

$$V_{DD} = I_{CQ1} \cdot R_{CQ1} + V \Rightarrow I_{CQ1} = \frac{V_{DD} - V}{R_{CQ1}}$$

$$V = V_{CEQ1}$$
$$\simeq Vt \cdot \ln\left[\frac{I_{CQ1} \cdot (\alpha r - 1) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + Ise \cdot (\alpha r \cdot \alpha f - 1)}{I_{CQ1} \cdot (1 - \alpha f) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$V = V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\left[\frac{V_{DD}-V}{R_{CQ1}}\right] \cdot (\alpha r - 1) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + Ise \cdot (\alpha r \cdot \alpha f - 1)}{\left[\frac{V_{DD}-V}{R_{CQ1}}\right] \cdot (1 - \alpha f) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$V = V_{CEQ1} \simeq Vt$$
$$\cdot \ln\left[\frac{-\frac{V}{R_{CQ1}} \cdot (\alpha r - 1) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + \frac{V_{DD}}{R_{CQ1}} \cdot (\alpha r - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{-\frac{V}{R_{CQ1}} \cdot (1 - \alpha f) - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + \frac{V_{DD}}{R_{CQ1}} \cdot (1 - \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

For simplicity we define new system global parameters: ξ_1, ξ_2, ξ_3, ξ_4.

$$\xi_1 = -\frac{(\alpha r - 1)}{R_{CQ1}};\ \xi_2 = \frac{V_{DD}}{R_{CQ1}} \cdot (\alpha r - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1);\ \xi_3 = -\frac{(1 - \alpha f)}{R_{CQ1}}$$

$$\xi_4 = \frac{V_{DD}}{R_{CQ1}} \cdot (1 - \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1);\ V = V_{CEQ1}$$
$$\simeq Vt \cdot \ln\left[\frac{V \cdot \xi_1 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + \xi_2}{V \cdot \xi_3 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + \xi_4}\right]$$

$e^{\left[\frac{V}{V_t}\right]} = \frac{V \cdot \xi_1 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + \xi_2}{V \cdot \xi_3 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + \xi_4}$; $e^{\left[\frac{V}{V_t}\right]} \approx \frac{V}{V_t} + 1$ (Taylor series approximation).

$$\frac{V}{V_t} + 1 = \frac{V \cdot \xi_1 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + \xi_2}{V \cdot \xi_3 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + \xi_4}$$

$$\left[\frac{V}{V_t} + 1\right] \cdot [V \cdot \xi_3 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] \cdot \alpha f + \xi_4]$$
$$= V \cdot \xi_1 - [k_1 \cdot \eta_1 \cdot \psi_1 + k_2 \cdot \eta_2 \cdot \psi_2] + \xi_2$$

$$\left[\frac{V}{V_t}+1\right]\cdot[V\cdot\xi_3-k_1\cdot\eta_1\cdot\psi_1\cdot\alpha f-k_2\cdot\eta_2\cdot\psi_2\cdot\alpha f+\xi_4]$$
$$=V\cdot\xi_1-k_1\cdot\eta_1\cdot\psi_1-k_2\cdot\eta_2\cdot\psi_2+\xi_2$$

$$\frac{V^2}{V_t}\cdot\xi_3-\frac{V}{V_t}\cdot k_1\cdot\eta_1\cdot\psi_1\cdot\alpha f-\frac{V}{V_t}\cdot k_2\cdot\eta_2\cdot\psi_2\cdot\alpha f+\frac{V}{V_t}\cdot\xi_4+V\cdot\xi_3$$
$$-k_1\cdot\eta_1\cdot\psi_1\cdot\alpha f-k_2\cdot\eta_2\cdot\psi_2\cdot\alpha f+\xi_4$$
$$=V\cdot\xi_1-k_1\cdot\eta_1\cdot\psi_1-k_2\cdot\eta_2\cdot\psi_2+\xi_2$$

$$\frac{V^2}{V_t}\cdot\xi_3-\frac{V}{V_t}\cdot k_1\cdot\eta_1\cdot\left[\frac{V}{R_{11}}+\frac{\dddot{V}}{R_{12}}+\frac{\ddot{V}}{R_{13}}+\frac{V\cdot\ddot{V}}{R_{14}}\right]\cdot\alpha f$$
$$-\frac{V}{V_t}\cdot k_2\cdot\eta_2\cdot\left[\frac{V\cdot\dot{V}}{R_{21}}+\frac{\ddot{V}\cdot\dot{V}}{R_{22}}+\frac{\dot{V}}{R_{23}}+\frac{[\dot{V}]^2}{R_{24}}\right]\cdot\alpha f$$
$$+\frac{V}{V_t}\cdot\xi_4+V\cdot\xi_3-k_1\cdot\eta_1\cdot\left[\frac{V}{R_{11}}+\frac{\dddot{V}}{R_{12}}+\frac{\ddot{V}}{R_{13}}+\frac{V\cdot\ddot{V}}{R_{14}}\right]\cdot\alpha f$$
$$-k_2\cdot\eta_2\cdot\left[\frac{V\cdot\dot{V}}{R_{21}}+\frac{\ddot{V}\cdot\dot{V}}{R_{22}}+\frac{\dot{V}}{R_{23}}+\frac{[\dot{V}]^2}{R_{24}}\right]\cdot\alpha f+\xi_4$$
$$=V\cdot\xi_1-k_1\cdot\eta_1\cdot\left[\frac{V}{R_{11}}+\frac{\dddot{V}}{R_{12}}+\frac{\ddot{V}}{R_{13}}+\frac{V\cdot\ddot{V}}{R_{14}}\right]$$
$$-k_2\cdot\eta_2\cdot\left[\frac{V\cdot\dot{V}}{R_{21}}+\frac{\ddot{V}\cdot\dot{V}}{R_{22}}+\frac{\dot{V}}{R_{23}}+\frac{[\dot{V}]^2}{R_{24}}\right]+\xi_2$$

$$\frac{V^2}{V_t}\cdot\xi_3-V^2\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{11}}-V\cdot\dddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{12}}-V\cdot\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{13}}$$
$$-V^2\cdot\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{14}}-V^2\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{21}}-V\cdot\ddot{V}\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{22}}$$
$$-V\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{23}}-V\cdot[\dot{V}]^2\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{24}}+\frac{V}{V_t}\cdot\xi_4+V\cdot\xi_3$$
$$-\frac{V\cdot k_1\cdot\eta_1\cdot\alpha f}{R_{11}}-\frac{\dddot{V}\cdot k_1\cdot\eta_1\cdot\alpha f}{R_{12}}-\frac{\ddot{V}\cdot k_1\cdot\eta_1\cdot\alpha f}{R_{13}}-\frac{V\cdot\ddot{V}\cdot k_1\cdot\eta_1\cdot\alpha f}{R_{14}}$$
$$-\frac{V\cdot\dot{V}\cdot k_2\cdot\eta_2\cdot\alpha f}{R_{21}}-\frac{\ddot{V}\cdot\dot{V}\cdot k_2\cdot\eta_2\cdot\alpha f}{R_{22}}-\frac{\dot{V}\cdot k_2\cdot\eta_2\cdot\alpha f}{R_{23}}-\frac{[\dot{V}]^2\cdot k_2\cdot\eta_2\cdot\alpha f}{R_{24}}+\xi_4$$
$$=V\cdot\xi_1-\frac{V\cdot k_1\cdot\eta_1}{R_{11}}-\frac{\dddot{V}\cdot k_1\cdot\eta_1}{R_{12}}-\frac{\ddot{V}\cdot k_1\cdot\eta_1}{R_{13}}-\frac{V\cdot\ddot{V}\cdot k_1\cdot\eta_1}{R_{14}}-\frac{k_2\cdot\eta_2\cdot V\cdot\dot{V}}{R_{21}}$$
$$-\frac{k_2\cdot\eta_2\cdot\ddot{V}\cdot\dot{V}}{R_{22}}-\frac{k_2\cdot\eta_2\cdot\dot{V}}{R_{23}}-\frac{k_2\cdot\eta_2\cdot[\dot{V}]^2}{R_{24}}+\xi_2$$

$$\frac{V^2}{V_t}\cdot\left(\xi_3-\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{11}}\right)-V\cdot\dddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{12}}-V\cdot\ddot{V}\cdot\left(\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{13}}+\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{14}}-\frac{k_1\cdot\eta_1}{R_{14}}\right)$$
$$-V^2\cdot\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{14}}-V^2\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{21}}-V\cdot\ddot{V}\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{22}}$$
$$-V\cdot\dot{V}\cdot\left(\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{23}}+\frac{k_2\cdot\eta_2\cdot\alpha f}{R_{21}}-\frac{k_2\cdot\eta_2}{R_{21}}\right)-V\cdot[\dot{V}]^2\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{24}}$$
$$+V\cdot\left(\frac{\xi_4}{V_t}+\xi_3-\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{11}}-\xi_1+\frac{k_1\cdot\eta_1}{R_{11}}\right)-\dddot{V}\cdot\left(\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{12}}-\frac{k_1\cdot\eta_1}{R_{12}}\right)-\ddot{V}\cdot\left(\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{13}}-\frac{k_1\cdot\eta_1}{R_{13}}\right)$$
$$-\ddot{V}\cdot\dot{V}\cdot\left(\frac{k_2\cdot\eta_2\cdot\alpha f}{R_{22}}-\frac{k_2\cdot\eta_2}{R_{22}}\right)-\dot{V}\cdot\left(\frac{k_2\cdot\eta_2\cdot\alpha f}{R_{23}}-\frac{k_2\cdot\eta_2}{R_{23}}\right)-[\dot{V}]^2\cdot\left(\frac{k_2\cdot\eta_2\cdot\alpha f}{R_{24}}-\frac{k_2\cdot\eta_2}{R_{24}}\right)$$
$$+\xi_4-\xi_2=0$$

$$\frac{V^2}{V_t}\cdot\left(\xi_3-\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{11}}\right)-V\cdot\dddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{12}}-V\cdot\ddot{V}\cdot k_1\cdot\eta_1\cdot\left(\frac{\alpha f}{V_t\cdot R_{13}}+\frac{(\alpha f-1)}{R_{14}}\right)$$
$$-V^2\cdot\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{14}}-V^2\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{21}}-V\cdot\ddot{V}\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{22}}$$
$$-V\cdot\dot{V}\cdot k_2\cdot\eta_2\cdot\left(\frac{\alpha f}{V_t\cdot R_{23}}+\frac{(\alpha f-1)}{R_{21}}\right)-V\cdot[\dot{V}]^2\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{24}}$$
$$+V\cdot\left(\frac{\xi_4}{V_t}+\xi_3-\xi_1+\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{11}}\right)-\dddot{V}\cdot\frac{k_1\cdot\eta_1\cdot(\alpha f-1)}{R_{12}}-\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot(\alpha f-1)}{R_{13}}$$
$$-\ddot{V}\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot(\alpha f-1)}{R_{22}}-\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot(\alpha f-1)}{R_{23}}-[\dot{V}]^2\cdot\frac{k_2\cdot\eta_2\cdot(\alpha f-1)}{R_{24}}+\xi_4-\xi_2=0$$

$$\frac{V^2}{V_t}\cdot\left(\xi_3-\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{11}}\right)-V\cdot\dddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{12}}+V\cdot\ddot{V}\cdot k_1\cdot\eta_1\cdot\left(-\frac{\alpha f}{V_t\cdot R_{13}}+\frac{(1-\alpha f)}{R_{14}}\right)$$
$$-V^2\cdot\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{14}}-V^2\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{21}}-V\cdot\ddot{V}\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{22}}$$
$$+V\cdot\dot{V}\cdot k_2\cdot\eta_2\cdot\left(-\frac{\alpha f}{V_t\cdot R_{23}}+\frac{(1-\alpha f)}{R_{21}}\right)-V\cdot[\dot{V}]^2\cdot\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{24}}$$
$$+V\cdot\left(\frac{\xi_4}{V_t}+\xi_3-\xi_1+\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{11}}\right)+\dddot{V}\cdot\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{12}}+\ddot{V}\cdot\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{13}}$$
$$+\ddot{V}\cdot\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot(1-\alpha f)}{R_{22}}+\dot{V}\cdot\frac{k_2\cdot\eta_2\cdot(1-\alpha f)}{R_{23}}+[\dot{V}]^2\cdot\frac{k_2\cdot\eta_2\cdot(1-\alpha f)}{R_{24}}+\xi_4-\xi_2=0$$

For simplicity we define the following global parameters:

$$\phi_1=\frac{1}{V_t}\cdot\left(\xi_3-\frac{k_1\cdot\eta_1\cdot\alpha f}{R_{11}}\right);\ \phi_2=-\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{12}};\ \phi_3=k_1\cdot\eta_1\cdot\left(-\frac{\alpha f}{V_t\cdot R_{13}}+\frac{(1-\alpha f)}{R_{14}}\right)$$
$$\phi_4=-\frac{k_1\cdot\eta_1\cdot\alpha f}{V_t\cdot R_{14}};\ \phi_5=-\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{21}};\ \phi_6=-\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{22}};\ \phi_7=k_2\cdot\eta_2\cdot\left(-\frac{\alpha f}{V_t\cdot R_{23}}+\frac{(1-\alpha f)}{R_{21}}\right)$$
$$\phi_8=-\frac{k_2\cdot\eta_2\cdot\alpha f}{V_t\cdot R_{24}};\ \phi_9=\frac{\xi_4}{V_t}+\xi_3-\xi_1+\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{11}};\ \phi_{10}=\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{12}}$$
$$\phi_{11}=\frac{k_1\cdot\eta_1\cdot(1-\alpha f)}{R_{13}};\ \phi_{12}=\frac{k_2\cdot\eta_2\cdot(1-\alpha f)}{R_{22}};\ \phi_{13}=\frac{k_2\cdot\eta_2\cdot(1-\alpha f)}{R_{23}};\ \phi_{14}=\frac{k_2\cdot\eta_2\cdot(1-\alpha f)}{R_{24}}$$
$$\phi_{15}=\xi_4-\xi_2.$$

We get very simple nonlinear and high order optoisolation circuit differential equation:

$$\begin{aligned} & V^2 \cdot \phi_1 + V \cdot \dddot{V} \cdot \phi_2 + V \cdot \ddot{V} \cdot \phi_3 + V^2 \cdot \ddot{V} \cdot \phi_4 + V^2 \cdot \dot{V} \cdot \phi_5 + V \cdot \ddot{V} \cdot \dot{V} \cdot \phi_6 + V \cdot \dot{V} \cdot \phi_7 \\ & + V \cdot [\dot{V}]^2 \cdot \phi_8 + V \cdot \phi_9 + \dddot{V} \cdot \phi_{10} + \ddot{V} \cdot \phi_{11} + \ddot{V} \cdot \dot{V} \cdot \phi_{12} + \dot{V} \cdot \phi_{13} + [\dot{V}]^2 \cdot \phi_{14} + \phi_{15} = 0 \end{aligned}$$

Another way to present our last differential equation:

$$\begin{aligned} & V^2 \cdot \phi_1 + \dddot{V} \cdot [V \cdot \phi_2 + \phi_{10}] + V \cdot \dot{V} \cdot [\phi_7 + V \cdot \phi_5] + [\dot{V}]^2 \cdot [\phi_{14} + V \cdot \phi_8] \\ & + \ddot{V} \cdot \dot{V} \cdot [\phi_{12} + V \cdot \phi_6] + \ddot{V} \cdot \phi_{11} + V \cdot \ddot{V} \cdot [\phi_3 + V \cdot \phi_4] + V \cdot \phi_9 + \dot{V} \cdot \phi_{13} + \phi_{15} = 0 \end{aligned}$$

The above result must fulfill our Rossler's prototype chaotic system differential equation (**):

$$\begin{aligned} & m_1 \cdot Y + m_2 \cdot \frac{dY}{dt} + m_3 \cdot \frac{d^2Y}{dt^2} + m_4 \cdot [\frac{dY}{dt}]^2 + m_5 \cdot \frac{dY}{dt} \cdot \frac{d^2Y}{dt^2} + m_6 \cdot Y^2 + m_7 \frac{dY}{dt} \cdot Y \\ & + m_8 \cdot Y \cdot \frac{d^2Y}{dt^2} + m_9 \cdot \frac{d^3Y}{dt^3} = 0 \\ & m_1 = -b \cdot a + c;\ m_2 = b - c \cdot a + 1;\ m_3 = c - a;\ m_4 = a;\ m_5 = -1 \\ & m_6 = a;\ m_7 = -\{a^2 + 1\};\ m_8 = a;\ m_9 = 1 \end{aligned}$$

We choose our system voltage (V) values in the band which fulfill our Rossler's prototype chaotic system differential equation (**).

$$\phi_{10} \gg V \cdot \phi_2 \Rightarrow V \cdot \phi_2 + \phi_{10} \approx \phi_{10}; \phi_7 \gg V \cdot \phi_5 \Rightarrow \phi_7 + V \cdot \phi_5 \approx \phi_7;$$
$$\phi_{14} \gg V \cdot \phi_8 \Rightarrow \phi_{14} + V \cdot \phi_8 \approx \phi_{14}$$
$$\phi_{12} \gg V \cdot \phi_6 \Rightarrow \phi_{12} + V \cdot \phi_6 \approx \phi_{12}; \phi_3 \gg V \cdot \phi_4 \Rightarrow \phi_3 + V \cdot \phi_4 \approx \phi_3$$

$$\phi_{10} \gg V \cdot \phi_2 \Rightarrow \frac{k_1 \cdot \eta_1 \cdot (1 - \alpha f)}{R_{12}} \gg V \cdot \left[-\frac{k_1 \cdot \eta_1 \cdot \alpha f}{V_t \cdot R_{12}} \right] \Rightarrow V \gg -\frac{(1 - \alpha f) \cdot V_t}{\alpha f}$$
$$\phi_7 \gg V \cdot \phi_5 \Rightarrow k_2 \cdot \eta_2 \cdot \left(-\frac{\alpha f}{V_t \cdot R_{23}} + \frac{(1 - \alpha f)}{R_{21}} \right) \gg V \cdot \left[-\frac{k_2 \cdot \eta_2 \cdot \alpha f}{V_t \cdot R_{21}} \right] \Rightarrow V \gg \left[\frac{R_{21}}{R_{23}} - \frac{(1 - \alpha f) \cdot V_t}{\alpha f} \right]$$

$$\text{If } \frac{R_{21}}{R_{23}} \gg \frac{(1 - \alpha f) \cdot V_t}{\alpha f} \Rightarrow \frac{R_{21}}{R_{23}} - \frac{(1 - \alpha f) \cdot V_t}{\alpha f} \approx \frac{R_{21}}{R_{23}} \Rightarrow V \Big|_{\frac{R_{21}}{R_{23}} \gg \frac{(1 - \alpha f) \cdot V_t}{\alpha f}} \gg \frac{R_{21}}{R_{23}}$$

$$\phi_{14} \gg V \cdot \phi_8 \Rightarrow \frac{k_2 \cdot \eta_2 \cdot (1 - \alpha f)}{R_{24}} \gg V \cdot \left[-\frac{k_2 \cdot \eta_2 \cdot \alpha f}{V_t \cdot R_{24}} \right] \Rightarrow V \gg -\frac{(1 - \alpha f) \cdot V_t}{\alpha f}$$
$$\phi_{12} \gg V \cdot \phi_6 \Rightarrow \frac{k_2 \cdot \eta_2 \cdot (1 - \alpha f)}{R_{22}} \gg V \cdot \left[-\frac{k_2 \cdot \eta_2 \cdot \alpha f}{V_t \cdot R_{22}} \right] \Rightarrow V \gg -\frac{(1 - \alpha f) \cdot V_t}{\alpha f}$$
$$\phi_3 \gg V \cdot \phi_4 \Rightarrow k_1 \cdot \eta_1 \cdot \left(-\frac{\alpha f}{V_t \cdot R_{13}} + \frac{(1 - \alpha f)}{R_{14}} \right) \gg V \cdot \left[-\frac{k_1 \cdot \eta_1 \cdot \alpha f}{V_t \cdot R_{14}} \right]$$
$$\Rightarrow V \gg \frac{R_{14}}{R_{13}} - \frac{(1 - \alpha f) \cdot V_t}{\alpha f}$$

$$\text{If } \frac{R_{14}}{R_{13}} \gg \frac{(1 - \alpha f) \cdot V_t}{\alpha f} \Rightarrow \frac{R_{14}}{R_{13}} - \frac{(1 - \alpha f) \cdot V_t}{\alpha f} \approx \frac{R_{14}}{R_{13}} \Rightarrow V \Big|_{\frac{R_{14}}{R_{13}} \gg \frac{(1 - \alpha f) \cdot V_t}{\alpha f}} \gg \frac{R_{14}}{R_{13}}$$

Table 3.3 Summary last results

$\phi_{10} \gg V \cdot \phi_2$	$V \cdot \phi_2 + \phi_{10} \approx \phi_{10}$	$V \gg -\frac{(1-\alpha f)\cdot V_t}{\alpha f}$
$\phi_7 \gg V \cdot \phi_5$	$\phi_7 + V \cdot \phi_5 \approx \phi_7$	$V \gg \left[\frac{R_{21}}{R_{23}} - \frac{(1-\alpha f)\cdot V_t}{\alpha f}\right]$; $V\vert_{\frac{R_{21}}{R_{23}} \gg \frac{(1-\alpha f)\cdot V_t}{\alpha f}} \gg \frac{R_{21}}{R_{23}}$
$\phi_{14} \gg V \cdot \phi_8$	$\phi_{14} + V \cdot \phi_8 \approx \phi_{14}$	$V \gg -\frac{(1-\alpha f)\cdot V_t}{\alpha f}$
$\phi_{12} \gg V \cdot \phi_6$	$\phi_{12} + V \cdot \phi_6 \approx \phi_{12}$	$V \gg -\frac{(1-\alpha f)\cdot V_t}{\alpha f}$
$\phi_3 \gg V \cdot \phi_4$	$\phi_3 + V \cdot \phi_4 \approx \phi_3$	$V \gg \frac{R_{14}}{R_{13}} - \frac{(1-\alpha f)\cdot V_t}{\alpha f}$; $V\vert_{\frac{R_{14}}{R_{13}} \gg \frac{(1-\alpha f)\cdot V_t}{\alpha f}} \gg \frac{R_{14}}{R_{13}}$

We can summarize our last results in Table 3.3:

If we choose $\alpha_f = 0.98$; $V_t = 0.026$ V then $-\frac{(1-\alpha f)\cdot V_t}{\alpha f} = -0.53\,\text{mV}$ and $V \gg -0.53$ m-volt. Since $\frac{R_{21}}{R_{23}} > 0$ and $\frac{R_{14}}{R_{13}} > 0$ then the global condition for our analysis is that the initial voltage $V(t = t_0)$ and $V(t)$ for $t > 0$ are $\gg \frac{R_{21}}{R_{23}}, \frac{R_{14}}{R_{13}}$.

Under above approximations/assumptions we get the following differential equation:

$$V^2 \cdot \phi_1 + \dddot{V} \cdot \phi_{10} + V \cdot \dot{V} \cdot \phi_7 + [\dot{V}]^2 \cdot \phi_{14} + \ddot{V} \cdot \dot{V} \cdot \phi_{12} + \ddot{V} \cdot \phi_{11} + V \cdot \ddot{V} \cdot \phi_3 + V \cdot \phi_9 + \dot{V} \cdot \phi_{13} + \phi_{15} = 0$$

And if we compare it with Rossler's prototype chaotic system differential equation (**), we get the following results: $V \rightleftharpoons Y$.

$$\phi_1 = m_6 \Rightarrow a = -\frac{1}{V_t} \cdot \left[\frac{(1-\alpha f)}{R_{CQ1}} + \frac{k_1 \cdot \left\{ \frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]} \right\} \cdot \alpha f}{R_{11}} \right]$$

$$\phi_{10} = m_9 \Rightarrow \frac{k_1 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]} \right] \cdot (1-\alpha f)}{R_{12}} = 1 \Rightarrow k_1 \cdot (1-\alpha f) = \left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right] \cdot R_{12}$$

$$\phi_7 = m_7 \Rightarrow a = \pm \sqrt{k_2 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right]} \right] \cdot \left(\frac{\alpha f}{V_t \cdot R_{23}} - \frac{(1-\alpha f)}{R_{21}} \right) - 1}$$

$$\phi_1 = m_6 \,\&\, \phi_7 = m_7 \Rightarrow \left\{ -\frac{1}{V_t} \cdot \left[\frac{(1-\alpha f)}{R_{CQ1}} + k_1 \cdot \left\{ \frac{1}{[1 + V_t \cdot \frac{1}{I_0} \cdot (\sum_{j=1}^{4} \frac{1}{R_{1j}})] \cdot R_{11}} \right\} \cdot \alpha f \right] \right\}^2$$

$$= k_2 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right]} \right] \cdot \left(\frac{\alpha f}{V_t \cdot R_{23}} - \frac{(1-\alpha f)}{R_{21}} \right) - 1$$

$$\phi_{14} = m_4 \Rightarrow a = k_2 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right] \cdot R_{24}} \right] \cdot (1-\alpha f)$$

$\phi_{12} = m_5 \Rightarrow k_2 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right] \cdot R_{22}} \right] \cdot (1-\alpha f) = -1$. The current result $\phi_{12} = m_5$ Can exist only if $k_2 < 0$ since $\frac{\eta_2 \cdot (1-\alpha f)}{R_{22}} > 0$ & $\phi_{12} = \frac{k_2 \cdot \eta_2 \cdot (1-\alpha f)}{R_{22}} < 0$. The meaning of $k_2 < 0$; $k_1 > 0$ is opposite sign coupling between $D1$ forward current and $Q1$ base dependent current source. $I_{BQ1} = |k_1| \cdot I_{D1} - |k_2| \cdot I_{D2}$ (Fig. 3.5)

$$\phi_{11} = m_3 \Rightarrow \frac{k_1 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]} \right] \cdot (1-\alpha f)}{R_{13}}$$

$$= c - a \Rightarrow c = k_1 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right] \cdot R_{13}} \right] \cdot (1-\alpha f) + a$$

$$c = (1-\alpha f) \cdot \left\{ \frac{k_1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right] \cdot R_{13}} + \frac{k_2}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right] \cdot R_{24}} \right\}$$

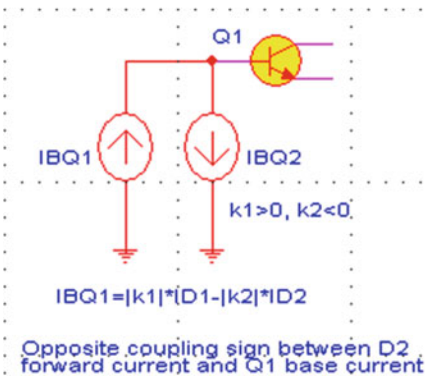

Fig. 3.5 Opposite coupling sign between $D2$ forward current and $Q1$ base current

$$\phi_3 = m_8 \Rightarrow a = k_1 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]}\right] \cdot \left(\frac{(1-\alpha f)}{R_{14}} - \frac{\alpha f}{V_t \cdot R_{13}}\right)$$

$$\phi_9 = m_1 \Rightarrow \frac{\xi_4}{V_t} + \xi_3 - \xi_1 + \frac{k_1 \cdot \eta_1 \cdot (1-\alpha f)}{R_{11}}$$
$$= -b \cdot a + c \Rightarrow b = \frac{1}{a} \cdot \left[c - \left\{\frac{\xi_4}{V_t} + \xi_3 - \xi_1 + \frac{k_1 \cdot \eta_1 \cdot (1-\alpha f)}{R_{11}}\right\}\right]$$

$$b = \frac{1}{a} \cdot \left[c - \left\{\frac{1}{V_t} \cdot \left[\frac{V_{DD}}{R_{CQ1}} \cdot (1-\alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)\right] - \frac{(1-\alpha f)}{R_{CQ1}} + \frac{(\alpha r - 1)}{R_{CQ1}} + \frac{k_1 \cdot (1-\alpha f)}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right] \cdot R_{11}}\right\}\right]$$

$$b = \frac{1}{k_1 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right]}\right] \cdot \left(\frac{(1-\alpha f)}{R_{14}} - \frac{\alpha f}{V_t \cdot R_{13}}\right)} \cdot \left[(1-\alpha f) \cdot \left\{\frac{k_1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right] \cdot R_{13}} + \frac{k_2}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right] \cdot R_{24}}\right\} - \left\{\frac{1}{V_t} \cdot \left[\frac{V_{DD}}{R_{CQ1}} \cdot (1-\alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)\right] - \frac{(1-\alpha f)}{R_{CQ1}} + \frac{(\alpha r - 1)}{R_{CQ1}} + \frac{k_1 \cdot (1-\alpha f)}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{1j}}\right)\right] \cdot R_{11}}\right\}\right]$$

$$\phi_{13} = m_2 \Rightarrow k_2 \cdot \left[\frac{1}{\left[1 + V_t \cdot \frac{1}{I_0} \cdot \left(\sum_{j=1}^{4} \frac{1}{R_{2j}}\right)\right] \cdot R_{23}}\right] \cdot (1-\alpha f) - 1 = b - c \cdot a$$

$$\phi_{15} = 0 \Rightarrow \xi_4 - \xi_2 = 0 \Rightarrow \frac{V_{DD}}{R_{CQ1}} \cdot [(1-\alpha f) - (\alpha r - 1)] + [Isc - Ise] \cdot (\alpha r \cdot \alpha f - 1) = 0$$

a, b, and c are Fold-Hopf bifurcation Rossler's prototype chaotic system parameters which establish system dynamic and bifurcation behavior. Our last results give as the desire constrains between optoisolation circuit parameters and those parameters as a function of optoisolation circuit parameters [25].

$$a(\alpha_r, \alpha_f, R_{1j}, R_{2j}, k_1, k_2, \text{etc.}); b(\alpha_r, \alpha_f, R_{1j}, R_{2j}, k_1, k_2, \text{etc.}); c(\alpha_r, \alpha_f, R_{1j}, R_{2j}, k_1, k_2, \text{etc.})$$

3.3 Hopf–Hopf Bifurcation System

We consider a continuous time dynamical system depending on parameters, given by $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), \alpha); f \in C^k(\Omega \times \Lambda, \mathbb{R}^n)$ with open sets $0 \in \Omega \subset \mathbb{R}^N$,

$0 \in \Lambda \subset \mathbb{R}^2, k \geq 1$ Sufficiently large, $N \geq 4$. The first used tool for exploring the dynamical behavior of system is numerical time integration. We can employ one step methods, which consists in approximating the evolution operator by a discrete time system $x \to \mathrm{g}(x, \alpha)$ with $g \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$, where the step size were assumed to be previously chosen. Hopf–Hopf bifurcations occur when the system presents equilibrium with two pairs of Hopf eigenvalues. The local bifurcation diagram near Hopf–Hopf point is known to present other phenomena, such as Neimark–Sacker bifurcation of cycles, and homoclinic bifurcations. A point $(x_0, \alpha_0) \in \Omega \times \Lambda$ is referred to as a Hopf–Hopf bifurcation (HH point). It is also called double Hopf, Hopf/Hopf mode interaction, and multiple Hopf in case that more than two pairs of Hopf eigenvalues are present. A point $(x_0, \alpha_0) \in \Omega \times \Lambda$ is referred to as a Hopf–Hopf bifurcation of $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), \alpha)$ If $f(x_0, \alpha_0) = 0$ and $f_x(x_0, \alpha_0) = \frac{\partial f}{\partial x}\Big|_{(x=x_0, \alpha=\alpha_0)}$ has the only critical eigenvalues $\{\lambda_{1,2} = \pm i \cdot \omega_1, \lambda_{3,4} = \pm i \cdot \omega_2\}, 0 < \omega_{1,2} \in \mathbb{R} \forall \omega_1 \neq \omega_2$.

The Hopf–Hopf bifurcation appears in many parameter dependent systems which describe the dynamic of various phenomena. We consider five dimensional, continuous time system [5–9].

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A - B \cdot X + X^2 \cdot Y - X; \frac{\mathrm{d}Y}{\mathrm{d}t} = B \cdot X - X^2 \cdot Y; \frac{\mathrm{d}Z}{\mathrm{d}t} = X - Z \cdot V$$

$\frac{\mathrm{d}V}{\mathrm{d}t} = B \cdot X - Z \cdot V + V^2 \cdot W - V; \frac{\mathrm{d}W}{\mathrm{d}t} = Z \cdot V - V^2 \cdot W$ with state variables $(X, Y, Z, V, W) \in \mathbb{R}^5; A, B \in \mathbb{R}$. This system can describe the behavior of oscillator which implemented by using optoisolation circuits. First, we need to find our system fixed points $E^* = (X^*, Y^*, Z^*, V^*, W^*)$ by setting $\frac{\mathrm{d}X}{\mathrm{d}t} = 0$

$\frac{\mathrm{d}Y}{\mathrm{d}t} = 0; \frac{\mathrm{d}Z}{\mathrm{d}t} = 0; \frac{\mathrm{d}V}{\mathrm{d}t} = 0; \frac{\mathrm{d}W}{\mathrm{d}t} = 0$. We get the following five equations:

$$A - B \cdot X^* + [X^*]^2 \cdot Y^* - X^* = 0; B \cdot X^* - [X^*]^2 \cdot Y^* = 0; X^* - Z^* \cdot V^* = 0$$

$$B \cdot X^* - Z^* \cdot V^* + [V^*]^2 \cdot W^* - V^* = 0; Z^* \cdot V^* - [V^*]^2 \cdot W^* = 0$$

$$\{A - B \cdot X^* + [X^*]^2 \cdot Y^* - X^* = 0\} + \{B \cdot X^* - [X^*]^2 \cdot Y^* = 0\} \Rightarrow A - X^* = 0 \\ \Rightarrow X^* = A$$

$$X^* = A \,\&\, B \cdot X^* - [X^*]^2 \cdot Y^* = 0 \Rightarrow B \cdot A - A^2 \cdot Y^* = 0 \Rightarrow Y^* = B/A$$

$$\left\{B \cdot X^* - Z^* \cdot V^* + [V^*]^2 \cdot W^* - V^* = 0\right\} + \left\{Z^* \cdot V^* - [V^*]^2 \cdot W^* = 0\right\}$$
$$\Rightarrow B \cdot X^* - V^* = 0 \Rightarrow|_{X^*=A} V^* = B \cdot A$$

$$X^* - Z^* \cdot V^* = 0 \Rightarrow|_{X^*=A, V^*=B \cdot A} \Rightarrow A - Z^* \cdot B \cdot A = 0 \Rightarrow Z^* = 1/B$$

$$Z^* \cdot V^* - [V^*]^2 \cdot W^* = 0 \Rightarrow|_{Z^*=\frac{1}{B}, V^*=B \cdot A} \Rightarrow \frac{1}{B} \cdot B \cdot A - B^2 \cdot A^2 \cdot W^* = 0 \Rightarrow W^* = \frac{1}{B^2 \cdot A}$$

$$E^* = (X^*, Y^*, Z^*, V^*, W^*) = \left(A, \frac{B}{A}, \frac{1}{B}, B \cdot A, \frac{1}{B^2 \cdot A}\right)$$

We define our system if the following way:

$$\frac{dX}{dt} = f_1(X, Y, Z, V, W) = f_1; \frac{dY}{dt} = f_2(X, Y, Z, V, W) = f_2; \frac{dZ}{dt} = f_3(X, Y, Z, V, W) = f_3$$
$$f_1 = A - B \cdot X + X^2 \cdot Y - X; f_2 = B \cdot X - X^2 \cdot Y; f_3 = X - Z \cdot V$$

$$\frac{dV}{dt} = f_4(X, Y, Z, V, W) = f_4; \frac{dW}{dt} = f_5(X, Y, Z, V, W) = f_5$$
$$f_4 = B \cdot X - Z \cdot V + V^2 \cdot W - V; f_5 = Z \cdot V - V^2 \cdot W$$

The next step is to find matrix A is called the jacobian matrix at fixed point $E^* = (X^*, Y^*, Z^*, V^*, W^*) = \left(A, \frac{B}{A}, \frac{1}{B}, B \cdot A, \frac{1}{B^2 \cdot A}\right)$.

$$A = \left.\begin{pmatrix} \frac{df_1}{dX} & \cdots & \frac{df_1}{dW} \\ \vdots & \ddots & \vdots \\ \frac{df_5}{dX} & \cdots & \frac{df_5}{dW} \end{pmatrix}\right|_{E^*=(X^*,Y^*,Z^*,V^*,W^*)=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}$$
$$= \begin{pmatrix} \left.\frac{df_1}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)} & \cdots & \left.\frac{df_1}{dW}\right|_{E^*=(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A})} \\ \vdots & \ddots & \vdots \\ \left.\frac{df_5}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)} & \cdots & \left.\frac{df_5}{dW}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)} \end{pmatrix}$$

$$\left.\frac{df_1}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)} = -B + 2 \cdot X^* \cdot Y^* - 1 = B - 1; \left.\frac{df_1}{dY}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)} = [X^*]^2 = A^2$$

$$\left.\frac{df_1}{dZ}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_1}{dV}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_1}{dW}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0$$

$$\left.\frac{df_2}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=B-2\cdot X^*\cdot Y^*=-B;\left.\frac{df_2}{dY}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=-[X^*]^2=-A^2$$

$$\left.\frac{df_2}{dZ}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_2}{dV}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_2}{dW}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0$$

$$\left.\frac{df_3}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=1;\left.\frac{df_3}{dY}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_3}{dZ}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=-V^*$$
$$=-B\cdot A$$

$$\left.\frac{df_3}{dV}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=-Z^*=-\frac{1}{B};\left.\frac{df_3}{dW}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_4}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=B$$

$$\left.\frac{df_4}{dY}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_4}{dZ}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=-V^*=-B\cdot A$$

$$\left.\frac{df_4}{dV}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=-Z^*+2\cdot V^*\cdot W^*-1=\frac{1}{B}-1;\left.\frac{df_4}{dW}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}$$
$$=B^2\cdot A^2$$

$$\left.\frac{df_5}{dX}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_5}{dY}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=0;\left.\frac{df_5}{dZ}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=V^*=B\cdot A$$

$$\left.\frac{df_5}{dV}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=Z^*-2\cdot V^*\cdot W^*=-\frac{1}{B};\left.\frac{df_5}{dW}\right|_{E^*=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}=-[V^*]^2$$
$$=-B^2\cdot A^2$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A-\lambda\cdot I)$, where I is the identity matrix.

Remark Do not confuse between system parameter A and matrix A.

$$A-\lambda\cdot I=\left.\begin{pmatrix}\frac{df_1}{dX} & \cdots & \frac{df_1}{dW}\\ \vdots & \ddots & \vdots\\ \frac{df_5}{dX} & \cdots & \frac{df_5}{dW}\end{pmatrix}\right|_{E^*=(X^*,Y^*,Z^*,V^*,W^*)=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}-\begin{pmatrix}\lambda & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda\end{pmatrix}$$

$$A-\lambda\cdot I=\left.\begin{pmatrix}\frac{df_1}{dX}-\lambda & \cdots & \frac{df_1}{dW}\\ \vdots & \ddots & \vdots\\ \frac{df_5}{dX} & \cdots & \frac{df_5}{dW}-\lambda\end{pmatrix}\right|_{E^*=(X^*,Y^*,Z^*,V^*,W^*)=\left(A,\frac{B}{A},\frac{1}{B},B\cdot A,\frac{1}{B^2\cdot A}\right)}$$

$$=\begin{pmatrix}B-1-\lambda & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & -B^2\cdot A^2-\lambda\end{pmatrix}$$

$$\det(A-\lambda\cdot I)=(B-1-\lambda)\cdot(-A^2-\lambda)\cdot\det\begin{pmatrix}-B\cdot A-\lambda & -\frac{1}{B} & 0\\ -B\cdot A & \frac{1}{B}-1-\lambda & B^2\cdot A^2\\ B\cdot A & -\frac{1}{B} & -B^2\cdot A^2-\lambda\end{pmatrix}$$
$$-A^2\cdot(-B)\cdot\det\begin{pmatrix}-B\cdot A-\lambda & -\frac{1}{B} & 0\\ -B\cdot A & \frac{1}{B}-1-\lambda & B^2\cdot A^2\\ B\cdot A & -\frac{1}{B} & -B^2\cdot A^2-\lambda\end{pmatrix}$$

$$\det(A-\lambda\cdot I)=\left[(B-1-\lambda)\cdot(-A^2-\lambda)+A^2\cdot B\right]\cdot\det\begin{pmatrix}-B\cdot A-\lambda & -\frac{1}{B} & 0\\ -B\cdot A & \frac{1}{B}-1-\lambda & B^2\cdot A^2\\ B\cdot A & -\frac{1}{B} & -B^2\cdot A^2-\lambda\end{pmatrix}$$

$$\det(A-\lambda\cdot I)=\left[(1+\lambda-B)\cdot(A^2+\lambda)+A^2\cdot B\right]\cdot\det\begin{pmatrix}-B\cdot A-\lambda & -\frac{1}{B} & 0\\ -B\cdot A & \frac{1}{B}-1-\lambda & B^2\cdot A^2\\ B\cdot A & -\frac{1}{B} & -B^2\cdot A^2-\lambda\end{pmatrix}$$

We define the following functions: $\det(A-\lambda\cdot I)=\eta_1(A,B,\lambda)\cdot\eta_2(A,B,\lambda)$

$$\eta_1(A,B,\lambda)=\left[(1+\lambda-B)\cdot(A^2+\lambda)+A^2\cdot B\right];\eta_2(A,B,\lambda)=\det\begin{pmatrix}-B\cdot A-\lambda & -\frac{1}{B} & 0\\ -B\cdot A & \frac{1}{B}-1-\lambda & B^2\cdot A^2\\ B\cdot A & -\frac{1}{B} & -B^2\cdot A^2-\lambda\end{pmatrix}$$

First, we find the final expression for $[(1+\lambda-B)\cdot(A^2+\lambda)+A^2\cdot B]$ when $\lambda=i\cdot\omega_{1,2}$. $\left\{\lambda_{1,2}=\pm i\cdot\omega_1,\lambda_{3,4}=\pm i\cdot\omega_2\right\},0<\omega_{1,2}\in\mathbb{R}\quad\forall\,\omega_1\neq\omega_2$.

$$\eta_1(A,B,\lambda)=\eta_1=[(1-B)+\lambda]\cdot[A^2+\lambda]+A^2\cdot B$$
$$=(1-B)\cdot A^2+\lambda\cdot[1-B+A^2]+\lambda^2$$

$$\eta_1(A,B,\lambda=i\cdot\omega_{1,2})=(1-B)\cdot A^2+\lambda\cdot[1-B+A^2]+\lambda^2\Big|_{\lambda=i\cdot\omega_{1,2}}$$
$$=[(1-B)\cdot A^2-\omega_{1,2}^2]+i\cdot\omega_{1,2}\cdot[1-B+A^2]$$

We define the following functions:

$$\phi_1 = (1-B)\cdot A^2 - \omega_{1,2}^2; \phi_2 = \omega_{1,2}\cdot[1-B+A^2] \Rightarrow \eta_1(A,B,\lambda = i\cdot\omega_{1,2})$$
$$= \phi_1 + i\cdot\phi_2\cdot\eta_2(A,B,\lambda)$$
$$= \det\begin{pmatrix} -B\cdot A-\lambda & -\frac{1}{B} & 0 \\ -B\cdot A & \frac{1}{B}-1-\lambda & B^2\cdot A^2 \\ B\cdot A & -\frac{1}{B} & -B^2\cdot A^2-\lambda \end{pmatrix}$$
$$= -(B\cdot A+\lambda)\cdot\begin{pmatrix} \frac{1}{B}-1-\lambda & B^2\cdot A^2 \\ -\frac{1}{B} & -[B^2\cdot A^2+\lambda] \end{pmatrix}$$
$$+\frac{1}{B}\cdot\begin{pmatrix} -B\cdot A & B^2\cdot A^2 \\ B\cdot A & -[B^2\cdot A^2+\lambda] \end{pmatrix}$$

We define the following functions: $\eta_2(A,B,\lambda) = \eta_{21}(A,B,\lambda) + \eta_{22}(A,B,\lambda)$

$$\eta_{21}(A,B,\lambda) = -(B\cdot A+\lambda)\cdot\begin{pmatrix} \frac{1}{B}-1-\lambda & B^2\cdot A^2 \\ -\frac{1}{B} & -[B^2\cdot A^2+\lambda] \end{pmatrix};$$
$$\eta_{22}(A,B,\lambda) = \frac{1}{B}\cdot\begin{pmatrix} -B\cdot A & B^2\cdot A^2 \\ B\cdot A & -[B^2\cdot A^2+\lambda] \end{pmatrix}$$

$$\eta_{21}(A,B,\lambda) = -(B\cdot A+\lambda)\cdot\begin{pmatrix} \frac{1}{B}-1-\lambda & B^2\cdot A^2 \\ -\frac{1}{B} & -[B^2\cdot A^2+\lambda] \end{pmatrix}$$
$$\eta_{21}(A,B,\lambda) = -(B\cdot A+\lambda)\cdot\left\{-[B^2\cdot A^2+\lambda]\cdot\left[\frac{1}{B}-1-\lambda\right]+\frac{1}{B}\cdot B^2\cdot A^2\right\}$$
$$\eta_{21}(A,B,\lambda) = (B\cdot A+\lambda)\cdot[B^2\cdot A^2+\lambda]\cdot\left[\frac{1}{B}-1-\lambda\right]-(B\cdot A+\lambda)\cdot B\cdot A^2$$

$$\eta_{21}(A,B,\lambda) = \left[B^3\cdot A^3+\lambda\cdot B\cdot A\cdot\{B\cdot A+1\}+\lambda^2\right]\cdot\left[\left(\frac{1}{B}-1\right)-\lambda\right]$$
$$-B^2\cdot A^3-\lambda\cdot B\cdot A^2$$

$$\eta_{21}(A,B,\lambda) = B^3\cdot A^3\cdot\left(\frac{1}{B}-1\right)+\lambda\cdot\left[B\cdot A\cdot\{B\cdot A+1\}\cdot\left(\frac{1}{B}-1\right)-B^3\cdot A^3\right]$$
$$+\lambda^2\cdot\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right]-\lambda^3-B^2\cdot A^3-\lambda\cdot B\cdot A^2$$

$$\eta_{21}(A,B,\lambda) = B^3\cdot A^3\cdot\left(\frac{1}{B}-1\right) - B^2\cdot A^3 + \lambda\cdot\left[B\cdot A\cdot\{B\cdot A+1\}\cdot\left(\frac{1}{B}-1\right) - B^3\cdot A^3 - B\cdot A^2\right] + \lambda^2\cdot\left[\left(\frac{1}{B}-1\right) - B\cdot A\cdot\{B\cdot A+1\}\right] - \lambda^3$$

We define the following functions: $\eta_{21}(A,B,\lambda) = \phi_3 + \lambda\cdot\phi_4 + \lambda^2\cdot\phi_5 + \lambda^3\cdot\phi_6$

$$\phi_3 = B^3\cdot A^3\cdot\left(\frac{1}{B}-1\right) - B^2\cdot A^3;$$
$$\phi_4 = \left[B\cdot A\cdot\{B\cdot A+1\}\cdot\left(\frac{1}{B}-1\right) - B^3\cdot A^3 - B\cdot A^2\right]$$
$$\phi_5 = \left[\left(\frac{1}{B}-1\right) - B\cdot A\cdot\{B\cdot A+1\}\right]; \phi_6 = -1$$

$$\eta_{22}(A,B,\lambda) = \frac{1}{B}\cdot\begin{pmatrix} -B\cdot A & B^2\cdot A^2 \\ B\cdot A & -[B^2\cdot A^2+\lambda]\end{pmatrix} = \frac{1}{B}\cdot\left\{B\cdot A\cdot[B^2\cdot A^2+\lambda] - B^3\cdot A^3\right\}$$

$$\eta_{22}(A,B,\lambda) = \frac{1}{B}\cdot\left\{B^3\cdot A^3 + B\cdot A\cdot\lambda - B^3\cdot A^3\right\} = A\cdot\lambda$$

$$\eta_2(A,B,\lambda) = \eta_{21}(A,B,\lambda) + \eta_{22}(A,B,\lambda) = \phi_3 + \lambda\cdot(\phi_4 + A) + \lambda^2\cdot\phi_5 + \lambda^3\cdot\phi_6$$

$$\eta_2(A,B,\lambda)|_{\lambda=i\cdot\omega_{1,2}} = \eta_{21}(A,B,\lambda=i\cdot\omega_{1,2}) + \eta_{22}(A,B,\lambda=i\cdot\omega_{1,2}) = \phi_3 + i\cdot\omega_{1,2}\cdot(\phi_4+A) - \omega_{1,2}^2\cdot\phi_5 - i\cdot\omega_{1,2}^3\cdot\phi_6$$

$$\eta_2(A,B,\lambda)|_{\lambda=i\cdot\omega_{1,2}} = [\phi_3 - \omega_{1,2}^2\cdot\phi_5] + i\cdot\left[\omega_{1,2}\cdot(\phi_4+A) - \omega_{1,2}^3\cdot\phi_6\right]$$

$$\det(A-\lambda\cdot I) = \eta_1(A,B,\lambda)\cdot\eta_2(A,B,\lambda) = \{\phi_1 + i\cdot\phi_2\}\cdot\left\{[\phi_3 - \omega_{1,2}^2\cdot\phi_5] + i\cdot[\omega_{1,2}\cdot(\phi_4+A) - \omega_{1,2}^3\cdot\phi_6]\right\}$$

$$\det(A-\lambda\cdot I) = \{\phi_1 + i\cdot\phi_2\}\cdot\left\{[\phi_3 - \omega_{1,2}^2\cdot\phi_5] + i\cdot\left[\omega_{1,2}\cdot(\phi_4+A) - \omega_{1,2}^3\cdot\phi_6\right]\right\} = \phi_1\cdot[\phi_3 - \omega_{1,2}^2\cdot\phi_5] + i\cdot[\omega_{1,2}\cdot(\phi_4+A) - \omega_{1,2}^3\cdot\phi_6]\cdot\phi_1 + i\cdot\phi_2\cdot[\phi_3 - \omega_{1,2}^2\cdot\phi_5] - \phi_2\cdot\left[\omega_{1,2}\cdot(\phi_4+A) - \omega_{1,2}^3\cdot\phi_6\right]$$

$$\det(A-\lambda\cdot I)=\phi_1\cdot[\phi_3-\omega_{1,2}^2\cdot\phi_5]-\phi_2\cdot\left[\omega_{1,2}\cdot(\phi_4+A)-\omega_{1,2}^3\cdot\phi_6\right]$$
$$+i\cdot\left\{[\omega_{1,2}\cdot(\phi_4+A)-\omega_{1,2}^3\cdot\phi_6]\cdot\phi_1+\phi_2\cdot[\phi_3-\omega_{1,2}^2\cdot\phi_5]\right\}$$

$$\det(A-\lambda\cdot I)=0\Rightarrow\phi_1\cdot[\phi_3-\omega_{1,2}^2\cdot\phi_5]-\phi_2\cdot[\omega_{1,2}\cdot(\phi_4+A)-\omega_{1,2}^3\cdot\phi_6]=0$$
$$\&\,[\omega_{1,2}\cdot(\phi_4+A)-\omega_{1,2}^3\cdot\phi_6]\cdot\phi_1+\phi_2\cdot[\phi_3-\omega_{1,2}^2\cdot\phi_5]=0$$

First we check the first expression:

$$\phi_1\cdot[\phi_3-\omega_{1,2}^2\cdot\phi_5]-\phi_2\cdot\left[\omega_{1,2}\cdot(\phi_4+A)-\omega_{1,2}^3\cdot\phi_6\right]=0$$
$$\Rightarrow\phi_1\cdot\phi_3-\omega_{1,2}^2\cdot\phi_5\cdot\phi_1-\omega_{1,2}\cdot\phi_2\cdot(\phi_4+A)+\omega_{1,2}^3\cdot\phi_2\cdot\phi_6=0$$

$$\phi_1=(1-B)\cdot A^2-\omega_{1,2}^2;\phi_2=\omega_{1,2}\cdot[1-B+A^2];\phi_3=-B^3\cdot A^3$$
$$\phi_4=A-B^2\cdot A^2-B\cdot A-B^3\cdot A^3;\phi_5=\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right];\phi_6=-1$$

$$\phi_1\cdot\phi_3=-\{(1-B)\cdot A^2-\omega_{1,2}^2\}\cdot B^3\cdot A^3=(B-1)\cdot B^3\cdot A^5+\omega_{1,2}^2\cdot B^3\cdot A^3$$
$$\sigma_1=(B-1)\cdot B^3\cdot A^5;\sigma_2=B^3\cdot A^3\Rightarrow\phi_1\cdot\phi_3=\sigma_1+\omega_{1,2}^2\cdot\sigma_2$$
$$\phi_5\cdot\phi_1=\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right]\cdot[(1-B)\cdot A^2-\omega_{1,2}^2]$$
$$=\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right]\cdot(1-B)\cdot A^2$$
$$-\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right]\cdot\omega_{1,2}^2$$
$$\sigma_3=\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right]\cdot(1-B)\cdot A^2;$$
$$\sigma_4=-\left[\left(\frac{1}{B}-1\right)-B\cdot A\cdot\{B\cdot A+1\}\right]\Rightarrow\phi_5\cdot\phi_1=\sigma_3+\sigma_4\cdot\omega_{1,2}^2$$
$$\phi_2\cdot(\phi_4+A)=\omega_{1,2}\cdot[1-B+A^2]\cdot(2\cdot A-B^2\cdot A^2-B\cdot A-B^3\cdot A^3)$$
$$\sigma_5=[1-B+A^2]\cdot(2\cdot A-B^2\cdot A^2-B\cdot A-B^3\cdot A^3)\Rightarrow\phi_2\cdot(\phi_4+A)=\omega_{1,2}\cdot\sigma_5$$
$$\phi_2\cdot\phi_6=\omega_{1,2}\cdot[B-A^2-1];\sigma_6=[B-A^2-1]\Rightarrow\phi_2\cdot\phi_6=\omega_{1,2}\cdot\sigma_6$$

$$\phi_1\cdot\phi_3-\omega_{1,2}^2\cdot\phi_5\cdot\phi_1-\omega_{1,2}\cdot\phi_2\cdot(\phi_4+A)+\omega_{1,2}^3\cdot\phi_2\cdot\phi_6=0\Rightarrow$$
$$\sigma_1+\omega_{1,2}^2\cdot\sigma_2-\omega_{1,2}^2\cdot(\sigma_3+\sigma_4\cdot\omega_{1,2}^2)-\omega_{1,2}\cdot\omega_{1,2}\cdot\sigma_5+\omega_{1,2}^3\cdot\omega_{1,2}\cdot\sigma_6=0$$
$$\sigma_1+\omega_{1,2}^2\cdot\sigma_2-\omega_{1,2}^2\cdot\sigma_3-\sigma_4\cdot\omega_{1,2}^4-\omega_{1,2}^2\cdot\sigma_5+\omega_{1,2}^4\cdot\sigma_6=0$$
$$\omega_{1,2}^4\cdot[\sigma_6-\sigma_4]+\omega_{1,2}^2\cdot[\sigma_2-\sigma_3-\sigma_5]+\sigma_1=0;\sigma'=\sigma_6-\sigma_4;\sigma''=\sigma_2-\sigma_3-\sigma_5$$
$$\omega_{1,2}^4\cdot[\sigma_6-\sigma_4]+\omega_{1,2}^2\cdot[\sigma_2-\sigma_3-\sigma_5]+\sigma_1=0\Rightarrow\omega_{1,2}^4\cdot\sigma'+\omega_{1,2}^2\cdot\sigma''+\sigma_1=0$$

$$\omega_{1,2}^4 \cdot \sigma' + \omega_{1,2}^2 \cdot \sigma'' + \sigma_1 = 0 \Rightarrow \omega_{1,2}^2 = \frac{-\sigma'' \pm \sqrt{[\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1}}{2 \cdot \sigma'}; \quad 0 < \omega_{1,2} \in \mathbb{R} \forall \omega_1 \neq \omega_2$$

$$\Rightarrow \omega_{1,2} = \pm \sqrt{\frac{-\sigma'' \pm \sqrt{[\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1}}{2 \cdot \sigma'}}\Bigg|_{\omega_{1,2} > 0} = +\sqrt{\frac{-\sigma'' \pm \sqrt{[\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1}}{2 \cdot \sigma'}}$$

$$\frac{-\sigma'' \pm \sqrt{[\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1}}{2 \cdot \sigma'} > 0 \ \& \ [\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1 > 0$$

We need to check for which A, B parameters values in the above two conditions can be fulfilled.

Second we check the second expression:

$$\left[\omega_{1,2} \cdot (\phi_4 + A) - \omega_{1,2}^3 \cdot \phi_6\right] \cdot \phi_1 + \phi_2 \cdot [\phi_3 - \omega_{1,2}^2 \cdot \phi_5] = 0$$
$$\Rightarrow \omega_{1,2} \cdot (\phi_4 + A) \cdot \phi_1 - \omega_{1,2}^3 \cdot \phi_6 \cdot \phi_1 + \phi_2 \cdot \phi_3 - \omega_{1,2}^2 \cdot \phi_5 \cdot \phi_2 = 0$$

$$(\phi_4 + A) \cdot \phi_1 = \left[2 \cdot A - B^2 \cdot A^2 - B \cdot A - B^3 \cdot A^3\right] \cdot [(1 - B) \cdot A^2 - \omega_{1,2}^2]$$
$$= \left[2 \cdot A - B^2 \cdot A^2 - B \cdot A - B^3 \cdot A^3\right] \cdot (1 - B) \cdot A^2$$
$$- \left[2 \cdot A - B^2 \cdot A^2 - B \cdot A - B^3 \cdot A^3\right] \cdot \omega_{1,2}^2$$
$$\varsigma_1 = \left[2 \cdot A - B^2 \cdot A^2 - B \cdot A - B^3 \cdot A^3\right] \cdot (1 - B) \cdot A^2;$$
$$\varsigma_2 = -\left[2 \cdot A - B^2 \cdot A^2 - B \cdot A - B^3 \cdot A^3\right]$$
$$(\phi_4 + A) \cdot \phi_1 = \varsigma_1 + \varsigma_2 \cdot \omega_{1,2}^2; \phi_6 \cdot \phi_1 = (B - 1) \cdot A^2 + \omega_{1,2}^2;$$
$$\varsigma_3 = (B - 1) \cdot A^2 \Rightarrow \phi_6 \cdot \phi_1 = \varsigma_3 + \omega_{1,2}^2$$
$$\phi_2 \cdot \phi_3 = \omega_{1,2} \cdot [B - A^2 - 1] \cdot B^3 \cdot A^3;$$
$$\varsigma_4 = [B - A^2 - 1] \cdot B^3 \cdot A^3 \Rightarrow \phi_2 \cdot \phi_3 = \omega_{1,2} \cdot \varsigma_4$$
$$\phi_5 \cdot \phi_2 = \omega_{1,2} \cdot \left[\left(\frac{1}{B} - 1\right) - B \cdot A \cdot \{B \cdot A + 1\}\right] \cdot [1 - B + A^2]$$
$$\varsigma_5 = \left[\left(\frac{1}{B} - 1\right) - B \cdot A \cdot \{B \cdot A + 1\}\right] \cdot [1 - B + A^2] \Rightarrow \phi_5 \cdot \phi_2 = \omega_{1,2} \cdot \varsigma_5$$

$$\omega_{1,2} \cdot (\phi_4 + A) \cdot \phi_1 - \omega_{1,2}^3 \cdot \phi_6 \cdot \phi_1 + \phi_2 \cdot \phi_3 - \omega_{1,2}^2 \cdot \phi_5 \cdot \phi_2 = 0$$
$$\Rightarrow \omega_{1,2} \cdot [\varsigma_1 + \varsigma_2 \cdot \omega_{1,2}^2] - \omega_{1,2}^3 \cdot [\varsigma_3 + \omega_{1,2}^2] + \omega_{1,2} \cdot \varsigma_4 - \omega_{1,2}^2 \cdot \omega_{1,2} \cdot \varsigma_5 = 0$$

$$\omega_{1,2} \cdot \varsigma_1 + \varsigma_2 \cdot \omega_{1,2}^3 - \omega_{1,2}^3 \cdot \varsigma_3 - \omega_{1,2}^5 + \omega_{1,2} \cdot \varsigma_4 - \omega_{1,2}^3 \cdot \varsigma_5 = 0$$
$$-\omega_{1,2}^5 + \omega_{1,2}^3 \cdot [\varsigma_2 - \varsigma_3 - \varsigma_5] + \omega_{1,2} \cdot [\varsigma_1 + \varsigma_4] = 0$$
$$\Rightarrow \omega_{1,2} \cdot \left\{-\omega_{1,2}^4 + \omega_{1,2}^2 \cdot [\varsigma_2 - \varsigma_3 - \varsigma_5] + \varsigma_1 + \varsigma_4\right\} = 0$$
$$\omega_{1,2} \neq 0 \Rightarrow -\omega_{1,2}^4 + \omega_{1,2}^2 \cdot [\varsigma_2 - \varsigma_3 - \varsigma_5] + \varsigma_1 + \varsigma_4 = 0$$

$$\omega_{1,2}^4 + \omega_{1,2}^2 \cdot [\varsigma_3 + \varsigma_5 - \varsigma_2] - [\varsigma_1 + \varsigma_4] = 0; \imath' = \varsigma_3 + \varsigma_5 - \varsigma_2; \imath' = -[\varsigma_1 + \varsigma_4]$$
$$\omega_{1,2}^4 + \omega_{1,2}^2 \cdot \imath' + \imath' = 0 \Rightarrow \omega_{1,2}^2 = \frac{-\imath' \pm \sqrt{[\imath']^2 - 4 \cdot \imath'}}{2}$$
$$\Rightarrow \omega_{1,2} = \pm\sqrt{\frac{-\imath' \pm \sqrt{[\imath']^2 - 4 \cdot \imath'}}{2}}$$

$$0 < \omega_{1,2} \in \mathbb{R} \forall \omega_1 \neq \omega_2 \Rightarrow \omega_{1,2} = +\sqrt{\frac{-\imath' \pm \sqrt{[\imath']^2 - 4 \cdot \imath''}}{2}; \frac{-\imath' \pm \sqrt{[\imath']^2 - 4 \cdot \imath''}}{2}}$$
$$> 0 \;\&\; [\imath']^2 - 4 \cdot \imath'' > 0$$

We need to check for which A, B parameters values in the above two conditions can be fulfilled.

The first and second expressions conditions give total *four* parameters (A, B) conditions which grantee that our five-dimensional system is a continuous time system with Hopf–Hopf bifurcation [45, 52].

$$\left\{\lambda_{1,2} = \pm i \cdot \omega_1, \lambda_{3,4} = \pm i \cdot \omega_2\right\}, 0 < \omega_{1,2} \in \mathbb{R} \forall \omega_1 \neq \omega_2$$

Remark The first and second expressions give two results for $\omega_{1,2}$ values. and the values must be the same. It is a reader exercise to check for $\lambda = -i \cdot \omega_{1,2}$.

$$\omega_{1,2} = +\sqrt{\frac{-\sigma'' \pm \sqrt{[\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1}}{2 \cdot \sigma'}} \;\&\; \omega_{1,2} = +\sqrt{\frac{-\imath' \pm \sqrt{[\imath']^2 - 4 \cdot \imath''}}{2}}$$
$$\Rightarrow \sqrt{\frac{-\sigma'' \pm \sqrt{[\sigma'']^2 - 4 \cdot \sigma' \cdot \sigma_1}}{2 \cdot \sigma'}} = \sqrt{\frac{-\imath' \pm \sqrt{[\imath']^2 - 4 \cdot \imath''}}{2}}$$

We plot our Hopf–Hopf bifurcation system behavior in time and phase portraits.

Matlab

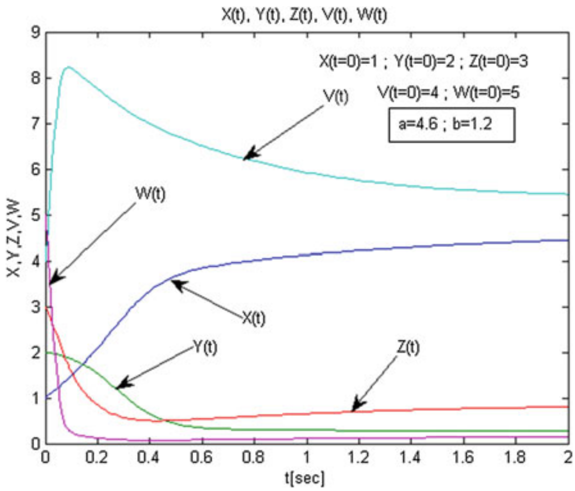

Fig. 3.6 Hopf–Hopf bifurcation system $X(t)$, $Y(t)$, $Z(t)$ and $W(t)$ functions

```
function h=hopfhopf1 (a,b,X0,Y0,Z0,V0,W0)
[t,x]=ODE45(@hopfhopf,[0,2],[X0,Y0,Z0,V0,W0],[],a,b);
%plot(t,x);
subplot(2,2,1);plot(x(:,1),x(:,2));subplot(2,2,2);plot(x(:,1),x(:,3))
;subplot(2,2,3);plot(x(:,1),x(:,4));subplot(2,2,4);plot(x(:,1),x(:,5)
)
%subplot(2,2,1);plot(x(:,2),x(:,3));subplot(2,2,2);plot(x(:,2),x(:,4)
);subplot(2,2,3);plot(x(:,2),x(:,5));subplot(2,2,4);plot(x(:,3),x(:,4
))
%subplot(2,1,1);plot(x(:,3),x(:,5));subplot(2,1,2);plot(x(:,4),x(:,5)
)

function g=hopfhopf(t,x,a,b)
g=zeros(5,1);
g(1)=a-b*x(1)+((x(1)).^2).*x(2)-x(1);
g(2)=b*x(1)-((x(1)).^2).*x(2);
g(3)=x(1)-x(3).*x(4);
g(4)=b*x(1)-x(3).*x(4)+((x(4)).^2).*x(5)-x(4);
g(5)=x(3).*x(4)-((x(4)).^2).*x(5);
```

In the following 10 system phase portrait figures, we plot all options for $a = 4.6$; $b = 1.2$; $X(t = 0) = 1$; $Y(t = 0) = 2$; $Z(t = 0) = 3$; $V(t = 0) = 4$; $W(t = 0) = 5$ (Figs. 3.6, 3.7, 3.8 and 3.9).

3.4 Optoisolation Circuits Hopf–Hopf Bifurcation System

We need to implement our Hopf–Hopf bifurcation system by using optoisolation elements, Op amps, resistors, capacitors, diodes, etc.

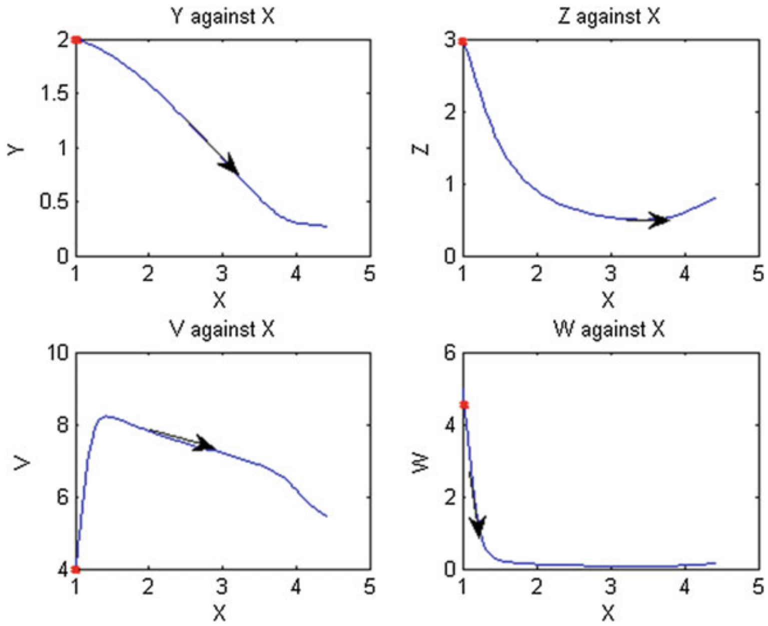

Fig. 3.7 10 system phase portrait figures

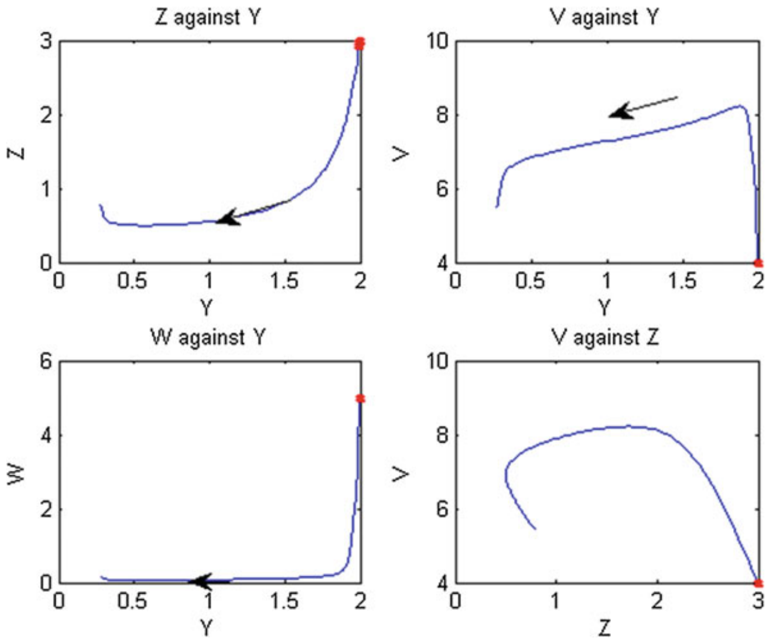

Fig. 3.8 10 system phase portrait figures

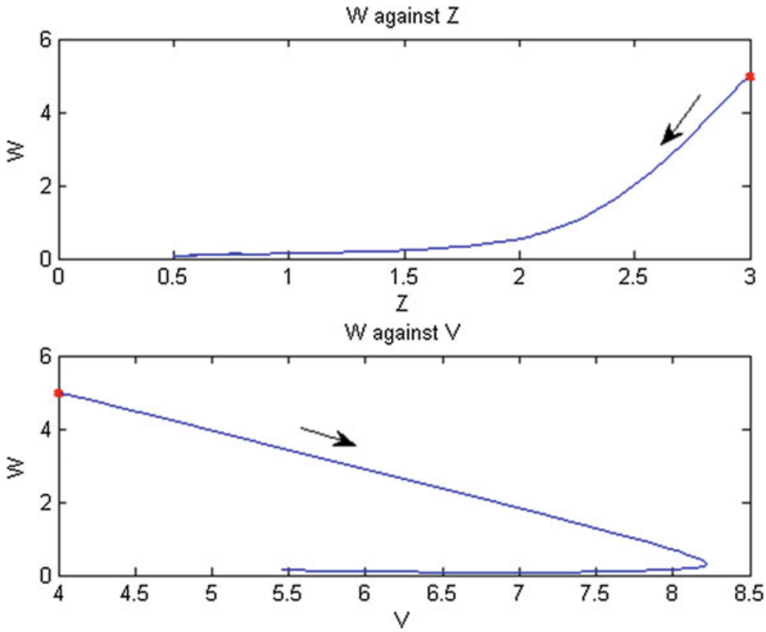

Fig. 3.9 10 system phase portrait figures W against Z and W against V

In this subchapter, we implement the circuit for X system variable [16, 25, 26]. We start with system first two differential equations:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A - B \cdot X + X^2 \cdot Y - X; \frac{\mathrm{d}Y}{\mathrm{d}t} = B \cdot X - X^2 \cdot Y$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A - B \cdot X + X^2 \cdot Y - X \Rightarrow Y = \left[\frac{\mathrm{d}X}{\mathrm{d}t} - A + B \cdot X + X\right] \cdot \frac{1}{X^2}$$

$$(*) \quad Y = \left[\frac{\mathrm{d}X}{\mathrm{d}t} - A + B \cdot X + X\right] \cdot \frac{1}{X^2} \Rightarrow Y = \frac{\frac{\mathrm{d}X}{\mathrm{d}t} - A}{X^2} + \frac{B+1}{X}$$

$$(**) \quad \frac{\mathrm{d}Y}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\frac{\mathrm{d}X}{\mathrm{d}t} - A}{X^2} + \frac{B+1}{X}\right\}$$

$$\Rightarrow \frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\frac{\mathrm{d}^2X}{\mathrm{d}t^2} \cdot X^2 - \left(\frac{\mathrm{d}X}{\mathrm{d}t} - A\right) \cdot 2 \cdot X \cdot \frac{\mathrm{d}X}{\mathrm{d}t}}{X^4} + \frac{-\frac{\mathrm{d}X}{\mathrm{d}t} \cdot (B+1)}{X^2}$$

$$(*) \,\&\, (*) \quad \Rightarrow \frac{\mathrm{d}Y}{\mathrm{d}t} = B \cdot X - X^2 \cdot Y; \frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\frac{\mathrm{d}^2X}{\mathrm{d}t^2}}{X^2} - \frac{(\frac{\mathrm{d}X}{\mathrm{d}t} - A) \cdot 2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t}}{X^3} - \frac{\frac{\mathrm{d}X}{\mathrm{d}t} \cdot (B+1)}{X^2}$$

$$\frac{\frac{d^2X}{dt^2}}{X^2} - \frac{\left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt}}{X^3} - \frac{\frac{dX}{dt} \cdot (B+1)}{X^2} = B \cdot X - X^2 \cdot \left\{\frac{\frac{dX}{dt} - A}{X^2} + \frac{B+1}{X}\right\}$$

$$\frac{\frac{d^2X}{dt^2}}{X^2} - \frac{\left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt}}{X^3} - \frac{\frac{dX}{dt} \cdot (B+1)}{X^2} = B \cdot X - \left\{\frac{dX}{dt} - A + X \cdot (B+1)\right\}$$

$$\frac{\frac{d^2X}{dt^2}}{X^2} - \frac{\left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt}}{X^3} - \frac{\frac{dX}{dt} \cdot (B+1)}{X^2} = -\frac{dX}{dt} + A - X$$

Multiple two sides of the above equation by $1/X^3$

$$X \cdot \frac{d^2X}{dt^2} - \left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt} - \frac{dX}{dt} \cdot (B+1) \cdot X = -\frac{dX}{dt} \cdot X^3 + A \cdot X^3 - X^4$$

$$X \cdot \left\{\frac{d^2X}{dt^2} - \frac{dX}{dt} \cdot (B+1)\right\} - \left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt} = -\frac{dX}{dt} \cdot X^3 + A \cdot X^3 - X^4$$

$$X \cdot \left\{\frac{d^2X}{dt^2} - \frac{dX}{dt} \cdot (B+1)\right\} = \left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt} - \frac{dX}{dt} \cdot X^3 + A \cdot X^3 - X^4$$

$$X = \frac{1}{\frac{d^2X}{dt^2} - \frac{dX}{dt} \cdot (B+1)} \cdot \left\{\left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt} - \frac{dX}{dt} \cdot X^3 + A \cdot X^3 - X^4\right\}$$

We define the following functions:

$$X = \frac{\chi_2\left(\frac{d^2X}{dt^2}, \frac{dX}{dt}\right)}{\chi_1\left(X, \frac{dX}{dt}\right)} = \frac{\chi_2}{\chi_1}; \chi_1 = \chi_1\left(X, \frac{dX}{dt}\right); \chi_2 = \chi_2\left(\frac{d^2X}{dt^2}, \frac{dX}{dt}\right)$$

$$\chi_1\left(X, \frac{dX}{dt}\right) = \left(\frac{dX}{dt} - A\right) \cdot 2 \cdot \frac{dX}{dt} - \frac{dX}{dt} \cdot X^3 + A \cdot X^3 - X^4; \chi_2\left(\frac{d^2X}{dt^2}, \frac{dX}{dt}\right) = \frac{d^2X}{dt^2} - \frac{dX}{dt} \cdot (B+1)$$

The following block diagram describes our system in terms of χ_1, χ_2 functions (Fig. 3.10).

Figure 3.11 block diagram describes our system in term of basic functional blocks $\left(\frac{d}{dt}, \frac{d^2}{dt^2}, [] \cdot [], []/[], []^n \quad \forall \quad n = 2, 3, 4\right)$.

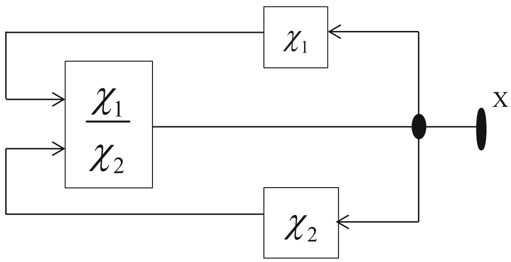

Fig. 3.10 System in terms of χ_1, χ_2 functions

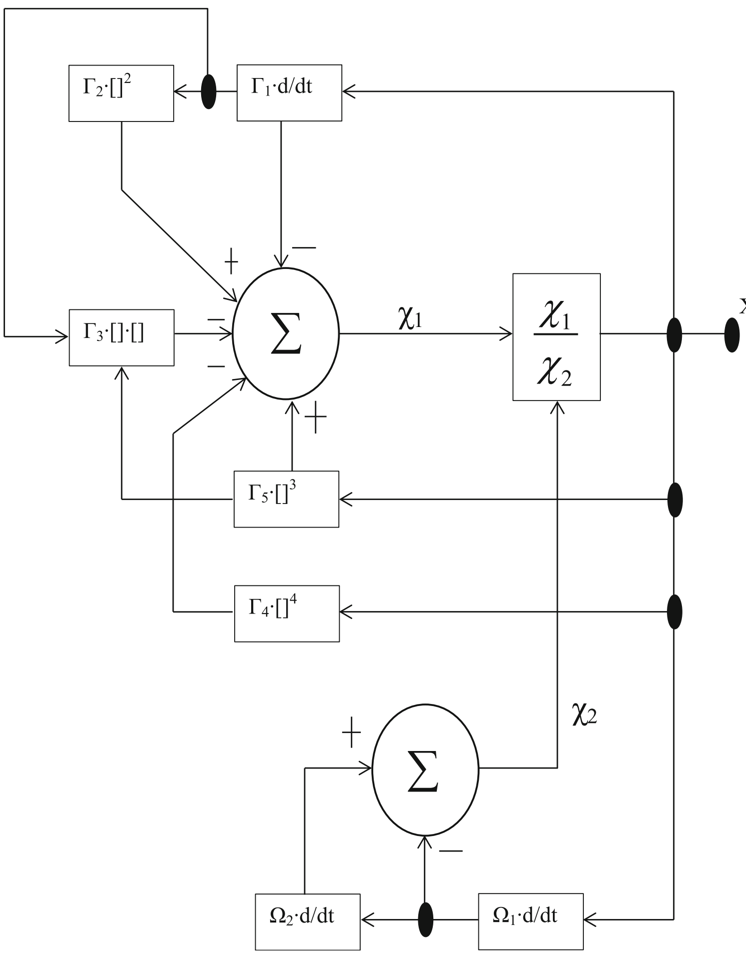

Fig. 3.11 system in term of basic functional blocks $\left(\frac{\mathrm{d}}{\mathrm{d}t}, \frac{\mathrm{d}^2}{\mathrm{d}t^2}, [] \cdot [], []/[], []^n \forall n = 2, 3, 4\right)$.

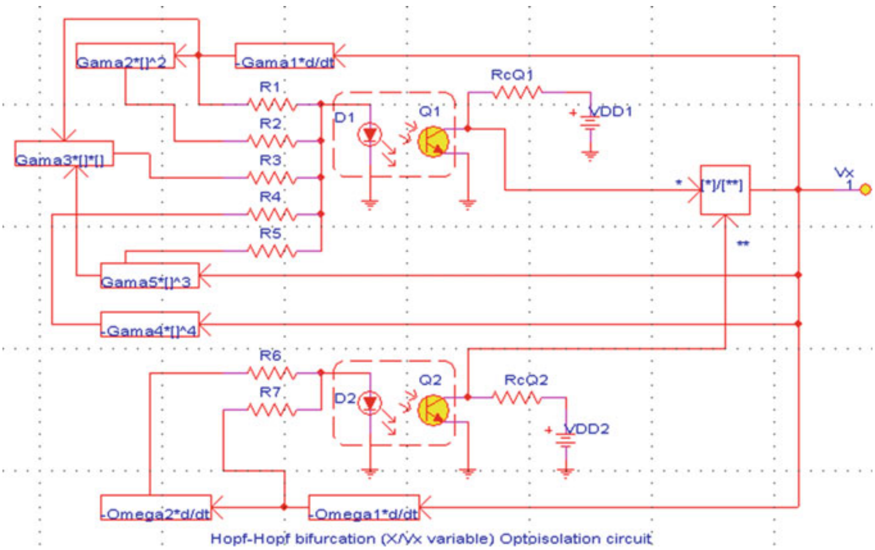

Fig. 3.12 Optoisolation circuit implementation to Hopf–Hopf bifurcation system for *X* variable

$$\Gamma_1, \ldots, \Gamma_5 \in \mathbb{R}; \Omega_1, \Omega_2$$

$$\chi_2 = \Omega_1 \cdot \Omega_2 \cdot \ddot{X} - \Omega_1 \cdot \dot{X}; \Omega_1 \cdot \Omega_2 = 1; \Omega_1 = B + 1 \Rightarrow \Omega_2 = \frac{1}{B+1}$$

$$\chi_1 = -\Gamma_1 \cdot \dot{X} + \Gamma_2 \cdot \Gamma_1^2 \cdot \dot{X}^2 - \Gamma_3 \cdot \Gamma_5 \cdot \Gamma_1 \cdot X^3 \cdot \dot{X} - \Gamma_4 \cdot X^4 + \Gamma_5 \cdot X^3$$

$$\Gamma_2 \cdot \Gamma_1^2 = 2; 2 \cdot A = \Gamma_1; A = \Gamma_5; \Gamma_4 = 1; \Gamma_5 \cdot \Gamma_1 \cdot \Gamma_3 = 1$$

$$\Gamma_1 = 2 \cdot A; \Gamma_2 = \frac{2}{\Gamma_1^2} = \frac{2}{4 \cdot A^2} = \frac{1}{2 \cdot A^2}; \Gamma_5 = A; \Gamma_3 = \frac{1}{\Gamma_5 \cdot \Gamma_1} = \frac{1}{2 \cdot A^2}; \Gamma_4 = 1$$

The next figure describes the optoisolation circuit implementation to Hopf–Hopf bifurcation system for *X* variable. We sign system variable *X* by voltage V_x then $X \rightleftharpoons V_X$ [5, 6].

Remark Reader exercise is to implement other system variables (*Y*, *Z*, *V*, *W*) by using optoisolation circuits $(Y \rightleftharpoons V_Y; Z \rightleftharpoons V_Z; V \rightleftharpoons V_V; W \rightleftharpoons V_W)$ (Fig. 3.12).

In the above circuit we sign the following parameters: Gama1 = Γ_1, Gama2 = Γ_2, Gama3 = Γ_3, Gama4 = Γ_4, Gama5 = Γ_5, Omega1 = Ω_1, Omega2 = Ω_2. Global functions: $\chi_1 \to (*); \chi_2 \to (**)$.

$$I_{D1} = \sum_{i=1}^{5} I_{Ri} = I_{R1} + \ldots + I_{R5}; I_{D2} = I_{R6} + I_{R7}; I_{BQ1} = k_1 \cdot I_{D1}; I_{BQ2} = k_2 \cdot I_{D2}$$

$V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D1}}{I_0}; V_{D2} = V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D2}}{I_0}$ (Taylor series approximation).

$$I_{R1} = \frac{-\Gamma_1 \cdot \dot{V}_X - V_{D1}}{R1}; I_{R2} = \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2 - V_{D1}}{R2} I_{R3} = \frac{-\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X - V_{D1}}{R3}$$

$$I_{R4} = \frac{-\Gamma_4 \cdot V_X^4 - V_{D1}}{R4}; I_{R5} = \frac{\Gamma_5 \cdot V_X^3 - V_{D1}}{R5}; I_{R6} = \frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X - V_{D1}}{R6}$$

$$I_{R7} = \frac{-\Omega_1 \cdot \dot{V}_X - V_{D1}}{R7}; I_{D2} = I_{R_6} + I_{R_7} = \frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X - V_{D2}}{R6} + \frac{-\Omega_1 \cdot \dot{V}_X - V_{D2}}{R7}$$

$$I_{D1} = \sum_{i=1}^{5} I_{R_i} = \frac{-\Gamma_1 \cdot \dot{V}_X - V_{D1}}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2 - V_{D1}}{R2} + \frac{-\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X - V_{D1}}{R3} + \frac{-\Gamma_4 \cdot V_X^4 - V_{D1}}{R4} + \frac{\Gamma_5 \cdot V_X^3 - V_{D1}}{R5}$$

$$I_{D2} = I_{R_6} + I_{R_7} = \frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} - \frac{\Omega_1 \cdot \dot{V}_X}{R7} - V_t \cdot \frac{I_{D2}}{I_0} \cdot \left(\frac{1}{R6} + \frac{1}{R7}\right)$$

$$I_{D2} \cdot \left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R6} + \frac{1}{R7}\right)\right] = I_{R_6} + I_{R_7} = \frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} - \frac{\Omega_1 \cdot \dot{V}_X}{R7}$$

$$I_{D2} = \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R6} + \frac{1}{R7}\right)\right]} \cdot \left\{\frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} - \frac{\Omega_1 \cdot \dot{V}_X}{R7}\right\}; \eta_2 = \frac{1}{\left[1 + \frac{V_t}{I_0} \cdot \left(\frac{1}{R6} + \frac{1}{R7}\right)\right]}$$

$$\psi_2(\ddot{V}_X, \dot{V}_X) = \frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} - \frac{\Omega_1 \cdot \dot{V}_X}{R7}; I_{D2} = \eta_2 \cdot \psi_2(\ddot{V}_X, \dot{V}_X); \psi_2 = \psi_2(\ddot{V}_X, \dot{V}_X)$$

$$I_{D1} = \sum_{i=1}^{5} I_{R_i} = \frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5} - V_{D1} \cdot \sum_{i=1}^{5} \frac{1}{Ri}$$

$$I_{D1} = \sum_{i=1}^{5} I_{R_i} = \frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5} - V_t \cdot \frac{I_{D1}}{I_0} \cdot \sum_{i=1}^{5} \frac{1}{Ri}$$

$$I_{D1} \cdot \left(1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{5} \frac{1}{Ri}\right) = \frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5}$$

$$I_{D1} = \frac{1}{\left(1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{5} \frac{1}{Ri}\right)} \cdot \left\{\frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5}\right\}$$

$$\eta_1 = \frac{1}{1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{5} \frac{1}{Ri}}; \psi_1(\dot{V}_X, V_X^2, V_X^3) = \frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5}$$

$$I_{D1} = \eta_1 \cdot \psi_1(\dot{V}_X, V_X^3, V_X^4); \psi_1 = \psi_1(\dot{V}_X, V_X^3, V_X^4); I_{D1} = \eta_1 \cdot \psi_1; \eta_1 > 0; \eta_2 > 0$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; I_{EQ2} = I_{BQ2} + I_{CQ2}; I_{EQ1} = k_1 \cdot I_{D1} + I_{CQ1} = k_1 \cdot \eta_1 \cdot \psi_1 + I_{CQ1}$$

$$I_{BQ1} = k_1 \cdot I_{D1} = k_1 \cdot \eta_1 \cdot \psi_1; I_{BQ2} = k_2 \cdot I_{D2} = k_2 \cdot \eta_2 \cdot \psi_2; I_{EQ2} = k_2 \cdot I_{D2} + I_{CQ2} = k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}$$

The Mathematical analysis is based on the basic Transistor Ebers–Moll equations. We need to implement the Regular Ebers–Moll Model to the above Opto Coupler circuit. Parameters k_1 and k_2 are coupling coefficients between LEDs $D1$, $D2$ and photo transistors $Q1$ and $Q2$, respectively.

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r_1 \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]; V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f_1}{Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]$$

$$V_{CEQ1} = V_{CBQ1} + V_{BEQ1}, \text{ but } V_{CBQ1} = -V_{BCQ1}, \text{ then } V_{CEQ1} = V_{BEQ1} - V_{BCQ1}$$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r_1 \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f_1}{Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r_1 \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f_1) + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right] + Vt \cdot \ln\left(\frac{Isc}{Ise}\right)$$

Since $Isc - Ise \to \varepsilon \Rightarrow \ln\left(\frac{Isc}{Ise}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.
$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r_1 \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f_1) + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right]$ and in the same manner

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{(\alpha r_2 \cdot I_{CQ2} - I_{EQ2}) + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha f_2) + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right]; V_X = \frac{[*]}{[**]} = \frac{V_{CEQ1}}{V_{CEQ2}}$$

$$\begin{aligned} \alpha r_1 \cdot I_{CQ1} - I_{EQ1} &= \alpha r_1 \cdot I_{CQ1} - [k_1 \cdot \eta_1 \cdot \psi_1 + I_{CQ1}] \\ &= I_{CQ1} \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 \end{aligned}$$

$$\begin{aligned} I_{CQ1} - I_{EQ1} \cdot \alpha f_1 &= I_{CQ1} - [k_1 \cdot \eta_1 \cdot \psi_1 + I_{CQ1}] \cdot \alpha f_1 \\ &= I_{CQ1} \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 \end{aligned}$$

$$\begin{aligned} I_{CQ1} &= \frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}} \Rightarrow \alpha r_1 \cdot I_{CQ1} - I_{EQ1} \\ &= \left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 \end{aligned}$$

$$\begin{aligned} I_{CQ1} &= \frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}} \Rightarrow I_{CQ1} - I_{EQ1} \cdot \alpha f_1 \\ &= \left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 \end{aligned}$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{\left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right]$$

$$\begin{aligned} V_{CEQ1} &\simeq Vt \\ &\cdot \ln\left[\frac{\frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - V_{CEQ1} \cdot \frac{(\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{\frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - V_{CEQ1} \cdot \frac{(1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right] \end{aligned}$$

$$\begin{aligned} V_{CEQ1} &\simeq Vt \\ &\cdot \ln\left[\frac{V_{CEQ1} \cdot \frac{(1 - \alpha r_1)}{R_{CQ1}} + \frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{-V_{CEQ1} \cdot \frac{(1 - \alpha f_1)}{R_{CQ1}} + \frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right] \end{aligned}$$

For simplicity we define the following functions:

$$\begin{aligned} \xi_1 &= \xi_1(V_{DD1}, k_1, \eta_1, \psi_1, \ldots) \\ &= \frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1) \end{aligned}$$

$$\xi_2 = \xi_2(V_{DD1}, k_1, \eta_1, \psi_1, \ldots) \\ = \frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{V_{CEQ1} \cdot \frac{(1-\alpha r_1)}{R_{CQ1}} + \xi_1}{-V_{CEQ1} \cdot \frac{(1-\alpha f_1)}{R_{CQ1}} + \xi_2}\right] \Rightarrow \mathrm{e}^{\left[\frac{V_{CEQ1}}{Vt}\right]} = \left[\frac{V_{CEQ1} \cdot \frac{(1-\alpha r_1)}{R_{CQ1}} + \xi_1}{-V_{CEQ1} \cdot \frac{(1-\alpha f_1)}{R_{CQ1}} + \xi_2}\right]$$

Taylor series approximation:

$$\mathrm{e}^{\left[\frac{V_{CEQ1}}{Vt}\right]} \approx \frac{V_{CEQ1}}{Vt} + 1 \Rightarrow \frac{V_{CEQ1}}{Vt} + 1 \approx \left[\frac{V_{CEQ1} \cdot \frac{(1-\alpha r_1)}{R_{CQ1}} + \xi_1}{-V_{CEQ1} \cdot \frac{(1-\alpha f_1)}{R_{CQ1}} + \xi_2}\right]$$

$$\Rightarrow \frac{V_{CEQ1}}{Vt} + 1 \approx \left[\frac{V_{CEQ1} \cdot (1 - \alpha r_1) + \xi_1 \cdot R_{CQ1}}{-V_{CEQ1} \cdot (1 - \alpha f_1) + \xi_2 \cdot R_{CQ1}}\right]$$

$$\left[\frac{V_{CEQ1}}{Vt} + 1\right] \cdot [-V_{CEQ1} \cdot (1 - \alpha f_1) + \xi_2 \cdot R_{CQ1}] = V_{CEQ1} \cdot (1 - \alpha r_1) + \xi_1 \cdot R_{CQ1}$$

$$-\frac{V_{CEQ1}^2 \cdot (1 - \alpha f_1)}{V_t} + \frac{V_{CEQ1} \cdot R_{CQ1}}{Vt} \cdot \xi_2 - V_{CEQ1} \cdot (1 - \alpha f_1) + \xi_2 \cdot R_{CQ1} \\ = V_{CEQ1} \cdot (1 - \alpha r_1) + \xi_1 \cdot R_{CQ1}$$

$$-\frac{V_{CEQ1}^2 \cdot (1 - \alpha f_1)}{V_t} + \frac{V_{CEQ1} \cdot R_{CQ1}}{Vt} \cdot \xi_2 - V_{CEQ1} \cdot (1 - \alpha f_1) \\ - V_{CEQ1} \cdot (1 - \alpha r_1) + \xi_2 \cdot R_{CQ1} - \xi_1 \cdot R_{CQ1} = 0$$

$$-\frac{V_{CEQ1}^2 \cdot (1 - \alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \xi_2 - (1 - \alpha f_1) - (1 - \alpha r_1)\right\} \\ + [\xi_2 - \xi_1] \cdot R_{CQ1} = 0$$

$$\frac{V_{CEQ1}^2 \cdot (1 - \alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{-\frac{R_{CQ1}}{Vt} \cdot \xi_2 + (1 - \alpha f_1) + (1 - \alpha r_1)\right\} \\ + [\xi_1 - \xi_2] \cdot R_{CQ1} = 0$$

$$\frac{V_{CEQ1}^2 \cdot (1 - \alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{-\frac{R_{CQ1}}{Vt} \cdot \xi_2 + 2 - \alpha f_1 - \alpha r_1\right\} + [\xi_1 - \xi_2] \cdot R_{CQ1} = 0$$

We define for simplicity new parameter A_1, when

$$A_1 = 2 - \alpha f_1 - \alpha r_1 \ \& \ 1 < \alpha f_1 + \alpha r_1 < 2 \Rightarrow 1 > A_1 > 0$$

$$\frac{V_{CEQ1}^{2}\cdot(1-\alpha f_1)}{V_t}+V_{CEQ1}\cdot\left\{A_1-\frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}+[\xi_1-\xi_2]\cdot R_{CQ1}=0$$

$$V_{CEQ1}^{\#,\#\#}=\frac{-A_1+\frac{R_{CQ1}}{Vt}\cdot\xi_2\pm\sqrt{\left\{A_1-\frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2-4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}}}{2\cdot\frac{(1-\alpha f_1)}{V_t}}$$

$$\begin{aligned}&\left\{A_1-\frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2-4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}\\&=\xi_2^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2-2\cdot A_1\cdot\frac{R_{CQ1}}{Vt}\cdot\xi_2+A_1^2-4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\xi_1+4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\xi_2\end{aligned}$$

$$\begin{aligned}&\left\{A_1-\frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2-4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}\\&=\xi_2^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2+\xi_2\cdot\left\{-2\cdot A_1\cdot\frac{R_{CQ1}}{Vt}+4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\right\}-4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\xi_1+A_1^2\end{aligned}$$

For simplicity we define the following functions:

$$\begin{aligned}\xi_1&=\xi_1(V_{DD1},k_1,\eta_1,\psi_1,\ldots)\\&=\frac{V_{DD1}\cdot(\alpha r_1-1)}{R_{CQ1}}-k_1\cdot\eta_1\cdot\psi_1+Ise\cdot(\alpha r_1\cdot\alpha f_1-1)\end{aligned}$$

$$\Xi_1=\frac{V_{DD1}\cdot(\alpha r_1-1)}{R_{CQ1}}+Ise\cdot(\alpha r_1\cdot\alpha f_1-1);\Xi_2=k_1\cdot\eta_1\Rightarrow\xi_1=\Xi_1-\Xi_2\cdot\psi_1$$

$$\begin{aligned}\xi_2=\xi_2(V_{DD1},k_1,\eta_1,\psi_1,\ldots)=&\frac{V_{DD1}\cdot(1-\alpha f_1)}{R_{CQ1}}-k_1\cdot\eta_1\cdot\psi_1\cdot\alpha f_1\\&+Isc\cdot(\alpha r_1\cdot\alpha f_1-1)\end{aligned}$$

$$\begin{aligned}\Xi_3&=\frac{V_{DD1}\cdot(1-\alpha f_1)}{R_{CQ1}}+Isc\cdot(\alpha r_1\cdot\alpha f_1-1);\Xi_4=k_1\cdot\eta_1\cdot\alpha f_1\Rightarrow\xi_2\\&=\Xi_3-\Xi_4\cdot\psi_1;\Xi_4=\alpha f_1\cdot\Xi_2\end{aligned}$$

$$\begin{aligned}&\left\{A_1-\tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2-4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}\\&=[\Xi_3-\Xi_4\cdot\psi_1]^2\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2+[\Xi_3-\Xi_4\cdot\psi_1]\cdot\left\{-2\cdot A_1\cdot\tfrac{R_{CQ1}}{Vt}+4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\right\}\\&\quad-4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot[\Xi_1-\Xi_2\cdot\psi_1]+A_1^2\end{aligned}$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}$$
$$= \left\{\Xi_3^2 - 2\cdot\Xi_3\cdot\Xi_4\cdot\psi_1 + \Xi_4^2\cdot\psi_1^2\right\}\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2 + [\Xi_3 - \Xi_4\cdot\psi_1]\cdot\tfrac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\}$$
$$-4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}\cdot\psi_1 + A_1^2$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}$$
$$= \Xi_3^2\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2 - 2\cdot\Xi_3\cdot\Xi_4\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2\cdot\psi_1 + \Xi_4^2\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2\cdot\psi_1^2 + \Xi_3\cdot\tfrac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\}$$
$$-\Xi_4\cdot\tfrac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\}\cdot\psi_1 - 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}\cdot\psi_1 + A_1^2$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}$$
$$= \Xi_4^2\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2\cdot\psi_1^2 + \psi_1\cdot\left\{-2\cdot\Xi_3\cdot\Xi_4\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2 - \Xi_4\cdot\tfrac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\} + 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}\right\}$$
$$+\Xi_3^2\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2 + \Xi_3\cdot\tfrac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\} - 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + A_1^2$$

For simplicity we define the following global parameters:

$$\Delta_1 = \Xi_4^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2;$$
$$\Delta_2 = -2\cdot\Xi_3\cdot\Xi_4\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 - \Xi_4\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\}$$
$$+4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}$$

$$\Delta_3 = \Xi_3^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 + \Xi_3\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\} - 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + A_1^2$$

$$\left\{A_1 - \frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1} = \Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1 + \Delta_3;$$
$$\Delta_1 = \Xi_4^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 > 0$$

We are using completing the square rule: $A\cdot x^2 + B\cdot x = \left(x\cdot\sqrt{A} + \frac{B}{2\cdot\sqrt{A}}\right)^2 - \frac{B^2}{4\cdot A}$
For our case: $\Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1$; $A = \Delta_1$; $B = \Delta_2$; $x = \psi_1$

$$\Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1 = \left(\psi_1\cdot\sqrt{\Delta_1} + \frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)^2 - \frac{\Delta_2^2}{4\cdot\Delta_1}$$

$$\left\{A_1 - \frac{R_{CQ1}}{Vt} \cdot \xi_2\right\}^2 - 4 \cdot \frac{(1-\alpha f_1)}{V_t} \cdot [\xi_1 - \xi_2] \cdot R_{CQ1}$$
$$= \Delta_1 \cdot \psi_1^2 + \Delta_2 \cdot \psi_1 + \Delta_3 = \left(\psi_1 \cdot \sqrt{\Delta_1} + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right)^2 - \frac{\Delta_2^2}{4 \cdot \Delta_1} + \Delta_3$$

For simplicity we choose circuit parameters constrain (first request):

$$-\frac{\Delta_2^2}{4 \cdot \Delta_1} + \Delta_3 \to \varepsilon \forall 0 < \varepsilon \ll 1$$

$$\left(\psi_1 \cdot \sqrt{\Delta_1} + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right)^2 - \frac{\Delta_2^2}{4 \cdot \Delta_1} + \Delta_3 \approx \left(\psi_1 \cdot \sqrt{\Delta_1} + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right)^2$$

Finally, we can state that

$$\left\{A_1 - \frac{R_{CQ1}}{Vt} \cdot \xi_2\right\}^2 - 4 \cdot \frac{(1-\alpha f_1)}{V_t} \cdot [\xi_1 - \xi_2] \cdot R_{CQ1} \simeq \left(\psi_1 \cdot \sqrt{\Delta_1} + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right)^2$$

$$V_{CEQ1}^{\#,\#\#} = \frac{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \xi_2 \pm \left(\psi_1 \cdot \sqrt{\Delta_1} + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right)}{2 \cdot \frac{(1-\alpha f_1)}{V_t}}; \xi_2 = \Xi_3 - \Xi_4 \cdot \psi_1$$

$$V_{CEQ1}^{\#,\#\#} = \frac{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{R_{CQ1}}{Vt} \cdot \Xi_4 \cdot \psi_1 \pm \left(\psi_1 \cdot \sqrt{\Delta_1} + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right)}{2 \cdot \frac{(1-\alpha f_1)}{V_t}}$$

We get possible two expressions for $V_{CEQ1} \cdot \chi_1^{\#} = V_{CEQ1}^{\#} = V_{CEQ1}^{(+)}; \chi_1^{\#\#} = V_{CEQ1}^{\#\#} = V_{CEQ1}^{(-)}$

$$V_{CEQ1}^{\#} = \frac{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} + \psi_1 \cdot \left\{\sqrt{\Delta_1} - \frac{R_{CQ1}}{Vt} \cdot \Xi_4\right\}}{2 \cdot \frac{(1-\alpha f_1)}{V_t}}$$

$$V_{CEQ1}^{\#} = \left[\frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - A_1\right] \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)}$$
$$+ \psi_1 \cdot \left\{\sqrt{\Delta_1} - \frac{R_{CQ1}}{Vt} \cdot \Xi_4\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)}$$

Since $\left\{\sqrt{\Delta_1} - \frac{R_{CQ1}}{Vt} \cdot \Xi_4\right\}\Big|_{\Delta_1 = \Xi_4^2 \cdot \left[\frac{R_{CQ1}}{Vt}\right]^2} = 0 \Rightarrow V_{CEQ1}^{\#} = \left[\frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - A_1\right] \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)}$

Our above result for $V_{CEQ1}^{\#}$ does not fulfill our demand for χ_1 differential equation $\chi_1 = -\Gamma_1 \cdot \dot{X} + \Gamma_2 \cdot \Gamma_1^2 \cdot \dot{X}^2 - \Gamma_3 \cdot \Gamma_5 \cdot \Gamma_1 \cdot X^3 \cdot \dot{X} - \Gamma_4 \cdot X^4 + \Gamma_5 \cdot X^3$.

$$V_{CEQ1}^{\#\#} = \frac{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - \psi_1 \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\}}{2 \cdot \frac{(1-\alpha f_1)}{V_t}}$$

$$V_{CEQ1}^{\#\#} = \left\{ -A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)} - \psi_1 \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)}$$

The second request we need to fulfill and get $V_{CEQ1} = [*] = \chi_1$ differential equation shape:

$$\left\{ -A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)} \to \varepsilon \quad \forall\, 0 < \varepsilon \ll 1$$

$$V_{CEQ1}^{\#\#} \approx -\psi_1 \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)}; \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)} < 0 \text{ (!!)}$$

$$\left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)} < 0 \Rightarrow \frac{R_{CQ1}}{Vt} \cdot \Xi_4 < -\sqrt{\Delta_1} \Rightarrow \Xi_4 < -\frac{Vt \cdot \sqrt{\Delta_1}}{R_{CQ1}}; \frac{Vt \cdot \sqrt{\Delta_1}}{R_{CQ1}} > 0$$

But $\Xi_4 = k_1 \cdot \eta_1 \cdot \alpha f_1$ is positive number so we need to fulfill the following:

$$\psi_1 = \frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5}$$

$$-\psi_1 = \frac{\Gamma_1 \cdot \dot{V}_X}{R1} - \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} + \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} + \frac{\Gamma_4 \cdot V_X^4}{R4} - \frac{\Gamma_5 \cdot V_X^3}{R5}$$

$V_{CEQ1}^{\#\#} \approx -\psi_1 \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)}$; By choosing $\Gamma_1 < 0$, $\Gamma_2 < 0$, $\Gamma_3 < 0$, $\Gamma_5 < 0$, and $\Gamma_4 < 0$ we get our χ_1 differential equation.

$$V_X \rightleftharpoons X; V_{CEQ1}^{\#\#} \rightleftharpoons \chi_1; \frac{1}{Ri} \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_1)} = 1 \forall i = 1, \ldots, 5$$

$$\chi_1 = -\Gamma_1 \cdot \dot{X} + \Gamma_2 \cdot \Gamma_1^2 \cdot \dot{X}^2 - \Gamma_3 \cdot \Gamma_5 \cdot \Gamma_1 \cdot X^3 \cdot \dot{X} - \Gamma_4 \cdot X^4 + \Gamma_5 \cdot X^3$$

The next step is to find our [**] circuit expression (V_{CEQ2}).

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{(\alpha r_2 \cdot I_{CQ2} - I_{EQ2}) + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha f_2) + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right]; I_{EQ2} = k_2 \cdot I_{D2} + I_{CQ2} \\ = k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}$$

$$\alpha r_2 \cdot I_{CQ2} - I_{EQ2} = \alpha r_2 \cdot I_{CQ2} - [k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}] = [\alpha r_2 - 1] \cdot I_{CQ2} - k_2 \cdot \eta_2 \cdot \psi_2$$

$$I_{CQ2} - I_{EQ2} \cdot \alpha f_2 = I_{CQ2} - [k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}] \cdot \alpha f_2 \\ = I_{CQ2} \cdot [1 - \alpha f_2] - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2$$

$$I_{CQ2} = \frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}} \Rightarrow \alpha r_2 \cdot I_{CQ2} - I_{EQ2} \\ = [\alpha r_2 - 1] \cdot \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] - k_2 \cdot \eta_2 \cdot \psi_2$$

$$I_{CQ2} = \frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}} \Rightarrow I_{CQ2} - I_{EQ2} \cdot \alpha f_2 \\ = \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] \cdot [1 - \alpha f_2] - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2$$

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{[\alpha r_2 - 1] \cdot \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] - k_2 \cdot \eta_2 \cdot \psi_2 + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{\left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] \cdot [1 - \alpha f_2] - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right]$$

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{[\alpha r_2 - 1] \cdot \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2}{\left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] \cdot [1 - \alpha f_2] + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2}\right]$$

For simplicity we define the following functions:

$$\xi_3 = \xi_3(V_{DD2}, k_2, \eta_2, \psi_2, \ldots) \\ = [\alpha r_2 - 1] \cdot \frac{V_{DD2}}{R_{CQ2}} + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2$$

$$\xi_4 = \xi_4(V_{DD2}, k_2, \eta_2, \psi_2, \ldots) \\ = \frac{V_{DD2}}{R_{CQ2}} \cdot [1 - \alpha f_2] + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2$$

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{V_{CEQ2} \cdot \frac{(1-\alpha r_2)}{R_{CQ2}} + \xi_3}{-V_{CEQ2} \cdot \frac{(1-\alpha f_2)}{R_{CQ2}} + \xi_4}\right] \Rightarrow e^{\left[\frac{V_{CEQ2}}{Vt}\right]} = \left[\frac{V_{CEQ2} \cdot \frac{(1-\alpha r_2)}{R_{CQ2}} + \xi_3}{-V_{CEQ2} \cdot \frac{(1-\alpha f_2)}{R_{CQ2}} + \xi_4}\right]$$

$$e^{\left[\frac{V_{CEQ2}}{Vt}\right]} \approx \frac{V_{CEQ2}}{Vt} + 1 \Rightarrow \frac{V_{CEQ2}}{Vt} + 1 \approx \left[\frac{V_{CEQ2} \cdot \frac{(1-\alpha r_2)}{R_{CQ2}} + \xi_3}{-V_{CEQ2} \cdot \frac{(1-\alpha f_2)}{R_{CQ2}} + \xi_4}\right]$$

$$\Rightarrow \frac{V_{CEQ2}}{Vt} + 1 \approx \frac{V_{CEQ2} \cdot (1 - \alpha r_2) + \xi_3 \cdot R_{CQ2}}{-V_{CEQ2} \cdot (1 - \alpha f_2) + \xi_4 \cdot R_{CQ2}}$$

$$\left[\frac{V_{CEQ2}}{Vt} + 1\right] \cdot [-V_{CEQ2} \cdot (1 - \alpha f_2) + \xi_4 \cdot R_{CQ2}] = V_{CEQ2} \cdot (1 - \alpha r_2) + \xi_3 \cdot R_{CQ2}$$

$$-\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + \frac{V_{CEQ2} \cdot R_{CQ2}}{Vt} \cdot \xi_4 - V_{CEQ2} \cdot (1 - \alpha f_2) + \xi_4 \cdot R_{CQ2}$$
$$= V_{CEQ2} \cdot (1 - \alpha r_2) + \xi_3 \cdot R_{CQ2}$$

$$-\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + \frac{V_{CEQ2} \cdot R_{CQ2}}{Vt} \cdot \xi_4 - V_{CEQ2} \cdot (1 - \alpha f_2) - V_{CEQ2} \cdot (1 - \alpha r_2)$$
$$+ \xi_4 \cdot R_{CQ2} - \xi_3 \cdot R_{CQ2} = 0$$

$$-\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \xi_4 - (1 - \alpha f_2) - (1 - \alpha r_2)\right\}$$
$$+ [\xi_4 - \xi_3] \cdot R_{CQ2} = 0$$

$$\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \xi_4 + (1 - \alpha f_2) + (1 - \alpha r_2)\right\}$$
$$+ [\xi_3 - \xi_4] \cdot R_{CQ2} = 0$$

$$\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{-\frac{R_{CQ2}}{Vt} \cdot \xi_4 + 2 - \alpha f_2 - \alpha r_2\right\}$$
$$+ [\xi_3 - \xi_4] \cdot R_{CQ2} = 0$$

We define for simplicity new parameter A_2, when

$$A_2 = 2 - \alpha f_2 - \alpha r_2 \;\&\; 1 < \alpha f_2 + \alpha r_2 < 2 \Rightarrow 1 > A_2 > 0$$

$$\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\} + [\xi_3 - \xi_4] \cdot R_{CQ2} = 0$$

$$V_{CEQ2}^{\#,\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \xi_4 \pm \sqrt{\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2}}}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= \xi_4^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2 - 2\cdot A_2\cdot\frac{R_{CQ2}}{Vt}\cdot\xi_4 + A_2^2$$
$$-4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\xi_3 + 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\xi_4$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= \xi_4^2\cdot[\frac{R_{CQ2}}{Vt}]^2 + \xi_4\cdot\left\{-2\cdot A_2\cdot\frac{R_{CQ2}}{Vt} + 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\right\}$$
$$-4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\xi_3 + A_2^2$$

For simplicity we define the following functions:

$$\xi_3 = \xi_3(V_{DD2}, k_2, \eta_2, \psi_2, \ldots)$$
$$= [\alpha r_2 - 1]\cdot\frac{V_{DD2}}{R_{CQ2}} + Ise\cdot(\alpha r_2\cdot\alpha f_2 - 1) - k_2\cdot\eta_2\cdot\psi_2$$

$$\Xi_5 = [\alpha r_2 - 1]\cdot\frac{V_{DD2}}{R_{CQ2}} + Ise\cdot(\alpha r_2\cdot\alpha f_2 - 1);\ \Xi_6 = k_2\cdot\eta_2 \Rightarrow \xi_3 = \Xi_5 - \Xi_6\cdot\psi_2$$

$$\xi_4 = \xi_4(V_{DD2}, k_2, \eta_2, \psi_2, \ldots)$$
$$= \frac{V_{DD2}}{R_{CQ2}}\cdot[1-\alpha f_2] + Isc\cdot(\alpha r_2\cdot\alpha f_2 - 1) - k_2\cdot\eta_2\cdot\alpha f_2\cdot\psi_2$$

$$\Xi_7 = \frac{V_{DD2}}{R_{CQ2}}\cdot[1-\alpha f_2] + Isc\cdot(\alpha r_2\cdot\alpha f_2 - 1);\ \Xi_8 = k_2\cdot\eta_2\cdot\alpha f_2 \Rightarrow \xi_4$$
$$= \Xi_7 - \Xi_8\cdot\psi_2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= [\Xi_7 - \Xi_8\cdot\psi_2]^2\cdot[\frac{R_{CQ2}}{Vt}]^2 + [\Xi_7 - \Xi_8\cdot\psi_2]\cdot\left\{-2\cdot A_2\cdot\frac{R_{CQ2}}{Vt} + 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\right\}$$
$$-4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot[\Xi_5 - \Xi_6\cdot\psi_2] + A_2^2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= \left\{\Xi_7^2 - 2\cdot\Xi_7\cdot\Xi_8\cdot\psi_2 + \Xi_8^2\cdot\psi_2^2\right\}\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2 + [\Xi_7 - \Xi_8\cdot\psi_2]$$
$$\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\}$$
$$-4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\Xi_5 + 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}\cdot\Xi_6}{V_t}\cdot\psi_2 + A_2^2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= \Xi_7^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2 - 2\cdot\Xi_7\cdot\Xi_8\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2\cdot\psi_2 + \Xi_8^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2\cdot\psi_2^2$$
$$+\Xi_7\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\} - \Xi_8\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\}\cdot\psi_2$$
$$-4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\Xi_5 + 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}\cdot\Xi_6}{V_t}\cdot\psi_2 + A_2^2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= \Xi_8^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2\cdot\psi_2^2 + \psi_2\cdot\left\{-2\cdot\Xi_7\cdot\Xi_8\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2 - \Xi_8\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\}\right.$$
$$\left. + 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}\cdot\Xi_6}{V_t}\right\} + \Xi_7\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\} - 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\Xi_5$$
$$+ A_2^2 + \Xi_7^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2$$

For simplicity we define the following global parameters:

$$\Delta_4 = \Xi_8^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2; \Delta_5 = -2\cdot\Xi_7\cdot\Xi_8\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2 - \Xi_8\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\}$$
$$+4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}\cdot\Xi_6}{V_t}$$

$$\Delta_6 = \Xi_7\cdot\frac{2\cdot R_{CQ2}}{Vt}\cdot\{2\cdot(1-\alpha f_2) - A_2\} - 4\cdot\frac{(1-\alpha f_2)\cdot R_{CQ2}}{V_t}\cdot\Xi_5 + A_2^2 + \Xi_7^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt}\cdot\xi_4\right\}^2 - 4\cdot\frac{(1-\alpha f_2)}{V_t}\cdot[\xi_3-\xi_4]\cdot R_{CQ2}$$
$$= \Delta_4\cdot\psi_2^2 + \Delta_5\cdot\psi_2 + \Delta_6; \Delta_4 = \Xi_8^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2 > 0$$

We are using completing the square rule: $A\cdot x^2 + B\cdot x = \left(x\cdot\sqrt{A} + \frac{B}{2\cdot\sqrt{A}}\right)^2 - \frac{B^2}{4\cdot A}$
For our case: $\Delta_4\cdot\psi_2^2 + \Delta_5\cdot\psi_2$; $A = \Delta_4$; $B = \Delta_5$; $x = \psi_2$

$$\Delta_4 \cdot \psi_2^2 + \Delta_5 \cdot \psi_2 = \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2 - \frac{\Delta_5^2}{4 \cdot \Delta_4}$$

$$\begin{aligned}\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} &= \Delta_4 \cdot \psi_2^2 + \Delta_5 \cdot \psi_2 + \Delta_6 \\ &= \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2 - \frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6\end{aligned}$$

For simplicity we choose circuit parameters constrain (first request):

$$-\frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6 \to \varepsilon \forall 0 < \varepsilon \ll 1$$

$$\left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2 - \frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6 \approx \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2$$

Finally, we can state that

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} \simeq \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2$$

$$V_{CEQ2}^{\#,\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \xi_4 \pm \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}; \xi_4 = \Xi_7 - \Xi_8 \cdot \psi_2$$

$$V_{CEQ2}^{\#,\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{R_{CQ2}}{Vt} \cdot \Xi_8 \cdot \psi_2 \pm \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

We get possible two expressions for $V_{CEQ1} \cdot \chi_2^{\#} = V_{CEQ2}^{\#} = V_{CEQ2}^{(+)}$; $\chi_2^{\#\#} = V_{CEQ2}^{\#\#} = V_{CEQ2}^{(-)}$

$$V_{CEQ2}^{\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} + \psi_2 \cdot \left\{\sqrt{\Delta_4} - \frac{R_{CQ2}}{Vt} \cdot \Xi_8\right\}}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

$$\begin{aligned}V_{CEQ2}^{\#} = &\left[\frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - A_2\right] \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)} \\ &+ \psi_2 \cdot \left\{\sqrt{\Delta_4} - \frac{R_{CQ2}}{Vt} \cdot \Xi_8\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}\end{aligned}$$

Since $\left\{\sqrt{\Delta_4} - \frac{R_{CQ2}}{Vt}\cdot\Xi_8\right\}\Big|_{\Delta_4=\Xi_8^2\cdot\left[\frac{R_{CQ2}}{Vt}\right]^2} = 0 \Rightarrow V^{\#}_{CEQ2} = \left[\frac{R_{CQ2}}{Vt}\cdot\Xi_7 + \frac{\Delta_5}{2\cdot\sqrt{\Delta_4}} - A_2\right]\cdot \frac{V_t}{2\cdot(1-\alpha f_2)}$

Our above result for $V^{\#}_{CEQ2}$ do not fulfill our demand for χ_2 differential equation $\chi_2 = \Omega_1\cdot\Omega_2\cdot\ddot{X} - \Omega_1\cdot\dot{X}$.

$$V^{\#\#}_{CEQ2} = \frac{-A_2 + \frac{R_{CQ2}}{Vt}\cdot\Xi_7 - \frac{\Delta_5}{2\cdot\sqrt{\Delta_4}} - \psi_2\cdot\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}}{2\cdot\frac{(1-\alpha f_2)}{V_t}}$$

$$V^{\#\#}_{CEQ2} = \left\{-A_2 + \frac{R_{CQ2}}{Vt}\cdot\Xi_7 - \frac{\Delta_5}{2\cdot\sqrt{\Delta_4}}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)} - \psi_2\cdot\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)}$$

$$V^{\#\#}_{CEQ2} \approx -\psi_2\cdot\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)};$$

$$\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)} < 0 \quad (!!)$$

$$\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)} < 0 \Rightarrow \frac{R_{CQ2}}{Vt}\cdot\Xi_8 < -\sqrt{\Delta_4}$$

$$\Rightarrow \Xi_8 < -\frac{Vt\cdot\sqrt{\Delta_4}}{R_{CQ2}}; \frac{Vt\cdot\sqrt{\Delta_4}}{R_{CQ2}} > 0$$

But $\Xi_8 = k_2\cdot\eta_2\cdot\alpha f_2$ is positive number so we need to fulfill the following:

$$\psi_2 = \frac{\Omega_1\cdot\Omega_2\cdot\ddot{V}_X}{R6} - \frac{\Omega_1\cdot\dot{V}_X}{R7} \Rightarrow -\psi_2 = -\frac{\Omega_1\cdot\Omega_2\cdot\ddot{V}_X}{R6} + \frac{\Omega_1\cdot\dot{V}_X}{R7}$$

$V^{\#\#}_{CEQ2} \approx -\psi_2\cdot\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)}$; By choosing $\Omega_1 < 0; \Omega_2 > 0$
We get our χ_2 differential equation $\chi_2 = \Omega_1\cdot\Omega_2\cdot\ddot{X} - \Omega_1\cdot\dot{X}$

$$V_X \rightleftharpoons X; V^{\#\#}_{CEQ2} \rightleftharpoons \chi_2; \frac{1}{Ri}\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8 + \sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)} = 1 \quad \forall\, i = 6,7$$

We can summarize our results for $\chi_1^{\#,\#\#} = V^{\#,\#\#}_{CEQ1} = V^{(+/-)}_{CEQ1}$ in Table 3.4.
We get some results for our Hopf–Hopf bifurcation system parameters:

Table 3.4 Summary results for $\chi_1^{\#,\#\#} = V_{CEQ1}^{\#,\#\#} = V_{CEQ1}^{(+/-)}$

Hopf–Hopf bifurcation system χ_1 expression which related to X variable	Hopf–Hopf bifurcation optoisolation circuit V_{CEQ1} expression which related to V_X variable
$\chi_1(X, \frac{dX}{dt}) = 2 \cdot \left[\frac{dX}{dt}\right]^2 - A \cdot 2 \cdot \frac{dX}{dt} - \frac{dX}{dt} \cdot X^3 + A \cdot X^3 - X^4$	$V_{CEQ1}^{\#\#} \approx -\psi_1 \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$ $\psi_1(\dot{V}_X, V_X^2, V_X^3) = \frac{-\Gamma_1 \cdot \dot{V}_X}{R1} + \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} - \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} - \frac{\Gamma_4 \cdot V_X^4}{R4} + \frac{\Gamma_5 \cdot V_X^3}{R5}$ $-\psi_1(\dot{V}_X, V_X^2, V_X^3) = \frac{\Gamma_1 \cdot \dot{V}_X}{R1} - \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} + \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} + \frac{\Gamma_4 \cdot V_X^4}{R4} - \frac{\Gamma_5 \cdot V_X^3}{R5}$
Results Hopf–Hopf bifurcation system block diagram: $\chi_1 = -\Gamma_1 \cdot \dot{X} + \Gamma_2 \cdot \Gamma_1^2 \cdot \dot{X}^2 - \Gamma_3 \cdot \Gamma_5 \cdot \Gamma_1 \cdot X^3 \cdot \dot{X} - \Gamma_4 \cdot X^4 + \Gamma_5 \cdot X^3$ $\Gamma_1 = 2 \cdot A; \Gamma_2 = \frac{2}{\Gamma_1^2} = \frac{2}{4 \cdot A^2} = \frac{1}{2 \cdot A^2}$ $\Gamma_5 = A; \Gamma_3 = \frac{1}{\Gamma_5 \cdot \Gamma_1} = \frac{1}{2 \cdot A^2}; \Gamma_4 = 1$	Circuit parameters constrain: $0 < \varepsilon \ll 1$ $-\frac{\Delta_2^2}{4 \cdot \Delta_1} + \Delta_3 \to \varepsilon \forall 0 < \varepsilon \ll 1$ $V_X \rightleftharpoons X; V_{CEQ1}^{\#\#} \rightleftharpoons \chi_1$ $\frac{1}{Ri} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)} = 1 \forall i = 1, \ldots, 5$ $V_{CEQ1}^{\#\#} \approx \left\{\frac{\Gamma_1 \cdot \dot{V}_X}{R1} - \frac{\Gamma_1^2 \cdot \Gamma_2 \cdot [\dot{V}_X]^2}{R2} + \frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \cdot V_X^3 \cdot \dot{V}_X}{R3} + \frac{\Gamma_4 \cdot V_X^4}{R4} - \frac{\Gamma_5 \cdot V_X^3}{R5}\right\} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$
2	$-\frac{\Gamma_1^2 \cdot \Gamma_2}{R_2} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$
−2 · A	$\frac{\Gamma_1}{R1} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$
−1	$\frac{\Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5}{R3} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$
A	$-\frac{\Gamma_5}{R5} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$
−1	$\frac{\Gamma_4}{R4} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2\cdot(1-\alpha f_1)}$

$$A = -\frac{\Gamma_1}{2 \cdot R1} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)};$$

$$A = -\frac{\Gamma_5}{R5} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} \Rightarrow$$

$$\frac{\Gamma_1}{2 \cdot R1} = \frac{\Gamma_5}{R5}; -\frac{\Gamma_4}{R4} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} = 1 \Rightarrow \Gamma_4 = -\frac{2 \cdot (1 - \alpha f_1) \cdot R4}{\left\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\right\} \cdot V_t}$$

$$-\frac{\Gamma_1^2\cdot\Gamma_2}{R_2}\cdot\left\{\frac{R_{CQ1}}{Vt}\cdot\Xi_4+\sqrt{\Delta_1}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_1)}=2\Rightarrow\Gamma_1^2\cdot\Gamma_2=-\frac{4\cdot(1-\alpha f_1)\cdot R_2}{\left\{\frac{R_{CQ1}}{Vt}\cdot\Xi_4+\sqrt{\Delta_1}\right\}\cdot V_t}$$

$$-\frac{\Gamma_1\cdot\Gamma_3\cdot\Gamma_5}{R3}\cdot\left\{\frac{R_{CQ1}}{Vt}\cdot\Xi_4+\sqrt{\Delta_1}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_1)}=1\Rightarrow\Gamma_1\cdot\Gamma_3\cdot\Gamma_5=-\frac{2\cdot(1-\alpha f_1)\cdot R3}{\left\{\frac{R_{CQ1}}{Vt}\cdot\Xi_4+\sqrt{\Delta_1}\right\}\cdot V_t}$$

$$\frac{\Gamma_1}{2\cdot R1}=\frac{\Gamma_5}{R5}\Rightarrow\Gamma_1=\frac{2\cdot R1}{R5}\cdot\Gamma_5\Rightarrow\Gamma_5^2\cdot\Gamma_3=-\frac{(1-\alpha f_1)\cdot R3\cdot R5}{R1\cdot\left\{\frac{R_{CQ1}}{Vt}\cdot\Xi_4+\sqrt{\Delta_1}\right\}\cdot V_t}$$

We can summarize our results for $\chi_2^{\#,\#\#}=V_{CEQ2}^{\#,\#\#}=V_{CEQ2}^{(+/-)}$ in Table 3.5. We get some results for our Hopf–Hopf bifurcation system parameters:

$$-\frac{\Omega_1\cdot\Omega_2}{R6}\cdot\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8+\sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)}=1$$
$$\Rightarrow\Omega_1\cdot\Omega_2=-\frac{2\cdot(1-\alpha f_2)\cdot R6}{\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8+\sqrt{\Delta_4}\right\}\cdot V_t}$$

$$B=-\frac{\Omega_1}{R7}\cdot\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8+\sqrt{\Delta_4}\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_2)}-1;\ \Omega_1=-\frac{(B+1)\cdot2\cdot(1-\alpha f_2)\cdot R7}{\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8+\sqrt{\Delta_4}\right\}\cdot V_t}$$

$$\Omega_2=-\frac{1}{\Omega_1}\cdot\frac{2\cdot(1-\alpha f_2)\cdot R6}{\left\{\frac{R_{CQ2}}{Vt}\cdot\Xi_8+\sqrt{\Delta_4}\right\}_t\cdot V_t}=\frac{R6}{(B+1)\cdot R7}$$

3.5 Neimark–Sacker (Torus) Bifurcation System

We have a discrete time dynamical system depending on a parameter $x\to f(x,\alpha), x\in\mathbb{R}^n, \alpha\in\mathbb{R}^1$. Where, the map f is smooth respect to both x and α. We can write this system as $\tilde{x}=f(x,\alpha), \tilde{x},x\in\mathbb{R}^n, \alpha\in\mathbb{R}^1$ where $\tilde{x}$ denotes the image of x under the action of the map. Let $x=x_0$ be a hyperbolic fixed point of the system for $\alpha=\alpha_0$. We monitor this fixed point and its multipliers while the parameter varies. There are three ways in which the hyperbolicity condition can be violated. The first:

Table 3.5 Summary results for $\chi_2^{\#,\#\#} = V_{CEQ2}^{\#,\#\#} = V_{CEQ2}^{(+/-)}$

Hopf–Hopf bifurcation system χ_2 expression which related to X variable	Hopf–Hopf bifurcation optoisolation circuit V_{CEQ2} expression which related to VX variable
$\chi_2\left(\frac{d^2X}{dt^2}, \frac{dX}{dt}\right) = \frac{d^2X}{dt^2} - \frac{dX}{dt} \cdot (B+1)$	$V_{CEQ2}^{\#\#} \approx -\psi_2 \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$ $\psi_2(\ddot{V}_X, \dot{V}_X) = \frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} - \frac{\Omega_1 \cdot \dot{V}_X}{R7}$ $-\psi_2(\ddot{V}_X, \dot{V}_X) = -\frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} + \frac{\Omega_1 \cdot \dot{V}_X}{R7}$
Results Hopf–Hopf bifurcation system block diagram: $\chi_2 = \Omega_1 \cdot \Omega_2 \cdot \ddot{X} - \Omega_1 \cdot \dot{X}; \Omega_1 \cdot \Omega_2 = 1$ $\Omega_1 = B+1 \Rightarrow \Omega_2 = \frac{1}{B+1}$	Circuit parameters constrains: $0 < \varepsilon \ll 1$ $-\frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6 \to \varepsilon \forall 0 < \varepsilon \ll 1$ $V_X \rightleftharpoons X; V_{CEQ2}^{\#\#} \rightleftharpoons \chi_2$ $\frac{1}{Ri}\left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)} \forall i = 6, 7$ $V_{CEQ2}^{\#\#} \approx \left\{-\frac{\Omega_1 \cdot \Omega_2 \cdot \ddot{V}_X}{R6} + \frac{\Omega_1 \cdot \dot{V}_X}{R7}\right\}$ $\cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$
1	$-\frac{\Omega_1 \cdot \Omega_2}{R6} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$
$-(B+1)$	$\frac{\Omega_1}{R7} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$

a simple positive multiplier approaches the unit cycle $\mu_1 = 1$, the second: a simple negative multiplier approaches the unit circle $\mu_1 = -1$, the third: a pair of simple complex multipliers reaches the unit circle $\mu_{1,2} = e_0^{\pm i \cdot \theta}$, $0 < \theta_0 < \pi$ for some value of parameter. The bifurcation corresponding to the presence of $\mu_{1,2} = e_0^{\pm i \cdot \theta}$, $0 < \theta_0 < \pi$ is called a Neimark–Sacker (or torus) bifurcation. The fold and flip bifurcations are possible if $n \geq 1$, but for the Neimark–Sacker bifurcation $n \geq 2$. Neimark–Sacker bifurcation is the birth of a closed invariant curve from a fixed point in dynamical systems with discrete time (iterated maps), when the fixed point changes stability via a pair of complex eigenvalues with unit modules. The bifurcation can be supercritical or subcritical, resulting in a stable or unstable within an invariant two-dimensional manifold closed invariant curve, respectively. The bifurcation generates an invariant two-dimensional torus in the corresponding ODE. We consider again a map $x \to f(x, \alpha), x \in \mathbb{R}^n, \alpha \in \mathbb{R}^1$ where f is smooth. Suppose that for all sufficiently small $|\alpha|$ the system has a family of fixed points $x^0(\alpha)$. The Jacobian matrix $A(\alpha) = f_x(x^0(\alpha), \alpha)$ has one pair of complex eigenvalues $\lambda_{1,2}(\alpha) = r(\alpha) \cdot e^{\pm i \cdot \theta(\alpha)}$ on the unit circle when $\alpha = 0$, i.e., $r(0) = 1$ and $0 < \theta(0) < \pi$. As α passes through $\alpha = 0$, the fixed point changes stability and a unique closed invariant curve bifurcates from it. This bifurcation is characterized by a single bifurcation condition $|\lambda_{1,2}| = 1$ has co-dimension one and appears generically in one parameter families of smooth map. For the two-dimensional case, we consider the map with $n = 2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} f(x_1, x_2, \alpha) \\ f(x_1, x_2, \alpha) \end{pmatrix}$. If the nondegeneracy hold: (NS.1) $e_0^{i \cdot k \cdot \theta} \neq 1$ for $k = 1, 2, 3, 4$, where $\theta_0 = \theta(0)$ which is no strong resonances, (NS.2) $r'(0) \neq 0$ then this map is locally conjugate near the fixed point to the normal form, and can be written using a complex coordinate $z = y_1 + i \cdot y_2$ as $z \to (1+\beta) \cdot e^{i\theta(\beta)} \cdot z + c(\beta) \cdot z \cdot |z|^2 + O(|z|^4)$ where $y = (y_1, y_2)^{\mathrm{T}} \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$ is the new parameter [5–9].

(NS.3) $d(0) = \mathrm{Re}[e^{-i \cdot \theta_0} \cdot c(0)] \neq 0$, where $d(0)$ is the first Lyapunov coefficient then if $d(0) < 0$ the normal form has a fixed point at the origin, which is asymptotically stable for $\beta \leq 0$, weakly at $\beta = 0$, and unstable for $\beta > 0$. There is a unique and stable closed invariant curve that exists for $\beta > 0$ and has a radius$O(\sqrt{\beta})$. This is a supercritical Neimark–Sacker bifurcation. When the Neimark–Sacker bifurcation occurs in the Poincare map of a limit cycle in ODE, the fixed point corresponding to the limit cycle has a pair of simple eigenvalues $\mu_{1,2} = e_0^{\pm i \cdot \theta}$, $0 < \theta_0 < \pi$ and all the formulated generality conditions hold. A unique two-dimensional invariant torus bifurcates from the cycle, while it changes stability. The intersection of the torus with the Poincare section corresponds to the closed invariant curve. The torus bifurcation is sometimes called the secondary Hopf bifurcation. The torus bifurcation can occur near the Fold-Hopf bifurcation and is always present near the Hopf Hopf bifurcation of equilibrium in ODEs. In a torus bifurcation, a spiral limit cycle reverses its stability and spawns a zero-amplitude torus in its immediate neighborhood, to which trajectories in the system are asymptotically attracted or repelled. The amplitude of the torus grows as the bifurcation parameter is pushed further beyond the bifurcation point. One starts off with a stable spiral limit cycle, whose Floquet multipliers are therefore a complex conjugate pair lying within the unit circle. Beyond the bifurcation point, the trajectories in the system are now asymptotically attracted to orbits on the two-dimensional surface of a torus. These orbits can be either periodic or quasiperiodic. Note that the original limit cycle still exists, but that it has become an unstable spiral cycle, with a complex conjugate pair of Floquet multipliers now lying outside of the unit circle. A torus bifurcation thus occurs when a complex conjugate pair of Floquet multipliers crosses the unit circle. At a torus bifurcation a pair of complex conjugate Floquet multipliers goes through the unit circle. The torus bifurcation of the orbit corresponds to a Hopf bifurcation in the map, which converts the stable spiral fixed point of the map into an unstable spiral point, spawning an invariant circle in its neighborhood. This invariant circle corresponds to the piercing of the Poincare plane of section by a quasiperiodic orbit lying in the surface of the torus that asymptotically visits all points on the circle. Because of its association with a Hopf bifurcation on the return map, the torus bifurcation is also called a Hopf bifurcation of periodic orbits or a secondary Hopf bifurcation. It is also referred to as the Hopf–Neimark or Neimark bifurcation. A limit cycle is stable if all of its nontrivial Floquet multipliers lie within the unit circle. If a parameter is gradually changed, the limit cycle can lose its stability in a "local" bifurcation if and only if one or more of these multipliers crosses through the unit cycle. This can happened in one of three generic ways as a single bifurcation parameter is changed: A single real multiplier goes through +1 (saddle–node bifurcation); a single real multiplier goes through -1 (period-doubling bifurcation); or a pair of complex conjugate multipliers crosses through the unit circle (torus bifurcation). There are other nonlocal bifurcations in which limit cycles can be created or destroyed. One of these "global" bifurcations: the homoclinic bifurcation. Floquet theory is the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points.

We consider the following model of an autonomous system, where X, Y, and Z are state variables and $\Gamma_1,\ldots,\Gamma_6$ are parameters [36, 38, 39].

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left[-(\Gamma_1+\Gamma_3)\cdot X + \Gamma_1\cdot Y - \Gamma_5\cdot X^3 + \Gamma_6\cdot (Y-X)^3\right]/\Gamma_4$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \Gamma_1\cdot X - (\Gamma_1+\Gamma_2)\cdot Y - Z - \Gamma_6\cdot (Y-X)^3; \frac{\mathrm{d}Z}{\mathrm{d}t} = Y$$

To find our system fixed points we set $\frac{\mathrm{d}X}{\mathrm{d}t}=0; \frac{\mathrm{d}Y}{\mathrm{d}t}=0; \frac{\mathrm{d}Z}{\mathrm{d}t}=0$

$\frac{\mathrm{d}Z}{\mathrm{d}t} = 0 \Rightarrow Y^{(i)} = 0; i = 0,1,2$ and we get two equations:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 0 \,\&\, Y^{(i)} = 0 \Rightarrow [-(\Gamma_1+\Gamma_3)\cdot X - \Gamma_5\cdot X^3 - \Gamma_6\cdot X^3]/\Gamma_4 = 0$$
$$\Gamma_4 \neq 0 \Rightarrow -X\cdot\left\{\Gamma_1+\Gamma_3+X^2\cdot(\Gamma_5+\Gamma_6)\right\} = 0 \Rightarrow X^{(0)} = 0$$

$$Y^{(i)} = 0 \,\&\, X^{(0)} = 0 \Rightarrow Z^{(0)} = 0;\ \mathrm{E}^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0,0,0)$$

$$\Gamma_1+\Gamma_3+[X^{(1,2)}]^2\cdot(\Gamma_5+\Gamma_6) = 0 \Rightarrow [X^{(1,2)}]^2 = \frac{-(\Gamma_1+\Gamma_3)}{\Gamma_5+\Gamma_6} \Rightarrow X^{(1,2)} = \pm\sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = 0 \Rightarrow \Gamma_1\cdot X^{(1,2)} - Z^{(1,2)} + \Gamma_6\cdot\left[X^{(1,2)}\right]^3 = 0 \Rightarrow Z^{(1,2)} = \Gamma_1\cdot X^{(1,2)} + \Gamma_6\cdot\left[X^{(1,2)}\right]^3 \Rightarrow Z^{(1,2)} = X^{(1,2)}\cdot\left\{\Gamma_1+\Gamma_6\cdot[X^{(1,2)}]^2\right\}$$

$$Z^{(1,2)} = \pm\sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}}\cdot\left\{\Gamma_1-\Gamma_6\cdot\left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\}.$$

We can summarize our system three fixed points.

$$\mathrm{E}^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0,0,0)$$

$$\mathrm{E}^{(1)} = (X^{(1)}, Y^{(1)}, Z^{(1)}) = \left(\sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}}, 0, \sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}}\cdot\left\{\Gamma_1-\Gamma_6\cdot\left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\}\right)$$

$$E^{(2)} = (X^{(2)}, Y^{(2)}, Z^{(2)}) = \left(-\sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}}, 0, -\sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}} \cdot \left\{\Gamma_1 - \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\}\right)$$

We define our system in the following way:

$$\frac{dX}{dt} = f_1(X,Y,Z) = \left[-(\Gamma_1+\Gamma_3)\cdot X + \Gamma_1 \cdot Y - \Gamma_5 \cdot X^3 + \Gamma_6 \cdot (Y-X)^3\right]/\Gamma_4$$

$$\frac{dY}{dt} = f_2(X,Y,Z) = \Gamma_1 \cdot X - (\Gamma_1+\Gamma_2)\cdot Y - Z - \Gamma_6 \cdot (Y-X)^3; \frac{dZ}{dt} = f_3(X,Y,Z) = Y$$

The next step is to find matrix A called the jacobian matrix at fixed point $A = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}$; For simplicity we define the following new parameters.

$$\Gamma_{13} = \Gamma_1 + \Gamma_3; \Gamma_{12} = \Gamma_1 + \Gamma_2;$$

$$\frac{\partial f_1}{\partial X} = -\frac{1}{\Gamma_4} \cdot \left[\Gamma_{13} + 3\cdot\Gamma_5 \cdot X^2 + 3\cdot\Gamma_6\cdot(Y-X)^2\right]$$

$$\frac{\partial f_1}{\partial Y} = \frac{1}{\Gamma_4}\cdot\left[\Gamma_1 + 3\cdot\Gamma_6\cdot(Y-X)^2\right]; \frac{\partial f_1}{\partial Z} = 0; \frac{\partial f_2}{\partial X} = \Gamma_1 + 3\cdot\Gamma_6\cdot(Y-X)^2$$

$$\frac{\partial f_2}{\partial Y} = -\Gamma_{12} - 3\Gamma_6\cdot(Y-X)^2 = -\left\{\Gamma_{12} + 3\Gamma_6\cdot(Y-X)^2\right\};$$

$$\frac{\partial f_2}{\partial Z} = -1;\ \frac{\partial f_3}{\partial X} = 0;\ \frac{\partial f_3}{\partial Y} = 1;\ \frac{\partial f_3}{\partial Z} = 0$$

We need to find the Jacobian matrix for the first fixed point:

$$E^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0,0,0)$$

$$\left.\frac{\partial f_1}{\partial X}\right|_{E^{(0)}=(0,0,0)} = -\frac{\Gamma_{13}}{\Gamma_4} = -\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4}; \left.\frac{\partial f_1}{\partial Y}\right|_{E^{(0)}=(0,0,0)} = \frac{\Gamma_1}{\Gamma_4}; \left.\frac{\partial f_1}{\partial Z}\right|_{E^{(0)}=(0,0,0)} = 0$$

$$\left.\frac{\partial f_2}{\partial X}\right|_{E^{(0)}=(0,0,0)} = \Gamma_1; \left.\frac{\partial f_2}{\partial Y}\right|_{E^{(0)}=(0,0,0)} = -\Gamma_{12} = -(\Gamma_1+\Gamma_2); \left.\frac{\partial f_2}{\partial Z}\right|_{E^{(0)}=(0,0,0)} = -1$$

$$\left.\frac{\partial f_3}{\partial X}\right|_{E^{(0)}=(0,0,0)} = 0; \left.\frac{\partial f_3}{\partial Y}\right|_{E^{(0)}=(0,0,0)} = 1; \left.\frac{\partial f_3}{\partial Z}\right|_{E^{(0)}=(0,0,0)} = 0$$

$$A|_{E^{(0)}=(0,0,0)} = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\Bigg|_{E^{(0)}=(0,0,0)}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial X}\Big|_{E^{(0)}=(0,0,0)} & \frac{\partial f_1}{\partial Y}\Big|_{E^{(0)}=(0,0,0)} & \frac{\partial f_1}{\partial Z}\Big|_{E^{(0)}=(0,0,0)} \\ \frac{\partial f_2}{\partial X}\Big|_{E^{(0)}=(0,0,0)} & \frac{\partial f_2}{\partial Y}\Big|_{E^{(0)}=(0,0,0)} & \frac{\partial f_2}{\partial Z}\Big|_{E^{(0)}=(0,0,0)} \\ \frac{\partial f_3}{\partial X}\Big|_{E^{(0)}=(0,0,0)} & \frac{\partial f_3}{\partial Y}\Big|_{E^{(0)}=(0,0,0)} & \frac{\partial f_3}{\partial Z}\Big|_{E^{(0)}=(0,0,0)} \end{pmatrix}$$

$$A|_{E^{(0)}=(0,0,0)} = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\Bigg|_{E^{(0)}=(0,0,0)} = \begin{pmatrix} -\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} & \frac{\Gamma_1}{\Gamma_4} & 0 \\ \Gamma_1 & -(\Gamma_1+\Gamma_2) & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\Bigg|_{E^{(0)}=(0,0,0)} - \lambda \cdot I$$

$$= \begin{pmatrix} -\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} - \lambda & \frac{\Gamma_1}{\Gamma_4} & 0 \\ \Gamma_1 & -(\Gamma_1+\Gamma_2) - \lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left[\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \lambda\right] \cdot \begin{pmatrix} -(\Gamma_1+\Gamma_2) - \lambda & -1 \\ 1 & -\lambda \end{pmatrix} - \frac{\Gamma_1}{\Gamma_4} \cdot \begin{pmatrix} \Gamma_1 & -1 \\ 0 & -\lambda \end{pmatrix}$$

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left[\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \lambda\right] \cdot \{[(\Gamma_1+\Gamma_2) + \lambda] \cdot \lambda + 1\} + \frac{\Gamma_1^2}{\Gamma_4} \cdot \lambda$$

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left[\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \lambda\right] \cdot \left\{(\Gamma_1+\Gamma_2) \cdot \lambda + \lambda^2 + 1\right\} + \frac{\Gamma_1^2}{\Gamma_4} \cdot \lambda$$

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left\{\frac{(\Gamma_1+\Gamma_3) \cdot (\Gamma_1+\Gamma_2)}{\Gamma_4} \cdot \lambda + \frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} \cdot \lambda^2 + \frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + (\Gamma_1+\Gamma_2) \cdot \lambda^2 + \lambda^3 + \lambda\right\} + \frac{\Gamma_1^2}{\Gamma_4} \cdot \lambda$$

Table 3.6 Neimark–Sacker (Torus) bifurcation system, Δ_1 possibilities and meanings

$\Delta_1 > 0$	$\Delta_1 < 0$	$\Delta_1 = 0$
The equation has three distinct real roots. When λ_1, λ_2, $\lambda_3 < 0$ stable node. λ_1, λ_2, $\lambda_3 > 0$ unstable node. When at least one eigenvalue is positive then we have saddle point	The equation has one real root and a pair of complex conjugate roots. We have stable and unstable spiral	At least two roots coincide. The system equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple roots

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left\{\left[\frac{(\Gamma_1+\Gamma_3)\cdot(\Gamma_1+\Gamma_2)}{\Gamma_4} - \frac{\Gamma_1^2}{\Gamma_4} + 1\right]\cdot\lambda + \left\{\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \Gamma_1 + \Gamma_2\right\}\cdot\lambda^2 + \frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \lambda^3\right\}$$

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left\{\lambda^3 + \left\{\frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \Gamma_1 + \Gamma_2\right\}\cdot\lambda^2 + \left[\frac{(\Gamma_1+\Gamma_3)\cdot(\Gamma_1+\Gamma_2)}{\Gamma_4} - \frac{\Gamma_1^2}{\Gamma_4} + 1\right]\cdot\lambda + \frac{(\Gamma_1+\Gamma_3)}{\Gamma_4}\right\}$$

For simplicity we define three global parameter functions: ξ_1, ξ_2, ξ_3

$$\xi_1 = \xi_1(\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4); \xi_2 = \xi_2(\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4); \xi_3 = \xi_3(\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4)$$

$$\xi_1 = \frac{(\Gamma_1+\Gamma_3)}{\Gamma_4} + \Gamma_1 + \Gamma_2; \xi_2 = \frac{(\Gamma_1+\Gamma_3)\cdot(\Gamma_1+\Gamma_2)}{\Gamma_4} - \frac{\Gamma_1^2}{\Gamma_4} + 1; \xi_3 = \frac{(\Gamma_1+\Gamma_3)}{\Gamma_4}$$

$$\det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = -\left\{\lambda^3 + \xi_1\cdot\lambda^2 + \xi_2\cdot\lambda + \xi_3\right\}; \det\left\{A|_{E^{(0)}=(0,0,0)} - \lambda \cdot I\right\} = 0$$
$$\Rightarrow \lambda^3 + \xi_1\cdot\lambda^2 + \xi_2\cdot\lambda + \xi_3 = 0$$

We have cubic function of system eigenvalues. ξ_1, ξ_2, ξ_3 are global parameters coefficients and are real numbers. We distinguish several possible cases using the discriminant [40–42].

$$\Delta_1 = -4\cdot\xi_1^3\cdot\xi_3 + \xi_1^2\cdot\xi_2^2 - 4\cdot\xi_2^3 + 18\cdot\prod_{i=1}^{3}\xi_i - 27\cdot\xi_3^2$$

The following cases need to be considered and present in Table 3.6.
We need to find the Jacobian matrix for the second fixed point:

$$\mathrm{E}^{(1)} = (X^{(1)}, Y^{(1)}, Z^{(1)}) = \left(\sqrt{-\frac{\Gamma+\Gamma_3}{\Gamma_5+\Gamma_6}}, 0, \sqrt{-\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}} \cdot \left\{\Gamma_1 - \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\}\right)$$

$$[X^{(1)}]^2 = -\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6};\ \frac{\partial f_1}{\partial X}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = -\frac{1}{\Gamma_4} \cdot \left[\Gamma_{13} + 3 \cdot [X^{(1)}]^2 \cdot (\Gamma_5+\Gamma_6)\right]$$

$$\frac{\partial f_1}{\partial X}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = -\frac{1}{\Gamma_4} \cdot \left[\Gamma_{13} - 3 \cdot \frac{(\Gamma_1+\Gamma_3)}{(\Gamma_5+\Gamma_6)} \cdot (\Gamma_5+\Gamma_6)\right] = -\frac{1}{\Gamma_4} \cdot [\Gamma_{13} - 3 \cdot (\Gamma_1+\Gamma_3)]$$

$$\frac{\partial f_1}{\partial X}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = -\frac{1}{\Gamma_4} \cdot [\Gamma_1+\Gamma_3 - 3 \cdot (\Gamma_1+\Gamma_3)] = \frac{2}{\Gamma_4} \cdot [\Gamma_1+\Gamma_3]$$

$$\frac{\partial f_1}{\partial Y}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = \frac{1}{\Gamma_4} \cdot \left[\Gamma_1 + 3 \cdot \Gamma_6 \cdot [X^{(1)}]^2\right] = \frac{1}{\Gamma_4} \cdot \left[\Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right]$$

$$\frac{\partial f_1}{\partial Z}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = 0; \frac{\partial f_2}{\partial X}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = \Gamma_1 + 3 \cdot \Gamma_6 \cdot [X^{(1)}]^2 = \Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)$$

$$\frac{\partial f_2}{\partial Y}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = -\left\{\Gamma_{12} + 3 \cdot \Gamma_6 \cdot [X^{(1)}]^2\right\} = -\left\{\Gamma_{12} - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\}$$

$$\frac{\partial f_2}{\partial Z}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = -1; \frac{\partial f_3}{\partial X}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = 0; \frac{\partial f_3}{\partial Y}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = 1$$

$$\frac{\partial f_3}{\partial Z}\bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = 0$$

$$A\big|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\Bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = \begin{pmatrix} \frac{\partial f_1}{\partial X}\big|_{\mathrm{E}^{(1)}} & \frac{\partial f_1}{\partial Y}\big|_{\mathrm{E}^{(1)}} & \frac{\partial f_1}{\partial Z}\big|_{\mathrm{E}^{(1)}} \\ \frac{\partial f_2}{\partial X}\big|_{\mathrm{E}^{(1)}} & \frac{\partial f_2}{\partial Y}\big|_{\mathrm{E}^{(1)}} & \frac{\partial f_2}{\partial Z}\big|_{\mathrm{E}^{(1)}} \\ \frac{\partial f_3}{\partial X}\big|_{\mathrm{E}^{(1)}} & \frac{\partial f_3}{\partial Y}\big|_{\mathrm{E}^{(1)}} & \frac{\partial f_3}{\partial Z}\big|_{\mathrm{E}^{(1)}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}\Bigg|_{\mathrm{E}^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} = \begin{pmatrix} \frac{2}{\Gamma_4} \cdot [\Gamma_1+\Gamma_3] & \frac{1}{\Gamma_4} \cdot \left[\Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right] & 0 \\ \Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right) & -\left\{\Gamma_{12} - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\} & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I = \begin{pmatrix} \frac{2}{\Gamma_4} \cdot [\Gamma_1 + \Gamma_3] - \lambda & \frac{1}{\Gamma_4} \cdot \left[\Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right] & 0 \\ \Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right) & -\left\{\Gamma_{12} - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\} - \lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

For simplicity we define the following parameters:

$$\varsigma_1 = \frac{2}{\Gamma_4} \cdot [\Gamma_1 + \Gamma_3];\ \varsigma_2 = \frac{1}{\Gamma_4} \cdot \left[\Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right];$$
$$\varsigma_3 = \Gamma_1 - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right);\ \varsigma_4 = -\left\{\Gamma_{12} - 3 \cdot \Gamma_6 \cdot \left(\frac{\Gamma_1+\Gamma_3}{\Gamma_5+\Gamma_6}\right)\right\}$$

$$A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I = \begin{pmatrix} \varsigma_1 - \lambda & \varsigma_2 & 0 \\ \varsigma_3 & \varsigma_4 - \lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix}; \det\left\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\right\} = 0$$

$$\det\left\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\right\} = (\varsigma_1 - \lambda) \cdot \begin{pmatrix} \varsigma_4 - \lambda & -1 \\ 1 & -\lambda \end{pmatrix} - \varsigma_2 \cdot \begin{pmatrix} \varsigma_3 & -1 \\ 0 & -\lambda \end{pmatrix}$$

$$\det\left\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\right\} = (\varsigma_1 - \lambda) \cdot \{(\varsigma_4 - \lambda) \cdot (-\lambda) + 1\} + \varsigma_2 \cdot \varsigma_3 \cdot \lambda$$

$$\det\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\} = (\varsigma_1 - \lambda) \cdot (-\lambda \cdot \varsigma_4 + \lambda^2 + 1) + \varsigma_2 \cdot \varsigma_3 \cdot \lambda$$

$$\det\left\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\right\} = -\lambda \cdot \varsigma_4 \cdot \varsigma_1 + \lambda^2 \cdot \varsigma_1 + \varsigma_1 + \lambda^2 \cdot \varsigma_4 - \lambda^3 - \lambda + \varsigma_2 \cdot \varsigma_3 \cdot \lambda$$

$$\det\left\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\right\} = -\lambda^3 + \lambda^2 \cdot (\varsigma_1 + \varsigma_4) + \lambda \cdot (\varsigma_2 \cdot \varsigma_3 - \varsigma_4 \cdot \varsigma_1 - 1) + \varsigma_1$$

$$\det\left\{A|_{E^{(1)}=(X^{(1)},Y^{(1)},Z^{(1)})} - \lambda \cdot I\right\} = 0 \Rightarrow -\lambda^3 + \lambda^2 \cdot (\varsigma_1 + \varsigma_4) + \lambda \cdot (\varsigma_2 \cdot \varsigma_3 - \varsigma_4 \cdot \varsigma_1 - 1) + \varsigma_1 = 0$$

For simplicity we define the following global parameters:

$$\phi_1 = \varsigma_1 + \varsigma_4; \phi_2 = \varsigma_2 \cdot \varsigma_3 - \varsigma_4 \cdot \varsigma_1 - 1 \Rightarrow -\lambda^3 + \lambda^2 \cdot \phi_1 + \lambda \cdot \phi_2 + \varsigma_1 = 0$$

We have cubic function of system eigenvalues. Φ_1, Φ_2, ζ_1 are global parameters coefficients and are real numbers. We distinguish several possible cases using the discriminant.

$$\Delta_2 = -4 \cdot \phi_1^3 \cdot \varsigma_1 + \phi_1^2 \cdot \phi_2^2 + 4 \cdot \phi_2^3 - 18 \cdot \phi_1 \cdot \phi_2 \cdot \varsigma_1 - 27 \cdot \varsigma_1^2$$

The following cases need to be considered and present in Table 3.7.
We need to find the Jacobian matrix for the third fixed point :

$$\begin{aligned} \mathrm{E}^{(2)} &= (X^{(2)}, Y^{(2)}, Z^{(2)}) \\ &= \left(-\sqrt{-\frac{\Gamma_1 + \Gamma_3}{\Gamma_5 + \Gamma_6}}, 0, -\sqrt{-\frac{\Gamma_1 + \Gamma_3}{\Gamma_5 + \Gamma_6}} \cdot \left\{\Gamma_1 - \Gamma_6 \cdot \left(\frac{\Gamma_1 + \Gamma_3}{\Gamma_5 + \Gamma_6}\right)\right\}\right) \end{aligned}$$

Since $\left[X^{(1,2)}\right]^2 = \frac{-(\Gamma_1 + \Gamma_3)}{\Gamma_5 + \Gamma_6} \Rightarrow X^{(1,2)} = \pm\sqrt{-\frac{\Gamma_1 + \Gamma_3}{\Gamma_5 + \Gamma_6}}; Y^{(i=1,2)} = 0$

$$\Gamma_{13} = \Gamma_1 + \Gamma_3; \Gamma_{12} = \Gamma_1 + \Gamma_2; \frac{\partial f_1}{\partial X} = -\frac{1}{\Gamma_4} \cdot [\Gamma_{13} + 3 \cdot [X^{(1,2)}]^2 \cdot (\Gamma_5 + \Gamma_6)]$$

$$\frac{\partial f_1}{\partial Y} = \frac{1}{\Gamma_4} \cdot \left\{\Gamma_1 + 3 \cdot \Gamma_6 \cdot \left[X^{(1,2)}\right]^2\right\}; \frac{\partial f_1}{\partial Z} = 0; \frac{\partial f_2}{\partial X} = \Gamma_1 + 3 \cdot \Gamma_6 \cdot \left[X^{(1,2)}\right]^2$$

$$\frac{\partial f_2}{\partial Y} = -\Gamma_{12} - 3\Gamma_6 \cdot \left[X^{(1,2)}\right]^2 = -\left\{\Gamma_{12} + 3 \cdot \Gamma_6 \cdot \left[X^{(1,2)}\right]^2\right\};$$

$$\frac{\partial f_2}{\partial Z} = -1; \frac{\partial f_3}{\partial X} = 0; \frac{\partial f_3}{\partial Y} = 1; \frac{\partial f_3}{\partial Z} = 0$$

Our Jacobian matrixes for the second and third fixed points are the same. We get the same eigenvalues cubic functions for the second and third fixed points. The stability discussions and the results for the second and third fixed points are the same. The discriminants Δ_2 (second fixed point) and Δ_3 (third fixed point) are the same. We can find a torus bifurcation in this system. It is found by starting an equilibrium curve from the trivial equilibrium point (first fixed point) $\mathrm{E}^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$ at Γ_1 = 0.5; Γ_2 = -0.6; Γ_4 = 0.6; Γ_5 = 0.32858; Γ_6 = 0.93358; Γ_3 = −0.9.

Table 3.7 Neimark–Sacker (Torus) bifurcation system, Δ_2 possibilities and meanings

$\Delta_2 > 0$	$\Delta_2 < 0$	$\Delta_2 = 0$
The equation has three distinct real roots. When λ_1, λ_2, $\lambda_3 < 0$ stable node. λ_1, λ_2, $\lambda_3 > 0$ unstable node. When at least one eigenvalue is positive then we have saddle point	The equation has one real root and a pair of complex conjugate roots. We have stable and unstable spiral	At least two roots coincide. The system equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple roots

The free parameter is Γ_3 and the branch is the trivial one $E^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$. On this branch a Hopf bifurcation is detected at $\Gamma_3 = -0.58934$. On the emerging branch of limit cycles a branch point of limit cycles is found; by continuing the newly found branch one detects a torus bifurcation of periodic orbits. We detect a torus bifurcation at $\Gamma_3 = -0.59575$. We can run Matlab script to find our system eigenvalues for torus and Hopf bifurcations.

$$\Gamma_1 \to \mathrm{a}; \Gamma_2 \to \mathrm{b}; \Gamma_3 \to \mathrm{c}; \Gamma_4 \to \mathrm{d}; \Gamma_5 \to \mathrm{e}; \Gamma_6 \to f; \xi_1 \to \mathrm{m}_1; \xi_2 \to \mathrm{m}_2; \xi_3 \to \mathrm{m}_3$$

$$EDU \gg a = 0.5; b = -0.6; d = 0.6; e = 0.32858; f = 0.93358; c = -0.9;$$

$$EDU \gg m1 = (a+c)/d + (a+b); m2 = ((a+c)*(a+b))/d - (a*a)/d + 1;$$
$$m3 = (a+c)/d;$$

$$EDU \gg p = [1\, m1\, m2\, m3]$$

$$p = 1.0000 \quad -0.7667 \quad 0.6500 \quad -0.6667$$

$$EDU \gg r = \mathrm{roots}(p)$$

Table 3.8 System parameters values

System parameters values	m_1	m_2	m_3	System eigenvalues
$\Gamma_1 = 0.5$; $\Gamma_2 = -0.6$; $\mathbf{\Gamma_3 = -0.9}$; $\Gamma_4 = 0.6$; $\Gamma_5 = 0.32858$; $\Gamma_6 = 0.93358$	−0.7667	0.6500	−0.6667	$\lambda_1 = 0.8842$ $\lambda_2 = -0.0588 + 0.8663*i$ $\lambda_3 = -0.0588 - 0.8663*i$
$\Gamma_1 = 0.5$; $\Gamma_2 = -0.6$; $\mathbf{\Gamma_3 = -0.58934}$; $\Gamma_4 = 0.6$; $\Gamma_5 = 0.32858$; $\Gamma_6 = 0.93358$	−0.2489	0.5982	−0.1489	$\lambda_1 = -0.00 + 0.7734*i$ $\lambda_2 = -0.00 - 0.7734*i$ $\lambda_3 = 0.2489$
$\Gamma_1 = 0.5$; $\Gamma_2 = -0.6$; $\mathbf{\Gamma_3 = -0.59575}$; $\Gamma_4 = 0.6$; $\Gamma_5 = 0.32858$; $\Gamma_6 = 0.93358$	−0.2596	0.5993	−0.1596	$\lambda_1 = -0.003 + 0.7752*i$ $\lambda_2 = -0.003 - 0.7752*i$ $\lambda_3 = 0.2656$
$\Gamma_1 = 0.5$; $\Gamma_2 = -0.6$; $\mathbf{\Gamma_3 = -0.69575}$; $\Gamma_4 = 0.6$; $\Gamma_5 = 0.32858$; $\Gamma_6 = 0.93358$	−0.4262	0.6160	−0.3262	$\lambda_1 = -0.0368 + 0.807*i$ $\lambda_2 = -0.036 - 0.8071*i$ $\lambda_3 = 0.4998$
$\Gamma_1 = 0.5$; $\Gamma_2 = -0.6$; $\mathbf{\Gamma_3 = -0.4}$; $\Gamma_4 = 0.6$; $\Gamma_5 = 0.32858$; $\Gamma_6 = 0.93358$	0.0667	0.5667	0.1667	$\lambda_1 = 0.1009 + 0.7814*i$ $\lambda_2 = 0.1009 - 0.7814*i$ $\lambda_3 = -0.2685$

$$r = 0.8842$$
$$-0.0588 + 0.8663i$$
$$-0.0588 - 0.8663i$$

Results We can control our system behavior by changing our free parameter Γ_3 values. Stable to unstable spiral point, periodic behavior ($\text{Re}[\lambda_i] = 0$), and detect Hope and Torus bifurcations (Table 3.8).

We need to plot system X versus Y, Y versus Z, X versus Z, X(t), Y(t), Z(t), and system 3D graph.

Matlab Script

```
function h=torus1(a,b,c,d,e,f,x0,y0,z0)
[t,x]=ODE45 (@torus,[0,50],[x0,y0,z0],[],a,b,c,d,e,f);
plot3 (x(:,1),x(:,2),x(:,3));
xlabel ('X')
ylabel ('Y')
zlabel ('Z')
grid on
axis square
%subplot(2,2,1);plot(x(:,1),x(:,2));subplot(2,2,2);plot(x(:,3),x(:,2)
);subplot(2,2,3);plot(x(:,1),x(:,3));subplot(2,2,4);plot(t,x);

function g=torus(t,x,a,b,c,d,e,f)
g=zeros(3,1);
g(1)=(-(a+c)*x(1)+a*x(2)-e*x(1).^3+f*(x(2)-x(1)).^3)/d;
g(2)=a*x(1)-(a+b)*x(2)-x(3)-f*(x(2)-x(1)).^3;
g(3)=x(2);
```

First, we choose the case when torus1 ($a = -0.5$, $b = -0.6$, $c = 0.6$, $d = 0.6$, $e = 0.32858$, $f = 0.93358$, $X0 = 5$, $Y0 = 2$, $Z0 = 8$) (Figs. 3.13 and 3.14)

$$\Gamma_1 \to a; \Gamma_2 \to b; \Gamma_3 \to c; \Gamma_4 \to d; \Gamma_5 \to e; \Gamma_6 \to f$$

torus1($a = 0.5$, $b = -0.6$, $c = 6$, $d = 0.6$, $e = 0.32858$, $f = 0.93358$, $X0 = 1$, $Y0 = 1.2$, $Z0 = 1.3$) (Figs. 3.15 and 3.16)

torus1($a = 0.5$, $b = -0.6$, $c = -0.6$, $d = 0.6$, $e = 0.032858$, $f = 0.093358$, $X0 = 10$, $Y0 = 2$, $Z0 = 8$) (Fig. 3.17 and 3.18)

The next example which we discuss is multifolded torus chaotic attractors system. We use a fold torus system for generating multifolded torus chaotic attractors. The folded torus system is described by the following differential and nonlinear equations. Our system variables are $X(t)$, $Y(t)$, $Z(t)$. System parameters $\Gamma_1,\ldots,\Gamma_5$.

$$\frac{dX}{dt} = -\Gamma_1 \cdot g(Y - X); \frac{dY}{dt} = -g(Y - X) - Z; \frac{dZ}{dt} = \Gamma_2 \cdot Y$$

$$g(Y - X) = \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|Y - X + \Gamma_5| - |Y - X - \Gamma_5|)$$

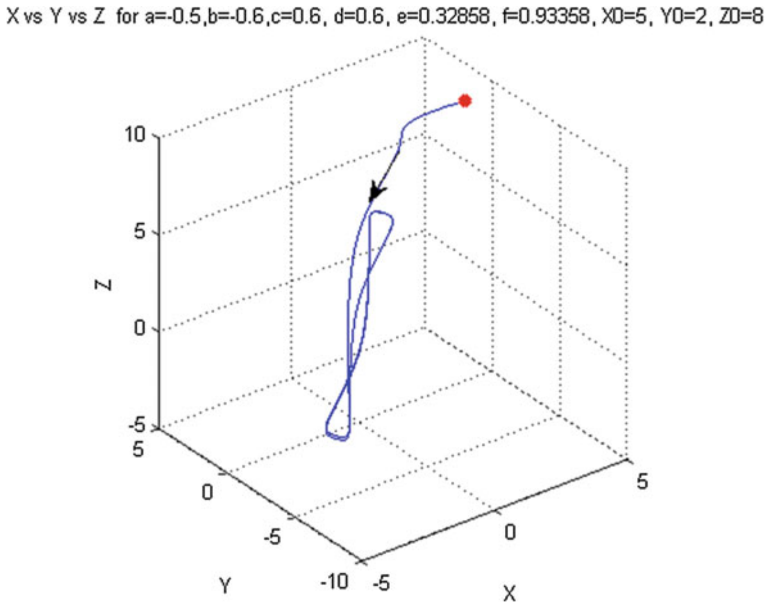

Fig. 3.13 *X* versus *Y* for $a = -0.5$, $b = -0.6$, $c = 0.6$, $d = 0.6$, $e = 0.32858$, $f = 0.93358$, $X0 = 5$, $Y0 = 2$, $Z0 = 8$

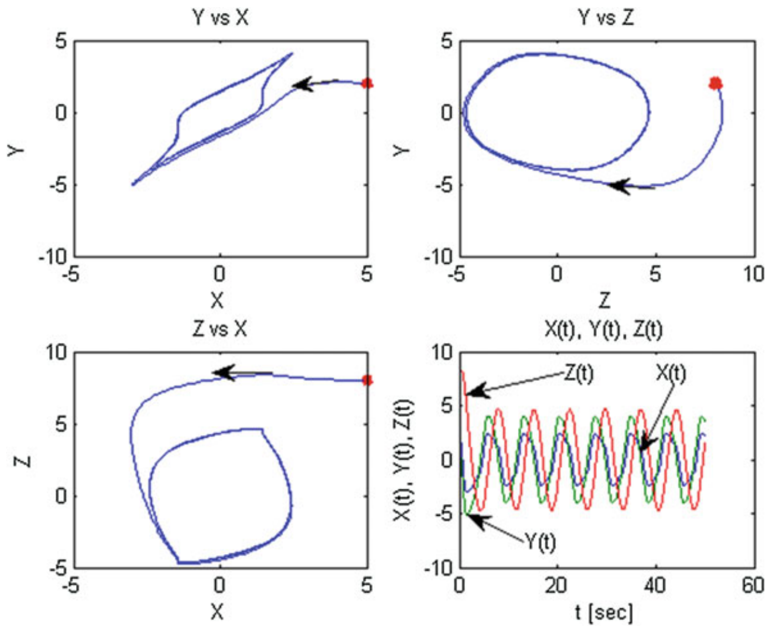

Fig. 3.14 $a = -0.5$, $b = -0.6$, $c = 0.6$, $d = 0.6$, $e = 0.32858$, $f = 0.93358$, $X0 = 5$, $Y0 = 2$, $Z0 = 8$

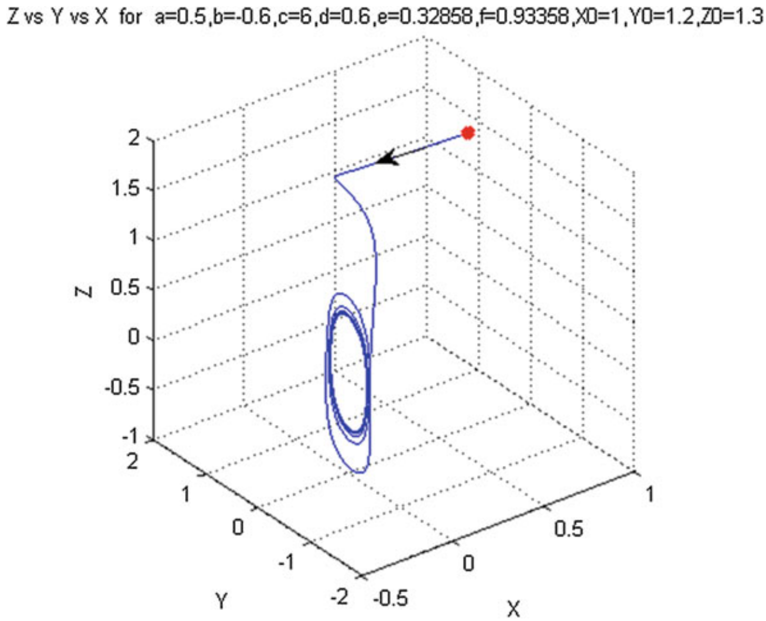

Fig. 3.15 *Z* versus *Y* versus *X* for $a = 0.5$, $b = -0.6$, $c = 6$, $d = 0.6$, $e = 0.32858$, $f = 0.93358$, $X0 = 1$, $Y0 = 1.2$, $Z0 = 1.3$

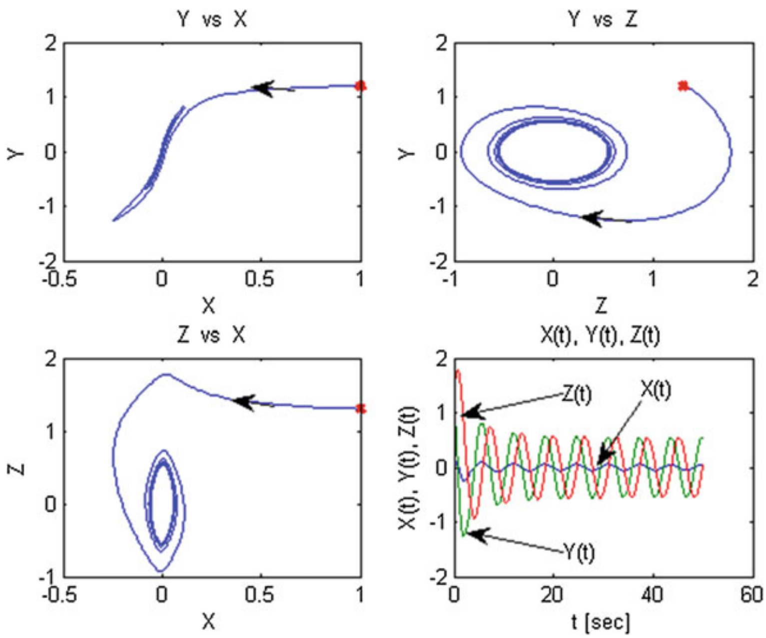

Fig. 3.16 $a = 0.5$, $b = -0.6$, $c = 6$, $d = 0.6$, $e = 0.32858$, $f = 0.93358$, $X0 = 1$, $Y0 = 1.2$, $Z0 = 1.3$

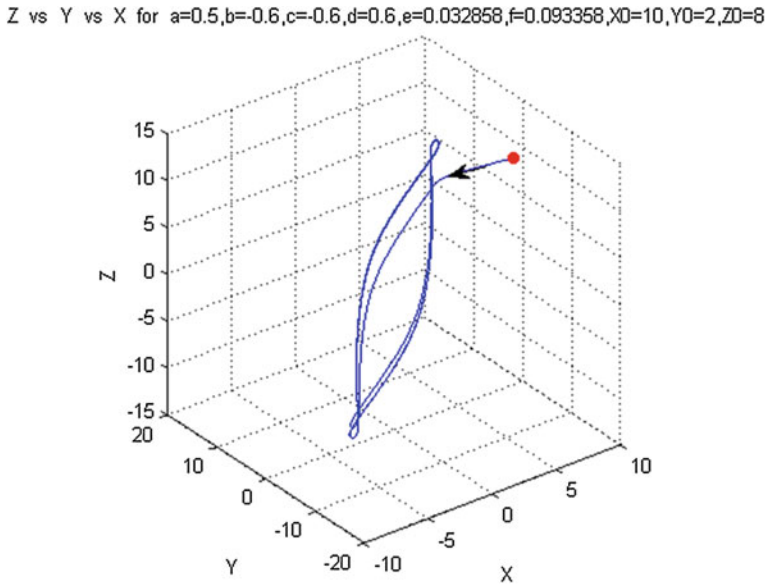

Fig. 3.17 Z versus Y versus X for $a = 0.5$, $b = -0.6$, $c = -0.6$, $d = 0.6$, $e = 0.032858$, $f = 0.093358$, $X0 = 10$, $Y0 = 2$, $Z0 = 8$

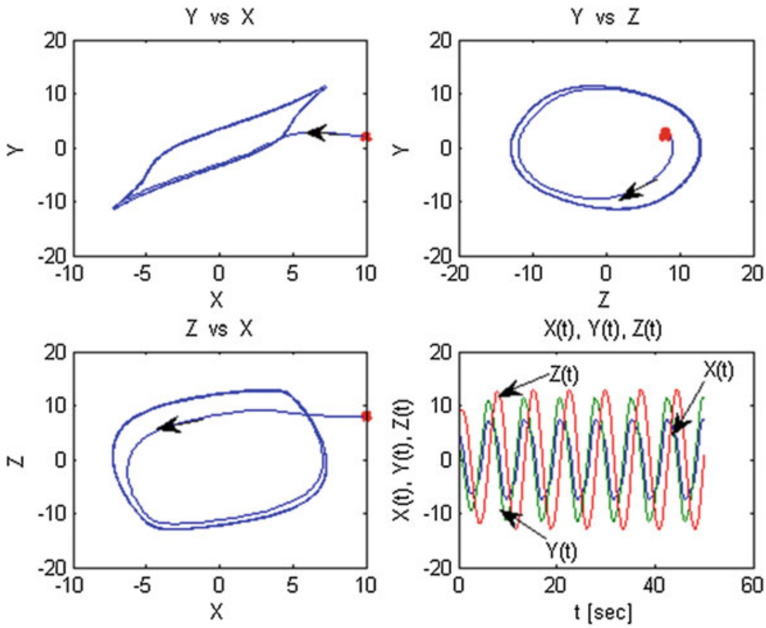

Fig. 3.18 $a = 0.5$, $b = -0.6$, $c = -0.6$, $d = 0.6$, $e = 0.032858$, $f = 0.093358$, $X0 = 10$, $Y0 = 2$, $Z0 = 8$

Where g(Y − X) is an odd function satisfying g(X − Y) = −g(Y − X). we get double-folded torus chaotic attractor.

First, we need to find our system fixed points: $\frac{dX}{dt} = 0; \frac{dY}{dt} = 0; \frac{dZ}{dt} = 0$

$$\frac{dX}{dt} = 0 \Rightarrow -\Gamma_1 \cdot g(Y^{(i)} - X^{(i)}) = 0; \frac{dY}{dt} = 0 \Rightarrow -g(Y^{(i)} - X^{(i)}) - Z^{(i)} = 0; \frac{dZ}{dt} = 0 \Rightarrow \Gamma_2 \cdot Y^{(i)} = 0$$

i—System fixed points index, $X^{(i)}$, $Y^{(i)}$, $Z^{(i)}$ are system fixed points coordinates.

$$-\Gamma_1 \cdot g(Y^{(i)} - X^{(i)}) = 0; \Gamma_1 \neq 0 \Rightarrow g(Y^{(i)} - X^{(i)}) = 0;$$
$$-g(Y^{(i)} - X^{(i)}) - Z^{(i)} = 0 \Rightarrow Z^{(i)} = -g(Y^{(i)} - X^{(i)})$$

$$\Gamma_2 \cdot Y^{(i)} = 0; \Gamma_2 \neq 0 \Rightarrow Y^{(i)} = 0; g(Y^{(i)} - X^{(i)})\big|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{\left|\Gamma_5 - X^{(i)}\right| - \left|-(X^{(i)} + \Gamma_5)\right|\right\}$$

We need to discuss all options for $g(Y^{(i)} - X^{(i)})\big|_{Y^{(i)}=0}$ expression.

We define $f_1(X^{(i)}) = \Gamma_5 - X^{(i)}; f_2(X^{(i)}) = -(X^{(i)} + \Gamma_5)$ and we get

$$g(Y^{(i)} - X^{(i)})\big|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{\left|f_1(X^{(i)})\right| - \left|f_2(X^{(i)})\right|\right\}$$

$$\left|f_1(X^{(i)})\right| = f_1(X^{(i)}) \forall f_1(X^{(i)}) > 0; \ \left|f_1(X^{(i)})\right| = -f_1(X^{(i)}) \forall f_1(X^{(i)}) < 0$$

$$\left|f_2(X^{(i)})\right| = f_2(X^{(i)}) \forall f_2(X^{(i)}) > 0; \ \left|f_2(X^{(i)})\right| = -f_2(X^{(i)}) \forall f_2(X^{(i)}) < 0$$

$$f_1(X^{(i)}) = \Gamma_5 - X^{(i)}; \ f_1(X^{(i)}) > 0 \Rightarrow \Gamma_5 - X^{(i)} > 0 \Rightarrow X^{(i)} < \Gamma_5$$

$$f_1(X^{(i)}) = \Gamma_5 - X^{(i)}; \ f_1(X^{(i)}) < 0 \Rightarrow \Gamma_5 - X^{(i)} < 0 \Rightarrow X^{(i)} > \Gamma_5$$

$$f_2(X^{(i)}) = -(X^{(i)} + \Gamma_5); \ f_2(X^{(i)}) > 0 \Rightarrow -X^{(i)} - \Gamma_5 > 0 \Rightarrow X^{(i)} < -\Gamma_5$$

$$f_2(X^{(i)}) = -(X^{(i)} + \Gamma_5); \ f_2(X^{(i)}) < 0 \Rightarrow -X^{(i)} - \Gamma_5 < 0 \Rightarrow X^{(i)} > -\Gamma_5$$

We can summarize $\left|f_1(X^{(i)})\right|$ and $\left|f_2(X^{(i)})\right|$ expressions in Table 3.9.

$$g(Y^{(i)} - X^{(i)})\big|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{\left|\Gamma_5 - X^{(i)}\right| - \left|-(X^{(i)} + \Gamma_5)\right|\right\}$$

$$\left|g(Y^{(i)} - X^{(i)})\right|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{\left|f_1(X^{(i)})\right| - \left|f_2(X^{(i)})\right|\right\}$$

Table 3.9 Summarize $\left|f_1(X^{(i)})\right|$ and $\left|f_2(X^{(i)})\right|$ expressions

Case number	$f_1(X^{(i)})$	$f_2(X^{(i)})$	Condition on $X^{(i)}$ values		$\|f_1(X^{(i)})\|$	$\|f_2(i^{(i)})\|$
			$\Gamma_5 > 0$	$\Gamma_5 < 0$		
Case I	>0 $\Gamma_5 - X^{(i)} > 0$ $X^{(i)} < \Gamma_5$	>0 $-X^{(i)} - \Gamma_5 > 0$ $X^{(i)} < -\Gamma_5$	$X^{(i)} < -\Gamma_5$	$X^{(i)} < \Gamma_5$	$\Gamma_5 - X^{(i)}$	$-X^{(i)} - \Gamma_5$
Case II	>0 $\Gamma_5 - X^{(i)} > 0$ $X^{(i)} < \Gamma_5$	<0 $-X^{(i)} - \Gamma_5 < 0$ $X^{(i)} > -\Gamma_5$	$-\Gamma_5 < X^{(i)}$ $\& X^{(i)} < \Gamma_5$	Can't exist	$\Gamma_5 - X^{(i)}$	$X^{(i)} + \Gamma_5$
Case III	<0 $\Gamma_5 - X^{(i)} < 0$ $X^{(i)} > \Gamma_5$	>0 $-X^{(i)} - \Gamma_5 > 0$ $X^{(i)} < -\Gamma_5$	Cannot exist	$-\Gamma_5 > X^{(i)}$ $\& X^{(i)} > \Gamma_5$	$-\Gamma_5 + X^{(i)}$	$-X^{(i)} - \Gamma_5$
Case IIII	<0 $\Gamma_5 - X^{(i)} < 0$ $X^{(i)} > \Gamma_5$	<0 $-X^{(i)} - \Gamma_5 < 0$ $X^{(i)} > -\Gamma_5$	$X^{(i)} > \Gamma_5$	$X^{(i)} > -\Gamma_5$	$-\Gamma_5 + X^{(i)}$	$X^{(i)} + \Gamma_5$

Case I $X^{(i)} < -\Gamma_5$ $(\Gamma_5 > 0)$; $X^{(i)} < \Gamma_5$ $(\Gamma_5 < 0)$

$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{\Gamma_5 - X^{(i)} + X^{(i)} + \Gamma_5\right\}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = 0 \Rightarrow -\Gamma_3 \cdot X^{(i)} + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5 = 0 \Rightarrow X^{(i)} = \left(\frac{\Gamma_4}{\Gamma_3} - 1\right) \cdot \Gamma_5$$

$$\Gamma_5 > 0 \Rightarrow X^{(i)} < -\Gamma_5 \Rightarrow \left(\frac{\Gamma_4}{\Gamma_3} - 1\right) \cdot \Gamma_5 < -\Gamma_5 \Rightarrow \frac{\Gamma_4}{\Gamma_3} - 1 < -1 \Rightarrow \frac{\Gamma_4}{\Gamma_3} < 0$$

$$\Gamma_5 < 0 \Rightarrow X^{(i)} < \Gamma_5 \Rightarrow \left(\frac{\Gamma_4}{\Gamma_3} - 1\right) \cdot \Gamma_5 < \Gamma_5 \Rightarrow \frac{\Gamma_4}{\Gamma_3} - 1 > 1 \Rightarrow \frac{\Gamma_4}{\Gamma_3} > 2$$

Case II $-\Gamma_5 < X^{(i)} < \Gamma_5$ $(\Gamma_5 > 0)$

$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{\Gamma_5 - X^{(i)} - X^{(i)} - \Gamma_5)|\right\}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} - (\Gamma_4 - \Gamma_3) \cdot X^{(i)} = -\Gamma_4 \cdot X^{(i)}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = 0 \Rightarrow -\Gamma_4 \cdot X^{(i)} = 0; \Gamma_4 \neq 0 \Rightarrow X^{(i)} = 0$$

Case III $-\Gamma_5 > X^{(i)} > \Gamma_5$ $(\Gamma_5 < 0)$

$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{-\Gamma_5 + X^{(i)} + X^{(i)} + \Gamma_5)|\right\}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + (\Gamma_4 - \Gamma_3) \cdot X^{(i)}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = 0 \Rightarrow [-2 \cdot \Gamma_3 + \Gamma_4] \cdot X^{(i)} = 0; \frac{\Gamma_4}{\Gamma_3} \neq 2 \Rightarrow X^{(i)} = 0$$

Case IV $X^{(i)} > \Gamma_5$ $(\Gamma_5 > 0)$; $X^{(i)} > -\Gamma_5$ $(\Gamma_5 < 0)$

$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot \left\{-\Gamma_5 + X^{(i)} - X^{(i)} - \Gamma_5|\right\}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = -\Gamma_3 \cdot X^{(i)} - (\Gamma_4 - \Gamma_3) \cdot \Gamma_5 = -\left\{\Gamma_3 \cdot X^{(i)} + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5\right\}$$
$$g(Y^{(i)} - X^{(i)})|_{Y^{(i)}=0} = 0 \Rightarrow \Gamma_3 \cdot X^{(i)} + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5 = 0 \Rightarrow X^{(i)} = (1 - \frac{\Gamma_4}{\Gamma_3}) \cdot \Gamma_5$$

$$\Gamma_5 > 0 \Rightarrow X^{(i)} > \Gamma_5 \Rightarrow (1 - \frac{\Gamma_4}{\Gamma_3}) \cdot \Gamma_5 > \Gamma_5 \Rightarrow 1 - \frac{\Gamma_4}{\Gamma_3} > 1 \Rightarrow \frac{\Gamma_4}{\Gamma_3} < 0$$

$$\Gamma_5 < 0 \Rightarrow X^{(i)} > -\Gamma_5 \Rightarrow (1 - \frac{\Gamma_4}{\Gamma_3}) \cdot \Gamma_5 > -\Gamma_5 \Rightarrow 1 - \frac{\Gamma_4}{\Gamma_3} < -1 \Rightarrow \frac{\Gamma_4}{\Gamma_3} > 2$$

We need to find $Z^{(i)} = -g(Y^{(i)} - X^{(i)})$ since $-\Gamma_1 \cdot g(Y^{(i)} - X^{(i)}) = 0$

$$\Gamma_1 \neq 0 \Rightarrow g(Y^{(i)} - X^{(i)}) = 0 \Rightarrow Z^{(i)} = 0.$$

Summary of our system fixed points (Table 3.10):
We discuss now system stability and define our system if the following way:

$$\frac{dX}{dt} = \psi_1 = -\Gamma_1 \cdot g(Y - X); \frac{dY}{dt} = \psi_2 = -g(Y - X) - Z; \frac{dZ}{dt} = \psi_3 = \Gamma_2 \cdot Y$$

$$g(Y - X) = \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|Y - X + \Gamma_5| - |Y - X - \Gamma_5|)$$

$$A = \begin{pmatrix} \frac{\partial\psi_1}{\partial X} & \frac{\partial\psi_1}{\partial Y} & \frac{\partial\psi_1}{\partial Z} \\ \frac{\partial\psi_2}{\partial X} & \frac{\partial\psi_2}{\partial Y} & \frac{\partial\psi_2}{\partial Z} \\ \frac{\partial\psi_3}{\partial X} & \frac{\partial\psi_3}{\partial Y} & \frac{\partial\psi_3}{\partial Z} \end{pmatrix}; \frac{\partial\psi_1}{\partial X} = -\Gamma_1 \cdot \frac{\partial g(Y - X)}{\partial X}; \frac{\partial\psi_1}{\partial Y} = -\Gamma_1 \cdot \frac{\partial g(Y - X)}{\partial Y}$$

$$\frac{\partial\psi_1}{\partial Z} = 0; \frac{\partial\psi_2}{\partial X} = -\frac{\partial g(Y - X)}{\partial X}; \frac{\partial\psi_2}{\partial Y} = -\frac{\partial g(Y - X)}{\partial Y}; \frac{\partial\psi_2}{\partial Z} = -1$$

$$\frac{\partial\psi_3}{\partial X} = 0; \frac{\partial\psi_3}{\partial Y} = \Gamma_2; \frac{\partial\psi_3}{\partial Z} = 0; \xi_1(X, Y) = Y - X + \Gamma_5; \xi_2(X, Y) = Y - X - \Gamma_5$$

$$\xi_1 = \xi_1(X, Y); \xi_2 = \xi_2(X, Y);$$

$$g(Y - X) = \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|\xi_1| - |\xi_2|)$$

$$|\xi_1| = \xi_1 \forall \xi_1 > 0; |\xi_1| = -\xi_1 \forall \xi_1 < 0; |\xi_1| = 0 \forall \xi_1 = 0$$

$$|\xi_1| = \xi_2 \forall \xi_2 > 0; |\xi_1| = -\xi_2 \forall \xi_2 < 0; |\xi_1| = 0 \forall \xi_2 = 0$$

Table 3.10 Summary of our system fixed points

Case number	Condition on $X^{(i)}$ values	Fixed point
Case II/III	$-\Gamma_5 < X^{(i)} < \Gamma_5 (\Gamma_5 > 0)$ $-\Gamma_5 > X^{(i)} > \Gamma_5 (\Gamma_5 < 0)$	$E^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0, 0, 0)$
Case I	$X^{(i)} < -\Gamma_5 (\Gamma_5 > 0)$ $X^{(i)} < \Gamma_5 (\Gamma_5 < 0)$	$E^{(1)} = (X^{(1)}, Y^{(1)}, Z^{(1)}) = \left(\left(\frac{\Gamma_4}{\Gamma_3} - 1\right) \cdot \Gamma_5, 0, 0\right)$
Case IIII	$X^{(i)} > \Gamma_5 (\Gamma_5 > 0)$ $X^{(i)} > -\Gamma_5 (\Gamma_5 < 0)$	$E^{(2)} = (X^{(2)}, Y^{(2)}, Z^{(2)}) = \left(\left(1 - \frac{\Gamma_4}{\Gamma_3}\right) \cdot \Gamma_5, 0, 0\right)$

Case A

$$\xi_1 > 0 \Rightarrow Y - X + \Gamma_5 > 0 \Rightarrow Y - X > -\Gamma_5 \ \& \ \xi_2 > 0$$
$$\Rightarrow Y - X - \Gamma_5 > 0 \Rightarrow Y - X > \Gamma_5$$

$$\Gamma_5 > 0 \Rightarrow Y - X > \Gamma_5; \Gamma_5 < 0 \Rightarrow Y - X > -\Gamma_5; \Gamma_5 = 0 \Rightarrow Y - X > 0$$

$$\xi_1 = Y - X + \Gamma_5; \xi_2 = Y - X - \Gamma_5;$$
$$g(Y-X) = \Gamma_3 \cdot (Y-X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|\xi_1| - |\xi_2|)$$

$$g(Y-X) = \Gamma_3 \cdot (Y-X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (2 \cdot \Gamma_5)$$
$$= \Gamma_3 \cdot (Y-X) + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$

$$\frac{\partial g(Y-X)}{\partial X} = -\Gamma_3; \frac{\partial g(Y-X)}{\partial Y} = \Gamma_3; \frac{\partial g(Y-X)}{\partial Z} = 0 \frac{\partial \psi_1}{\partial X} = -\Gamma_1 \cdot \frac{\partial g(Y-X)}{\partial X}$$
$$= \Gamma_1 \cdot \Gamma_3$$

$$\frac{\partial \psi_1}{\partial Y} = -\Gamma_1 \cdot \frac{\partial g(Y-X)}{\partial Y} = -\Gamma_1 \cdot \Gamma_3; \frac{\partial \psi_1}{\partial Z} = 0; \frac{\partial \psi_2}{\partial X} = -\frac{\partial g(Y-X)}{\partial X} = \Gamma_3$$
$$\frac{\partial \psi_2}{\partial Y} = -\frac{\partial g(Y-X)}{\partial Y} = -\Gamma_3; \frac{\partial \psi_2}{\partial Z} = -1; \frac{\partial \psi_3}{\partial X} = 0; \frac{\partial \psi_3}{\partial Y} = \Gamma_2; \frac{\partial \psi_3}{\partial Z} = 0$$

$$A = \begin{pmatrix} \frac{\partial \psi_1}{\partial X} & \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial Z} \\ \frac{\partial \psi_2}{\partial X} & \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial Z} \\ \frac{\partial \psi_3}{\partial X} & \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial Z} \end{pmatrix} = \begin{pmatrix} \Gamma_1 \cdot \Gamma_3 & -\Gamma_1 \cdot \Gamma_3 & 0 \\ \Gamma_3 & -\Gamma_3 & -1 \\ 0 & \Gamma_2 & 0 \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$A - \lambda \cdot I = \begin{pmatrix} \frac{\partial \psi_1}{\partial X} & \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial Z} \\ \frac{\partial \psi_2}{\partial X} & \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial Z} \\ \frac{\partial \psi_3}{\partial X} & \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial Z} \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \Gamma_1 \cdot \Gamma_3 - \lambda & -\Gamma_1 \cdot \Gamma_3 & 0 \\ \Gamma_3 & -(\Gamma_3 + \lambda) & -1 \\ 0 & \Gamma_2 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = (\Gamma_1 \cdot \Gamma_3 - \lambda) \cdot \begin{pmatrix} -(\Gamma_3 + \lambda) & -1 \\ \Gamma_2 & -\lambda \end{pmatrix} + \Gamma_1 \cdot \Gamma_3 \cdot \begin{pmatrix} \Gamma_3 & -1 \\ 0 & -\lambda \end{pmatrix}$$

Table 3.11 Neimark–Sacker (Torus) bifurcation system, Δ_1 possibilities and meanings

$\Delta_1 > 0$	$\Delta_1 < 0$	$\Delta_1 = 0$
The equation has three distinct real roots. When λ_1, λ_2, $\lambda_3 < 0$ stable node. λ_1, λ_2, $\lambda_3 > 0$ unstable node. When at least one eigenvalue is positive then we have saddle point	The equation has one real root and a pair of complex conjugate roots. We have stable and unstable spiral	At least two roots coincide. The system equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple roots

$$\det(A - \lambda \cdot I) = (\Gamma_1 \cdot \Gamma_3 - \lambda) \cdot \{(\Gamma_3 + \lambda) \cdot \lambda + \Gamma_2\} - \Gamma_1 \cdot \Gamma_3^2 \cdot \lambda$$
$$= -\lambda^3 + \lambda^2 \cdot \Gamma_3 \cdot (\Gamma_1 - 1) - \lambda \cdot \Gamma_2 + \prod_{j=1}^{3} \Gamma_j$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot \Gamma_3 \cdot (1 - \Gamma_1) + \lambda \cdot \Gamma_2 - \prod_{j=1}^{3} \Gamma_j = 0$$

We have cubic function of system eigenvalues. $\Gamma_1, \Gamma_2, \Gamma_3, \prod_{j=1}^{3} \Gamma_j$ are global parameters coefficients and are real numbers. We distinguish several possible cases using the discriminant [43, 44].

$$\Delta_1 = 4 \cdot \Gamma_3^3 \cdot (1 - \Gamma_1)^3 \cdot \prod_{j=1}^{3} \Gamma_j + \Gamma_3^2 \cdot (1 - \Gamma_1)^2 \cdot \Gamma_2^2$$
$$- 4 \cdot \Gamma_2^3 - 18 \cdot \Gamma_3 \cdot (1 - \Gamma_1) \cdot \Gamma_2 \cdot \prod_{j=1}^{3} \Gamma_j - 27 \cdot \prod_{j=1}^{3} \Gamma_j^2$$

The following cases need to be considered and present in Table 3.11.

Case B

$$\xi_1 > 0 \Rightarrow Y - X + \Gamma_5 > 0 \Rightarrow Y - X > -\Gamma_5 \; \& \; \xi_2 < 0$$
$$\Rightarrow Y - X - \Gamma_5 < 0 \Rightarrow Y - X < \Gamma_5$$

$\Gamma_5 > 0 \Rightarrow \Gamma_5 > [Y - X] > -\Gamma_5; \Gamma_5 < 0$ and $\Gamma_5 = 0$ are not possible in that case.

$$\xi_1 = Y - X + \Gamma_5; \xi_2 = Y - X - \Gamma_5;$$
$$g(Y - X) = \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|\xi_1| - |\xi_2|)$$

$$g(Y-X)=\Gamma_3\cdot(Y-X)+(\Gamma_4-\Gamma_3)\cdot(Y-X)=\Gamma_4\cdot(Y-X)$$

$$\frac{\partial g(Y-X)}{\partial X}=-\Gamma_4;\ \frac{\partial g(Y-X)}{\partial Y}=\Gamma_4;\ \frac{\partial g(Y-X)}{\partial Z}=0\ \frac{\partial\psi_1}{\partial X}=-\Gamma_1\cdot\frac{\partial g(Y-X)}{\partial X}=\Gamma_1\cdot\Gamma_4$$

$$\frac{\partial\psi_1}{\partial Y}=-\Gamma_1\cdot\frac{\partial g(Y-X)}{\partial Y}=-\Gamma_1\cdot\Gamma_4;\ \frac{\partial\psi_1}{\partial Z}=0;\ \frac{\partial\psi_2}{\partial X}=-\frac{\partial g(Y-X)}{\partial X}=\Gamma_4$$

$$\frac{\partial\psi_2}{\partial Y}=-\frac{\partial g(Y-X)}{\partial Y}=-\Gamma_4;\ \frac{\partial\psi_2}{\partial Z}=-1;\ \frac{\partial\psi_3}{\partial X}=0;\ \frac{\partial\psi_3}{\partial Y}=\Gamma_2;\ \frac{\partial\psi_3}{\partial Z}=0$$

$$A=\begin{pmatrix}\frac{\partial\psi_1}{\partial X} & \frac{\partial\psi_1}{\partial Y} & \frac{\partial\psi_1}{\partial Z}\\ \frac{\partial\psi_2}{\partial X} & \frac{\partial\psi_2}{\partial Y} & \frac{\partial\psi_2}{\partial Z}\\ \frac{\partial\psi_3}{\partial X} & \frac{\partial\psi_3}{\partial Y} & \frac{\partial\psi_3}{\partial Z}\end{pmatrix}=\begin{pmatrix}\Gamma_1\cdot\Gamma_4 & -\Gamma_1\cdot\Gamma_4 & 0\\ \Gamma_4 & -\Gamma_4 & -1\\ 0 & \Gamma_2 & 0\end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A-\lambda\cdot I)$, where I is the identity matrix.

$$A-\lambda\cdot I=\begin{pmatrix}\frac{\partial\psi_1}{\partial X} & \frac{\partial\psi_1}{\partial Y} & \frac{\partial\psi_1}{\partial Z}\\ \frac{\partial\psi_2}{\partial X} & \frac{\partial\psi_2}{\partial Y} & \frac{\partial\psi_2}{\partial Z}\\ \frac{\partial\psi_3}{\partial X} & \frac{\partial\psi_3}{\partial Y} & \frac{\partial\psi_3}{\partial Z}\end{pmatrix}-\lambda\cdot\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}$$
$$=\begin{pmatrix}\Gamma_1\cdot\Gamma_4-\lambda & -\Gamma_1\cdot\Gamma_4 & 0\\ \Gamma_4 & -(\Gamma_4+\lambda) & -1\\ 0 & \Gamma_2 & -\lambda\end{pmatrix}$$

$$\det(A-\lambda\cdot I)=(\Gamma_1\cdot\Gamma_4-\lambda)\cdot\begin{pmatrix}-(\Gamma_4+\lambda) & -1\\ \Gamma_2 & -\lambda\end{pmatrix}+\Gamma_1\cdot\Gamma_4\cdot\begin{pmatrix}\Gamma_4 & -1\\ 0 & -\lambda\end{pmatrix}$$

$$\det(A-\lambda\cdot I)=(\Gamma_1\cdot\Gamma_4-\lambda)\cdot\{(\Gamma_4+\lambda)\cdot\lambda+\Gamma_2\}-\Gamma_1\cdot\Gamma_4^2\cdot\lambda$$
$$=-\lambda^3+\lambda^2\cdot\Gamma_4\cdot(\Gamma_1-1)-\lambda\cdot\Gamma_2+\Gamma_1\cdot\Gamma_4\cdot\Gamma_2$$

$$\det(A-\lambda\cdot I)=0\Rightarrow\lambda^3+\lambda^2\cdot\Gamma_4\cdot(1-\Gamma_1)+\lambda\cdot\Gamma_2-\Gamma_1\cdot\Gamma_4\cdot\Gamma_2=0$$

We have cubic function of system eigenvalues. $\Gamma_1,\Gamma_2,\Gamma_4$ are global parameters coefficients and are real numbers. We distinguish several possible cases using the discriminant.

$$\Delta_2=4\cdot\Gamma_4^4\cdot(1-\Gamma_1)^3\cdot\Gamma_1\cdot\Gamma_2+\Gamma_4^2\cdot(1-\Gamma_1)^2\cdot\Gamma_2^2-4\cdot\Gamma_2^3$$
$$-18\cdot\Gamma_4^2\cdot(1-\Gamma_1)\cdot\Gamma_2^2\cdot\Gamma_1-27\cdot\Gamma_2\cdot\Gamma_1\cdot\Gamma_4$$

The following cases need to be considered and present in Table 3.12.

Table 3.12 Neimark–Sacker (Torus) bifurcation system, Δ_2 possibilities and meanings

$\Delta_2 > 0$	$\Delta_2 < 0$	$\Delta_2 = 0$
The equation has three distinct real roots. When λ_1, λ_2, $\lambda_3 < 0$ stable node. λ_1, λ_2, $\lambda_3 > 0$ unstable node. When at least one eigenvalue is positive then we have saddle point	The equation has one real root and a pair of complex conjugate roots. We have stable and unstable spiral	At least two roots coincide. The system equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple roots

Case C

$$\begin{aligned}\xi_1 < 0 &\Rightarrow Y - X + \Gamma_5 < 0 \Rightarrow Y - X < -\Gamma_5 \;\&\; \xi_2 > 0 \\ &\Rightarrow Y - X - \Gamma_5 > 0 \Rightarrow Y - X > \Gamma_5\end{aligned}$$

$\Gamma_5 < 0 \Rightarrow -\Gamma_5 > [Y - X] > \Gamma_5; \Gamma_5 > 0$ and $\Gamma_5 = 0$ are not possible in that case.

$$\begin{aligned}\xi_1 &= Y - X + \Gamma_5; \xi_2 = Y - X - \Gamma_5; g(Y - X) \\ &= \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|\xi_1| - |\xi_2|)\end{aligned}$$

$$g(Y - X) = \Gamma_3 \cdot (Y - X) + (\Gamma_4 - \Gamma_3) \cdot (X - Y) = (2 \cdot \Gamma_3 - \Gamma_4) \cdot (Y - X)$$

$$\begin{aligned}&\frac{\partial g(Y - X)}{\partial X} = \Gamma_4 - 2 \cdot \Gamma_3; \frac{\partial g(Y - X)}{\partial Y} = 2 \cdot \Gamma_3 - \Gamma_4; \frac{\partial g(Y - X)}{\partial Z} = 0 \\ &\frac{\partial \psi_1}{\partial X} = -\Gamma_1 \cdot \frac{\partial g(Y - X)}{\partial X} = \Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4)\end{aligned}$$

$$\begin{aligned}\frac{\partial \psi_1}{\partial Y} &= -\Gamma_1 \cdot \frac{\partial g(Y - X)}{\partial Y} = \Gamma_1 \cdot (\Gamma_4 - 2 \cdot \Gamma_3); \frac{\partial \psi_1}{\partial Z} = 0; \frac{\partial \psi_2}{\partial X} = -\frac{\partial g(Y - X)}{\partial X} \\ &= 2 \cdot \Gamma_3 - \Gamma_4\end{aligned}$$

$$\frac{\partial \psi_2}{\partial Y} = -\frac{\partial g(Y - X)}{\partial Y} = \Gamma_4 - 2 \cdot \Gamma_3; \frac{\partial \psi_2}{\partial Z} = -1; \frac{\partial \psi_3}{\partial X} = 0; \frac{\partial \psi_3}{\partial Y} = \Gamma_2; \frac{\partial \psi_3}{\partial Z} = 0$$

$$A = \begin{pmatrix} \frac{\partial \psi_1}{\partial X} & \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial Z} \\ \frac{\partial \psi_2}{\partial X} & \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial Z} \\ \frac{\partial \psi_3}{\partial X} & \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial Z} \end{pmatrix} = \begin{pmatrix} \Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4) & \Gamma_1 \cdot (\Gamma_4 - 2 \cdot \Gamma_3) & 0 \\ 2 \cdot \Gamma_3 - \Gamma_4 & \Gamma_4 - 2 \cdot \Gamma_3 & -1 \\ 0 & \Gamma_2 & 0 \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det $(A - \lambda \cdot I)$, where I is the identity matrix.

$$A - \lambda \cdot I = \begin{pmatrix} \frac{\partial \psi_1}{\partial X} & \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial Z} \\ \frac{\partial \psi_2}{\partial X} & \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial Z} \\ \frac{\partial \psi_3}{\partial X} & \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial Z} \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4) - \lambda & \Gamma_1 \cdot (\Gamma_4 - 2 \cdot \Gamma_3) & 0 \\ 2 \cdot \Gamma_3 - \Gamma_4 & \Gamma_4 - 2 \cdot \Gamma_3 - \lambda & -1 \\ 0 & \Gamma_2 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = [\Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4) - \lambda] \cdot \begin{pmatrix} \Gamma_4 - 2 \cdot \Gamma_3 - \lambda & -1 \\ \Gamma_2 & -\lambda \end{pmatrix}$$
$$+ \Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4) \cdot \begin{pmatrix} 2 \cdot \Gamma_3 - \Gamma_4 & -1 \\ 0 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = [\Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4) - \lambda] \cdot \{(2 \cdot \Gamma_3 - \Gamma_4 + \lambda) \cdot \lambda + \Gamma_2\} - \Gamma_1 \cdot (2 \cdot \Gamma_3 - \Gamma_4)^2 \cdot \lambda$$

For simplicity we define $\Gamma_{\#} = 2 \cdot \Gamma_3 - \Gamma_4$

$$\Gamma_{\#} = 2 \cdot \Gamma_3 - \Gamma_4 \Rightarrow \det(A - \lambda \cdot I) = [\Gamma_1 \cdot \Gamma_{\#} - \lambda] \cdot \{(\Gamma_{\#} + \lambda) \cdot \lambda + \Gamma_2\} - \Gamma_1 \cdot \Gamma_{\#}^2 \cdot \lambda$$

$$\Gamma_{\#} = 2 \cdot \Gamma_3 - \Gamma_4 \Rightarrow \det(A - \lambda \cdot I) = [\Gamma_1 \cdot \Gamma_{\#} - \lambda] \cdot \{\Gamma_{\#} \cdot \lambda + \lambda^2 + \Gamma_2\} - \Gamma_1 \cdot \Gamma_{\#}^2 \cdot \lambda$$

$$\det(A - \lambda \cdot I) = -\lambda^3 + \lambda^2 \cdot \Gamma_{\#} \cdot (\Gamma_1 - 1) - \lambda \cdot \Gamma_2 + \Gamma_2 \cdot \Gamma_1 \cdot \Gamma_{\#}$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot \Gamma_{\#} \cdot (1 - \Gamma_1) + \lambda \cdot \Gamma_2 - \Gamma_2 \cdot \Gamma_1 \cdot \Gamma_{\#} = 0$$

We have cubic function of system eigenvalues. $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are global parameters coefficients and are real numbers. We distinguish several possible cases using the discriminant.

$$\Delta_3 = 4 \cdot \Gamma_{\#}^4 \cdot (1 - \Gamma_1)^3 \cdot \Gamma_1 \cdot \Gamma_2 + \Gamma_{\#}^2 \cdot (1 - \Gamma_1)^2 \cdot \Gamma_2^2 - 4 \cdot \Gamma_2^3 - 18 \cdot \Gamma_{\#}^2 \cdot (1 - \Gamma_1) \cdot \Gamma_2^2 \cdot \Gamma_1 - 27 \cdot \Gamma_2 \cdot \Gamma_1 \cdot \Gamma_{\#}$$

The following cases need to be considered and present in Table 3.13.

Case D

$$\xi_1 < 0 \Rightarrow Y - X + \Gamma_5 < 0 \Rightarrow Y - X < -\Gamma_5 \;\&\; \xi_2 < 0 \Rightarrow Y - X - \Gamma_5 < 0 \Rightarrow Y - X < \Gamma_5$$

Table 3.13 Neimark–Sacker (Torus) bifurcation system, Δ_3 possibilities and meanings

$\Delta_3 > 0$	$\Delta_3 < 0$	$\Delta_3 = 0$
The equation has three distinct real roots. When λ_1, λ_2, $\lambda_3 < 0$ stable node. λ_1, λ_2, $\lambda_3 > 0$ unstable node. When at least one eigenvalue is positive then we have saddle point	The equation has one real root and a pair of complex conjugate roots. We have stable and unstable spiral	At least two roots coincide. The system equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple roots

$$\Gamma_5 > 0 \Rightarrow Y - X < -\Gamma_5; \Gamma_5 < 0 \Rightarrow Y - X < \Gamma_5; \Gamma_5 = 0 \Rightarrow Y - X < 0$$

$$\begin{aligned}\xi_1 &= Y - X + \Gamma_5; \xi_2 = Y - X - \Gamma_5; g(Y - X) \\ &= \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|\xi_1| - |\xi_2|)\end{aligned}$$

$$\begin{aligned}g(Y - X) &= \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (-2 \cdot \Gamma_5) \\ &= \Gamma_3 \cdot (Y - X) + (\Gamma_3 - \Gamma_4) \cdot \Gamma_5\end{aligned}$$

$$\begin{aligned}&\frac{\partial g(Y - X)}{\partial X} = -\Gamma_3; \quad \frac{\partial g(Y - X)}{\partial Y} = \Gamma_3; \\ &\frac{\partial g(Y - X)}{\partial Z} = 0 \ \frac{\partial \psi_1}{\partial X} = -\Gamma_1 \cdot \frac{\partial g(Y - X)}{\partial X} = \Gamma_1 \cdot \Gamma_3\end{aligned}$$

$$\frac{\partial \psi_1}{\partial Y} = -\Gamma_1 \cdot \frac{\partial g(Y - X)}{\partial Y} = -\Gamma_1 \cdot \Gamma_3; \frac{\partial \psi_1}{\partial Z} = 0; \frac{\partial \psi_2}{\partial X} = -\frac{\partial g(Y - X)}{\partial X} = \Gamma_3$$

$$\frac{\partial \psi_2}{\partial Y} = -\frac{\partial g(Y - X)}{\partial Y} = -\Gamma_3; \frac{\partial \psi_2}{\partial Z} = -1; \frac{\partial \psi_3}{\partial X} = 0; \frac{\partial \psi_3}{\partial Y} = \Gamma_2; \frac{\partial \psi_3}{\partial Z} = 0$$

$$A = \begin{pmatrix} \frac{\partial \psi_1}{\partial X} & \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial Z} \\ \frac{\partial \psi_2}{\partial X} & \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial Z} \\ \frac{\partial \psi_3}{\partial X} & \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial Z} \end{pmatrix} = \begin{pmatrix} \Gamma_1 \cdot \Gamma_3 & -\Gamma_1 \cdot \Gamma_3 & 0 \\ \Gamma_3 & -\Gamma_3 & -1 \\ 0 & \Gamma_2 & 0 \end{pmatrix}$$

The eigenvalues of a matrix A are given by the characteristic equation det ($A - \lambda \cdot I$), where I is the identity matrix.

$$\begin{aligned}A - \lambda \cdot I &= \begin{pmatrix} \frac{\partial \psi_1}{\partial X} & \frac{\partial \psi_1}{\partial Y} & \frac{\partial \psi_1}{\partial Z} \\ \frac{\partial \psi_2}{\partial X} & \frac{\partial \psi_2}{\partial Y} & \frac{\partial \psi_2}{\partial Z} \\ \frac{\partial \psi_3}{\partial X} & \frac{\partial \psi_3}{\partial Y} & \frac{\partial \psi_3}{\partial Z} \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_1 \cdot \Gamma_3 - \lambda & -\Gamma_1 \cdot \Gamma_3 & 0 \\ \Gamma_3 & -(\Gamma_3 + \lambda) & -1 \\ 0 & \Gamma_2 & -\lambda \end{pmatrix}\end{aligned}$$

$$\det(A - \lambda \cdot I) = (\Gamma_1 \cdot \Gamma_3 - \lambda) \cdot \begin{pmatrix} -(\Gamma_3 + \lambda) & -1 \\ \Gamma_2 & -\lambda \end{pmatrix} + \Gamma_1 \cdot \Gamma_3 \cdot \begin{pmatrix} \Gamma_3 & -1 \\ 0 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = (\Gamma_1 \cdot \Gamma_3 - \lambda) \cdot \{(\Gamma_3 + \lambda) \cdot \lambda + \Gamma_2\} - \Gamma_1 \cdot \Gamma_3^2 \cdot \lambda$$
$$= -\lambda^3 + \lambda^2 \cdot \Gamma_3 \cdot (\Gamma_1 - 1) - \lambda \cdot \Gamma_2 + \prod_{j=1}^{3} \Gamma_j$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot \Gamma_3 \cdot (1 - \Gamma_1) + \lambda \cdot \Gamma_2 - \prod_{j=1}^{3} \Gamma_j = 0$$

We have cubic function of system eigenvalues. $\Gamma_1, \Gamma_2, \Gamma_3, \prod_{j=1}^{3} \Gamma_j$ are global parameters coefficients and are real numbers. We distinguish several possible cases using the discriminant.

$$\Delta_4 = 4 \cdot \Gamma_3^3 \cdot (1 - \Gamma_1)^3 \cdot \prod_{j=1}^{3} \Gamma_j + \Gamma_3^2 \cdot (1 - \Gamma_1)^2 \cdot \Gamma_2^2 - 4 \cdot \Gamma_2^3$$
$$- 18 \cdot \Gamma_3 \cdot (1 - \Gamma_1) \cdot \Gamma_2 \cdot \prod_{j=1}^{3} \Gamma_j - 27 \cdot \prod_{j=1}^{3} \Gamma_j^2$$

The following cases need to be considered and present in Table 3.14.

Results Discussion We got the same characteristic equations and discriminants for case A and case D ($\Delta_1 = \Delta_4$). All our Jacobians (four cases) are independent on specific system fixed point coordinates. We got four cases for system fixed points (I/II/III/IIII) and four cases for system characteristic equations (A/B/C/D). We need to adapt between each fixed point and characteristic equation and discuss stability.

We need to plot our system 2D phase portrait behavior, 3D phase, $X(t)$, $Y(t)$, $Z(t)$ for different Γ_i values ($i = 1,\ldots,5$) and different initial values $X(t = 0)$, $Y(t = 0)$, $Z(t = 0)$.

Table 3.14 Neimark–Sacker (Torus) bifurcation system, Δ_4 possibilities and meanings

$\Delta_4 > 0$	$\Delta_4 < 0$	$\Delta_4 = 0$
The equation has three distinct real roots. When λ_1, λ_2, $\lambda_3 < 0$ stable node. λ_1, λ_2, $\lambda_3 > 0$ unstable node. When at least one eigenvalue is positive then we have saddle point	The equation has one real root and a pair of complex conjugate roots. We have stable and unstable spiral	At least two roots coincide. The system equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple roots

Matlab Script

$$\Gamma_1 \to a;\ \Gamma_2 \to b;\ \Gamma_3 \to c;\ \Gamma_4 \to d;\ \Gamma_5 \to e;\ X(t=0) \to x0;\ Y(t=0) \to y0;\ Z(t=0) \to z0.$$

```
function h=doubletorus1(a,b,c,d,e,x0,y0,z0)
[t,x]=ODE45 (@doubletorus,[0,500],[x0,y0,z0],[],a,b,c,d,e);
%[X,Y]=pol2cart(x(2),x(1));
%Z=abs(x(1).*exp(1i.*x(2)));
%plot3 (x(:,1),x(:,2),x(:,3));
xlabel ('X')
ylabel ('Y')
zlabel ('Z')
grid on
axis square
subplot(2,2,1);plot(x(:,1),x(:,2));subplot(2,2,2);plot(x(:,3),x(:,2))
;subplot(2,2,3);plot(x(:,1),x(:,3));subplot(2,2,4);plot(t,x);
function g=doubletorus(t,x,a,b,c,d,e)
g=zeros(3,1);
g(1)=-a*(c*(x(2)-x(1))+((d-c)/2)*(abs(x(2)-x(1)+e)-abs(x(2)-x(1)-
e)));
g(2)=-(c*(x(2)-x(1))+((d-c)/2)*(abs(x(2)-x(1)+e)-abs(x(2)-x(1)-e)))-
x(3);
g(3)=b*x(2);
```

doubletorus1($a = 2$, $b = 1$, $c = -0.07$, $d = 0.1$, $e = 1$, $x0 = 0.8$, $y0 = 0.7$, $z0 = 0.9$) (Fig. 3.19)

When $\Gamma_1 = 15$, $\Gamma_2 = 1$, $\Gamma_3 = -0.07$, $\Gamma_4 = 0.1$, $\Gamma_5 = 1$ the system has a double-folded torus chaotic attractor. Our system has three equilibrium points [53, 65–69]. $E^{(0)} = (X^{(0)}, Y^{(0)}, Z^{(0)}) = (0,0,0)$, $E^{(1)} = (X^{(1)}, Y^{(1)}, Z^{(1)}) = \left(\left(\frac{\Gamma_4}{\Gamma_3} - 1\right) \cdot \Gamma_5, 0, 0\right)$ and

$$E^{(2)} = (X^{(2)}, Y^{(2)}, Z^{(2)}) = \left(\left(1 - \frac{\Gamma_4}{\Gamma_3}\right) \cdot \Gamma_5, 0, 0\right) \cdot \frac{\Gamma_4}{\Gamma_3} - 1\right) \cdot \Gamma_5|_{\Gamma_4=0.1,\Gamma_3=-0.07,\Gamma_5=1} = -\frac{17}{7} = -2.42$$

$\left(1 - \frac{\Gamma_4}{\Gamma_3}\right) \cdot \Gamma_5\Big|_{\Gamma_4=0.1,\Gamma_3=-0.07,\Gamma_5=1} = \frac{17}{7} = 2.42.$ $E^{(0)} = (0,0,0); E^{(1,2)} = \left(\pm\frac{17}{7}, 0, 0\right).$

Linearization system at equilibrium point $E^{(0)}$, first fixed point gives the corresponding eigenvalues: $\lambda_1^{(0)} \simeq 1.43; \lambda_{2,3}^{(0)} \simeq -0.0164 \pm 1.023 \cdot i$. Similarly, linearizing system at the equilibrium points $E^{(1,2)}$ gets the following eigenvalues: $\lambda_1^{(1,2)} \simeq -1.0145$; $\lambda_{2,3}^{(1,2)} \simeq 0.0172 \pm 1.0172 \cdot i$ (Figs. 3.20 and 3.21)

doubletorus1($a = 0.15$, $b = 1$, $c = -0.07$, $d = 0.1$, $e = 1$, $x0 = 10$, $y0 = 2$, $z0 = 3$) (Figs. 3.22 and 3.23)

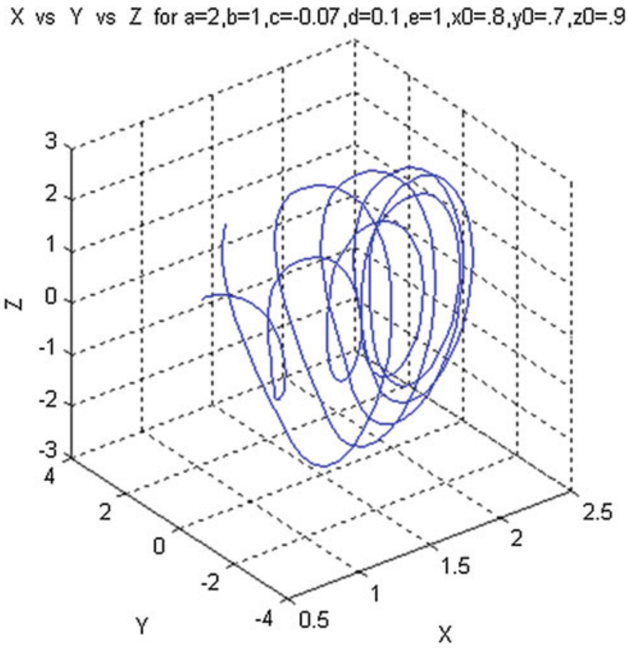

Fig. 3.19 X versus Y versus Z for a = 2, b = 1, c = −0.07, d = 0.1, e = 1, $x0$ = 0.8, $y0$ = 0.7, $z0$ = 0.9

doubletorus1(a = 15, b = 1, c = −0.07, d = 0.1, e = 1, $x0$ = 0.1, $y0$ = 0.2, $z0$ = 0.3) (Fig. 3.24)

doubletorus1(a = 15, b = 1, c = −0.07, d = 0.1, e = 1, $x0$ = 10, $y0$ = 2, $z0$ = 3) (Fig. 3.25)

3.6 Optoisolation Circuits Neimark–Sacker (Torus) Bifurcation

We need to implement our multifolded torus chaotic attractors system by using optoisolation circuits. Since we differentiated our system according all options for $g(Y - X)$ function, there are four options (A, B, C, D) of block diagram and optoisolation circuits implementation [16, 25, 26].

Case A

$$\xi_1 > 0 \Rightarrow Y - X + \Gamma_5 > 0 \Rightarrow Y - X > -\Gamma_5 \; \& \; \xi_2 > 0$$
$$\Rightarrow Y - X - \Gamma_5 > 0 \Rightarrow Y - X > \Gamma_5$$

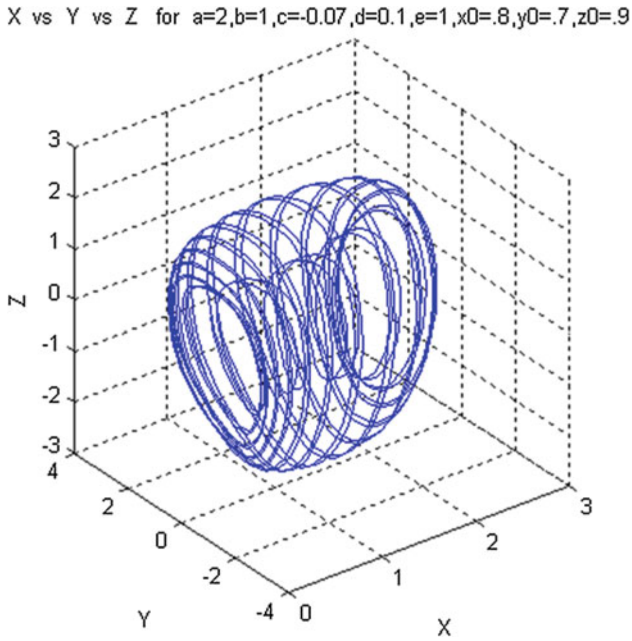

Fig. 3.20 *X* versus *Y* versus *Z* for $a = 2$, $b = 1$, $c = -0.07$, $d = 0.1$, $e = 1$, $x0 = 0.8$, $y0 = 0.7$, $z0 = 0.9$

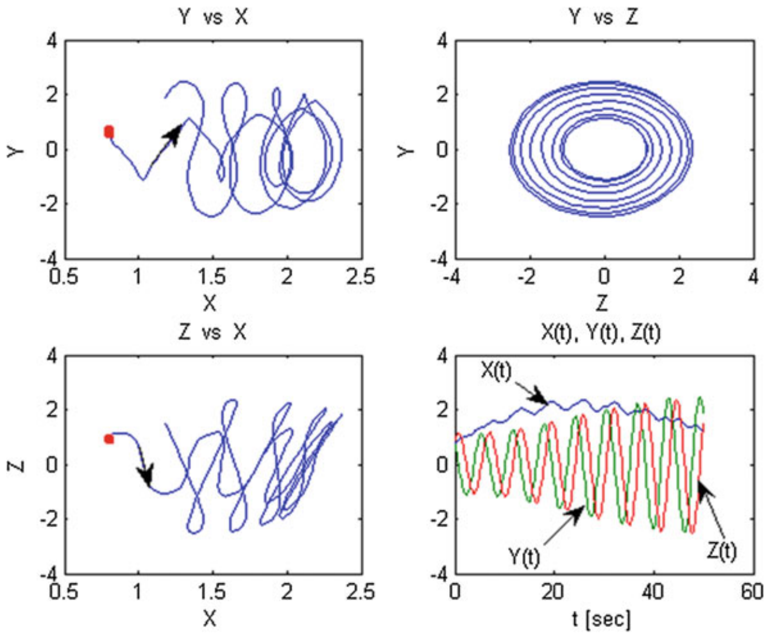

Fig. 3.21 *X* versus *Y* versus *Z* for $a = 2$, $b = 1$, $c = -0.07$, $d = 0.1$, $e = 1$, $x0 = 0.8$, $y0 = 0.7$, $z0 = 0.9$

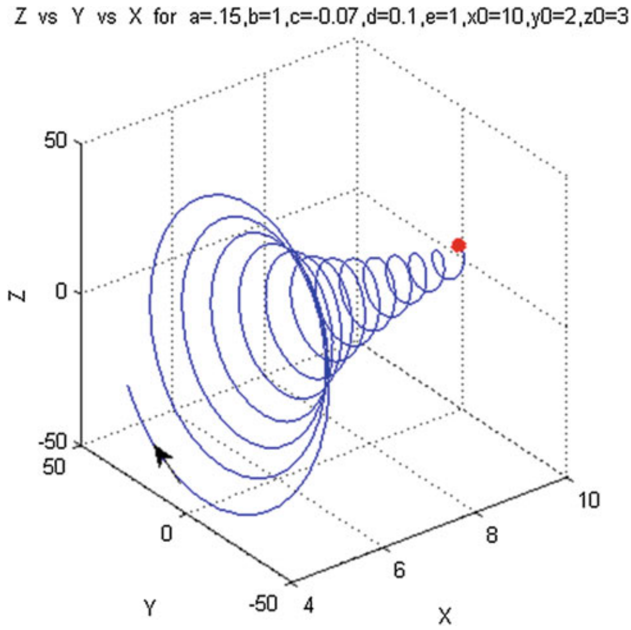

Fig. 3.22 $a = 2$, $b = 1$, $c = -0.07$, $d = 0.1$, $e = 1$, $x0 = 0.8$, $y0 = 0.7$, $z0 = 0.9$

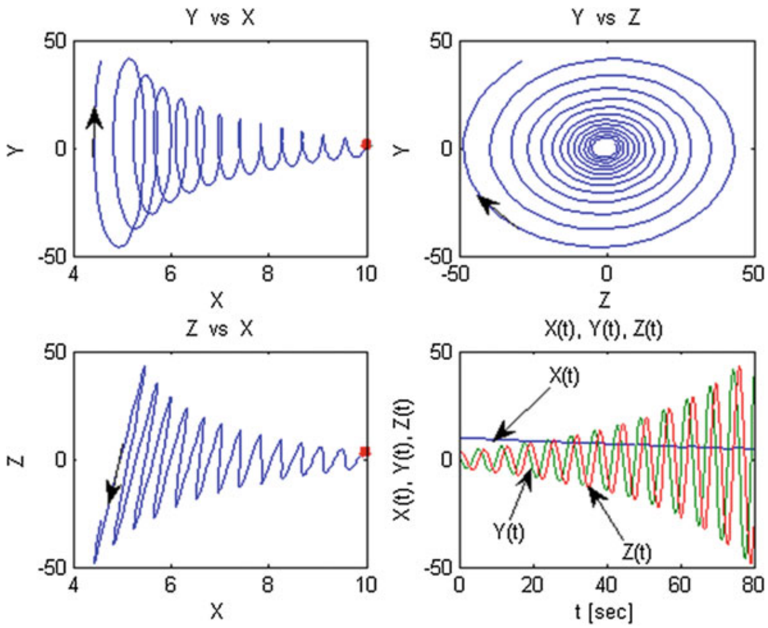

Fig. 3.23 Z versus Y versus X for $a = 0.15$, $b = 1$, $c = -0.07$, $d = 0.1$, $e = 1$, $x0 = 10$, $y0 = 2$, $z0 = 3$

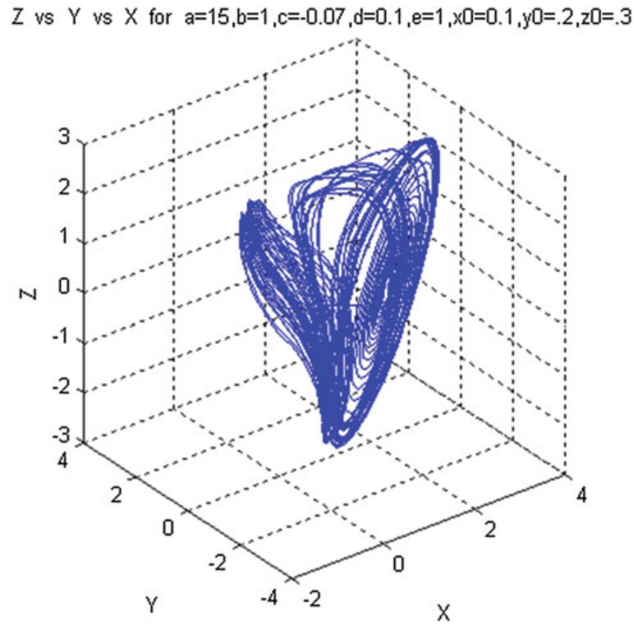

Fig. 3.24 a = 0.15, b = 1, c = −0.07, d = 0.1, e = 1, $x0$ = 10, $y0$ = 2, $z0$ = 3

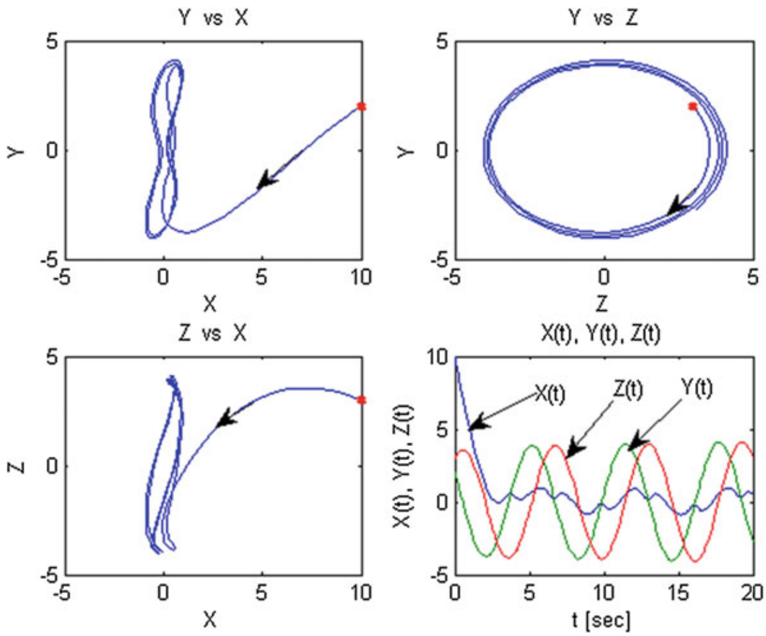

Fig. 3.25 Z versus Y versus X for a = 15, b = 1, c = −0.07, d = 0.1, e = 1, $x0$ = 0.1, $y0$ = 0.2, $z0$ = 0.3

$$\Gamma_5 > 0 \Rightarrow Y - X > \Gamma_5; \quad \Gamma_5 < 0 \Rightarrow Y - X > -\Gamma_5; \quad \Gamma_5 = 0 \Rightarrow Y - X > 0$$

$$\xi_1 = Y - X + \Gamma_5; \quad \xi_2 = Y - X - \Gamma_5;$$
$$g(Y - X) = \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (|\xi_1| - |\xi_2|)$$

$$g(Y - X) = \Gamma_3 \cdot (Y - X) + \frac{(\Gamma_4 - \Gamma_3)}{2} \cdot (2 \cdot \Gamma_5)$$
$$= \Gamma_3 \cdot (Y - X) + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$

$$\frac{dY}{dt} = -\Gamma_1 \cdot g(Y - X) = -\Gamma_1 \cdot \{\Gamma_3 \cdot (Y - X) + (\Gamma_4 - \Gamma_3) \cdot \Gamma_5\}$$
$$= -\Gamma_1 \cdot \Gamma_3 \cdot (Y - X) - \Gamma_1 \cdot (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$
$$\frac{dY}{dt} = -g(Y - X) - Z = -\Gamma_3 \cdot (Y - X) - (\Gamma_4 - \Gamma_3) \cdot \Gamma_5 - Z; \quad \frac{dZ}{dt} = \Gamma_2 \cdot Y$$

$$\frac{dX}{dt} = -\Gamma_1 \cdot \Gamma_3 \cdot Y + \Gamma_1 \cdot \Gamma_3 \cdot X - \Gamma_1 \cdot (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$
$$\Rightarrow X = \frac{1}{\Gamma_1 \cdot \Gamma_3} \cdot \frac{dX}{dt} + Y + \Gamma_5 \cdot \left(\frac{\Gamma_4}{\Gamma_3} - 1\right)$$
$$\frac{dY}{dt} = -\Gamma_3 \cdot Y + \Gamma_3 \cdot X - (\Gamma_4 - \Gamma_3) \cdot \Gamma_5 - Z$$
$$\Rightarrow Z = -\frac{dY}{dt} - \Gamma_3 \cdot Y + \Gamma_3 \cdot X - (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$

$\frac{dZ}{dt} = \Gamma_2 \cdot Y \Rightarrow Y = \frac{1}{\Gamma_2} \cdot \frac{dZ}{dt}$. The below system block diagram describe our system (Fig. 3.26).

$$f_1(\Gamma_3, \Gamma_4, \Gamma_5) = \Gamma_5 \cdot (\frac{\Gamma_4}{\Gamma_3} - 1); f_2(\Gamma_3, \Gamma_4, \Gamma_5) = (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$$

We implement our system by using optoisolation circuits. The circuit variables are V_X, V_Y, V_Z which are equivalent to system variables *X*, *Y*, *Z*.

$V_X \rightleftharpoons X; V_Y \rightleftharpoons Y; V_Z \rightleftharpoons Z$. The below optoisolation circuits implement our multifolded torus chaotic attractors system (Case A). Gama2 → Γ_2 [16, 26] (Fig. 3.27).

$$I_{D1} = \sum_{i=1}^{3} I_{R1i} = I_{R11} + I_{R12}; I_{D2} = \sum_{i=1}^{4} I_{R2i} = I_{R21} + I_{R22} + I_{R23}$$

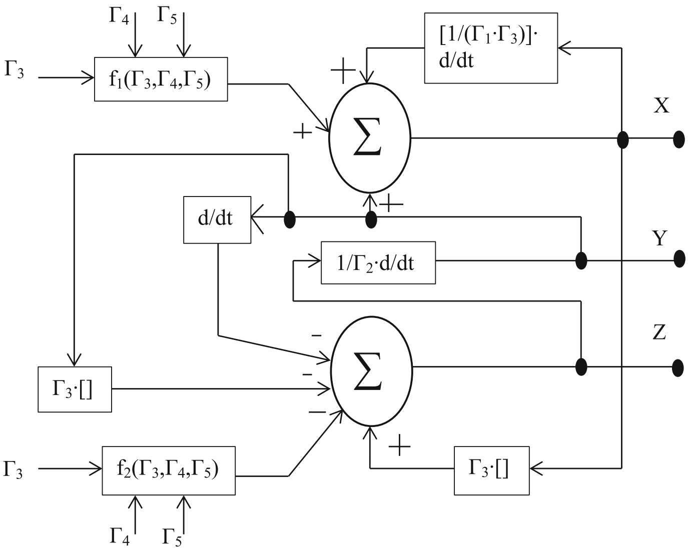

Fig. 3.26 *Z* versus *Y* versus *X* for

Using Taylor series approximation: $I_{BQ1} = k_1 \cdot I_{D1}; I_{BQ2} = k_2 \cdot I_{D2}$

$$V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D1}}{I_0}; V_{D2} = V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right] \approx V_t \cdot \frac{I_{D2}}{I_0}$$

$$I_{R11} = \frac{-\dot{V}_X - V_{D1}}{R11}; I_{R12} = \frac{-V_Y - V_{D1}}{R12}; I_{D1} = \sum_{i=1}^{3} I_{R1i}$$
$$= \frac{-\dot{V}_X - V_{D1}}{R11} + \frac{-V_Y - V_{D1}}{R12}$$

$$I_{D1} = \sum_{i=1}^{2} I_{R1i} = \frac{-\dot{V}_X}{R11} + \frac{-V_Y}{R12} - V_{D1} \cdot \sum_{i=1}^{2} \frac{1}{R1i} = \frac{-\dot{V}_X}{R11} + \frac{-V_Y}{R12} - I_{D1} \cdot \frac{V_t}{I_0} \cdot \sum_{i=1}^{2} \frac{1}{R1i}$$

$$I_{D1} \cdot \left\{1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{2} \frac{1}{R1i}\right\} = -\frac{\dot{V}_X}{R11} - \frac{V_Y}{R12} \Rightarrow I_{D1}$$
$$= \frac{1}{1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{2} \frac{1}{R1i}} \cdot \left\{\frac{-\dot{V}_X}{R11} - \frac{V_Y}{R12}\right\}$$

$$\eta_1 = \frac{1}{1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{2} \frac{1}{R1i}}; \psi_1 = \psi_1(V_X, V_Y) = \frac{-\dot{V}_X}{R11} - \frac{V_Y}{R12}; I_{D1} = \eta_1 \cdot \psi_1$$

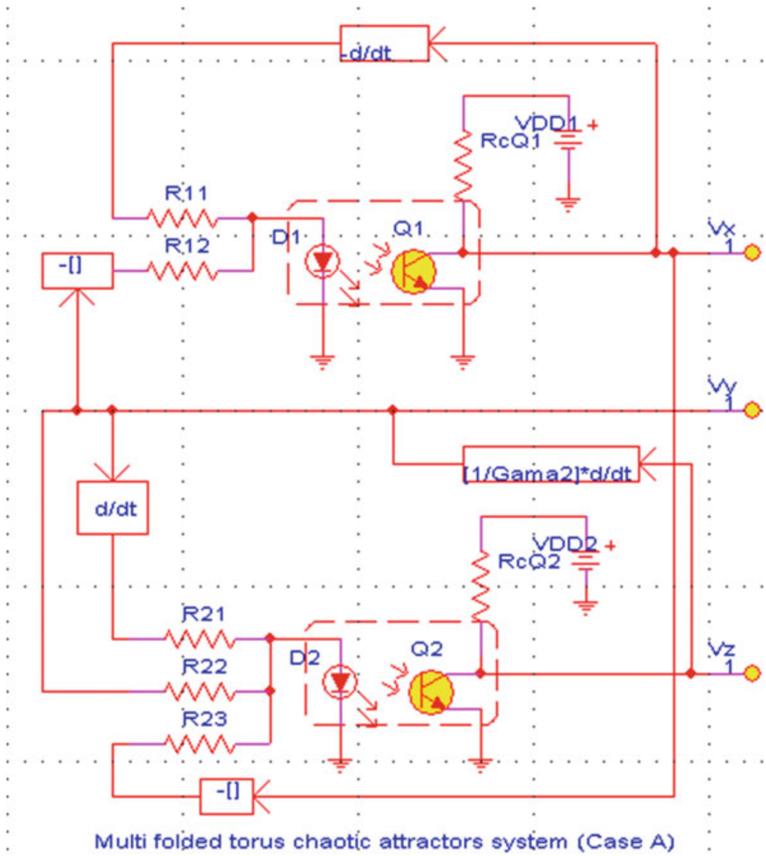

Fig. 3.27 Multifolded torus chaotic attractors system (Case A)

$$I_{R21} = \frac{\dot{V}_Y - V_{D2}}{R_{21}}; I_{R22} = \frac{V_Y - V_{D2}}{R_{22}}; I_{R23} = \frac{-V_X - V_{D2}}{R_{23}}$$

$$I_{D2} = \sum_{i=1}^{3} I_{R2i} = \frac{\dot{V}_Y - V_{D2}}{R_{21}} + \frac{V_Y - V_{D2}}{R_{22}} + \frac{-V_X - V_{D2}}{R_{23}}$$

$$\begin{aligned} I_{D2} &= \sum_{i=1}^{3} I_{R2i} = \frac{\dot{V}_Y}{R_{21}} + \frac{V_Y}{R_{22}} - \frac{V_X}{R_{23}} - V_{D2} \cdot \sum_{i=1}^{3} R_{2i} \\ &= \frac{\dot{V}_Y}{R_{21}} + \frac{V_Y}{R_{22}} - \frac{V_X}{R_{23}} - I_{D2} \cdot \frac{V_t}{I_0} \cdot \sum_{i=1}^{3} R_{2i} \end{aligned}$$

$$I_{D2} \cdot \left(1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{3} R_{2i}\right) = \frac{\dot{V}_Y}{R_{21}} + \frac{V_Y}{R_{22}} - \frac{V_X}{R_{23}}$$

$$\Rightarrow I_{D2} = \frac{1}{1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{3} R_{2i}} \cdot \left\{\frac{\dot{V}_Y}{R_{21}} + \frac{V_Y}{R_{22}} - \frac{V_X}{R_{23}}\right\}$$

$$\eta_2 = \frac{1}{1 + \frac{V_t}{I_0} \cdot \sum_{i=1}^{3} R_{2i}}; \psi_2 = \psi_2(V_X, V_Y) = \frac{\dot{V}_Y}{R_{21}} + \frac{V_Y}{R_{22}} - \frac{V_X}{R_{23}}; I_{D2} = \eta_2 \cdot \psi_2$$

$$\eta_1 > 0; \eta_2 > 0; V_X = V_{CEQ1}; V_Z = V_{CEQ2}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; I_{EQ2} = I_{BQ2} + I_{CQ2};$$
$$I_{EQ1} = k_1 \cdot I_{D1} + I_{CQ1} = k_1 \cdot \eta_1 \cdot \psi_1 + I_{CQ1}$$

$$I_{BQ1} = k_1 \cdot I_{D1} = k_1 \cdot \eta_1 \cdot \psi_1; I_{BQ2} = k_2 \cdot I_{D2} = k_2 \cdot \eta_2 \cdot \psi_2;$$
$$I_{EQ2} = k_2 \cdot I_{D2} + I_{CQ2} = k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}$$

The Mathematical analysis is based on the basic Transistor Ebers–Moll equations. We need to implement the Regular Ebers–Moll Model to the above Opto Coupler circuit. Parameters k_1 and k_2 are coupling coefficients between LEDs $D1$, $D2$, and photo transistors $Q1$ and $Q2$, respectively. $V_X = V_{CEQ1}; V_Z = V_{CEQ2}$

$$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r_1 \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right];$$
$$V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f_1}{Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]$$

$V_{CEQ1} = V_{CBQ1} + V_{BEQ1}$, but $V_{CBQ1} = -V_{BCQ1}$, then $V_{CEQ_1} = V_{BEQ_1} - V_{BCQ1}$

$$V_{CEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r_1 \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f_1}{Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r_1 \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f_1) + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right] + Vt \cdot \ln\left(\frac{Isc}{Ise}\right)$$

Since $Isc - Ise \to \varepsilon \Rightarrow \ln\left(\frac{Isc}{Ise}\right) \to \varepsilon$ then we can write V_{CEQ1} expression.
$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r_1 \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f_1) + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right]$ and in the same manner

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{(\alpha r_2 \cdot I_{CQ2} - I_{EQ2}) + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha f_2) + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right]$$

$$\alpha r_1 \cdot I_{CQ1} - I_{EQ1} = \alpha r_1 \cdot I_{CQ1} - [k_1 \cdot \eta_1 \cdot \psi_1 + I_{CQ1}] \\ = I_{CQ1} \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f_1 = I_{CQ1} - [k_1 \cdot \eta_1 \cdot \psi_1 + I_{CQ1}] \cdot \alpha f_1 \\ = I_{CQ1} \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1$$

$$I_{CQ1} = \frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}} \Rightarrow \alpha r_1 \cdot I_{CQ1} - I_{EQ1} \\ = \left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1$$

$$I_{CQ1} = \frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}} \Rightarrow I_{CQ1} - I_{EQ1} \cdot \alpha f_1 \\ = \left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (\alpha r_1 - 1) - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{\left[\frac{V_{DD1} - V_{CEQ1}}{R_{CQ1}}\right] \cdot (1 - \alpha f_1) - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right]$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{\frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - V_{CEQ1} \cdot \frac{(\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{\frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - V_{CEQ1} \cdot \frac{(1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right]$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{V_{CEQ1} \cdot \frac{(1 - \alpha r_1)}{R_{CQ1}} + \frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}{-V_{CEQ1} \cdot \frac{(1 - \alpha f_1)}{R_{CQ1}} + \frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)}\right]$$

For simplicity we define the following functions:

$$\xi_1 = \xi_1(V_{DD1}, k_1, \eta_1, \psi_1, \ldots) = \frac{V_{DD1} \cdot (\alpha r_1 - 1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 + Ise \cdot (\alpha r_1 \cdot \alpha f_1 - 1)$$

$$\xi_2 = \xi_2(V_{DD1}, k_1, \eta_1, \psi_1, \ldots) \\ = \frac{V_{DD1} \cdot (1 - \alpha f_1)}{R_{CQ1}} - k_1 \cdot \eta_1 \cdot \psi_1 \cdot \alpha f_1 + Isc \cdot (\alpha r_1 \cdot \alpha f_1 - 1)$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{V_{CEQ1} \cdot \frac{(1 - \alpha r_1)}{R_{CQ1}} + \xi_1}{-V_{CEQ1} \cdot \frac{(1 - \alpha f_1)}{R_{CQ1}} + \xi_2}\right] \Rightarrow e^{\left[\frac{V_{CEQ1}}{Vt}\right]} = \left[\frac{V_{CEQ1} \cdot \frac{(1 - \alpha r_1)}{R_{CQ1}} + \xi_1}{-V_{CEQ1} \cdot \frac{(1 - \alpha f_1)}{R_{CQ1}} + \xi_2}\right]$$

Taylor series approximation:

$$e^{\left[\frac{V_{CEQ1}}{Vt}\right]} \approx \frac{V_{CEQ1}}{Vt} + 1 \Rightarrow \frac{V_{CEQ1}}{Vt} + 1 \approx \left[\frac{V_{CEQ1} \cdot \frac{(1-\alpha r_1)}{R_{CQ1}} + \xi_1}{-V_{CEQ1} \cdot \frac{(1-\alpha f_1)}{R_{CQ1}} + \xi_2}\right]$$

$$\Rightarrow \frac{V_{CEQ1}}{Vt} + 1 \approx \left[\frac{V_{CEQ1} \cdot (1-\alpha r_1) + \xi_1 \cdot R_{CQ1}}{-V_{CEQ1} \cdot (1-\alpha f_1) + \xi_2 \cdot R_{CQ1}}\right]$$

$$\left[\frac{V_{CEQ1}}{Vt} + 1\right] \cdot [-V_{CEQ1} \cdot (1-\alpha f_1) + \xi_2 \cdot R_{CQ1}] = V_{CEQ1} \cdot (1-\alpha r_1) + \xi_1 \cdot R_{CQ1}$$

$$-\frac{V_{CEQ1}^2 \cdot (1-\alpha f_1)}{V_t} + \frac{V_{CEQ1} \cdot R_{CQ1}}{Vt} \cdot \xi_2 - V_{CEQ1} \cdot (1-\alpha f_1) + \xi_2 \cdot R_{CQ1}$$
$$= V_{CEQ1} \cdot (1-\alpha r_1) + \xi_1 \cdot R_{CQ1}$$

$$-\frac{V_{CEQ1}^2 \cdot (1-\alpha f_1)}{V_t} + \frac{V_{CEQ1} \cdot R_{CQ1}}{Vt} \cdot \xi_2 - V_{CEQ1} \cdot (1-\alpha f_1)$$
$$- V_{CEQ1} \cdot (1-\alpha r_1) + \xi_2 \cdot R_{CQ1} - \xi_1 \cdot R_{CQ1} = 0$$

$$-\frac{V_{CEQ1}^2 \cdot (1-\alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{\frac{R_{CQ1}}{Vt} \cdot \xi_2 - (1-\alpha f_1) - (1-\alpha r_1)\right\}$$
$$+ [\xi_2 - \xi_1] \cdot R_{CQ1} = 0$$

$$\frac{V_{CEQ1}^2 \cdot (1-\alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{-\frac{R_{CQ1}}{Vt} \cdot \xi_2 + (1-\alpha f_1) + (1-\alpha r_1)\right\} + [\xi_1 - \xi_2] \cdot R_{CQ1} = 0$$

$$\frac{V_{CEQ1}^2 \cdot (1-\alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{-\frac{R_{CQ1}}{Vt} \cdot \xi_2 + 2 - \alpha f_1 - \alpha r_1\right\} + [\xi_1 - \xi_2] \cdot R_{CQ1} = 0$$

We define for simplicity new parameter A_1, when

$$A_1 = 2 - \alpha f_1 - \alpha r_1 \;\&\; 1 < \alpha f_1 + \alpha r_1 < 2 \Rightarrow 1 > A_1 > 0$$

$$\frac{V_{CEQ1}^2 \cdot (1-\alpha f_1)}{V_t} + V_{CEQ1} \cdot \left\{A_1 - \frac{R_{CQ1}}{Vt} \cdot \xi_2\right\} + [\xi_1 - \xi_2] \cdot R_{CQ1} = 0$$

$$V_{CEQ1}^{\#,\#\#} = \frac{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \xi_2 \pm \sqrt{\left\{A_1 - \frac{R_{CQ1}}{Vt} \cdot \xi_2\right\}^2 - 4 \cdot \frac{(1-\alpha f_1)}{V_t} \cdot [\xi_1 - \xi_2] \cdot R_{CQ1}}}{2 \cdot \frac{(1-\alpha f_1)}{V_t}}$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1} = \xi_2^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 - 2\cdot A_1\cdot\frac{R_{CQ1}}{Vt}\cdot\xi_2 + A_1^2 - 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\xi_1 + 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\xi_2$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1} = \xi_2^2\cdot[\frac{R_{CQ1}}{Vt}]^2 + \xi_2\cdot\left\{-2\cdot A_1\cdot\frac{R_{CQ1}}{Vt} + 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\right\} - 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\xi_1 + A_1^2$$

For simplicity we define the following functions:

$$\xi_1 = \xi_1(V_{DD1}, k_1, \eta_1, \psi_1, \ldots) = \frac{V_{DD1}\cdot(\alpha r_1 - 1)}{R_{CQ1}} - k_1\cdot\eta_1\cdot\psi_1 + Ise\cdot(\alpha r_1\cdot\alpha f_1 - 1)$$

$$\Xi_1 = \frac{V_{DD1}\cdot(\alpha r_1 - 1)}{R_{CQ1}} + Ise\cdot(\alpha r_1\cdot\alpha f_1 - 1); \Xi_2 = k_1\cdot\eta_1 \Rightarrow \xi_1 = \Xi_1 - \Xi_2\cdot\psi_1$$

$$\xi_2 = \xi_2(V_{DD1}, k_1, \eta_1, \psi_1, \ldots) = \frac{V_{DD1}\cdot(1-\alpha f_1)}{R_{CQ1}} - k_1\cdot\eta_1\cdot\psi_1\cdot\alpha f_1 + Isc\cdot(\alpha r_1\cdot\alpha f_1 - 1)$$

$$\Xi_3 = \frac{V_{DD1}\cdot(1-\alpha f_1)}{R_{CQ1}} + Isc\cdot(\alpha r_1\cdot\alpha f_1 - 1); \Xi_4 = k_1\cdot\eta_1\cdot\alpha f_1 \Rightarrow \xi_2 = \Xi_3 - \Xi_4\cdot\psi_1; \Xi_4 = \alpha f_1\cdot\Xi_2$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1} = [\Xi_3 - \Xi_4\cdot\psi_1]^2\cdot[\tfrac{R_{CQ1}}{Vt}]^2 + [\Xi_3 - \Xi_4\cdot\psi_1]\cdot\left\{-2\cdot A_1\cdot\tfrac{R_{CQ1}}{Vt} + 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\right\} - 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot[\Xi_1 - \Xi_2\cdot\psi_1] + A_1^2$$

$$\left\{A_1 - \tfrac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\tfrac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1} = \left\{\Xi_3^2 - 2\cdot\Xi_3\cdot\Xi_4\cdot\psi_1 + \Xi_4^2\cdot\psi_1^2\right\}\cdot\left[\tfrac{R_{CQ1}}{Vt}\right]^2 + [\Xi_3 - \Xi_4\cdot\psi_1]\cdot\tfrac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1) - A_1\} - 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + 4\cdot\tfrac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}\cdot\psi_1 + A_1^2$$

$$\left\{A_1 - \frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}$$
$$= \Xi_3^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 - 2\cdot\Xi_3\cdot\Xi_4\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2\cdot\psi_1 + \Xi_4^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2\cdot\psi_1^2$$
$$+\Xi_3\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1)-A_1\} - \Xi_4\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1)-A_1\}\cdot\psi_1$$
$$-4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}\cdot\psi_1 + A_1^2$$

$$\left\{A_1 - \frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}$$
$$= \Xi_4^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2\cdot\psi_1^2 + \psi_1\cdot\left\{-2\cdot\Xi_3\cdot\Xi_4\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 - \Xi_4\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1)-A_1\} + 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}\right\}$$
$$+\Xi_3^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 + \Xi_3\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1)-A_1\} - 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + A_1^2$$

For simplicity we define the following global parameters:

$$\Delta_1 = \Xi_4^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2;\ \Delta_2 = -2\cdot\Xi_3\cdot\Xi_4\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 - \Xi_4\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1)-A_1\}$$
$$+4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}\cdot\Xi_2}{V_t}$$

$$\Delta_3 = \Xi_3^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 + \Xi_3\cdot\frac{2\cdot R_{CQ1}}{Vt}\cdot\{2\cdot(1-\alpha f_1)-A_1\} - 4\cdot\frac{(1-\alpha f_1)\cdot R_{CQ1}}{V_t}\cdot\Xi_1 + A_1^2$$

$$\{A_1 - \frac{R_{CQ1}}{Vt}\cdot\xi_2\}^2 - 4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1} = \Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1 + \Delta_3;$$
$$\Delta_1 = \Xi_4^2\cdot\left[\frac{R_{CQ1}}{Vt}\right]^2 > 0$$

We are using completing the square rule: $A\cdot x^2 + B\cdot x = (x\cdot\sqrt{A} + \frac{B}{2\cdot\sqrt{A}})^2 - \frac{B^2}{4\cdot A}$
For our case: $\Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1$; $A = \Delta_1$; $B = \Delta_2$; $x = \psi_1$

$$\Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1 = \left(\psi_1\cdot\sqrt{\Delta_1} + \frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)^2 - \frac{\Delta_2^2}{4\cdot\Delta_1}$$

$$\left\{A_1 - \frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2 - 4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}$$
$$= \Delta_1\cdot\psi_1^2 + \Delta_2\cdot\psi_1 + \Delta_3 = \left(\psi_1\cdot\sqrt{\Delta_1} + \frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)^2 - \frac{\Delta_2^2}{4\cdot\Delta_1} + \Delta_3$$

For simplicity we choose circuit parameters constrain (first request):

$$-\frac{\Delta_2^2}{4\cdot\Delta_1}+\Delta_3\to\varepsilon\forall 0<\varepsilon\ll 1$$

$$\left(\psi_1\cdot\sqrt{\Delta_1}+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)^2-\frac{\Delta_2^2}{4\cdot\Delta_1}+\Delta_3\approx\left(\psi_1\cdot\sqrt{\Delta_1}+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)^2$$

Finally, we can state that

$$\left\{A_1-\frac{R_{CQ1}}{Vt}\cdot\xi_2\right\}^2-4\cdot\frac{(1-\alpha f_1)}{V_t}\cdot[\xi_1-\xi_2]\cdot R_{CQ1}\simeq\left(\psi_1\cdot\sqrt{\Delta_1}+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)^2$$

$$V_{CEQ1}^{\#,\#\#}=\frac{-A_1+\frac{R_{CQ1}}{Vt}\cdot\xi_2\pm\left(\psi_1\cdot\sqrt{\Delta_1}+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)}{2\cdot\frac{(1-\alpha f_1)}{V_t}};\xi_2=\Xi_3-\Xi_4\cdot\psi_1$$

$$V_{CEQ1}^{\#,\#\#}=\frac{-A_1+\frac{R_{CQ1}}{Vt}\cdot\Xi_3-\frac{R_{CQ1}}{Vt}\cdot\Xi_4\cdot\psi_1\pm\left(\psi_1\cdot\sqrt{\Delta_1}+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}\right)}{2\cdot\frac{(1-\alpha f_1)}{V_t}}$$

We get possible two expressions for V_{CEQ1}. $V_{CEQ1}^{\#}=V_{CEQ1}^{(+)};V_{CEQ1}^{\#\#}=V_{CEQ1}^{(-)}$

$$V_{CEQ1}^{\#}=\frac{-A_1+\frac{R_{CQ1}}{Vt}\cdot\Xi_3+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}+\psi_1\cdot\left\{\sqrt{\Delta_1}-\frac{R_{CQ1}}{Vt}\cdot\Xi_4\right\}}{2\cdot\frac{(1-\alpha f_1)}{V_t}}$$

$$V_{CEQ1}^{\#}=\left[\frac{R_{CQ1}}{Vt}\cdot\Xi_3+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}-A_1\right]\cdot\frac{V_t}{2\cdot(1-\alpha f_1)}+\psi_1\cdot\left\{\sqrt{\Delta_1}-\frac{R_{CQ1}}{Vt}\cdot\Xi_4\right\}\cdot\frac{V_t}{2\cdot(1-\alpha f_1)}$$

The second request we need to fulfill and get $V_{CEQ1}=V_X$ differential equation shape:

$$\left\{\sqrt{\Delta_1}-\frac{R_{CQ1}}{Vt}\cdot\Xi_4\right\}\Bigg|_{\Delta_1=\Xi_4^2\cdot[\frac{R_{CQ1}}{Vt}]^2}=0$$
$$\Rightarrow V_{CEQ1}^{\#}=\left[\frac{R_{CQ1}}{Vt}\cdot\Xi_3+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}-A_1\right]\cdot\frac{V_t}{2\cdot(1-\alpha f_1)}$$

$\left[\frac{R_{CQ1}}{Vt}\cdot\Xi_3+\frac{\Delta_2}{2\cdot\sqrt{\Delta_1}}-A_1\right]\cdot\frac{V_t}{2\cdot(1-\alpha f_1)}\to\varepsilon\;\forall\;0<\varepsilon\ll 1$ For the first fixed point

$$V_{CEQ1}^{\#\#} = \frac{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - \psi_1 \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\}}{2 \cdot \frac{(1-\alpha f_1)}{V_t}}$$

$$V_{CEQ1}^{\#\#} = \left\{ -A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} - \psi_1 \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)}$$

$$V_{CEQ1}^{\#\#} = \left\{ -A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} - \left\{ \frac{-\dot{V}_X}{R11} - \frac{V_Y}{R12} \right\} \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)}$$

$$V_{CEQ1}^{\#\#} = \left\{ -A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} + \frac{\dot{V}_X}{R11} \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} + \frac{V_Y}{R12} \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)}$$

We need to adapt between our system block diagram and optoisolation implementation circuit. $V_{CEQ1}^{\#\#} \rightleftharpoons X \rightleftharpoons V_X$; $X = \frac{1}{\Gamma_1 \cdot \Gamma_3} \cdot \frac{\mathrm{d}X}{\mathrm{d}t} + Y + \Gamma_5 \cdot (\frac{\Gamma_4}{\Gamma_3} - 1)$ $Y \rightleftharpoons V_Y$.

$$\Gamma_5 \cdot (\frac{\Gamma_4}{\Gamma_3} - 1) = \left\{ -A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)}; \frac{1}{\Gamma_1 \cdot \Gamma_3} = \frac{\left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot V_t}{2 \cdot (1 - \alpha f_1) \cdot R11}$$

$$\frac{1}{R12} \cdot \left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} = 1 \Rightarrow \frac{\left\{ \frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1} \right\} \cdot V_t}{2 \cdot (1 - \alpha f_1)} = R12$$

The next step is to find our V_Z circuit expression ($V_Z = V_{CEQ2}$).

$$V_{CEQ2} \simeq Vt \cdot \ln \left[\frac{(\alpha r_2 \cdot I_{CQ2} - I_{EQ2}) + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha f_2) + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1)} \right];$$

$$I_{EQ2} = k_2 \cdot I_{D2} + I_{CQ2} = k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}$$

$$\alpha r_2 \cdot I_{CQ2} - I_{EQ2} = \alpha r_2 \cdot I_{CQ2} - [k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}] = [\alpha r_2 - 1] \cdot I_{CQ2} - k_2 \cdot \eta_2 \cdot \psi_2$$

$$\begin{aligned} I_{CQ2} - I_{EQ2} \cdot \alpha f_2 &= I_{CQ2} - [k_2 \cdot \eta_2 \cdot \psi_2 + I_{CQ2}] \cdot \alpha f_2 \\ &= I_{CQ2} \cdot [1 - \alpha f_2] - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 \end{aligned}$$

$$\begin{aligned} I_{CQ2} = \frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}} &\Rightarrow \alpha r_2 \cdot I_{CQ2} - I_{EQ2} \\ &= [\alpha r_2 - 1] \cdot \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] - k_2 \cdot \eta_2 \cdot \psi_2 \end{aligned}$$

$$\begin{aligned} I_{CQ2} = \frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}} &\Rightarrow I_{CQ2} - I_{EQ2} \cdot \alpha f_2 \\ &= \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] \cdot [1 - \alpha f_2] - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 \end{aligned}$$

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{[\alpha r_2 - 1] \cdot \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] - k_2 \cdot \eta_2 \cdot \psi_2 + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}{\left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] \cdot [1 - \alpha f_2] - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1)}\right]$$

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{[\alpha r_2 - 1] \cdot \left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2}{\left[\frac{V_{DD2} - V_{CEQ2}}{R_{CQ2}}\right] \cdot [1 - \alpha f_2] + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2}\right]$$

For simplicity we define the following functions:

$$\begin{aligned} \xi_3 &= \xi_3(V_{DD2}, k_2, \eta_2, \psi_2, \ldots) \\ &= [\alpha r_2 - 1] \cdot \frac{V_{DD2}}{R_{CQ2}} + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2 \end{aligned}$$

$$\begin{aligned} \xi_4 &= \xi_4(V_{DD2}, k_2, \eta_2, \psi_2, \ldots) \\ &= \frac{V_{DD2}}{R_{CQ2}} \cdot [1 - \alpha f_2] + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2 \end{aligned}$$

$$V_{CEQ2} \simeq Vt \cdot \ln\left[\frac{V_{CEQ2} \cdot \frac{(1-\alpha r_2)}{R_{CQ2}} + \xi_3}{-V_{CEQ2} \cdot \frac{(1-\alpha f_2)}{R_{CQ2}} + \xi_4}\right] \Rightarrow e^{\left[\frac{V_{CEQ2}}{Vt}\right]} = \left[\frac{V_{CEQ2} \cdot \frac{(1-\alpha r_2)}{R_{CQ2}} + \xi_3}{-V_{CEQ2} \cdot \frac{(1-\alpha f_2)}{R_{CQ2}} + \xi_4}\right]$$

$$\begin{aligned} e^{\left[\frac{V_{CEQ2}}{Vt}\right]} \approx \frac{V_{CEQ2}}{Vt} + 1 \Rightarrow \frac{V_{CEQ2}}{Vt} + 1 \approx \left[\frac{V_{CEQ2} \cdot \frac{(1-\alpha r_2)}{R_{CQ2}} + \xi_3}{-V_{CEQ2} \cdot \frac{(1-\alpha f_2)}{R_{CQ2}} + \xi_4}\right] \Rightarrow \frac{V_{CEQ2}}{Vt} + 1 \\ \approx \frac{V_{CEQ2} \cdot (1 - \alpha r_2) + \xi_3 \cdot R_{CQ2}}{-V_{CEQ2} \cdot (1 - \alpha f_2) + \xi_4 \cdot R_{CQ2}} \end{aligned}$$

$$[\frac{V_{CEQ2}}{Vt} + 1] \cdot [-V_{CEQ2} \cdot (1 - \alpha f_2) + \xi_4 \cdot R_{CQ2}] = V_{CEQ2} \cdot (1 - \alpha r_2) + \xi_3 \cdot R_{CQ2}$$

$$-\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + \frac{V_{CEQ2} \cdot R_{CQ2}}{Vt} \cdot \xi_4 - V_{CEQ2} \cdot (1 - \alpha f_2) + \xi_4 \cdot R_{CQ2}$$
$$= V_{CEQ2} \cdot (1 - \alpha r_2) + \xi_3 \cdot R_{CQ2}$$

$$-\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + \frac{V_{CEQ2} \cdot R_{CQ2}}{Vt} \cdot \xi_4 - V_{CEQ2} \cdot (1 - \alpha f_2) - V_{CEQ2} \cdot (1 - \alpha r_2)$$
$$+ \xi_4 \cdot R_{CQ2} - \xi_3 \cdot R_{CQ2} = 0$$

$$-\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \xi_4 - (1 - \alpha f_2) - (1 - \alpha r_2)\right\}$$
$$+ [\xi_4 - \xi_3] \cdot R_{CQ2} = 0$$

$$\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{-\frac{R_{CQ2}}{Vt} \cdot \xi_4 + (1 - \alpha f_2) + (1 - \alpha r_2)\right\} + [\xi_3 - \xi_4] \cdot R_{CQ2} = 0$$

$$\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{-\frac{R_{CQ2}}{Vt} \cdot \xi_4 + 2 - \alpha f_2 - \alpha r_2\right\} + [\xi_3 - \xi_4] \cdot R_{CQ2} = 0$$

We define for simplicity new parameter A_2, when

$$A_2 = 2 - \alpha f_2 - \alpha r_2 \;\&\; 1 < \alpha f_2 + \alpha r_2 < 2 \Rightarrow 1 > A_2 > 0$$

$$\frac{V_{CEQ2}^2 \cdot (1 - \alpha f_2)}{V_t} + V_{CEQ2} \cdot \left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\} + [\xi_3 - \xi_4] \cdot R_{CQ2} = 0$$

$$V_{CEQ2}^{\#,\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \xi_4 \pm \sqrt{\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2}}}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = \xi_4^2 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 - 2 \cdot A_2 \cdot \frac{R_{CQ2}}{Vt} \cdot \xi_4 + A_2^2$$
$$-4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \xi_3 + 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \xi_4$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2}$$
$$= \xi_4^2 \cdot [\frac{R_{CQ2}}{Vt}]^2 + \xi_4 \cdot \left\{-2 \cdot A_2 \cdot \frac{R_{CQ2}}{Vt} + 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t}\right\} - 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \xi_3 + A_2^2$$

For simplicity we define the following functions:

$$\xi_3 = \xi_3(V_{DD2}, k_2, \eta_2, \psi_2, \ldots) = [\alpha r_2 - 1] \cdot \frac{V_{DD2}}{R_{CQ2}} + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \psi_2$$

$$\Xi_5 = [\alpha r_2 - 1] \cdot \frac{V_{DD2}}{R_{CQ2}} + Ise \cdot (\alpha r_2 \cdot \alpha f_2 - 1); \Xi_6 = k_2 \cdot \eta_2 \Rightarrow \xi_3 = \Xi_5 - \Xi_6 \cdot \psi_2$$

$$\xi_4 = \xi_4(V_{DD2}, k_2, \eta_2, \psi_2, \ldots) = \frac{V_{DD2}}{R_{CQ2}} \cdot [1 - \alpha f_2] + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1) - k_2 \cdot \eta_2 \cdot \alpha f_2 \cdot \psi_2$$

$$\Xi_7 = \frac{V_{DD2}}{R_{CQ2}} \cdot [1 - \alpha f_2] + Isc \cdot (\alpha r_2 \cdot \alpha f_2 - 1); \Xi_8 = k_2 \cdot \eta_2 \cdot \alpha f_2 \Rightarrow \xi_4 = \Xi_7 - \Xi_8 \cdot \psi_2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = [\Xi_7 - \Xi_8 \cdot \psi_2]^2 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 + [\Xi_7 - \Xi_8 \cdot \psi_2] \cdot \left\{-2 \cdot A_2 \cdot \frac{R_{CQ2}}{Vt} + 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t}\right\} - 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot [\Xi_5 - \Xi_6 \cdot \psi_2] + A_2^2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = \{\Xi_7^2 - 2 \cdot \Xi_7 \cdot \Xi_8 \cdot \psi_2 + \Xi_8^2 \cdot \psi_2^2\} \cdot [\frac{R_{CQ2}}{Vt}]^2 + [\Xi_7 - \Xi_8 \cdot \psi_2] \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} - 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \Xi_5 + 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2} \cdot \Xi_6}{V_t} \cdot \psi_2 + A_2^2$$

$$\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = \Xi_7^2 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 - 2 \cdot \Xi_7 \cdot \Xi_8 \cdot [\frac{R_{CQ2}}{Vt}]^2 \cdot \psi_2 + \Xi_8^2 \cdot [\frac{R_{CQ2}}{Vt}]^2 \cdot \psi_2^2 + \Xi_7 \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} - \Xi_8 \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} \cdot \psi_2 - 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \Xi_5 + 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2} \cdot \Xi_6}{V_t} \cdot \psi_2 + A_2^2$$

$$\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\}^2 - 4 \cdot \frac{(1-\alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = \Xi_8^2 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 \cdot \psi_2^2 + \psi_2 \cdot \{-2 \cdot \Xi_7 \cdot \Xi_8 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 - \Xi_8 \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} + 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2} \cdot \Xi_6}{V_t}\} + \Xi_7 \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} - 4 \cdot \frac{(1-\alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \Xi_5 + A_2^2 + \Xi_7^2 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2$$

For simplicity we define the following global parameters:

$$\Delta_4 = \Xi_8^2 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 ; \Delta_5 = -2 \cdot \Xi_7 \cdot \Xi_8 \cdot \left[\frac{R_{CQ2}}{Vt}\right]^2 - \Xi_8 \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} + 4 \cdot \frac{(1 - \alpha f_2) \cdot R_{CQ2} \cdot \Xi_6}{V_t}$$

$$\Delta_6 = \Xi_7 \cdot \frac{2 \cdot R_{CQ2}}{Vt} \cdot \{2 \cdot (1 - \alpha f_2) - A_2\} - 4 \cdot \frac{(1 - \alpha f_2) \cdot R_{CQ2}}{V_t} \cdot \Xi_5 + A_2^2 + \Xi_7^2 \cdot [\frac{R_{CQ2}}{Vt}]^2$$

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1 - \alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = \Delta_4 \cdot \psi_2^2 + \Delta_5 \cdot \psi_2 + \Delta_6;$$

$$\Delta_4 = \Xi_8^2 \cdot [\frac{R_{CQ2}}{Vt}]^2 > 0$$

We are using completing the square rule: $A \cdot x^2 + B \cdot x = \left(x \cdot \sqrt{A} + \frac{B}{2 \cdot \sqrt{A}}\right)^2 - \frac{B^2}{4 \cdot A}$

For our case: $\Delta_4 \cdot \psi_2^2 + \Delta_5 \cdot \psi_2$; $A = \Delta_4$; $B = \Delta_5$; $x = \psi_2$

$$\Delta_4 \cdot \psi_2^2 + \Delta_5 \cdot \psi_2 = \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2 - \frac{\Delta_5^2}{4 \cdot \Delta_4}$$

$$\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\}^2 - 4 \cdot \frac{(1 - \alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} = \Delta_4 \cdot \psi_2^2 + \Delta_5 \cdot \psi_2 + \Delta_6 = \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2 - \frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6$$

For simplicity we choose circuit parameters constrain (first request):

$$-\frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6 \to \varepsilon \forall 0 < \varepsilon \ll 1$$

$$\left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2 - \frac{\Delta_5^2}{4 \cdot \Delta_4} + \Delta_6 \approx \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2$$

Finally we can state that

$$\left\{A_2 - \frac{R_{CQ2}}{Vt} \cdot \xi_4\right\}^2 - 4 \cdot \frac{(1 - \alpha f_2)}{V_t} \cdot [\xi_3 - \xi_4] \cdot R_{CQ2} \simeq \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)^2$$

$$V_{CEQ2}^{\#,\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \xi_4 \pm \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}; \xi_4 = \Xi_7 - \Xi_8 \cdot \psi_2$$

$$V_{CEQ2}^{\#,\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{R_{CQ2}}{Vt} \cdot \Xi_8 \cdot \psi_2 \pm \left(\psi_2 \cdot \sqrt{\Delta_4} + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right)}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

We get possible two expressions for V_{CEQ2}. $\chi_2^{\#} = V_{CEQ2}^{\#} = V_{CEQ2}^{(+)}; \chi_2^{\#\#} = V_{CEQ2}^{\#\#} = V_{CEQ2}^{(-)}$

$$V_{CEQ2}^{\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} + \psi_2 \cdot \left\{\sqrt{\Delta_4} - \frac{R_{CQ2}}{Vt} \cdot \Xi_8\right\}}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

$$V_{CEQ2}^{\#} = \left[\frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - A_2\right] \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)} + \psi_2 \cdot \left\{\sqrt{\Delta_4} - \frac{R_{CQ2}}{Vt} \cdot \Xi_8\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$$

$$\left\{\sqrt{\Delta_4} - \frac{R_{CQ2}}{Vt} \cdot \Xi_8\right\}\Bigg|_{\Delta_4 = \Xi_8^2 \cdot [\frac{R_{CQ2}}{Vt}]^2} = 0 \Rightarrow V_{CEQ2}^{\#} = \left[\frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - A_2\right] \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$$

$\left[\frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - A_2\right] \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)} \to \varepsilon \forall 0 < \varepsilon \ll 1$ it fulfill our system fixed points Z coordinate for all fixed points.

$$V_{CEQ2}^{\#\#} = \frac{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - \psi_2 \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\}}{2 \cdot \frac{(1-\alpha f_2)}{V_t}}$$

$$V_{CEQ2}^{\#\#} = \left\{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)} - \psi_2 \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$$

$$V_{CEQ2}^{\#\#} = \left\{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)} - \left\{\frac{\dot{V}_Y}{R_{21}} + \frac{V_Y}{R_{22}} - \frac{V_X}{R_{23}}\right\} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1-\alpha f_2)}$$

$$V_{CEQ2}^{\#\#} = \left\{-A_2 + \tfrac{R_{CQ2}}{Vt} \cdot \Xi_7 - \tfrac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right\} \cdot \tfrac{V_t}{2 \cdot (1 - \alpha f_2)} - \frac{\dot{V}_Y}{R_{21}} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} - \frac{V_Y}{R_{22}} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} + \frac{V_X}{R_{23}} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$$

We need to adapt between our system block diagram and optoisolation implementation circuit. $V_{CEQ2}^{\#\#} \rightleftharpoons Z \rightleftharpoons V_Z$; $Z = -\frac{dY}{dt} - \Gamma_3 \cdot Y + \Gamma_3 \cdot X - (\Gamma_4 - \Gamma_3) \cdot \Gamma_5 Y \rightleftharpoons V_Y$ [65–69].

$$\frac{1}{R_{21}} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} = 1 \Rightarrow \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} = R_{21}$$

$$\Gamma_3 = \frac{1}{R_{22}} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)};$$
$$\Gamma_3 = \frac{1}{R_{23}} \cdot \left\{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$$

$$-(\Gamma_4 - \Gamma_3) \cdot \Gamma_5 = \left\{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\right\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$$

Results Discussion We got two options for V_X expression and two options for V_Z expression. The first option $V_X = V_{CEQ1}^{\#}/V_Z = V_{CEQ2}^{\#}$ gives a real number (circuit parameters dependent) which can represent a situation of V_X/V_Z coordinate at one of the system fixed point . For $V_X = V_{CEQ1}^{\#}$ it can be zero (→ε) for the first fixed point or ± real value for the second and third fixed point X coordinates and for $V_z = V_{CEQ2}^{\#}$ it can be zero (→ε) for all fixed points Z coordinates. The second option $V_X = V_{CEQ1}^{\#\#}/V_Z = V_{CEQ2}^{\#\#}$ gives the related X and Z differential equations which describe system dynamic behavior for case A. we adapt between optoisolation circuit parameters and system block diagram parameters (Table 3.15).

Remark We discuss multifolded torus chaotic attractors system by using optoisolation circuits only for Case A. it is reader exercise to discuss it for other cases B, C, D [65–69].

Table 3.15 Optoisolation circuit parameters and system

	$V_X = V^{\#}_{CEQ1}/V^{\#\#}_{CEQ1}$	$V_Z = V^{\#}_{CEQ2}/V^{\#\#}_{CEQ2}$
First option $V_X = V^{\#}_{CEQ1}$ $V_Z = V^{\#}_{CEQ2}$	$V_X^{(0)} = 0; V_X^{(1,2)} = \pm(\frac{\Gamma_4}{\Gamma_3} - 1) \cdot \Gamma_5$ $V^{\#}_{CEQ1} = \left[\frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - A_1\right] \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)}$ $V_X^{(0)} = 0 \Rightarrow \left[\frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - A_1\right]$ $\cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} = 0 \Rightarrow \frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} = A_1$ $V_X^{(1,2)} = \pm(\frac{\Gamma_4}{\Gamma_3} - 1) \cdot \Gamma_5 \Rightarrow$ $\left[\frac{R_{CQ1}}{Vt} \cdot \Xi_3 + \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}} - A_1\right]$ $\cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} = \pm(\frac{\Gamma_4}{\Gamma_3} - 1) \cdot \Gamma_5$	$V_Z^{(i)} = 0 \forall$ i = 0,1,2 $V^{\#}_{CEQ2} = \left[\frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - A_2\right]$ $\cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$ $V_Z^{(i)} = 0 \Rightarrow \left[\frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} - A_2\right]$ $\cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} = 0$ $\Rightarrow \frac{R_{CQ2}}{Vt} \cdot \Xi_7 + \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}} = A_2$
Second option $V_X = V^{\#\#}_{CEQ1}$ $V_Z = V^{\#\#}_{CEQ2}$	$V^{\#\#}_{CEQ1} \rightleftharpoons X \rightleftharpoons V_X; Y \rightleftharpoons V_Y.$ $X = \frac{1}{\Gamma_1 \cdot \Gamma_3} \cdot \frac{dX}{dt} + Y + \Gamma_5 \cdot (\frac{\Gamma_4}{\Gamma_3} - 1)$ $\Gamma_5 \cdot (\frac{\Gamma_4}{\Gamma_3} - 1) = \left\{-A_1 + \frac{R_{CQ1}}{Vt} \cdot \Xi_3 - \frac{\Delta_2}{2 \cdot \sqrt{\Delta_1}}\right\}$ $\cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)}$ $\frac{1}{\Gamma_1 \cdot \Gamma_3} = \frac{\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\} \cdot V_t}{2 \cdot (1 - \alpha f_1) \cdot R11}$ $\frac{1}{R12} \cdot \{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_1)} = 1$ $\Rightarrow \frac{\{\frac{R_{CQ1}}{Vt} \cdot \Xi_4 + \sqrt{\Delta_1}\} \cdot V_t}{2 \cdot (1 - \alpha f_1)} = R12$	$V^{\#\#}_{CEQ2} \rightleftharpoons Z \rightleftharpoons V_Z; Y \rightleftharpoons V_Y$ $Z = -\frac{dY}{dt} - \Gamma_3 \cdot Y + \Gamma_3 \cdot X - (\Gamma_4 - \Gamma_3) \cdot \Gamma_5$ $\frac{1}{R_{21}} \cdot \{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} = 1$ $\Rightarrow \{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)} = R_{21}$ $\Gamma_3 = \frac{1}{R_{22}} \cdot \{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$ $\Gamma_3 = \frac{1}{R_{23}} \cdot \{\frac{R_{CQ2}}{Vt} \cdot \Xi_8 + \sqrt{\Delta_4}\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$ $-(\Gamma_4 - \Gamma_3) \cdot \Gamma_5$ $= \{-A_2 + \frac{R_{CQ2}}{Vt} \cdot \Xi_7 - \frac{\Delta_5}{2 \cdot \sqrt{\Delta_4}}\} \cdot \frac{V_t}{2 \cdot (1 - \alpha f_2)}$

3.7 Exercises

1. We have Rossler's prototype chaotic system in which some of the variables replaced by theabsolute values.

$$\frac{dX}{dt} = -|Y| - Z; \frac{dY}{dt} = |X| + a \cdot Y;$$
$$\frac{dZ}{dt} = b \cdot X - c \cdot Z + X \cdot |Z| \quad \forall\, a, b, c \in \mathbb{R}^3$$

1.1 Find system fixed points and discuss all possible options.
1.2 Find system Jacobian matrix for all options and discuss stability.
1.3 Try to describe the system block diagram.
1.4 Implement it by using optoisolation circuits. (Hint: try to differentiate each option in the circuit implementation which fit system block diagram).

2. We have Rossler's chaotic system variation model which includes sign() function.

$$\frac{dX}{dt} = -|Y| - Z; \frac{dY}{dt} = \text{sign}\{|X| + a \cdot Y\}$$
$$\frac{dZ}{dt} = b \cdot X - c \cdot Z + \text{sign}\{X \cdot |Z|\} \quad \forall\, a, b, c \in \mathbb{R}^3$$

2.1 Find system fixed points and discuss all possible options.
2.2 Find system Jacobian matrix for all options and discuss stability.
2.3 Try to describe the system block diagram.
2.4 Implement it by using optoisolation circuits. (Hint: try to differentiate each option in the circuit implementation which fit system block diagram).

3. We have the following planar system (truncated amplitude system).

$$\dot{r}_1 = r_1 \cdot (\mu_1 + p_{11} \cdot r_1^2 + p_{12} \cdot r_2^2 + s_1 \cdot r_2^4)$$
$$\dot{r}_2 = r_2 \cdot (\mu_2 + p_{21} \cdot r_1^2 + p_{22} \cdot r_2^2 + s_2 \cdot r_1^4)$$

$\mu_1, \mu_2, p_{ij}\ \forall\, i = 1, 2; j = 1, 2; s_1, s_2$ are system parameters.

3.1 Find system fixed points and discuss all possible options.
3.2 Find system Jacobian matrix for all options and discuss stability.
3.3 We have the following connections between our system parameters:

$$\mu_1 = \chi_1 \cdot \mu_2; \mu_2 = \mu; p_{11} = \chi_2 \cdot p_{12}; p_{12} = p^{\#}$$
$$p_{21} = \chi_3 \cdot p_{22}; p_{22} = p^{\#\#}; s_1 = s_2 = s$$

How system stability is dependent on these new parameters: $\mu, p^{\#}, p^{\#\#}, s$. $\chi_1, \ldots, \chi_3$ are Fibonacci series numbers, where

$$\chi_n = \chi_{n-1} + \chi_{n-2}.$$

3.4 Implement the system by using optoisolation circuits, find the equivalent circuit parameters which depend on system parameters.

4. We consider Hopf–Hopf bifurcation system with five dimensional, continuous time system. A, B system parameters. X,…, W system variables.

$$\frac{dX}{dt} = A - B \cdot |X| + X^2 \cdot Y - X; \frac{dY}{dt} = B \cdot X - sign\{X^2 \cdot Y\}; \frac{dZ}{dt} = X - |Z| \cdot V$$

$$\frac{dV}{dt} = B \cdot X - sign\{Z \cdot V\} + V^2 \cdot |W| - V; \frac{dW}{dt} = Z \cdot |V| - V^2 \cdot W$$

4.1 Find system fixed points and discuss all possible options.
4.2 Find system Jacobian matrix for all options and discuss stability.
4.3 How system stability depend on Δ parameter when $A = \Delta + B$.
4.4 Implement the system by using optoisolation circuits, find the equivalent circuit parameters which depend on system parameters.
4.5 We restrict our system to four dimensional by deleting the last differential equation $dW/dt = \ldots$ and set W variable to zero. Discuss system stability and find optoisolation circuit implementation.

5. We have system model which exhibits fold and torus bifurcation. The model is known as a chemical model (peroxidase-oxidase reaction). $\eta_1, \ldots, \eta_4$ are system state variables. $\Gamma_1, \ldots, \Gamma_9$ are system parameters.

$$\frac{d\eta_1}{dt} = -\Gamma_1 \cdot \prod_{j=1}^{3} \eta_j - \Gamma_3 \cdot \eta_1 \cdot \eta_2 \cdot \eta_4 + \Gamma_7 - \Gamma_9 \cdot \eta_1$$

$$\frac{d\eta_2}{dt} = -\Gamma_1 \cdot \prod_{j=1}^{3} \eta_j - \Gamma_3 \cdot \eta_1 \cdot \eta_2 \cdot \eta_4 + \Gamma_8$$

$$\frac{d\eta_3}{dt} = \Gamma_1 \cdot \prod_{j=1}^{3} \eta_j - 2 \cdot \Gamma_2 \cdot \eta_3^2 + 2 \cdot \Gamma_3 \cdot \eta_1 \cdot \eta_2 \cdot \eta_4 - \Gamma_4 \cdot \eta_3 + \Gamma_6$$

$$\frac{d\eta_4}{dt} = -\Gamma_3 \cdot \eta_1 \cdot \eta_2 \cdot \eta_4 + 2 \cdot \Gamma_2 \cdot \eta_3^2 - \Gamma_5 \cdot \eta_4$$

5.1 Find system fixed points and discuss all possible options.
5.2 Find system Jacobian matrix and how stability is dependent on $\Gamma_1,\ldots,\Gamma_9$ parameters.
5.3 What happened when all system parameters have the same value $\Gamma_1 = \ldots. = \Gamma_9 = \Gamma$. Find system fixed points and discuss stability.
5.4 Find the conditions for Torus bifurcation and the related phase plots.
5.5 Implement the system by using optoisolation circuits. How the bifurcation is dependent on circuit parameters?

6. We consider a system of two coupled oscillation elements, when one element is used to suppress vibrations of the other. The system is given by the following differential equations.

$$\frac{d^2V_1}{dt^2} + \Gamma_1 \cdot \left(\frac{dV_1}{dt} - \frac{dV_2}{dt}\right) + \Gamma_2^2 \cdot (1 + \varepsilon \cdot \cos(\eta \cdot t)) \cdot (V_1 - V_2) = 0$$

$$\frac{d^2V_2}{dt^2} - \Gamma_3 \cdot \Gamma_1 \cdot \left(\frac{dV_1}{dt} - \frac{dV_2}{dt}\right) - \Gamma_3 \cdot \Gamma_2^2 \cdot (1 + \varepsilon \cdot \cos(\eta \cdot t)) \cdot (V_1 - V_2)$$
$$+ \Gamma_4 \cdot \frac{dV_2}{dt} + V_2 - \Gamma_5 \cdot \Gamma_6^2 \cdot \left(1 - \gamma \cdot \left[\frac{dV_2}{dt}\right]^2\right) \cdot V_2 = 0$$

V_1, V_2 are system variables and $\Gamma_1,\ldots,\Gamma_6$, ε, η, γ are system parameters. The natural frequencies of this system are

$$\Omega_{1,2}^2 = \frac{1}{2} \cdot \left(1 + \Gamma_2^2 \cdot (1 + \Gamma_3)\right) \mp \sqrt{\frac{1}{4} \cdot (1 + \Gamma_2^2 \cdot (1 + \Gamma_3))^2 - \Gamma_2^2}$$

Near the parametric resonance $\eta_0 = \Omega_2 - \Omega_1$, the trivial solution exhibits a double NS bifurcation when $\Gamma_1 = (\Gamma_5 \cdot \Gamma_6^2 - \Gamma_4)/(1 + \Gamma_3)$.

6.1 Represent our system by set of first order and nonlinear differential equations. Find fixed points and discuss stability.
6.2 Detect for which parameters values NS bifurcation happened.
6.3 What happened to the system when η parameter is equal to zero ?
6.4 How ε parameter influences system behavior.
6.5 Implement the system by using optoisolation circuits.
6.6 Find to which circuit parameters values bifurcation happened.

7. We consider the following system X, Y, Z are variables, α is a parameter.

$$\frac{dX}{dt} = -Y + X \cdot (1 - X^2 - Y^2); \frac{dY}{dt} = X + Y \cdot (1 - X^2 - Y^2); \frac{dZ}{dt} = \alpha$$

Has the z-axis and the cylinder $X^2 + Y^2 = 1$ as invariant sets. The cylinder is an attracting set. We identify the points $(X, Y, 0)$ and $(X, Y, 2\pi)$ in the planes $Z = 0$ and $Z = 2\pi$, we get a flow in R^3 with a two-dimensional invariant torus T^2 as an attracting set. The Z-axis gets mapped onto an unstable cycle Γ. If α is an irrational multiple of π then the torus T^2 is an attractor and it is the ω-limit set of every trajectory except the cycle Γ.

7.1 Find system fixed points and discuss stability.
7.2 Parameter $\alpha = \alpha(X, Y)$, How it influences system behavior.
7.3 Implement the system by using optoisolation circuits.
7.4 The implementation optoisolation circuit must contain optocoupler which is coupler with three LEDs $I_{BQ1=}k_1 \cdot I_{D1} + k_2 \cdot I_{D2} + k_3 \cdot I_{D3}$. Draw the circuit and find the related differential equation. Discuss Torus bifurcation.
7.5 We transfer some of differential equation's variables to their absolute values. How it influence system behavior and stability.

$$\frac{dX}{dt} = -Y + |X| \cdot (1 - X^2 - Y^2); \frac{dY}{dt} = X + |Y| \cdot (1 - X^2 - Y^2); \frac{dZ}{dt} = \alpha$$

Discuss all possible options.

8. Homoclinic bifurcation is a scenario, part of a limit cycle moves closer and closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic orbit. This is kind of infinite period bifurcation. It is also call saddle loop or homoclinic bifurcation.

We have the system $\frac{dX}{d_t} = Y; \frac{dY}{d_t} = \mu \cdot Y + X - X^2 + X \cdot Y$

8.1 Find system fixed points and discuss stability. How μ values influence system behavior and which is the critical value μ_c which homoclinic bifurcation happened.
8.2 Plots a series of phase portraits before, during, and after bifurcation. How μ parameter value influence phase portrait.
8.3 We define variation system:

$$\frac{dX}{dt} = |Y|; \frac{dY}{dt} = \mu \cdot Y + X - X^2 + X \cdot Y \cdot sign\{X \cdot Y\}$$

Discuss fixed points and stability for all options.
8.4 Implement the system by using optoisolation circuits. How circuit parameters cause to homoclinic bifurcation ?

9. We have coupled chaotic system, two identical chaotic systems $\frac{dX}{dt} = f(X); \frac{dY}{dt} = f(Y); X, Y \in \mathbb{R}^n, n \geq 3$, evolving on an asymptotically stable chaotic attractor A, when one to one coupling (see below)

$$\frac{dX}{dt} = f(X) + d_1 \cdot (Y - X); \frac{dY}{dt} = f(Y) + d_2 \cdot (X - Y); X, Y \in \mathbb{R}^n, n \geq 3$$

Is introduced, can be synchronized for some ranges of $d_{1,2}$ when $d_{1,2} \in \mathbb{R}$. We consider the dynamics of two different Lorenz systems coupled by nonsymmetrical one to one coupling.

$$\frac{dX_1}{dt} = -\sigma \cdot (X_1 - Y_1) + d_1 \cdot (X_2 - X_1);$$
$$\frac{dY_1}{dt} = -X_1 \cdot Z_1 + r_1 \cdot X_1 - Y_1 + d_1 \cdot (Y_2 - Y_1)$$

$$\frac{dZ_1}{dt} = X_1 \cdot Y_1 - b \cdot Z_1 + d_1 \cdot (Z_2 - Z_1);$$
$$\frac{dX_2}{dt} = -\sigma \cdot (X_2 - Y_2) + d_2 \cdot (X_1 - X_2)$$

$$\frac{dY_2}{dt} = -X_2 \cdot Z_2 + r_2 \cdot X_2 - Y_2 + d_2 \cdot (Y_1 - Y_2);$$
$$\frac{dZ_2}{dt} = X_2 \cdot Y_2 - b \cdot Z_2 + d_2 \cdot (Z_1 - Z_2)$$

$\sigma, B, r_1, r_2, d_1, d_2 \in \mathbb{R}$. We assume that each uncoupled system ($d_1 = d_2 = 0$) evolves on chaotic attractors.

9.1 Find system fixed points for the coupled case ($d_{1,2} \neq 0$).
9.2 Find Jacobian matrix and discuss stability.
9.3 Discuss system evolution on three-dimensional torus T.
9.4 Describe evolution transverse to the Torus T.
9.5 Implement the system by using optoisolation circuits.
9.6 Discuss bifurcation upon circuit's parameters variation.
9.7 Discuss on-off intermittency linked with the destruction of the torus attractor of a pair coupled chaotic Lorenz systems.

10. We have system which characterize by two variables X, Y and two parameters A, B.

$$\frac{dX}{dt} = A \cdot \frac{dY}{dX} - \left\{ \frac{\sum_{k=1}^{3} \sum_{l=k}^{k^2} X^k \cdot Y^l}{|X|} \right\} / (|Y - B|);$$
$$\frac{dY}{dt} = B \cdot X - X^2 \cdot Y + \frac{\prod_{k=1}^{2} \sum_{l=k}^{k^2} X^k \cdot Y^l}{\left(A + \frac{dX}{dY}\right)}$$

10.1 Find system fixed points and discuss stability for the following cases ($A = B = 0$; $A = 0$ & $B \neq 0$; $A \neq 0$ & $B = 0$; $A \neq 0$ & $B \neq 0$), analyze all possible options.
10.2 Discuss system bifurcation based on different values of A and B.

10.3 How the system behavior changes for $A = B \cdot \Delta$? Discuss bifurcation for different values of Δ parameter.
10.4 Implement the system by using optoisolation circuits.
10.5 Discuss bifurcation upon circuit's parameters variation.
10.6 We have system constrain $A/B = [A/B]^2 - 1$, How it influence system behavior, fixed points and stability.

Chapter 4
Optoisolation Circuits Analysis Floquet Theory

Floquet theory is the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points. Another way to describe Floquet theory: it is the study of linear systems of differential equations with periodic coefficients. Floquet theory can be used for anything you would use linear stability analysis for, when dealing with a periodic system.

Although Floquet theory is a linear theory, nonlinear models can be linearized near limit cycle solutions to enable the use of Floquet theory. Floquet theory deals with continuous-time systems. The theory of periodic discrete-time systems is closely analogous. In that case, one can multiply the T transition matrices together to determine how a perturbation changes over a period, which is similar to finding the fundamental matrix. One limitation of Floquet theory is that it applies only to periodic systems. Although many systems experience periodic forcing, others experience stochastic or chaotic forcing. In these cases, the more general Lyapunov exponents described play the role of Floquet exponents. Conceptually similar to Floquet exponents, Lyapunov exponents are more challenging to compute numerically because, instead of calculating how a perturbation grows or shrinks over one period, this must be done in the limit at $T \to \infty$. Many optoisolation systems are periodic and continuous in time, therefore Floquet theory is an ideal way for behavior and stability analysis. There are many optoisolation systems which contain periodic forcing source (voltage or current sources) and can be analyzed by this theory [46–48].

4.1 Floquet Theory Basic Assumptions and Definitions

We consider a set of linear, homogeneous, time periodic differential equations: $\mathrm{d}x/\mathrm{d}t = A(t) \cdot x$; where x is a n-dimensional vector and $A(t)$ is an $n \times n$ matrix with minimal period T. $A(t)$ varies periodically and the solutions are typically not periodic,

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and despite its linearity, closed from the solutions of $dx/dt = A(t) \cdot x$ typically cannot be found. The general solution of $dx/dt = A(t) \cdot x$ takes the form $x(t) = \sum_i^n c_i \cdot e^{\mu_i \cdot t} \cdot p_i(t)$, where c_i are constants that depend on initial conditions, $p_i(t)$ are vector value functions with period T, and μ_i are complex numbers called characteristic or Floquet exponents. Characteristic or Floquet multipliers are related to the Floquet exponents by the relationship $\rho_i = e^{\mu_i \cdot T}$ (μ_i—Floquet exponents; ρ_i—Floquet multipliers). The solution to $dx/dt = A(t) \cdot x$ is the sum of n periodic functions multiplied by exponentially growing or shrinking terms $(e^{\mu_i \cdot t})$. The long-term behavior of the system is determined by the Floquet exponents. The zero equilibrium is stable if all Floquet exponents have negative real parts or, equivalently, all Floquet multipliers have real parts between −1 and +1. If any Floquet exponent has a positive real part which is equivalent to a Floquet multiplier with modulus greater than one, then the zero equilibrium is unstable and $||x(t)||_{t \to \infty} \to \infty$. Thus, Floquet exponents or Floquet multipliers can be interpreted in the same way as eigenvalues are in models with constant coefficients in continuous/discrete time, respectively and they represent the growth rate of different perturbations averaged over a cycle. Floquet exponents are rates with unit's 1/time, and Floquet multipliers are dimensionless numbers that give the period to period increase/decrease of the perturbation. Typically Floquet exponents and multipliers must be calculated numerically while eigenvalues of a matrix can be calculated analytically. The way is to solve the matrix differential equation $dX/dt = A(t) \cdot X$ over one period ($t = 0$ to $t = T$) with the identity matrix as an initial condition ($X(t = 0) = I$). $X(T)$ is known as a fundamental matrix and Floquet multipliers ρ_i are the eigenvalues of $X(T)$ and Floquet exponents, μ_i can calculated as $(\ln[\rho_i])/T$ $\left(\rho_i = e^{\mu_i \cdot T} \Rightarrow \mu_i = \frac{1}{T} \cdot \ln(\rho_i)\right)$.

$$x(t) = \sum_i^n c_i \cdot e^{\mu_i \cdot t} \cdot p_i(t) \Rightarrow \frac{dx(t)}{dt} = \sum_i^n c_i \cdot \left\{\mu_i \cdot e^{\mu_i \cdot t} \cdot p_i(t) + e^{\mu_i \cdot t} \cdot \frac{\partial p_i(t)}{\partial t}\right\}$$

$$\frac{dx(t)}{dt} = \sum_i^n c_i \cdot \left\{\mu_i \cdot e^{\mu_i \cdot t} \cdot p_i(t) + e^{\mu_i \cdot t} \cdot \frac{\partial p_i(t)}{\partial t}\right\} = \frac{dx(t)}{dt} = \sum_i^n c_i \cdot \left\{\mu_i \cdot p_i(t) + \frac{\partial p_i(t)}{\partial t}\right\} \cdot e^{\mu_i \cdot t}$$

$$\frac{dx(t)}{dt} = A(t) \cdot x(t) \Rightarrow \sum_i^n c_i \cdot \left\{\mu_i \cdot p_i(t) + \frac{\partial p_i(t)}{\partial t}\right\} \cdot e^{\mu_i \cdot t} = A(t) \cdot \sum_i^n c_i \cdot e^{\mu_i \cdot t} \cdot p_i(t)$$

We can summarize our last discussion in Table 4.1:

$$\frac{dx}{dt} = A(t) \cdot x;\ A(t) = A(t+T);\ x \sim e^{\mu \cdot t} \cdot p(t);\ p(t) = p(t+T)$$

We define $tr(A)$ as the trace of an $n \times n$ square matrix A. Trace A, $tr(A)$ is defined to be the sum of the elements on the main diagonal which is the diagonal from the upper left to the lower right of A [78, 79].

$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^n a_{ii}$, where a_{ii} represents the entry on the ith row and ith column of A. The trace of a matrix is the sum of the complex eigenvalues and its invariant with respect to a change of basis. This characterization

Table 4.1 Floquet exponents

Floquet exponents (characteristic exponents)	Floquet multipliers
Rated with units [1/time]	Dimensionless numbers
$\mu_i = \alpha_i + i \cdot \beta_i$; μ_i—complex numbers $\rho_i = e^{\mu_i \cdot T} \Rightarrow \mu_i = \frac{1}{T} \cdot \ln(\rho_i)$	$\rho_i = e^{\mu_i \cdot T} = \rho_i\|_{\mu_i = \alpha_i + i \cdot \beta_i} = e^{(\alpha_i + i \cdot \beta_i) \cdot T}$ Using Euler's formula $\rho_i = e^{\alpha_i \cdot T} \cdot e^{i \cdot \beta_i \cdot T}$ $\rho_i = e^{\alpha_i \cdot T} \cdot \{\cos(\beta_i \cdot T) + i \cdot \sin(\beta_i \cdot T)\}$
Stability issue at zero equilibrium: *Stable* for $\text{Re}[\mu_i] = \alpha_i < 0$ *Unstable* for $\text{Re}[\mu_i] = \alpha_i > 0$ $\|\|x(t)\|\|_{t \to \infty} \to \infty$	Stability issue at zero equilibrium: *Stable* for $-1 < \text{Re}[\rho_i] < 1$ $-1 < e^{\alpha_i \cdot T} \cdot \cos(\beta_i \cdot T) < 1$; T—period *Unstable* if Floquet multipliers modulus (absolute value, magnitude) greater than one $\|\rho_i\| = e^{\alpha_i \cdot T} > 1$; $\|\|x(t)\|\|_{t \to \infty} \to \infty$
Typically must be calculated numerically	Typically must be calculated numerically
Floquet exponents μ_i can calculated as $\rho_i = e^{\mu_i \cdot T} \Rightarrow \mu_i = \frac{1}{T} \cdot \ln(\rho_i)$	Floquet multipliers ρ_i are eigenvalues of $X(T)$; $X(T)$ is a fundamental matrix

can be used to define the trace of a linear operator in general. The matrix trace is only defined for a square matrix (i.e., $n \times n$).

$$e^{\mu_1 \cdot T} \cdot e^{\mu_2 \cdot T} \cdot \ldots \cdot e^{\mu_n \cdot T} = \prod_{i=1}^{n} e^{\mu_i \cdot T} = e^{T \cdot \sum_{i=1}^{n} \mu_i} = e^{\int_0^T tr(A(s)) \cdot \mathrm{d}s}$$

Let $x_1(t), \ldots, x_n(t)$ be n solutions of $\mathrm{d}x/\mathrm{d}t = A(t) \cdot x$ then $X(t) = [[x_1]\ldots[x_n]]$; $X(t)$ is an $n \times n$ matrix solution of $\frac{\mathrm{d}X}{\mathrm{d}t} = A \cdot X$. If $x_1(t), \ldots, x_n(t)$ are linearly independent, then $X(t)$ is nonsingular and is called a fundamental matrix. If $X(t_0) = I$, then $X(t)$ is the principal fundamental matrix.

Lemma 1 *if $X(t)$ is a fundamental matrix then so is $Y(t) = X(t) \cdot B$ for any non-singular constant matrix B.*

Lemma 2 *If $W(t)$ of $X(t)$ be the determinant of $X(t)$ then* $W(t) = W(t_0) \cdot e^{\int_{t_0}^{t} tr(A(s)) \cdot \mathrm{d}s}$.

Theorem 1 *Let $A(t)$ be a T-period matrix. If $X(t)$ is a fundamental matrix then so is $X(t + T)$ and there exists a nonsingular constant matrix B such that*

(I) $$X(t + T) = X(t) \cdot B \, \forall \, t$$

(II) $$\det(B) = e^{\int_0^T tr(A(s)) \cdot \mathrm{d}s} = e^{\mu_1 \cdot T} \cdot e^{\mu_2 \cdot T} \cdot \ldots \cdot e^{\mu_n \cdot T} = \prod_{i=1}^{n} e^{\mu_i \cdot T} = e^{T \cdot \sum_{i=1}^{n} \mu_i}$$

Definition 1 The eigenvalues $\rho_1,\ldots,\rho_n$ of B are called the characteristic multipliers (Floquet multipliers) for $dX/dt = A(t) \cdot X$. The characteristic exponent (Floquet exponents) are $\mu_1,\ldots,\mu_n$ satisfying $\rho_1 = e^{\mu_1 \cdot T}, \rho_2 = e^{\mu_2 \cdot T}, \ldots, \rho_n = e^{\mu_n \cdot T}$
μ_j for $j \in \mathbb{N}$ may be complex.

(I) The characteristic multipliers (eigenvalues) $\rho_1,\ldots,\rho_n$ of $B = X(T)$ with $X(T = 0) = I$ satisfy $\det(B) = \rho_1 \cdot \rho_2 \cdot \ldots \cdot \rho_n = \prod_{j=1}^{n} \rho_j = e^{\int_0^T tr(A(s)) \cdot ds}$
(II) Trace is the sum of the eigenvalues $tr(B) = \rho_1 + \rho_2 + \cdots + \rho_n = \sum_{j=1}^{n} \rho_j$
(III) The characteristic exponents are not unique since if $\rho_j = e^{\mu_j \cdot T} \Rightarrow \rho_j = e^{\left(\mu_j + \frac{2 \cdot \pi \cdot i}{T}\right) \cdot T}$
(IV) The characteristic multipliers ρ_j are an intrinsic property of the equation $dX/dt = A(t) \cdot X$ and do not depend on the choice of the fundamental matrix.

Theorem 2 *Let ρ be a characteristic multiplier and let μ be the corresponding characteristic exponent so that $\rho = e^{\mu \cdot T}$. Then there exists a solution $dx/dt = A \cdot x$ such that $x(t + T) = \rho \cdot x(t)$ and there exists a periodic solution $p(t)$ with period T such that $x(t) = e^{\mu \cdot T} \cdot p(t)$.*

(I) If μ is replaced by $\mu + 2 \cdot \pi \cdot i/T$ ($\mu \to \mu + 2 \cdot \pi \cdot i/T$) then we get $x(t) = p(t) \cdot e^{\left(\mu + \frac{2 \cdot \pi \cdot i}{T}\right) \cdot t} = p(t) \cdot e^{\mu \cdot t} \cdot e^{\frac{2 \cdot \pi \cdot i}{T} \cdot t}$, where $p(t) \cdot e^{\frac{2 \cdot \pi \cdot i}{T} \cdot t}$ is a periodic with period T. The fact that μ is not unique does not alter the results.
(II) $x_j(t + T) = \rho_j \cdot x_j(t); x_j(t + N \cdot T) = \rho_j^N \cdot x_j(t)$ each characteristic multipliers falls into one of the following categories:

(II.1) if $|\rho| < 1$ then $\mathrm{Re}(\mu) < 0$ and $x(t) \to 0$ for $t \to \infty$.

$$\rho_j = e^{\mu_j \cdot T} \Rightarrow \mu_j = \frac{1}{T} \cdot \ln[\rho_j]; \quad \rho_j = e^{(\alpha_j + i \cdot \beta_j) \cdot T} = e^{\alpha_j \cdot T} \cdot e^{i \cdot \beta_j \cdot T}$$
$$\rho_j = e^{\alpha_j \cdot T} \cdot [\cos(\beta_j \cdot t) + i \cdot \sin(\beta_j \cdot t)] \Rightarrow |\rho_j| = e^{\alpha_j \cdot T}; \quad |\rho_j| 1 \Rightarrow e^{\alpha_j \cdot T} < 1$$
$$e^{\alpha_j \cdot T} < 1 \Rightarrow \alpha_j \cdot T < 0; \quad \mathrm{T} > 0 \Rightarrow \alpha_j < 0 \Rightarrow \mathrm{Re}(\mu_j) = \alpha_j < 0$$
$$x(t) = p(t) \cdot e^{\mu \cdot t} = p(t) \cdot e^{(\alpha + i \cdot \beta) \cdot t} = p(t) \cdot e^{\alpha \cdot t} \cdot e^{i \cdot \beta \cdot t}|_{\substack{\alpha > 0 \\ t \to \infty \\ e^{\alpha \cdot t} \to \varepsilon}} \to \varepsilon = 0$$

(II.2) if $|\rho| = 1$ then $\mathrm{Re}(\mu) = 0$ and we have pseudo-periodic solution. If $\rho = \pm 1$ then the solution is periodic with period T.

$$\rho_j = e^{\mu_j \cdot T} \Rightarrow \mu_j = \frac{1}{T} \cdot \ln[\rho_j]; \quad \rho_j = e^{(\alpha_j + i \cdot \beta_j) \cdot T} = e^{\alpha_j \cdot T} \cdot e^{i \cdot \beta_j \cdot T}$$
$$|\rho_j| = e^{\alpha_j \cdot T} \Rightarrow |\rho_j| = 1 \Rightarrow e^{\alpha_j \cdot T} = 1 \Rightarrow \alpha_j \cdot T = 0$$
$$\mathrm{T} \neq 0 \Rightarrow \alpha_j = 0 \Rightarrow \mathrm{Re}(\mu_j) = 0$$

(II.3) if $|\rho| > 1$ then $\mathrm{Re}(\mu) > 0$ and $x(t) \to \infty$ for $t \to \infty$.

$$\rho_j = \mathrm{e}^{\mu_j \cdot T} \Rightarrow \mu_j = \frac{1}{T} \cdot \ln[\rho_j]; \rho_j = \mathrm{e}^{(\alpha_j + i \cdot \beta_j) \cdot T} = \mathrm{e}^{\alpha_j \cdot T} \cdot \mathrm{e}^{i \cdot \beta_j \cdot T}$$
$$\rho_j = \mathrm{e}^{\alpha_j \cdot T} \cdot [\cos(\beta_j \cdot t) + i \cdot \sin(\beta_j \cdot t)] \Rightarrow |\rho_j| = \mathrm{e}^{\alpha_j \cdot T}; |\rho_j| > 1 \Rightarrow \mathrm{e}^{\alpha_j \cdot T} > 1$$
$$\mathrm{e}^{\alpha_j \cdot T} > 1 \Rightarrow \alpha_j \cdot T > 0; \mathrm{T} > 0 \Rightarrow \alpha_j > 0 \Rightarrow \mathrm{Re}(\mu_j) = \alpha_j > 0$$
$$x(t) = p(t) \cdot \mathrm{e}^{\mu \cdot t} = p(t) \cdot \mathrm{e}^{(\alpha + i \cdot \beta) \cdot t} = p(t) \cdot \mathrm{e}^{\alpha \cdot t} \cdot \mathrm{e}^{i \cdot \beta \cdot t}|_{\substack{\alpha > 0 \\ t \to \infty \\ \mathrm{e}^{\alpha \cdot t} \to \varepsilon}} \to \infty$$

The entire solution is stable if all the characteristic multipliers satisfy$|\rho_j| \le 1$

(III) If $b_1,\ldots,b_n$ are n linearly independent eigenvectors of B corresponding to distinct eigenvalues $\rho_1,\ldots,\rho_n$ then there are n linearly independent solutions to $\mathrm{d}x/\mathrm{d}t = A \cdot x$, which can be given by $x_j(t) = \mathrm{e}^{\mu_j \cdot t} \cdot p_j(t)$.
$p_j(t)$ is T periodic. We can define the following:

$$X_0(t) = [[x_1]\ldots[x_n]]; P_0(t) = [[p_1]\ldots[p_n]]; D_0(t) = \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{pmatrix}$$
$$D_0(t) = \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{pmatrix} \& Y_0(t) = \begin{pmatrix} \mathrm{e}^{\mu_1 \cdot t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{\mu_n \cdot t} \end{pmatrix}$$
$$\Rightarrow X_0 = P_0 \cdot Y_0; \frac{\mathrm{d}Y_0}{\mathrm{d}t} = D_0 \cdot Y_0$$

(IV) What happened if $\rho < 0$ and is real then $\rho = \mathrm{e}^{\mu \cdot T} = \mathrm{e}^{(\alpha + i \cdot \beta) \cdot T}$; $\rho = \mathrm{e}^{(\alpha + i \cdot \beta) \cdot T} = \mathrm{e}^{\alpha \cdot T} \cdot \mathrm{e}^{i \cdot \beta \cdot T}|_{\beta = 0} = \mathrm{e}^{\alpha \cdot T}$;
$\rho = \mathrm{e}^{\left(\alpha + \frac{i\pi}{T}\right) \cdot T}; \rho = -\mathrm{e}^{\alpha \cdot T} x(t) = p(t) \cdot \mathrm{e}^{\mu \cdot t} = p(t) \cdot \mathrm{e}^{\alpha \cdot t} \cdot \mathrm{e}^{\frac{i \cdot \pi \cdot t}{T}} = \mathrm{e}^{\alpha \cdot t} \cdot q(t)$;
$q(t)$ has period T since $p(t)$ has period T. Since we can choose x to be real, without loss of generality, we can choose q to be real. For the general solution if $\rho_j < 0$, we can replace p_j and μ_j with α_j and get the following results:

$$P_0 = [[p_1]\ldots[q_j]\ldots[p_n]]; \quad Y_0 = \begin{pmatrix} \mathrm{e}^{\mu_1 \cdot T} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{\mu_n \cdot T} \end{pmatrix};$$
$$X_0(t) = P_0(t) \cdot Y_0(t)$$

(V) Suppose that ρ is complex; $\rho_j = \mathrm{e}^{\alpha_j \cdot T} \cdot [\cos(\beta_j \cdot t) + i \cdot \sin(\beta_j \cdot t)]$ then since ρ is an eigenvalue of the real matrix B, $\bar{\rho}$ is as well. The characteristic exponents are μ and $\bar{\mu}$. Let $\mu = \alpha + i \cdot \beta$ and $p(t) = q(t) + i \cdot r(t)$, where $q(t)$ and $r(t)$ must both have period T since $p(t)$ does. Since $x(t) = p(t) \cdot e^{\mu \cdot t}$ is a solution to $dx/dt = A(t) \cdot x$ then by taking the complex conjugate, $\bar{x}(t) = \bar{p}(t) \cdot e^{\bar{\mu} \cdot t}$. We can write these as

$$x(t) = e^{[\alpha + i \cdot \beta] \cdot t} \cdot \{q(t) + i \cdot r(t)\} = e^{\alpha \cdot t} \cdot e^{i \cdot \beta \cdot t} \cdot \{q(t) + i \cdot r(t)\}$$
$$x(t) = e^{[\alpha + i \cdot \beta] \cdot t} \cdot \{q(t) + i \cdot r(t)\} = e^{\alpha \cdot t} \cdot \{\cos(\beta \cdot t) + i \cdot \sin(\beta \cdot t)\} \cdot \{q(t) + i \cdot r(t)\}$$
$$x(t) = e^{\alpha \cdot t} \cdot \{q(t) \cdot \cos(\beta \cdot t) - r(t) \cdot \sin(\beta \cdot t) + i \cdot [r(t) \cdot \cos(\beta \cdot t) + q(t) \cdot \sin(\beta \cdot t)\}$$
$$\bar{x}(t) = e^{\alpha \cdot t} \cdot \{q(t) \cdot \cos(\beta \cdot t) - r(t) \cdot \sin(\beta \cdot t) - i \cdot [r(t) \cdot \cos(\beta \cdot t) + q(t) \cdot \sin(\beta \cdot t)]\}$$

We can alternately write the linearly independent real solutions:

$$x_R(t) = \mathrm{Re}[e^{\mu \cdot t} \cdot p(t)] = e^{\alpha \cdot t} \cdot \{q(t) \cdot \cos(\beta \cdot t) - r(t) \cdot \sin(\beta \cdot t)\}$$
$$x_I(t) = \mathrm{Im}[e^{\mu \cdot t} \cdot p(t)] = e^{\alpha \cdot t} \cdot \{q(t) \cdot \sin(\beta \cdot t) + r(t) \cdot \cos(\beta \cdot t)\}$$

$$X_0 = [[x_1] \ldots [x_R][x_I] \ldots [x_n]]; P_0 = [[p_1] \ldots [q][r] \ldots [p_n]]; Y_0 = \begin{pmatrix} e^{\mu_1 \cdot t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\mu_n \cdot t} \end{pmatrix}$$

$X_0(t) = P_0(t) \cdot Y_0(t)$. If we consider system $dx/dt = f(x)$ with $x \in \mathbb{R}^2$, where there is a periodic solution $x(t) = \Phi(t)$ with period T. We must have $\rho_1 = 1$ then we get $\rho_1 \cdot \rho_2 = e^{\int_0^T tr(A(s)) \cdot ds}|_{\rho_1 = 1} \Rightarrow \rho_2 = e^{\int_0^T tr(A(s)) \cdot ds}$. The perturbation should be bounded and hence for the solution to be stable, $\rho_1 \leq 1$; $\rho_2 \leq 1$ and since we know $\rho_1 = 1$ and we wish ρ_1 and ρ_2 to be distinct, we must have $\rho_2 < 1$; $\rho_1 = 1$

$$\rho_2 < 1 \Rightarrow e^{\int_0^T tr(A(s)) \cdot ds} < 1 \Rightarrow \int_0^T tr(A(s)) \cdot ds < 0 \Rightarrow \int_0^T tr\left(\frac{\partial f_i}{\partial x_j}|_{\phi(s)}\right) \cdot ds < 0$$
$$\Rightarrow \int_0^T \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)|_{\phi(s)} \cdot ds < 0$$

$\int_0^T \nabla \cdot f|_{x=\phi} \cdot ds < 0$. We get instability when $\int_0^T \nabla \cdot f|_{x=\phi} \cdot ds > 0$. If we consider n x n linear equation $dy(t)/dt = A(t) \cdot y(t)$; $t \in \mathbb{R}$, where $A(t)$ is a T-periodic function. Let $\Phi(t)$ be a fundamental of $dy(t)/dt = A(t) \cdot y(t)$ and we can write $d\Phi(t)/dt = A(t) \cdot \Phi(t)$ and Φ is nonsingular. The periodicity of $A(\cdot)$, $\Phi(t + T)$ is satisfies a fundamental solution. Since there are n independent solutions of $dy(t)/dt = A(t) \cdot y(t)$, thus there exists a constant nonsingular matrix Ω such that $\Phi(t + T) = \Phi(t) \cdot \Omega$. Since any solution of y can be expressed as $\Phi(t) \cdot \omega$ for some constant vector ω, we get $y(t + T) = \Phi(t) \cdot \Omega \cdot \omega$. If we choose ω to be an eigenvector of Ω with eigenvalue ρ then the corresponding $y(t) = \Phi(t) \cdot \omega$ and satisfies $y(t + T) = \rho \cdot y(t)$. Let Ω has eigenvalues $\rho_1,\ldots,\rho_n$ with eigenvectors $\omega_1,\ldots,\omega_n$ then the solution $y_i(t) = \Phi(t) \cdot \omega_i$ satisfies $y_i(t + T) = \rho_i \cdot y_i(t)$. The above definition of ρ_i is independent of the choice of a particular choice of the fundamental solution. If $\psi(t)$ is another fundamental solution, then there exists a constant D such that $\psi(t) = \Phi(t) \cdot D$ and $\Omega = \Phi(T) \cdot \Phi(0)^{-1} = \psi(T) \cdot \psi(0)^{-1}$. The eigenvalues ρ_i are called the characteristic values of $\mathrm{d}y(t)/\mathrm{d}t = A(t) \cdot y(t)$. The Wronskian $W(t) = \det \Phi(t)$ satisfies $\mathrm{d}W(t)/\mathrm{d}t = [tr(At)] \cdot W(t)$. Hence there exists $W(t)$, $W(t) = \mathrm{e}^{\int_0^t [tr(A(s))]\cdot \mathrm{d}s} \neq 0 \Rightarrow \rho_i \neq 0 \, \forall \, i$. We can express $\rho_i = \mathrm{e}^{T\cdot\mu_i}$ and the constants μ_i, $i = 1,2,\ldots,n$ are called Floquet exponents. If we define $\phi_i(t) = y(t) \cdot \mathrm{e}^{-\mu_i \cdot t}$ then Φ_i is a T-periodic function.

$$
\begin{aligned}
\phi_i(t+T) &= y(t+T) \cdot \mathrm{e}^{-\mu_i\cdot(t+T)}|_{y(t+T)=\rho_i\cdot y_i(t)} \\
&= \rho_i \cdot y_i(t) \cdot \mathrm{e}^{-\mu_i\cdot(t+T)}|_{\rho_i=\mathrm{e}^{T\cdot\mu_i}} = \mathrm{e}^{T\cdot\mu_i} \cdot y_i(t) \cdot \mathrm{e}^{-\mu_i\cdot(t+T)} \\
\phi_i(t+T) &= \mathrm{e}^{T\cdot\mu_i} \cdot y_i(t) \cdot \mathrm{e}^{-\mu_i\cdot(t+T)} \\
&= \mathrm{e}^{T\cdot\mu_i} \cdot y_i(t) \cdot \mathrm{e}^{-T\cdot\mu_i} \cdot \mathrm{e}^{-\mu_i\cdot t} = y_i(t) \cdot \mathrm{e}^{-\mu_i\cdot t} = \phi_i(t)
\end{aligned}
$$

We may express $y_i(t) \cdot \mathrm{e}^{-\mu_i\cdot t} = \phi_i(t) \Rightarrow y_i(t) = \phi_i(t) \cdot \mathrm{e}^{\mu_i\cdot t}$. If the real part of $\mu_i \leq 0$, then the corresponding $y_i(t)$ is bounded for $t \geq 0$ [5, 6, 46, 47].

We can inspect our Floquet theory in the glance of periodic linear differential equation. In general, theory of time-varying differential equations like $\mathrm{d}x(t)/\mathrm{d}t = A(t) \cdot x(t)$ only for certain classes of functions $A : \mathbb{R} \rightarrow gl(d, \mathbb{R})$ we need to understand of the qualitative behavior of the solutions. Floquet's theory and its relation to the idea of Lyapunov exponents and Lyapunov spaces belong to the theory for a class of time-varying linear systems which was initiated by Floquet for the periodic case [80].

Definition 2 A periodic linear differential equation $\mathrm{d}x/\mathrm{d}t = A(t) \cdot x$ is given by a matrix function $A : \mathbb{R} \rightarrow gl(d, \mathbb{R})$ that is continuous and periodic (of period $T > 0$). We use the shift $\theta(t, \tau) = t + \tau \bmod T$. The $\mathrm{d}x/\mathrm{d}t = A(\theta(t, 0)) \cdot x$ and the solutions define a dynamical system via $\Phi : \mathbb{R} \times S^1 \times \mathbb{R}^d \rightarrow S^1 \times \mathbb{R}^d$, if we identify $R \bmod T$ with the circle S^1. The results concern the fundamental matrix of a periodic linear system.

Lemma 3 *For every invertible matrix $S \in Gl(d, \mathbb{C})$ there is a matrix $R \in gl(d, \mathbb{C})$* such that $S = e^R$. For every invertible matrix $S \in Gl(d, \mathbb{R})$, *there is a real $Q \in gl(d, \mathbb{R})$ such that $S^2 = e^Q$. The eigenvalues of R and Q are mapped onto the eigenvalues of S and S^2, respectively.*

Remark 1 We can construct Q by observing that the real parts of the eigenvalues of S^2 are all positive. Then the real logarithm of these real parts exists and one can discuss the Jordan blocks similarly as above noting that a real logarithm of $r \cdot \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \rightarrow (\ln r) \cdot I + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$.

Remark 2 The real parts of the eigenvalues of R and Q, respectively, are uniquely determined by S. The imaginary parts are unique up to the addition of $2 \cdot k \cdot \pi \cdot i; k \in k \cdot \mathbb{Z}$. In particular, several eigenvalues of R and Q may be mapped to the same eigenvalue of e^R and e^Q, respectively.

The principal fundamental solution $X(t)$, $t \in \mathbb{R}$ is a unique solution of the matrix differential equation $dX(t)/dt = A(t) \cdot X(t)$ with initial value $X(t = 0) = I$. Then the solutions of $dx/dt = A(t) \cdot x$, $x(t = 0) = x_0$, are given by $x(t) = X(t) \cdot x_0$. We need to discuss the consequences of the periodicity assumption for $A(t)$ for the fundamental solution.

Lemma 4 *The principal fundamental solution* $X(t)$ of $dx/dt = A(t) \cdot x$ *with T-periodic* $A(\cdot)$ *satisfies* $X(k \cdot T + t) = X(t) \cdot X(T)^k$ *for all* $t \in \mathbb{R}\,\&\, k \in \mathbb{N}$.

There is a matrix $Q \in gl(d, \mathbb{R})$ such that the fundamental solution $X(\cdot)$ satisfies $X(2 \cdot T) = e^{2 \cdot T \cdot Q}$. The real parts λ_i of the eigenvalues Q are uniquely determined by this condition, and are called Floquet exponents. Furthermore, the eigenvalues α_j of $X(2 \cdot T) = X(T)^2$ satisfy $|\alpha_j| = e^{\lambda j}$.

Theorem 3 *Let* $\Phi = (\theta, \varphi) : \mathbb{R} \times S^1 \times \mathbb{R}^d \to S^1 \times \mathbb{R}^d$ *be the flow associated with a periodic linear differential equation* $dx/dt = A(t) \cdot x$. *The system has a finite number of Lyapunov exponents and they coincide with the Floquet exponent's* $\lambda_j \cdot j = 1,\ldots,l \le d$. *For each exponent* λ_j *and each* $\tau \in S^1$, *there exists a splitting* $\mathbb{R}^d = \oplus_{j=1}^{l} L(\lambda_j, \tau)$ *of* $\mathbb{R}^d$ *into linear subspaces with the following properties*:

(I) *The subspaces* $L(\lambda_j, \tau)$ *have the same dimension independent of* τ, *i.e., for each* $j = 1,\ldots,l$ *it holds that* $\dim L(\lambda_j, \sigma) = \dim L(\lambda_j, \tau) = d_i$ *for all* $\sigma, \tau \in S^1$.

(II) *The subspaces* $L(\lambda_j, \tau)$ *are invariant under the flow* Φ. *i.e., for* $j = 1,\ldots,l$ *it holds that* $\varphi(t, \tau) \cdot L(\lambda_j, \tau) = L(\lambda_j, \theta(t, \tau)) = L(\lambda_j, t + \tau)$ *for all* $t \in \mathbb{R}\ \&\ \tau \in S^1$.

(III) $\lambda(x, \tau) = \lim_{t \to \pm\infty} \frac{1}{t} \cdot \ln ||\varphi(t, \tau, x)|| = \lambda_j$ *if and only if* $x \in L(\lambda_j, \tau) \backslash \{0\}$.

For each $j = 1, \ldots, l \le d$ the map $L_j : S^1 \to G_{dj}$ defined by $\tau \to L(\lambda_j, \tau)$ is continuous. The linear subspaces $L(\lambda_j, \cdot)$ are called the Lyapunov spaces (Floquet spaces) of the periodic matrix function $A(t)$.

Definition 3 The stable, center and unstable subspaces associated with the periodic matrix function $A : \mathbb{R} \to gl(d, \mathbb{R})$ are defined as $L^-(\tau) = \oplus\{L(\lambda_j, \tau), \lambda_j < 0\}$ $L^0(\tau) = \oplus\{L(\lambda_j, \tau), \lambda_j = 0\}$, and $L^+(\tau) = \oplus\{L(\lambda_j, \tau), \lambda_j > 0\}$ respectively for $\tau \in S^1$.

Theorem 4 *The zero solution* $x(t, 0) \equiv 0$ *of the periodic linear differential equation* $dx/dt = A(t) \cdot x$ *is asymptotically stable if and only if it is exponentially stable if and only if all Lyapunov exponents are negative if and only if* $L^-(\tau) = \mathbb{R}^d$ *for some and hence for all* $\tau \in S^1$.

4.2 Optoisolation Circuit's Two Variables with Periodic Sources

We have an optoisolation circuits which are characterized by two main variables $V_1(t)$ and $V_2(t)$. The first variable $V_1(t)$ characterizes the output voltage in time for the first optocoupler's output voltage (V_{CEQ1}) and the second variable $V_2(t)$ characterizes the output voltage in time for the second optocoupler's output voltage (V_{CEQ2}). Each optoisolation circuit includes feedback loop which couples with the other circuit by LED and photo transistor coupling elements. The first and the second optoisolation circuits have input periodic sources $X_1(t)$ and $X_2(t)$. These periodic sources $X_i(t)$ for $i = 1, 2$ with $X_1(t) = \sin(2 \cdot \pi \cdot t)$ and $X_2(t) = -\sin(2 \cdot \pi \cdot t)$ so that two circuit output variables can change from sources ($X > 0$) to sinks ($X < 0$) perfectly out of phase with period $T = 1(2 \cdot \pi \cdot t = \omega \cdot t = 2 \cdot \pi \cdot f \cdot t \Rightarrow 2 \cdot \pi \cdot t = \frac{2 \cdot \pi}{T} \cdot t \Rightarrow T = 1)$. We inspect system stability, limit cycle, and limit cycle stability by using Floquet theory. We inspect the cases for $X_i(t) = 0$ for $i = 1, 2$ and $X_i(t) = (-1)^{i+1} \cdot \sin(2 \cdot \pi \cdot t)$ for $i = 1, 2$. Additionally we consider two special cases. If the two optoisolation circuits are completely uncoupled ($k_1 = 0$; $k_2 = 0$) or coupled ($k_1 \neq 0$; $k_2 \neq 0$) [1, 2].

k_1 and k_2 are the coupling coefficients between D_2 forward current and phototransistor $Q1$ partial base current and D_1 forward current and phototransistor $Q2$ partial base current, respectively. The circuits include two multiplication elements which are implemented by using Op Amps and discrete components (capacitors, resistors, diodes, etc.,). The output of each multiplication elements is $X_1(t) \cdot V_1(t)$ and $X_2(t) \cdot V_2(t)$, respectively. Circuits multiplication elements input ports impedances are infinite, then the input ports current is zero ($R_{in-1} \to \infty \Rightarrow I_{in-1} \to \varepsilon$ and $R_{in-2} \to \infty \Rightarrow I_{in-2} \to \varepsilon$). The below optoisolation circuits implement our two variables with periodic sources system. Our circuit includes two sub-circuits, V_1 circuit ($D2$–$Q1$) and V_2 circuit ($D1$–$Q2$) [10, 15] (Fig. 4.1).

$$X_1(t) = \sin(2 \cdot \pi \cdot t); \quad X_2(t) = -\sin(2 \cdot \pi \cdot t);$$
$$X_i(t) = (-1)^{i+1} \cdot \sin(2 \cdot \pi \cdot t) \forall i = 1, 2$$

$$V_2 = I_{D2} \cdot R_2 + V_{D2} = I_{D2} \cdot R_2 + V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right]\Bigg|_{\ln\left[\frac{I_{D2}}{I_0}+1\right] \approx \frac{I_{D2}}{I_0}} = I_{D2} \cdot R_2 + V_t \cdot \frac{I_{D2}}{I_0}$$

$$V_1 = I_{D1} \cdot R_1 + V_{D1} = I_{D1} \cdot R_2 + V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right]\Bigg|_{\ln\left[\frac{I_{D1}}{I_0}+1\right] \approx \frac{I_{D1}}{I_0}} = I_{D1} \cdot R_1 + V_t \cdot \frac{I_{D1}}{I_0}$$

(Taylor series approximation). $V_2 = I_{D2} \cdot \left(R_2 + \frac{V_t}{I_0}\right) \Rightarrow I_{D2} = \frac{V_2}{R_2 + \frac{V_t}{I_0}} = \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2$

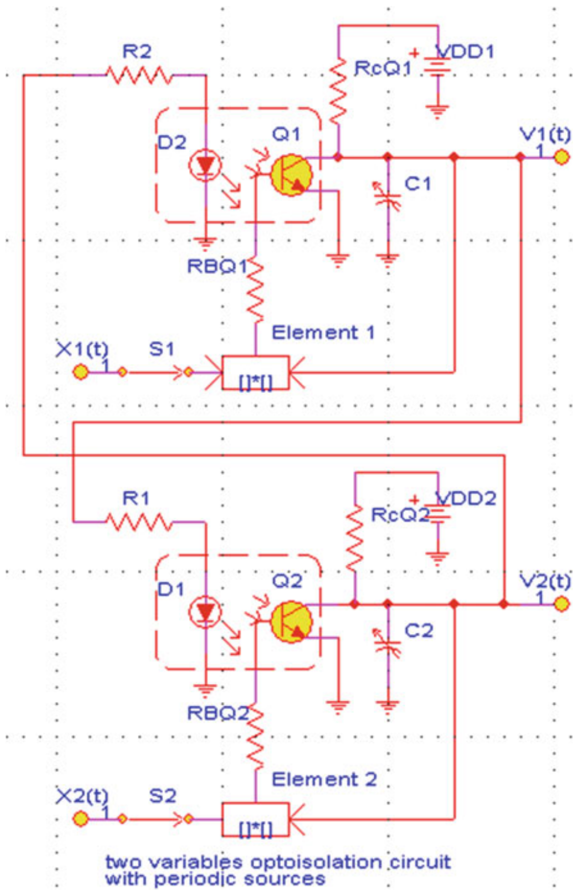

Fig. 4.1 Two variables optoisolation circuit with periodic source

V_1 circuit ($D2$–$Q1$):

$$V_1 = I_{D1} \cdot \left(R_1 + \frac{V_t}{I_0}\right) \Rightarrow I_{D1} = \frac{V_1}{R_1 + \frac{V_t}{I_0}} = \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1;\ I_{BQ1} = k_1 \cdot I_{D2} + \frac{V_1 \cdot X_1(t) - V_{BEQ1}}{R_{BQ1}}$$

$$V_{DD1} = I_{RcQ1} \cdot R_{cQ1} + V_1;\ \ V_1 = V_{CEQ1};\ \ I_{RcQ1} = I_{CQ1} + I_{C1} + I_{D1};\ \ I_{CQ1} = I_{CQ1}(V_1)$$

$$V_{DD1} = (I_{CQ1} + I_{C1} + I_{D1}) \cdot R_{cQ1} + V_1;\ \ I_{C1} = C_1 \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t};\ \ V_1 = \frac{1}{C_1} \cdot \int I_{C1} \cdot \mathrm{d}t$$

$$V_{DD1} = \left(I_{CQ1}(V_1) + C_1 \cdot \frac{dV_1}{dt} + \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1\right) \cdot R_{cQ1} + V_1; V_{BEQ1} = V_t \cdot \ln\left\{\left[\frac{\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right] + 1\right\}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; I_{BQ1} = k_1 \cdot I_{D2} + \frac{V_1 \cdot X_1(t) - V_{BEQ1}}{R_{BQ1}} = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t) - V_{BEQ1}}{R_{BQ1}}$$

$$V_{CEQ1} = V_t \cdot \ln\left\{\frac{(\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\} + V_t \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right); \frac{I_{sc}}{I_{se}} \to 1 \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$V_{CEQ1} = V_1 \simeq V_t \cdot \ln\left\{\frac{(\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}; I_{BQ1} = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_{BEQ1}}{R_{BQ1}}$$

$$I_{BQ1} = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t}{R_{BQ1}} \cdot \ln\left\{\left[\frac{\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right] + 1\right\}$$

$$V_{CEQ1} = V_1 = V_{BEQ1} + V_{CBQ1}|_{V_{CBQ1} = -V_{BCQ1}} = V_{BEQ1} - V_{BCQ1}; V_{BCQ1} = V_t \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}}{I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right) + 1\right]$$

$$I_{BQ1} = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_{BEQ1}}{R_{BQ1}} \Rightarrow V_{BEQ1} = \left[k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - I_{BQ1}\right] \cdot R_{BQ1}$$

$$V_{CEQ1} = V_1 \Rightarrow V_1 = V_{BEQ1} - V_{BCQ1}$$

$$V_1 = \left[k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - I_{BQ1}\right] \cdot R_{BQ1} - V_t \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}}{I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right) + 1\right]$$

$$V_{DD1} = \left(I_{CQ1} + C_1 \cdot \frac{dV_1}{dt} + \left\{ \frac{I_0}{I_0 \cdot R_1 + V_t} \right\} \cdot V_1 \right) \cdot R_{cQ1} + V_1$$
$$\Rightarrow I_{CQ1} = \frac{V_{DD1} - V_1}{R_{cQ1}} - C_1 \cdot \frac{dV_1}{dt} - \left\{ \frac{I_0}{I_0 \cdot R_1 + V_t} \right\} \cdot V_1$$

$$I_{CQ1} = \frac{V_{DD1} - V_1}{R_{cQ1}} - C_1 \cdot \frac{dV_1}{dt} - \left\{ \frac{I_0}{I_0 \cdot R_1 + V_t} \right\} \cdot V_1$$
$$= \frac{V_{DD1}}{R_{cQ1}} - V_1 \cdot \left[\frac{1}{R_{cQ1}} + \frac{I_0}{I_0 \cdot R_1 + V_t} \right] - C_1 \cdot \frac{dV_1}{dt}$$

We get a result that $I_{EQ1} = I_{BQ1} + I_{CQ1}$ and $I_{CQ1} = I_{CQ1}\left(V_1, \frac{dV_1}{dt}\right)$.

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \alpha_{r1} \cdot I_{CQ1} - (I_{BQ1} + I_{CQ1}) = I_{CQ1} \cdot (\alpha_{r1} - 1) - I_{BQ1}$$
$$= I_{CQ1}\left(V_1, \frac{dV_1}{dt}\right) \cdot (\alpha_{r1} - 1) - I_{BQ1}$$

For simplicity, we define two functions: $\eta_2 = I_{BQ1}$; $\eta_1 = I_{CQ1}$

$$I_{BQ1} = \eta_2$$
$$= k_1 \cdot \left\{ \frac{I_0}{I_0 \cdot R_2 + V_t} \right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t}{R_{BQ1}}$$
$$\cdot \ln\left\{ \left[\frac{I_{CQ1}(V_1, \frac{dV_1}{dt}) \cdot (\alpha_{r1} - 1) - I_{BQ1}}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \right] + 1 \right\}$$

$$I_{BQ1} = \eta_2$$
$$= k_1 \cdot \left\{ \frac{I_0}{I_0 \cdot R_2 + V_t} \right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t}{R_{BQ1}}$$
$$\cdot \ln\left\{ \left[\frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \right] + 1 \right\}$$

$$\eta_1 = I_{CQ1} = \frac{V_{DD1}}{R_{cQ1}} - V_1 \cdot \left[\frac{1}{R_{cQ1}} + \frac{I_0}{I_0 \cdot R_1 + V_t} \right] - C_1 \cdot \frac{dV_1}{dt}$$

We get two expressions for $\eta_2 = I_{BQ1}$; $\eta_1 = I_{CQ1}$ which depend on $V_1, V_2, \frac{dV_1}{dt}$ and circuit parameters.

$$I_{CQ1} = I_{CQ1}\left(V_1, \frac{dV_1}{dt}\right) = \eta_1; \quad I_{BQ1} = I_{BQ1}\left(V_1, V_2, \frac{dV_1}{dt}, X_1(t)\right) = \eta_2 \eta_1 = \eta_1\left(V_1, \frac{dV_1}{dt}\right);$$
$$\eta_2 = \eta_2\left(V_1, V_2, \frac{dV_1}{dt}, X_1(t)\right)$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \alpha_{r1} \cdot I_{CQ1} - (I_{BQ1} + I_{CQ1}) = I_{CQ1} \cdot (\alpha_{r1} - 1) - I_{BQ1}$$
$$= \eta_1 \cdot (\alpha_{r1} - 1) - \eta_2$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = I_{CQ1} - (I_{BQ1} + I_{CQ1}) \cdot \alpha_{f1} = I_{CQ1} \cdot (1 - \alpha_{f1}) - I_{BQ1} \cdot \alpha_{f1}$$
$$= \eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1}$$

$$V_{CEQ1} = V_1 \simeq V_t \cdot \ln\left\{\frac{(\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}$$
$$= V_t \cdot \ln\left\{\frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2 + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}$$

$$V_1 \simeq V_t \cdot \ln\left\{\frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2 + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\} \Rightarrow e^{\left[\frac{V_1}{V_t}\right]}$$
$$= \frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2 + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}$$

Taylor series approximation:

$$e^{\left[\frac{V_1}{V_t}\right]} \approx \frac{V_1}{V_t} + 1;$$
$$\frac{V_1}{V_t} + 1 = \frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2 + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}$$

$$\eta_1 = \frac{V_{DD1}}{R_{cQ1}} - V_1 \cdot \left[\frac{1}{R_{cQ1}} + \frac{I_0}{I_0 \cdot R_1 + V_t}\right] - C_1 \cdot \frac{dV_1}{dt}$$

$$\eta_2 = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t}{R_{BQ1}} \cdot \ln\left\{\left[\frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right] + 1\right\}$$

We use Taylor series approximation: $\ln\left\{\left[\frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right] + 1\right\} \approx \frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}$

We define $\phi_1 = \phi_1(\eta_1, \eta_2) = \frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \Rightarrow \ln[\phi_1 + 1] \approx \phi_1$; $\ln[\phi_1 + 1] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \phi_1^n$

For $|\phi_1| \leq 1$ unless $\phi_1 = -1$. The Taylor polynomials for $\ln[\phi_1 + 1]$ only provide accurate approximations in the range $-1 < \phi_1 \leq 1$ and for $\phi_1 > 1$ the Taylor polynomials of higher degree provide worse approximation. The low degree approximations give $\ln[\phi_1 + 1] \approx \phi_1$.

$$\eta_2 = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t}{R_{BQ1}} \cdot \left[\frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2}{I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right]$$

$$\eta_2 = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t \cdot (\alpha_{r1} - 1)}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \cdot \eta_1 + \frac{V_t}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \cdot \eta_2$$

$$\eta_2 \cdot \left\{1 - \frac{V_t}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\} = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t \cdot (\alpha_{r1} - 1)}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \cdot \eta_1$$

$$\eta_2 = \frac{1}{\left\{1 - \frac{V_t}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}} \cdot \left[k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - \frac{V_t \cdot (\alpha_{r1} - 1)}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \cdot \eta_1\right]$$

For simplicity we define the following parameters:

$$A_1 = \frac{1}{\left\{1 - \frac{V_t}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}}; \quad A_2 = k_1 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\};$$

$$A_3 = \frac{V_t \cdot (\alpha_{r1} - 1)}{R_{BQ1} \cdot I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}$$

$$I_{BQ1} = \eta_2 = A_1 \cdot \left[A_2 \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - A_3 \cdot \eta_1\right];$$

$$\eta_1 = \frac{V_{DD1}}{R_{cQ1}} - V_1 \cdot \left[\frac{1}{R_{cQ1}} + \frac{I_0}{I_0 \cdot R_1 + V_t}\right] - C_1 \cdot \frac{dV_1}{dt}$$

$$A_4 = \frac{V_{DD1}}{R_{cQ1}}; \; A_5 = \frac{1}{R_{cQ1}} + \frac{I_0}{I_0 \cdot R_1 + V_t}; \; \eta_1 = A_4 - V_1 \cdot A_5 - C_1 \cdot \frac{dV_1}{dt}$$

$$\eta_2 = A_1 \cdot \left[A_2 \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - A_3 \cdot \eta_1\right] = A_1 \cdot \left[A_2 \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - A_3 \cdot \left(A_4 - V_1 \cdot A_5 - C_1 \cdot \frac{dV_1}{dt}\right)\right]$$

$$\eta_2 = A_1 \cdot \left[A_2 \cdot V_2 + \frac{V_1 \cdot X_1(t)}{R_{BQ1}} - A_3 \cdot \left(A_4 - V_1 \cdot A_5 - C_1 \cdot \frac{dV_1}{dt}\right)\right] = A_1 \cdot \left[A_2 \cdot V_2 + A_3 \cdot C_1 \cdot \frac{dV_1}{dt} + V_1 \cdot \left\{\frac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5\right\} - A_3 \cdot A_4\right]$$

Back to last result equation: $\frac{V_1}{V_t} + 1 = \frac{\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2 + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}$

$$\eta_1 \cdot (\alpha_{r1} - 1) - \eta_2 + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) = \left(\frac{V_1}{V_t} + 1\right) \cdot \left\{ \begin{array}{c} \eta_1 \cdot (1 - \alpha_{f1}) - \eta_2 \cdot \alpha_{f1} \\ + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \end{array} \right\}$$

$$\begin{aligned} &\eta_1 \cdot \left\{ (\alpha_{r1} - 1) - \left(\frac{V_1}{V_t} + 1\right) \cdot (1 - \alpha_{f1}) \right\} + \eta_2 \cdot \left\{ \left(\frac{V_1}{V_t} + 1\right) \cdot \alpha_{f1} - 1 \right\} \\ &\quad + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) - \left(\frac{V_1}{V_t} + 1\right) \cdot I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) = 0 \end{aligned}$$

$$\begin{aligned} &\eta_1 \cdot \left\{ (\alpha_{r1} - 1) - \left(\frac{V_1}{V_t} + 1\right) \cdot (1 - \alpha_{f1}) \right\} + \eta_2 \cdot \left\{ \left(\frac{V_1}{V_t} + 1\right) \cdot \alpha_{f1} - 1 \right\} \\ &\quad + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{se} - \left(\frac{V_1}{V_t} + 1\right) \cdot I_{sc} \right\} = 0 \end{aligned}$$

For simplicity, we define the following functions: $\psi_1 = \psi_1(V_1); \psi_2 = \psi_2(V_1); \psi_3 = \psi_3(V_1)$

$$\begin{aligned} \psi_1 = \psi_1(V_1) = (\alpha_{r1} - 1) - \left(\frac{V_1}{V_t} + 1\right) \cdot (1 - \alpha_{f1}); \psi_2 = \psi_2(V_1) \\ = \left(\frac{V_1}{V_t} + 1\right) \cdot \alpha_{f1} - 1 \end{aligned}$$

$$\psi_3 = \psi_3(V_1) = (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{se} - \left(\frac{V_1}{V_t} + 1\right) \cdot I_{sc} \right\}; \ \eta_1 \cdot \psi_1 + \eta_2 \cdot \psi_2 + \psi_3 = 0$$

$$\psi_1 = -\left\{ (2 - \alpha_{r1} - \alpha_{f1}) + \frac{V_1}{V_t} \cdot (1 - \alpha_{f1}) \right\}; \ \psi_2 = \frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})$$

$$\begin{aligned} \psi_3 = (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{se} - I_{sc} - \frac{V_1}{V_t} \cdot I_{sc} \right\} = (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot (I_{se} - I_{sc}) \\ - \frac{V_1}{V_t} \cdot I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \end{aligned}$$

$$\eta_1 \cdot \psi_1 = \left(A_4 - V_1 \cdot A_5 - C_1 \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t}\right) \cdot \psi_1 = A_4 \cdot \psi_1 - V_1 \cdot A_5 \cdot \psi_1 - C_1 \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t} \cdot \psi_1$$

$$\eta_2 \cdot \psi_2 = A_1 \cdot \left[\begin{array}{r} A_2 \cdot V_2 \cdot \psi_2 + A_3 \cdot C_1 \cdot \psi_2 \cdot \dfrac{\mathrm{d}V_1}{\mathrm{d}t} \\ + V_1 \cdot \psi_2 \cdot \left\{ \dfrac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5 \right\} - \psi_2 \cdot A_3 \cdot A_4 \end{array} \right]$$

$$\eta_1 \cdot \psi_1 + \eta_2 \cdot \psi_2 + \psi_3 = 0 \Rightarrow A_4 \cdot \psi_1 - V_1 \cdot A_5 \cdot \psi_1 - C_1 \cdot \frac{dV_1}{dt} \cdot \psi_1 + A_1 \cdot [A_2 \cdot V_2 \cdot \psi_2 + A_3 \cdot C_1 \cdot \psi_2 \cdot \frac{dV_1}{dt} + V_1 \cdot \psi_2 \cdot \left\{ \frac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5 \right\} - \psi_2 \cdot A_3 \cdot A_4] + \psi_3 = 0$$

$$\eta_1 \cdot \psi_1 + \eta_2 \cdot \psi_2 + \psi_3 = 0 \Rightarrow A_4 \cdot \psi_1 - V_1 \cdot A_5 \cdot \psi_1 - C_1 \cdot \frac{dV_1}{dt} \cdot \psi_1 + A_1 \cdot A_2 \cdot V_2 \cdot \psi_2 + A_1 \cdot A_3 \cdot C_1 \cdot \psi_2 \cdot \frac{dV_1}{dt} + A_1 \cdot V_1 \cdot \psi_2 \cdot \left\{ \frac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5 \right\} - \psi_2 \cdot A_1 \cdot A_3 \cdot A_4 + \psi_3 = 0$$

$$C_1 \cdot \frac{dV_1}{dt} \cdot \{\psi_1 - A_1 \cdot A_3 \cdot \psi_2\} = A_4 \cdot \psi_1 - V_1 \cdot A_5 \cdot \psi_1 + A_1 \cdot A_2 \cdot V_2 \cdot \psi_2 + A_1 \cdot V_1 \cdot \psi_2 \cdot \left\{ \frac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5 \right\} - \psi_2 \cdot A_1 \cdot A_3 \cdot A_4 + \psi_3$$

$$C_1 \cdot \frac{dV_1}{dt} \cdot \{\psi_1 - A_1 \cdot A_3 \cdot \psi_2\} = \psi_1 \cdot (A_4 - V_1 \cdot A_5) + \psi_2 \cdot A_1 \left\{ A_2 \cdot V_2 + V_1 \cdot \left[\frac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5 \right] - A_3 \cdot A_4 \right\} + \psi_3$$

$$C_1 \cdot \frac{dV_1}{dt} \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = \psi_1 \cdot (V_1 \cdot A_5 - A_4) + \psi_2 \cdot A_1 \left\{ A_3 \cdot A_4 - A_2 \cdot V_2 - V_1 \cdot \left[\frac{X_1(t)}{R_{BQ1}} + A_3 \cdot A_5 \right] \right\} - \psi_3$$

$$C_1 \cdot \frac{dV_1}{dt} \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = \psi_1 \cdot (V_1 \cdot A_5 - A_4) + \psi_2 \cdot A_1 \{A_3 \cdot A_4 - A_2 \cdot V_2 - V_1 \cdot A_3 \cdot A_5\} - \psi_3 - \psi_2 \cdot A_1 \cdot V_1 \cdot \frac{X_1(t)}{R_{BQ1}}$$

$$\psi_2 \cdot A_1 \cdot V_1 \cdot \frac{X_1(t)}{R_{BQ1}} = \left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1}) \right] \cdot A_1 \cdot V_1 \cdot \frac{X_1(t)}{R_{BQ1}} = A_1 \cdot V_1^2 \cdot \alpha_{f1} \cdot \frac{X_1(t)}{R_{BQ1} \cdot V_t} - (1 - \alpha_{f1}) \cdot A_1 \cdot V_1 \cdot \frac{X_1(t)}{R_{BQ1}}$$

$$C_1 \cdot \frac{\mathrm{d}V_1}{\mathrm{d}t} \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = \psi_1 \cdot (V_1 \cdot A_5 - A_4) + \psi_2 \cdot A_1\{A_3 \cdot A_4 - A_2 \cdot V_2 - V_1 \cdot A_3 \cdot A_5\} - \psi_3 - \left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot A_1 \cdot V_1 \cdot \frac{X_1(t)}{R_{BQ1}}$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = \frac{\psi_1 \cdot (V_1 \cdot A_5 - A_4) + \psi_2 \cdot A_1\{A_3 \cdot A_4 - A_2 \cdot V_2 - V_1 \cdot A_3 \cdot A_5\} - \psi_3}{C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}} - \frac{\left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot \frac{A_1 \cdot V_1}{R_{BQ1}}}{C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}} \cdot X_1(t)$$

$$g_1 = g_1(V_1, V_2) = \frac{\psi_1 \cdot (V_1 \cdot A_5 - A_4) + \psi_2 \cdot A_1\{A_3 \cdot A_4 - A_2 \cdot V_2 - V_1 \cdot A_3 \cdot A_5\} - \psi_3}{C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}}$$

$$g_1 = g_1(V_1, V_2) = \frac{\psi_1 \cdot (V_1 \cdot A_5 - A_4) + \psi_2 \cdot A_1\{A_3 \cdot A_4 - A_2 \cdot V_2\} - [\psi_2 \cdot A_1 \cdot V_1 \cdot A_3 \cdot A_5 + \psi_3]}{C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}}$$

$$g_2 = g_2(V_1) = -\frac{\left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot \frac{A_1 \cdot V_1}{R_{BQ1}}}{C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}}; \ \frac{\mathrm{d}V_1}{\mathrm{d}t} = g_1(V_1, V_2) + g_2(V_1) \cdot X_1(t)$$

We need to calculate the following expressions (inside g_1 and g_2 functions):

$$\psi_1 \cdot (V_1 \cdot A_5 - A_4); \ \psi_2 \cdot A_1 \cdot \{A_3 \cdot A_4 - A_2 \cdot V_2\}; \ \psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3; \ C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}$$

and $\dfrac{\left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot \frac{A_1 \cdot V_1}{R_{BQ1}}}{C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}}$.

$$\psi_1 \cdot (V_1 \cdot A_5 - A_4) = \psi_1 \cdot V_1 \cdot A_5 - \psi_1 \cdot A_4 = -\left\{(2 - \alpha_{r1} - \alpha_{f1}) + \frac{V_1}{V_t} \cdot (1 - \alpha_{f1})\right\} \cdot V_1 \cdot A_5 + \left\{(2 - \alpha_{r1} - \alpha_{f1}) + \frac{V_1}{V_t} \cdot (1 - \alpha_{f1})\right\} \cdot A_4$$

$$\psi_1 \cdot (V_1 \cdot A_5 - A_4) = -(2 - \alpha_{r1} - \alpha_{f1}) \cdot V_1 \cdot A_5 - \frac{V_1^2}{V_t} \cdot (1 - \alpha_{f1}) \cdot A_5 + (2 - \alpha_{r1} - \alpha_{f1}) \quad \cdot A_4 + \frac{V_1}{V_t} \cdot (1 - \alpha_{f1}) \cdot A_4$$

$$\psi_1 \cdot (V_1 \cdot A_5 - A_4) = -V_1^2 \cdot \frac{(1 - \alpha_{f1}) \cdot A_5}{V_t} + V_1 \cdot \left\{ \frac{A_4}{V_t} \cdot (1 - \alpha_{f1}) - (2 - \alpha_{r1} - \alpha_{f1}) \cdot A_5 \right\} + (2 - \alpha_{r1} - \alpha_{f1}) \cdot A_4$$

We define the following parameters: $\psi_1 \cdot (V_1 \cdot A_5 - A_4) = V_1^2 \cdot \Gamma_{11} + V_1 \cdot \Gamma_{12} + \Gamma_{13}$

$$\Gamma_{11} = -\frac{(1 - \alpha_{f1}) \cdot A_5}{V_t}; \; \Gamma_{12} = \frac{A_4}{V_t} \cdot (1 - \alpha_{f1}) - (2 - \alpha_{r1} - \alpha_{f1}) \cdot A_5;$$
$$\Gamma_{13} = (2 - \alpha_{r1} - \alpha_{f1}) \cdot A_4$$

$$\psi_1 \cdot (V_1 \cdot A_5 - A_4) = V_1^2 \cdot \Gamma_{11} + V_1 \cdot \Gamma_{12} + \Gamma_{13} = \sum_{n=1}^{3} V_1^{3-n} \cdot \Gamma_{1n}$$

$$\psi_2 \cdot A_1 \cdot \{A_3 \cdot A_4 - A_2 \cdot V_2\} = \psi_2 \cdot A_1 \cdot A_3 \cdot A_4 - \psi_2 \cdot V_2 \cdot A_1 \cdot A_2 = \left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot A_1 \cdot A_3 \cdot A_4 - \left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot V_2 \cdot A_1 \cdot A_2$$

$$\psi_2 \cdot A_1 \cdot \{A_3 \cdot A_4 - A_2 \cdot V_2\} = \frac{V_1}{V_t} \cdot \alpha_{f1} \cdot A_1 \cdot A_3 \cdot A_4 - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_4 - \frac{V_1 \cdot V_2}{V_t} \cdot \alpha_{f1} \cdot A_1 \cdot A_2 + (1 - \alpha_{f1}) \cdot A_1 \cdot A_2 \cdot V_2$$

$$\psi_2 \cdot A_1 \cdot \{A_3 \cdot A_4 - A_2 \cdot V_2\} = \frac{\alpha_{f1} \cdot A_1 \cdot A_3 \cdot A_4}{V_t} \cdot V_1 + (1 - \alpha_{f1}) \cdot A_1 \cdot A_2 \cdot V_2 - V_1 \cdot V_2 \cdot \frac{\alpha_{f1} \cdot A_1 \cdot A_2}{V_t} - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_4$$

We define the following parameters:

$$\psi_2 \cdot A_1 \cdot \{A_3 \cdot A_4 - A_2 \cdot V_2\} = \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}$$

$$\Gamma_{21} = \frac{\alpha_{f1} \cdot A_1 \cdot A_3 \cdot A_4}{V_t}; \; \Gamma_{22} = (1 - \alpha_{f1}) \cdot A_1 \cdot A_2; \; \Gamma_{23} = -\frac{\alpha_{f1} \cdot A_1 \cdot A_2}{V_t};$$
$$\Gamma_{24} = -(1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_4$$

$$\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3 = \left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot (I_{se} - I_{sc}) - \frac{V_1}{V_t} \cdot I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)$$

$$\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3 = V_1^2 \cdot \frac{A_1 \cdot A_3 \cdot A_5 \cdot \alpha_{f1}}{V_t} - V_1 \cdot (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_5 - V_1 \cdot \frac{I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{V_t} + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot (I_{se} - I_{sc})$$

$$\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3 = V_1^2 \cdot \frac{A_1 \cdot A_3 \cdot A_5 \cdot \alpha_{f1}}{V_t} - V_1 \cdot (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_5 + V_1 \cdot \frac{I_{sc} \cdot (1 - \alpha_{r1} \cdot \alpha_{f1})}{V_t} + (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se})$$

$$\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3 = V_1^2 \cdot \frac{A_1 \cdot A_3 \cdot A_5 \cdot \alpha_{f1}}{V_t} + V_1 \cdot \left\{\frac{I_{sc} \cdot (1 - \alpha_{r1} \cdot \alpha_{f1})}{V_t} - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_5\right\} + (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se})$$

We define the following parameters: $\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3 = V_1^2 \cdot \Gamma_{31} + V_1 \cdot \Gamma_{32} + \Gamma_{33}$

$$\Gamma_{31} = \frac{A_1 \cdot A_3 \cdot A_5 \cdot \alpha_{f1}}{V_t};\ \Gamma_{32} = \frac{I_{sc} \cdot (1 - \alpha_{r1} \cdot \alpha_{f1})}{V_t} - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_5;$$
$$\Gamma_{33} = (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se})$$

$$\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3 = V_1^2 \cdot \Gamma_{31} + V_1 \cdot \Gamma_{32} + \Gamma_{33} = \sum_{n=1}^{3} V_1^{3-n} \cdot \Gamma_{3n};\ (I_{sc} - I_{se}) \neq \varepsilon \to 0$$

$$C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = C_1 \cdot \left\{ \begin{array}{l} A_1 \cdot A_3 \cdot \left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \\ \quad + (2 - \alpha_{r1} - \alpha_{f1}) + \frac{V_1}{V_t} \cdot (1 - \alpha_{f1}) \end{array} \right\}$$

$$C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = C_1 \cdot \left\{ \begin{array}{l} \frac{V_1}{V_t} \cdot A_1 \cdot A_3 \cdot \alpha_{f1} + \frac{V_1}{V_t} \cdot (1 - \alpha_{f1}) \\ \quad - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3 + (2 - \alpha_{r1} - \alpha_{f1}) \end{array} \right\}$$

$$C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = \frac{V_1}{V_t} \cdot C_1 \cdot \{A_1 \cdot A_3 \cdot \alpha_{f1} + 1 - \alpha_{f1}\} + C_1 \cdot \{2 - \alpha_{r1} - \alpha_{f1} - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3\}$$

$$C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = \frac{V_1}{V_t} \cdot C_1 \cdot \{1 + \alpha_{f1} \cdot (A_1 \cdot A_3 - 1)\} + C_1 \cdot \{2 - \alpha_{r1} - \alpha_{f1} - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3\}$$

$$\Gamma_{41} = \frac{C_1}{V_t} \cdot \{1 + \alpha_{f1} \cdot (A_1 \cdot A_3 - 1)\};$$
$$\Gamma_{42} = C_1 \cdot \{2 - \alpha_{r1} - \alpha_{f1} - (1 - \alpha_{f1}) \cdot A_1 \cdot A_3\}$$

$$C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\} = V_1 \cdot \Gamma_{41} + \Gamma_{42}.$$

$$\left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot \frac{A_1 \cdot V_1}{R_{BQ1}} = \frac{\alpha_{f1} \cdot A_1}{V_t \cdot R_{BQ1}} \cdot V_1^2 - (1 - \alpha_{f1}) \cdot \frac{A_1}{R_{BQ1}} \cdot V_1;$$
$$\Gamma_{51} = \frac{\alpha_{f1} \cdot A_1}{V_t \cdot R_{BQ1}}; \quad \Gamma_{52} = -(1 - \alpha_{f1}) \cdot \frac{A_1}{R_{BQ1}}$$

$$\left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1 - \alpha_{f1})\right] \cdot \frac{A_1 \cdot V_1}{R_{BQ1}} = \Gamma_{51} \cdot V_1^2 + \Gamma_{52} \cdot V_1 = \sum_{n=1}^{2} V_1^{3-n} \cdot \Gamma_{5n}$$

We can summarize our last results in Table 4.2.
We get the following expressions:

$$g_1 = g_1(V_1, V_2) = \frac{\sum_{n=1}^{3} V_1^{3-n} \cdot \Gamma_{1n} + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24} - \sum_{n=1}^{3} V_1^{3-n} \cdot \Gamma_{3n}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}}$$

$$g_1 = g_1(V_1, V_2) = \frac{\sum_{n=1}^{3} V_1^{3-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}}$$
$$g_2 = g_2(V_1) = -\frac{\sum_{n=1}^{2} V_1^{3-n} \cdot \Gamma_{5n}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}}; \quad \frac{dV_1}{dt} = g_1(V_1, V_2) + g_2(V_1) \cdot X_1(t)$$

$$\frac{dV_1}{dt} = \frac{\sum_{n=1}^{3} V_1^{3-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}} - \frac{\sum_{n=1}^{2} V_1^{3-n} \cdot \Gamma_{5n}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}} \cdot X_1(t)$$

We define the following functions: $\xi_1 = \xi_1(V_1, V_2)$; $\xi_2 = \xi_2(V_1)$; $\xi_3 = \xi_3(V_1)$

Table 4.2 Summary of our last results

$\psi_1 \cdot (V_1 \cdot A_5 - A_4)$	$V_1^2 \cdot \Gamma_{11} + V_1 \cdot \Gamma_{12} + \Gamma_{13} = \sum_{n=1}^{3} V_1^{3-n} \cdot \Gamma_{1n}$ $\Gamma_{11} = -\frac{(1-\alpha_{f1}) \cdot A_5}{V_t}$ $\Gamma_{12} = \frac{A_4}{V_t} \cdot (1-\alpha_{f1}) - (2 - \alpha_{r1} - \alpha_{f1}) \cdot A_5$ $\Gamma_{13} = (2 - \alpha_{r1} - \alpha_{f1}) \cdot A_4$
$\psi_2 \cdot A_1 \cdot \{A_3 \cdot A_4 - A_2 \cdot V_2\}$	$\Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}$ $\Gamma_{21} = \frac{\alpha_{f1} \cdot A_1 \cdot A_3 \cdot A_4}{V_t}$ $\Gamma_{22} = (1-\alpha_{f1}) \cdot A_1 \cdot A_2$ $\Gamma_{23} = -\frac{\alpha_{f1} \cdot A_1 \cdot A_2}{V_t}$ $\Gamma_{24} = -(1-\alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_4$
$\psi_2 \cdot V_1 \cdot A_1 \cdot A_3 \cdot A_5 + \psi_3$	$V_1^2 \cdot \Gamma_{31} + V_1 \cdot \Gamma_{32} + \Gamma_{33} = \sum_{n=1}^{3} V_1^{3-n} \cdot \Gamma_{3n}$ $(I_{sc} - I_{se}) \neq \varepsilon \to 0$ $\Gamma_{31} = \frac{A_1 \cdot A_3 \cdot A_5 \cdot \alpha_{f1}}{V_t}$ $\Gamma_{32} = \frac{I_{sc} \cdot (1 - \alpha_{r1} \cdot \alpha_{f1})}{V_t} - (1-\alpha_{f1}) \cdot A_1 \cdot A_3 \cdot A_5$ $\Gamma_{33} = (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se})$
$C_1 \cdot \{A_1 \cdot A_3 \cdot \psi_2 - \psi_1\}$	$V_1 \cdot \Gamma_{41} + \Gamma_{42}$ $\Gamma_{41} = \frac{C_1}{V_t} \cdot \{1 + \alpha_{f1} \cdot (A_1 \cdot A_3 - 1)\}$ $\Gamma_{42} = C_1 \cdot \{2 - \alpha_{r1} - \alpha_{f1} - (1-\alpha_{f1}) \cdot A_1 \cdot A_3\}$
$\left[\frac{V_1}{V_t} \cdot \alpha_{f1} - (1-\alpha_{f1})\right] \cdot \frac{A_1 \cdot V_1}{R_{BQ1}}$	$\Gamma_{51} \cdot V_1^2 + \Gamma_{52} \cdot V_1 = \sum_{n=1}^{2} V_1^{3-n} \cdot \Gamma_{5n}$ $\Gamma_{51} = \frac{\alpha_{f1} \cdot A_1}{V_t \cdot R_{BQ1}}; \Gamma_{52} = -(1-\alpha_{f1}) \cdot \frac{A_1}{R_{BQ1}}$

$$\xi_1 = \sum_{n=1}^{3} V_1^{3-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}; \xi_2 = V_1 \cdot \Gamma_{41} + \Gamma_{42}$$

$$\xi_3 = \sum_{n=1}^{2} V_1^{3-n} \cdot \Gamma_{5n};\ g_1(V_1, V_2) = \frac{\xi_1(V_1, V_2)}{\xi_2(V_1)};\ g_2(V_1) = -\frac{\xi_3(V_1)}{\xi_2(V_1)};\ \frac{dV_1}{dt} = \frac{\xi_1}{\xi_2} - \frac{\xi_3}{\xi_2} \cdot X_1(t)$$

$$\frac{\partial \xi_1}{\partial V_1} = \sum_{n=1}^{3} (3-n) \cdot V_1^{2-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} + \Gamma_{23} \cdot V_2;\ \frac{\partial \xi_1}{\partial V_2} = \Gamma_{22} + \Gamma_{23} \cdot V_1$$

$$\frac{\partial \xi_2}{\partial V_1} = \Gamma_{41};\ \frac{\partial \xi_2}{\partial V_2} = 0;\ \frac{\partial \xi_3}{\partial V_1} = \sum_{n=1}^{2} (3-n) \cdot V_1^{2-n} \cdot \Gamma_{5n}$$
$$= 2 \cdot V_1 \cdot \Gamma_{51} + \Gamma_{52};\ \frac{\partial \xi_3}{\partial V_2} = 0$$

V_2 circuit (D1–Q2):

$$V_2 = I_{D2} \cdot \left(R_2 + \frac{V_t}{I_0}\right) \Rightarrow I_{D2} = \frac{V_2}{R_2 + \frac{V_t}{I_0}} = \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2;\ I_{BQ2}$$
$$= k_2 \cdot I_{D1} + \frac{V_2 \cdot X_2(t) - V_{BEQ2}}{R_{BQ2}}$$

$$V_{DD2} = I_{RcQ2} \cdot R_{cQ2} + V_2;\ V_2 = V_{CEQ2};\ I_{RcQ2} = I_{CQ2} + I_{C2} + I_{D2};\ I_{CQ2} = I_{CQ2}(V_2)$$

$$V_{DD2} = (I_{CQ2} + I_{C2} + I_{D2}) \cdot R_{cQ1} + V_2;\ I_{C2} = C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t};\ V_2 = \frac{1}{C_2} \cdot \int I_{C2} \cdot \mathrm{d}t$$

$$V_{DD2} = \left(I_{CQ2}(V_2) + C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} + \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2\right) \cdot R_{cQ2} + V_2;$$
$$V_{BEQ2} = V_t \cdot \ln\left\{\left[\frac{\alpha_{r2} \cdot I_{CQ2} - I_{EQ2}}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right] + 1\right\}$$

$$I_{EQ2} = I_{BQ2} + I_{CQ2};\ I_{BQ2} = k_2 \cdot I_{D1} + \frac{V_2 \cdot X_2(t) - V_{BEQ2}}{R_{BQ2}}$$
$$= k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t) - V_{BEQ2}}{R_{BQ2}}$$

$$V_{CEQ2} = V_t \cdot \ln\left\{\frac{(\alpha_{r2} \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_{f2}) + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\} + V_t \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right); \frac{I_{sc}}{I_{se}} \to 1$$
$$\Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$V_{CEQ2} = V_2 \simeq V_t \cdot \ln\left\{\frac{(\alpha_{r2} \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_{f2}) + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\};$$
$$I_{BQ2} = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_{BEQ2}}{R_{BQ2}}$$

$$I_{BQ2} = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t}{R_{BQ2}} \cdot \ln\left\{\left[\frac{\alpha_{r2} \cdot I_{CQ2} - I_{EQ2}}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right] + 1\right\}$$

$$V_{CEQ2} = V_2 = V_{BEQ2} + V_{CBQ2}|_{V_{CBQ2} = -V_{BCQ2}} = V_{BEQ2} - V_{BCQ2}; V_{BCQ2} = V_t \cdot \ln\left[\left(\frac{I_{CQ2} - I_{EQ2} \cdot \alpha_{f1}}{I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right) + 1\right]$$

$$I_{BQ2} = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_{BEQ2}}{R_{BQ2}} \Rightarrow V_{BEQ2} = \left[k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - I_{BQ2}\right] \cdot R_{BQ2}$$

$$V_{CEQ2} = V_2 \Rightarrow V_2 = V_{BEQ2} - V_{BCQ2}$$
$$V_2 = \left[k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - I_{BQ2}\right] \cdot R_{BQ2} - V_t \cdot \ln\left[\left(\frac{I_{CQ2} - I_{EQ2} \cdot \alpha_{f2}}{I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right) + 1\right]$$

$$V_{DD2} = \left(I_{CQ2} + C_2 \cdot \frac{dV_2}{dt} + \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2\right) \cdot R_{cQ2} + V_2 \Rightarrow I_{CQ2} = \frac{V_{DD2} - V_2}{R_{cQ2}} - C_2 \cdot \frac{dV_2}{dt} - \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2$$

$$I_{CQ2} = \frac{V_{DD2} - V_2}{R_{cQ2}} - C_2 \cdot \frac{dV_2}{dt} - \left\{\frac{I_0}{I_0 \cdot R_2 + V_t}\right\} \cdot V_2 = \frac{V_{DD2}}{R_{cQ2}} - V_2 \cdot \left[\frac{1}{R_{cQ2}} + \frac{I_0}{I_0 \cdot R_2 + V_t}\right] - C_2 \cdot \frac{dV_2}{dt}$$

We get a result that $I_{EQ2} = I_{BQ2} + I_{CQ2}$ and $I_{CQ2} = I_{CQ2}\left(V_2, \frac{dV_2}{dt}\right)$.

$$\alpha_{r2} \cdot I_{CQ2} - I_{EQ2} = \alpha_{r2} \cdot I_{CQ2} - (I_{BQ2} + I_{CQ2}) = I_{CQ2} \cdot (\alpha_{r2} - 1) - I_{BQ2} = I_{CQ2}\left(V_2, \frac{dV_2}{dt}\right) \cdot (\alpha_{r2} - 1) - I_{BQ2}$$

For simplicity, we define two functions: $\varsigma_2 = I_{BQ2}$; $\varsigma_1 = I_{CQ2}$

$$\begin{aligned} I_{BQ2} &= \varsigma_2 \\ &= k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t}{R_{BQ2}} \\ &\quad \cdot \ln\left\{\left[\frac{I_{CQ2}(V_2, \frac{dV_2}{dt}) \cdot (\alpha_{r2} - 1) - I_{BQ2}}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right] + 1\right\} \end{aligned}$$

$$\begin{aligned} I_{BQ2} = \varsigma_2 = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} \\ - \frac{V_t}{R_{BQ2}} \cdot \ln\left\{\left[\frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right] + 1\right\} \end{aligned}$$

$$\varsigma_1 = I_{CQ2} = \frac{V_{DD2}}{R_{cQ2}} - V_2 \cdot \left[\frac{1}{R_{cQ2}} + \frac{I_0}{I_0 \cdot R_2 + V_t}\right] - C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t}$$

We get two expressions for $\varsigma_2 = I_{BQ2}$; $\varsigma_1 = I_{CQ2}$ which depend on $V_2, V_1, \frac{\mathrm{d}V_2}{\mathrm{d}t}$ and circuit parameters [16, 25].

$$\begin{aligned} I_{CQ2} = I_{CQ2}\left(V_2, \frac{\mathrm{d}V_2}{\mathrm{d}t}\right) = \varsigma_1;\ I_{BQ2} = I_{BQ2}\left(V_1, V_2, \frac{\mathrm{d}V_2}{\mathrm{d}t}, X_2(t)\right) = \varsigma_2 \\ \varsigma_1 = \varsigma_1\left(V_2, \frac{\mathrm{d}V_2}{\mathrm{d}t}\right);\ \varsigma_2 = \varsigma_2\left(V_1, V_2, \frac{\mathrm{d}V_2}{\mathrm{d}t}, X_2(t)\right) \end{aligned}$$

$$\begin{aligned} \alpha_{r2} \cdot I_{CQ2} - I_{EQ2} &= \alpha_{r2} \cdot I_{CQ2} - (I_{BQ2} + I_{CQ2}) = I_{CQ2} \cdot (\alpha_{r2} - 1) - I_{BQ2} \\ &= \varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 \end{aligned}$$

$$\begin{aligned} I_{CQ2} - I_{EQ2} \cdot \alpha_{f2} &= I_{CQ2} - (I_{BQ2} + I_{CQ2}) \cdot \alpha_{f2} = I_{CQ2} \cdot (1 - \alpha_{f2}) - I_{BQ2} \cdot \alpha_{f2} \\ &= \varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} \end{aligned}$$

$$\begin{aligned} V_{CEQ2} = V_2 &\simeq V_t \cdot \ln\left\{\frac{(\alpha_{r2} \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_{f2}) + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\} \\ &= V_t \cdot \ln\left\{\frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{\varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\} \end{aligned}$$

$$\begin{aligned} V_2 &\simeq V_t \cdot \ln\left\{\frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{\varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\} \\ \Rightarrow \mathrm{e}^{\left[\frac{V_2}{V_t}\right]} &= \frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{\varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)} \end{aligned}$$

Taylor series approximation: $\mathrm{e}^{\left[\frac{V_2}{V_t}\right]} \approx \frac{V_2}{V_t} + 1$; $\frac{V_2}{V_t} + 1 = \frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{\varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}$

$$\varsigma_1 = \frac{V_{DD2}}{R_{cQ2}} - V_2 \cdot \left[\frac{1}{R_{cQ2}} + \frac{I_0}{I_0 \cdot R_2 + V_t}\right] - C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t}$$

$$\varsigma_2 = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t}{R_{BQ2}} \cdot \ln\left\{\left[\frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right] + 1\right\}$$

We use Taylor series approximation: $\ln\left\{\left[\frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right] + 1\right\} \approx \frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}$

We define $\phi_2 = \phi_2(\varsigma_1, \varsigma_2) = \frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)} \Rightarrow \ln[\phi_2 + 1] \approx \phi_2$; $\ln[\phi_2 + 1] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \phi_2^n$

For $|\phi_2| \leq 1$ unless $\phi_2 = -1$. The Taylor polynomials for $\ln[\phi_2 + 1]$ only provide accurate approximations in the range $-1 < \phi_2 \leq 1$ and for $\phi_2 > 1$ the Taylor polynomials of higher degree provide worse approximation. The low degree approximations give $\ln[\phi_2 + 1] \approx \phi_2$.

$$\varsigma_2 = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t}{R_{BQ2}} \cdot \left[\frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2}{I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right]$$

$$\varsigma_2 = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t \cdot (\alpha_{r2} - 1)}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)} \cdot \varsigma_1 + \frac{V_t}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)} \cdot \varsigma_2$$

$$\varsigma_2 \cdot \left\{1 - \frac{V_t}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\} = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t \cdot (\alpha_{r2} - 1)}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)} \cdot \varsigma_1$$

$$\varsigma_2 = \frac{1}{\left\{1 - \frac{V_t}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\}} \cdot \left[k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\} \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - \frac{V_t \cdot (\alpha_{r2} - 1)}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)} \cdot \varsigma_1\right]$$

For simplicity, we define the following parameters:

$$B_1 = \frac{1}{\left\{1 - \frac{V_t}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}\right\}};\ B_2 = k_2 \cdot \left\{\frac{I_0}{I_0 \cdot R_1 + V_t}\right\};\ B_3 = \frac{V_t \cdot (\alpha_{r2} - 1)}{R_{BQ2} \cdot I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}$$

$$I_{BQ2} = \varsigma_2 = B_1 \cdot \left[B_2 \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - B_3 \cdot \varsigma_1\right]; \ \varsigma_1 = \frac{V_{DD2}}{R_{cQ2}} - V_2 \cdot \left[\frac{1}{R_{cQ2}} + \frac{I_0}{I_0 \cdot R_2 + V_t}\right] - C_2 \cdot \frac{dV_2}{dt}$$

$$B_4 = \frac{V_{DD2}}{R_{cQ2}}; \ B_5 = \frac{1}{R_{cQ2}} + \frac{I_0}{I_0 \cdot R_2 + V_t}; \ \varsigma_1 = B_4 - V_2 \cdot B_5 - C_2 \cdot \frac{dV_2}{dt}$$

$$\varsigma_2 = B_1 \cdot \left[B_2 \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - B_3 \cdot \varsigma_1\right] = B_1 \cdot \left[B_2 \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - B_3 \cdot \left(B_4 - V_2 \cdot B_5 - C_2 \cdot \frac{dV_2}{dt}\right)\right]$$

$$\varsigma_2 = B_1 \cdot \left[B_2 \cdot V_1 + \frac{V_2 \cdot X_2(t)}{R_{BQ2}} - B_3 \cdot \left(B_4 - V_2 \cdot B_5 - C_2 \cdot \frac{dV_2}{dt}\right)\right] = B_1 \cdot \left[B_2 \cdot V_1 + B_3 \cdot C_2 \cdot \frac{dV_2}{dt} + V_2 \cdot \left\{\frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5\right\} - B_3 \cdot B_4\right]$$

Back to last result equation: $\frac{V_2}{V_t} + 1 = \frac{\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{\varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}$

$$\varsigma_1 \cdot (\alpha_{r2} - 1) - \varsigma_2 + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1) = \left(\frac{V_2}{V_t} + 1\right) \cdot \{\varsigma_1 \cdot (1 - \alpha_{f2}) - \varsigma_2 \cdot \alpha_{f2} + I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)\}$$

$$\varsigma_1 \cdot \left\{(\alpha_{r2} - 1) - \left(\frac{V_2}{V_t} + 1\right) \cdot (1 - \alpha_{f2})\right\} + \varsigma_2 \cdot \left\{\left(\frac{V_2}{V_t} + 1\right) \cdot \alpha_{f2} - 1\right\} + I_{se} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1) - \left(\frac{V_2}{V_t} + 1\right) \cdot I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1) = 0$$

$$\varsigma_1 \cdot \left\{(\alpha_{r2} - 1) - \left(\frac{V_2}{V_t} + 1\right) \cdot (1 - \alpha_{f2})\right\} + \varsigma_2 \cdot \left\{\left(\frac{V_2}{V_t} + 1\right) \cdot \alpha_{f2} - 1\right\} + (\alpha_{r2} \cdot \alpha_{f2} - 1) \cdot \left\{I_{se} - \left(\frac{V_2}{V_t} + 1\right) \cdot I_{sc}\right\} = 0$$

For simplicity, we define the following functions: $\vartheta_1 = \vartheta_1(V_2)$; $\vartheta_2 = \vartheta_2(V_2)$; $\vartheta_3 = \vartheta_3(V_2)$

$$\vartheta_1 = \vartheta_1(V_2) = (\alpha_{r2} - 1) - \left(\frac{V_2}{V_t} + 1\right) \cdot (1 - \alpha_{f2}); \quad \vartheta_2 = \vartheta_2(V_2) = \left(\frac{V_2}{V_t} + 1\right) \cdot \alpha_{f2} - 1$$

$$\vartheta_3 = \vartheta_3(V_2) = (\alpha_{r2} \cdot \alpha_{f2} - 1) \cdot \left\{ I_{se} - \left(\frac{V_2}{V_t} + 1 \right) \cdot I_{sc} \right\}; \; \varsigma_1 \cdot \vartheta_1 + \varsigma_2 \cdot \vartheta_2 + \vartheta_3 = 0$$

$$\vartheta_1 = -\left\{ (2 - \alpha_{r2} - \alpha_{f2}) + \frac{V_2}{V_t} \cdot (1 - \alpha_{f2}) \right\}; \; \vartheta_2 = \frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2})$$

$$\begin{aligned} \vartheta_3 &= (\alpha_{r2} \cdot \alpha_{f2} - 1) \cdot \left\{ I_{se} - I_{sc} - \frac{V_2}{V_t} \cdot I_{sc} \right\} \\ &= (\alpha_{r2} \cdot \alpha_{f2} - 1) \cdot (I_{se} - I_{sc}) - \frac{V_2}{V_t} \cdot I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1) \end{aligned}$$

$$\varsigma_1 \cdot \vartheta_1 = \left(B_4 - V_2 \cdot B_5 - C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \right) \cdot \vartheta_1 = B_4 \cdot \vartheta_1 - V_2 \cdot B_5 \cdot \vartheta_1 - C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \vartheta_1$$

$$\varsigma_2 \cdot \vartheta_2 = B_1 \cdot \left[\begin{array}{r} B_2 \cdot V_1 \cdot \vartheta_2 + B_3 \cdot C_2 \cdot \vartheta_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \\ + V_2 \cdot \vartheta_2 \cdot \left\{ \frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5 \right\} - \vartheta_2 \cdot B_3 \cdot B_4 \end{array} \right]$$

$$\begin{aligned} \varsigma_1 \cdot \vartheta_1 + \varsigma_2 \cdot \vartheta_2 + \vartheta_3 = 0 \Rightarrow & B_4 \cdot \vartheta_1 - V_2 \cdot B_5 \cdot \vartheta_1 - C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \vartheta_1 \\ & + B_1 \cdot \left[B_2 \cdot V_1 \cdot \vartheta_2 + B_3 \cdot C_2 \cdot \vartheta_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \right. \\ & \left. + V_2 \cdot \vartheta_2 \cdot \left\{ \frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5 \right\} - \vartheta_2 \cdot B_3 \cdot B_4 \right] + \vartheta_3 = 0 \end{aligned}$$

$$\begin{aligned} \varsigma_1 \cdot \vartheta_1 + \varsigma_2 \cdot \vartheta_2 + \vartheta_3 = 0 \Rightarrow & B_4 \cdot \vartheta_1 - V_2 \cdot B_5 \cdot \vartheta_1 - C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \vartheta_1 \\ & + B_1 \cdot B_2 \cdot V_1 \cdot \vartheta_2 + B_1 \cdot B_3 \cdot C_2 \cdot \vartheta_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \\ & + B_1 \cdot V_2 \cdot \vartheta_2 \cdot \left\{ \frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5 \right\} - \vartheta_2 \cdot B_1 \cdot B_3 \cdot B_4 + \vartheta_3 = 0 \end{aligned}$$

$$\begin{aligned} C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \{\vartheta_1 - B_1 \cdot B_3 \cdot \vartheta_2\} = & B_4 \cdot \vartheta_1 - V_2 \cdot B_5 \cdot \vartheta_1 + B_1 \cdot B_2 \cdot V_1 \cdot \vartheta_2 \\ & + B_1 \cdot V_2 \cdot \vartheta_2 \cdot \left\{ \frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5 \right\} \\ & - \vartheta_2 \cdot B_1 \cdot B_3 \cdot B_4 + \vartheta_3 \end{aligned}$$

$$\begin{aligned} C_2 \cdot \frac{\mathrm{d}V_2}{\mathrm{d}t} \cdot \{\vartheta_1 - B_1 \cdot B_3 \cdot \vartheta_2\} = & \vartheta_1 \cdot (B_4 - V_2 \cdot B_5) + \vartheta_2 \\ & \cdot B_1 \left\{ B_2 \cdot V_1 + V_2 \cdot \left[\frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5 \right] - B_3 \cdot B_4 \right\} + \vartheta_3 \end{aligned}$$

$$C_2 \cdot \frac{dV_2}{dt} \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = \vartheta_1 \cdot (V_2 \cdot B_5 - B_4) + \vartheta_2 \cdot B_1 \left\{ B_3 \cdot B_4 - B_2 \cdot V_1 - V_2 \cdot \left[\frac{X_2(t)}{R_{BQ2}} + B_3 \cdot B_5 \right] \right\} - \vartheta_3$$

$$C_2 \cdot \frac{dV_2}{dt} \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = \vartheta_1 \cdot (V_2 \cdot B_5 - B_4) + \vartheta_2 \cdot B_1\{B_3 \cdot B_4 - B_2 \cdot V_1 - V_2 \cdot B_3 \cdot B_5\} - \vartheta_3 - \vartheta_2 \cdot B_1 \cdot V_2 \cdot \frac{X_2(t)}{R_{BQ2}}$$

$$\vartheta_2 \cdot B_1 \cdot V_2 \cdot \frac{X_2(t)}{R_{BQ2}} = \left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2}) \right] \cdot B_1 \cdot V_2 \cdot \frac{X_2(t)}{R_{BQ2}} = B_1 \cdot V_2^2 \cdot \alpha_{f2} \cdot \frac{X_2(t)}{R_{BQ2} \cdot V_t} - (1 - \alpha_{f2}) \cdot B_1 \cdot V_2 \cdot \frac{X_2(t)}{R_{BQ2}}$$

$$C_2 \cdot \frac{dV_2}{dt} \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = \vartheta_1 \cdot (V_2 \cdot B_5 - B_4) + \vartheta_2 \cdot B_1\{B_3 \cdot B_4 - B_2 \cdot V_1 - V_2 \cdot B_3 \cdot B_5\} - \vartheta_3 - \left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2}) \right] \cdot B_1 \cdot V_2 \cdot \frac{X_2(t)}{R_{BQ2}}$$

$$\frac{dV_2}{dt} = \frac{\vartheta_1 \cdot (V_2 \cdot B_5 - B_4) + \vartheta_2 \cdot B_1\{B_3 \cdot B_4 - B_2 \cdot V_1 - V_2 \cdot B_3 \cdot B_5\} - \vartheta_3}{C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}} - \frac{\left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2}) \right] \cdot \frac{B_1 \cdot V_2}{R_{BQ2}}}{C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}} \cdot X_2(t)$$

$$h_1 = h_1(V_1, V_2) = \frac{\vartheta_1 \cdot (V_2 \cdot B_5 - B_4) + \vartheta_2 \cdot B_1\{B_3 \cdot B_4 - B_2 \cdot V_1 - V_2 \cdot B_3 \cdot B_5\} - \vartheta_3}{C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}}$$

$$h_1 = h_1(V_1, V_2) = \frac{\vartheta_1 \cdot (V_2 \cdot B_5 - B_4) + \vartheta_2 \cdot B_1\{B_3 \cdot B_4 - B_2 \cdot V_1\} - [\vartheta_2 \cdot B_1 \cdot V_2 \cdot B_3 \cdot B_5 + \vartheta_3]}{C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}}$$

$$h_2 = h_2(V_2) = -\frac{\left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2}) \right] \cdot \frac{B_1 \cdot V_2}{R_{BQ2}}}{C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}}; \ \frac{dV_2}{dt} = h_1(V_1, V_2) + h_2(V_2) \cdot X_2(t)$$

We need to calculate the following expressions (inside h_1 and h_2 functions):

$$\vartheta_1 \cdot (V_2 \cdot B_5 - B_4); \ \vartheta_2 \cdot B_1 \cdot \{B_3 \cdot B_4 - B_2 \cdot V_1\}; \ \vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3; \ C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}$$

and $\frac{\left[\frac{V_2}{V_t}\cdot\alpha_{f2}-(1-\alpha_{f2})\right]\cdot\frac{B_1\cdot V_2}{R_{BQ2}}}{C_2\cdot\{B_1\cdot B_3\cdot\vartheta_2-\vartheta_1\}}$.

$$\vartheta_1\cdot(V_2\cdot B_5-B_4)=\vartheta_1\cdot V_2\cdot B_5-\vartheta_1\cdot B_4=-\left\{(2-\alpha_{r2}-\alpha_{f2})+\frac{V_2}{V_t}\cdot(1-\alpha_{f2})\right\}\cdot V_2\cdot B_5+\left\{(2-\alpha_{r2}-\alpha_{f2})+\frac{V_2}{V_t}\cdot(1-\alpha_{f2})\right\}\cdot B_4$$

$$\vartheta_1\cdot(V_2\cdot B_5-B_4)=-(2-\alpha_{r2}-\alpha_{f2})\cdot V_2\cdot B_5-\frac{V_2^2}{V_t}\cdot(1-\alpha_{f2})\cdot B_5+(2-\alpha_{r2}-\alpha_{f2})\cdot B_4+\frac{V_2}{V_t}\cdot(1-\alpha_{f2})\cdot B_4$$

$$\vartheta_1\cdot(V_2\cdot B_5-B_4)=-V_2^2\cdot\frac{(1-\alpha_{f2})\cdot B_5}{V_t}+V_2\cdot\left\{\frac{B_4}{V_t}\cdot(1-\alpha_{f2})-(2-\alpha_{r2}-\alpha_{f2})\cdot B_5\right\}+(2-\alpha_{r2}-\alpha_{f2})\cdot B_4$$

We define the following parameters: $\vartheta_1\cdot(V_2\cdot B_5-B_4)=V_2^2\cdot\Xi_{11}+V_2\cdot\Xi_{12}+\Xi_{13}$

$$\Xi_{11}=-\frac{(1-\alpha_{f2})\cdot B_5}{V_t};\ \Xi_{12}=\frac{B_4}{V_t}\cdot(1-\alpha_{f2})-(2-\alpha_{r2}-\alpha_{f2})\cdot B_5;$$
$$\Xi_{13}=(2-\alpha_{r2}-\alpha_{f2})\cdot B_4$$

$$\vartheta_1\cdot(V_2\cdot B_5-B_4)=V_2^2\cdot\Xi_{11}+V_2\cdot\Xi_{12}+\Xi_{13}=\sum_{n=1}^{3}V_2^{3-n}\cdot\Xi_{1n}$$

$$\vartheta_2\cdot B_1\cdot\{B_3\cdot B_4-B_2\cdot V_1\}=\vartheta_2\cdot B_1\cdot B_3\cdot B_4-\vartheta_2\cdot V_1\cdot B_1\cdot B_2=\left[\frac{V_2}{V_t}\cdot\alpha_{f2}-(1-\alpha_{f2})\right]\cdot B_1\cdot B_3\cdot B_4-\left[\frac{V_2}{V_t}\cdot\alpha_{f2}-(1-\alpha_{f2})\right]\cdot V_1\cdot B_1\cdot B_2$$

$$\vartheta_2\cdot B_1\cdot\{B_3\cdot B_4-B_2\cdot V_1\}=\frac{V_2}{V_t}\cdot\alpha_{f2}\cdot B_1\cdot B_3\cdot B_4-(1-\alpha_{f2})\cdot B_1\cdot B_3\cdot B_4-\frac{V_1\cdot V_2}{V_t}\cdot\alpha_{f2}\cdot B_1\cdot B_2+(1-\alpha_{f2})\cdot B_1\cdot B_2\cdot V_1$$

$$\vartheta_2\cdot B_1\cdot\{B_3\cdot B_4-B_2\cdot V_1\}=\frac{\alpha_{f2}\cdot B_1\cdot B_3\cdot B_4}{V_t}\cdot V_2+(1-\alpha_{f2})\cdot B_1\cdot B_2\cdot V_1-V_1\cdot V_2\cdot\frac{\alpha_{f2}\cdot B_1\cdot B_2}{V_t}-(1-\alpha_{f2})\cdot B_1\cdot B_3\cdot B_4$$

We define the following parameters:

$$\vartheta_2 \cdot B_1 \cdot \{B_3 \cdot B_4 - B_2 \cdot V_1\} = \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}$$

$$\Xi_{21} = \frac{\alpha_{f2} \cdot B_1 \cdot B_3 \cdot B_4}{V_t}; \; \Xi_{22} = (1 - \alpha_{f2}) \cdot B_1 \cdot B_2; \; \Xi_{23} = -\frac{\alpha_{f2} \cdot B_1 \cdot B_2}{V_t};$$
$$\Xi_{24} = -(1 - \alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_4$$

$$\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3 = \left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2})\right] \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + (\alpha_{r2} \cdot \alpha_{f2} - 1) \cdot (I_{se} - I_{sc}) - \frac{V_2}{V_t} \cdot I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)$$

$$\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3 = V_2^2 \cdot \frac{B_1 \cdot B_3 \cdot B_5 \cdot \alpha_{f2}}{V_t} - V_2 \cdot (1 - \alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_5 - V_2 \cdot \frac{I_{sc} \cdot (\alpha_{r2} \cdot \alpha_{f2} - 1)}{V_t} + (\alpha_{r2} \cdot \alpha_{f2} - 1) \cdot (I_{se} - I_{sc})$$

$$\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3 = V_2^2 \cdot \frac{B_1 \cdot B_3 \cdot B_5 \cdot \alpha_{f2}}{V_t} - V_2 \cdot (1 - \alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_5 + V_2 \cdot \frac{I_{sc} \cdot (1 - \alpha_{r2} \cdot \alpha_{f2})}{V_t} + (1 - \alpha_{r2} \cdot \alpha_{f2}) \cdot (I_{sc} - I_{se})$$

$$\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3 = V_2^2 \cdot \frac{B_1 \cdot B_3 \cdot B_5 \cdot \alpha_{f2}}{V_t} + V_2 \cdot \left\{ \begin{array}{l} \frac{I_{sc} \cdot (1 - \alpha_{r2} \cdot \alpha_{f2})}{V_t} \\ - (1 - \alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_5 \end{array} \right\} + (1 - \alpha_{r2} \cdot \alpha_{f2}) \cdot (I_{sc} - I_{se})$$

We define the following parameters: $\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3 = V_2^2 \cdot \Xi_{31} + V_2 \cdot \Xi_{32} + \Xi_{33}$

$$\Xi_{31} = \frac{B_1 \cdot B_3 \cdot B_5 \cdot \alpha_{f2}}{V_t}; \; \Xi_{32} = \frac{I_{sc} \cdot (1 - \alpha_{r2} \cdot \alpha_{f2})}{V_t} - (1 - \alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_5;$$
$$\Xi_{33} = (1 - \alpha_{r2} \cdot \alpha_{f2}) \cdot (I_{sc} - I_{se})$$

$$\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3 = V_2^2 \cdot \Xi_{31} + V_2 \cdot \Xi_{32} + \Xi_{33} = \sum_{n=1}^{3} V_2^{3-n} \cdot \Xi_{3n};$$
$$(I_{sc} - I_{se}) \quad \varepsilon \to 0$$

$$C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = C_2 \cdot \left\{ \begin{array}{l} B_1 \cdot B_3 \cdot \left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2})\right] \\ \quad + (2 - \alpha_{r2} - \alpha_{f2}) + \frac{V_2}{V_t} \cdot (1 - \alpha_{f2}) \end{array} \right\}$$

$$C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = C_2 \cdot \left\{ \begin{array}{l} \frac{V_2}{V_t} \cdot B_1 \cdot B_3 \cdot \alpha_{f2} + \frac{V_2}{V_t} \cdot (1 - \alpha_{f2}) \\ \quad - (1 - \alpha_{f2}) \cdot B_1 \cdot B_3 + (2 - \alpha_{r2} - \alpha_{f2}) \end{array} \right\}$$

$$\begin{aligned} C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = {} & \frac{V_2}{V_t} \cdot C_2 \cdot \{B_1 \cdot B_3 \cdot \alpha_{f2} + 1 - \alpha_{f2}\} \\ & + C_2 \cdot \{2 - \alpha_{r2} - \alpha_{f2} - (1 - \alpha_{f2}) \cdot B_1 \cdot B_3\} \end{aligned}$$

$$\begin{aligned} C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = {} & \frac{V_2}{V_t} \cdot C_2 \cdot \{1 + \alpha_{f2} \cdot (B_1 \cdot B_3 - 1)\} \\ & + C_2 \cdot \{2 - \alpha_{r2} - \alpha_{f2} - (1 - \alpha_{f2}) \cdot B_1 \cdot B_3\} \end{aligned}$$

$$\Xi_{41} = \frac{C_2}{V_t} \cdot \{1 + \alpha_{f2} \cdot (B_1 \cdot B_3 - 1)\};$$
$$\Xi_{42} = C_2 \cdot \{2 - \alpha_{r2} - \alpha_{f2} - (1 - \alpha_{f2}) \cdot B_1 \cdot B_3\}$$

$$C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\} = V_2 \cdot \Xi_{41} + \Xi_{42}.$$

$$\left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2})\right] \cdot \frac{B_1 \cdot V_2}{R_{BQ2}} = \frac{\alpha_{f2} \cdot B_1}{V_t \cdot R_{BQ2}} \cdot V_2^2 - (1 - \alpha_{f2}) \cdot \frac{B_1}{R_{BQ2}} \cdot V_2;$$
$$\Xi_{51} = \frac{\alpha_{f2} \cdot B_1}{V_t \cdot R_{BQ2}}; \quad \Xi_{52} = -(1 - \alpha_{f2}) \cdot \frac{B_1}{R_{BQ2}}$$

$$\left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1 - \alpha_{f2})\right] \cdot \frac{B_1 \cdot V_2}{R_{BQ2}} = \Xi_{51} \cdot V_2^2 + \Xi_{52} \cdot V_2 = \sum_{n=1}^{2} V_2^{3-n} \cdot \Xi_{5n}$$

We can summarize our last results in Table 4.3.
We get the following expressions:

$$\begin{aligned} h_1 &= h_1(V_1, V_2) \\ &= \frac{\sum_{n=1}^{3} V_2^{3-n} \cdot \Xi_{1n} + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24} - \sum_{n=1}^{3} V_2^{3-n} \cdot \Xi_{3n}}{V_2 \cdot \Xi_{41} + \Xi_{42}} \end{aligned}$$

Table 4.3 Summary of our last results

$\vartheta_1 \cdot (V_2 \cdot B_5 - B_4)$	$V_2^2 \cdot \Xi_{11} + V_2 \cdot \Xi_{12} + \Xi_{13} = \sum_{n=1}^{3} V_2^{3-n} \cdot \Xi_{1n}$ $\Xi_{11} = -\frac{(1-\alpha_{f2}) \cdot B_5}{V_t}$ $\Xi_{12} = \frac{B_4}{V_t} \cdot (1-\alpha_{f2}) - (2 - \alpha_{r2} - \alpha_{f2}) \cdot B_5$ $\Xi_{13} = (2 - \alpha_{r2} - \alpha_{f2}) \cdot B_4$
$\vartheta_2 \cdot B_1 \cdot \{B_3 \cdot B_4 - B_2 \cdot V_1\}$	$\Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}$ $\Xi_{21} = \frac{\alpha_{f2} \cdot B_1 \cdot B_3 \cdot B_4}{V_t}$ $\Xi_{22} = (1-\alpha_{f2}) \cdot B_1 \cdot B_2$ $\Xi_{23} = -\frac{\alpha_{f2} \cdot B_1 \cdot B_2}{V_t}$ $\Xi_{24} = -(1-\alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_4$
$\vartheta_2 \cdot V_2 \cdot B_1 \cdot B_3 \cdot B_5 + \vartheta_3$	$V_2^2 \cdot \Xi_{31} + V_2 \cdot \Xi_{32} + \Xi_{33} = \sum_{n=1}^{3} V_2^{3-n} \cdot \Xi_{3n}$ $(I_{sc} - I_{se}) \neq \varepsilon \to 0$ $\Xi_{31} = \frac{B_1 \cdot B_3 \cdot B_5 \cdot \alpha_{f2}}{V_t}$ $\Xi_{32} = \frac{I_{sc} \cdot (1 - \alpha_{r2} \cdot \alpha_{f2})}{V_t} - (1-\alpha_{f2}) \cdot B_1 \cdot B_3 \cdot B_5$ $\Xi_{33} = (1 - \alpha_{r2} \cdot \alpha_{f2}) \cdot (I_{sc} - I_{se})$
$C_2 \cdot \{B_1 \cdot B_3 \cdot \vartheta_2 - \vartheta_1\}$	$V_2 \cdot \Xi_{41} + \Xi_{42}$ $\Xi_{41} = \frac{C_2}{V_t} \cdot \{1 + \alpha_{f2} \cdot (B_1 \cdot B_3 - 1)\}$ $\Xi_{42} = C_2 \cdot \{2 - \alpha_{r2} - \alpha_{f2} - (1-\alpha_{f2}) \cdot B_1 \cdot B_3\}$
$\left[\frac{V_2}{V_t} \cdot \alpha_{f2} - (1-\alpha_{f2})\right] \cdot \frac{B_1 \cdot V_2}{R_{BQ2}}$	$\Xi_{51} \cdot V_2^2 + \Xi_{52} \cdot V_2 = \sum_{n=1}^{2} V_2^{3-n} \cdot \Xi_{5n}$ $\Xi_{51} = \frac{\alpha_{f2} \cdot B_1}{V_t \cdot R_{BQ2}}; \Xi_{52} = -(1-\alpha_{f2}) \cdot \frac{B_2}{R_{BQ2}}$

$$h_1 = h_1(V_1, V_2) = \frac{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}}{V_2 \cdot \Xi_{41} + \Xi_{42}}$$

$$h_2 = h_2(V_2) = -\frac{\sum_{n=1}^{2} V_2^{3-n} \cdot \Xi_{5n}}{V_2 \cdot \Xi_{41} + \Xi_{42}}; \frac{dV_2}{dt} = h_1(V_1, V_2) + h_2(V_2) \cdot X_2(t)$$

$$\frac{dV_2}{dt} = \frac{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}}{V_2 \cdot \Xi_{41} + \Xi_{42}} - \frac{\sum_{n=1}^{2} V_2^{3-n} \cdot \Xi_{5n}}{V_2 \cdot \Xi_{41} + \Xi_{42}} \cdot X_2(t)$$

We define the following functions: $\xi_4 = \xi_4(V_1, V_2)$; $\xi_5 = \xi_5(V_2)$; $\xi_6 = \xi_6(V_2)$

$$\xi_4 = \sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24};\ \xi_5 = V_2 \cdot \Xi_{41} + \Xi_{42}$$

$$\xi_6 = \sum_{n=1}^{2} V_2^{3-n} \cdot \Xi_{5n};\ \mathrm{h}_1(V_1, V_2) = \frac{\xi_4(V_1, V_2)}{\xi_5(V_2)};\ \mathrm{h}_2(V_2) = -\frac{\xi_6(V_2)}{\xi_5(V_2)};\ \frac{dV_2}{dt} = \frac{\xi_4}{\xi_5} - \frac{\xi_6}{\xi_5} \cdot X_2(t)$$

$$\frac{\partial \xi_4}{\partial V_2} = \sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1;\ \frac{\partial \xi_4}{\partial V_1} = \Xi_{22} + \Xi_{23} \cdot V_2$$

$$\frac{\partial \xi_5}{\partial V_2} = \Xi_{41};\ \frac{\partial \xi_5}{\partial V_1} = 0;\ \frac{\partial \xi_6}{\partial V_2} = \sum_{n=1}^{2} (3-n) \cdot V_2^{2-n} \cdot \Xi_{5n} = 2 \cdot V_2 \cdot \Xi_{51} + \Xi_{52};\ \frac{\partial \xi_6}{\partial V_1} = 0$$

We can summarize our system discussion and present system's differential equations with periodic sources $X_1(t)$ and $X_2(t)$. The Table 4.4 presents our results.

4.3 Optoisolation Circuit's Two Variables with Periodic Sources Limit Cycle Stability

We discuss in this subchapter, optoisolation circuit's two variables with periodic sources, limit cycle stability. First, we need to prove that the system has periodic orbits and it is done by changing system cartesian coordinates ($V_1(t)$, $V_2(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. We approve that there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers [79, 80]. One of them is unity. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z-axis. In our system, we refer to Cartesian V_1–V_2 plane (with equation $V_3 = 0$). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (V_1, V_2) are the same as for polar coordinates, namely $V_1(t) = r(t) \cdot \cos[\theta(t)]$; $V_2(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{V_1^2 + V_2^2}$. $\theta(t) = 0$ if $V_1 = 0$ and $V_2 = 0$. $\theta(t) = \arcsin(V_2/r)$ if $V_1 \geq 0$.

$\theta(t) = -\arcsin(V_2/r) + \pi$ if $V_1 < 0$. We represent our system equation as

$\frac{dV_1}{dt} = g_1(V_1, V_2) + g_2(V_1) \cdot X_1(t)$; $\frac{dV_2}{dt} = h_1(V_1, V_2) + h_2(V_2) \cdot X_2(t)$ by using cylindrical coordinates ($r(t)$, $\theta(t)$).

Table 4.4 Differential equations with periodic sources $X_1(t)$ and $X_2(t)$

$\frac{dV_1}{dt} = g_1(V_1, V_2) + g_2(V_1) \cdot X_1(t)$	$\frac{dV_2}{dt} = h_1(V_1, V_2) + h_2(V_2) \cdot X_2(t)$
$g_1 = g_1(V_1, V_2) = \frac{\sum_{n=1}^{3} V_1^{3-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}}$	$h_1 = h_1(V_1, V_2) = \frac{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}}{V_2 \cdot \Xi_{41} + \Xi_{42}}$
$g_2 = g_2(V_1) = -\frac{\sum_{n=1}^{2} V_1^{3-n} \cdot \Gamma_{5n}}{V_1 \cdot \Gamma_{41} + \Gamma_{42}}$	$h_2 = h_2(V_2) = -\frac{\sum_{n=1}^{2} V_2^{3-n} \cdot \Xi_{5n}}{V_2 \cdot \Xi_{41} + \Xi_{42}}$

$$V_1(t) = r(t)\cdot\cos[\theta(t)] \Rightarrow \frac{\mathrm{d}V_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t}\cdot\cos[\theta(t)] - r(t)\cdot\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\cdot\sin[\theta(t)]$$
$$V_2(t) = r(t)\cdot\sin[\theta(t)] \Rightarrow \frac{\mathrm{d}V_2(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t}\cdot\sin[\theta(t)] + r(t)\cdot\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\cdot\cos[\theta(t)]$$
$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}V_1}{\mathrm{d}t};\ \frac{\mathrm{d}r(t)}{\mathrm{d}t} = r';\ \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \theta';\ \theta(t) = \theta;\ r(t) = r$$

We get the equations: $\frac{\mathrm{d}V_1}{\mathrm{d}t} = r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta$; $\frac{\mathrm{d}V_2}{\mathrm{d}t} = r'\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta$.

We do our analysis first for system without periodic sources $X_1(t) = X_2(t) = 0$.

Our system equations become $\frac{\mathrm{d}V_1}{\mathrm{d}t} = g_1(V_1, V_2)$; $\frac{\mathrm{d}V_2}{\mathrm{d}t} = h_1(V_1, V_2)$ and we need to convert the representation to $\frac{\mathrm{d}r}{\mathrm{d}t} = l(r,\theta,\theta')$; $\frac{\mathrm{d}\theta}{\mathrm{d}t} = N(\alpha f_1, \alpha r_1, \alpha f_2, \alpha r_2, \ldots)$

$\forall N \in \mathbb{R}$.

$$g_1 = g_1(V_1, V_2) = \frac{\sum_{n=1}^{3} V_1^{3-n}\cdot(\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21}\cdot V_1 + \Gamma_{22}\cdot V_2 + \Gamma_{23}\cdot V_1\cdot V_2 + \Gamma_{24}}{V_1\cdot\Gamma_{41} + \Gamma_{42}}$$

$$g_1 = g_1(r,\theta) = \frac{\sum_{n=1}^{3} r^{(3-n)}\cdot\cos^{(3-n)}[\theta]\cdot(\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21}\cdot r\cdot\cos\theta + \Gamma_{22}\cdot r\cdot\sin\theta + \Gamma_{23}\cdot r^2\cdot\sin\theta\cdot\cos\theta + \Gamma_{24}}{r\cdot\cos\theta\cdot\Gamma_{41} + \Gamma_{42}}$$

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = g_1(V_1, V_2) \Rightarrow r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta = \frac{\sum_{n=1}^{3} r^{(3-n)}\cdot\cos^{(3-n)}[\theta]\cdot(\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21}\cdot r\cdot\cos\theta + \Gamma_{22}\cdot r\cdot\sin\theta + \Gamma_{23}\cdot r^2\cdot\sin\theta\cdot\cos\theta + \Gamma_{24}}{r\cdot\cos\theta\cdot\Gamma_{41} + \Gamma_{42}}$$

$$(r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta)\cdot(r\cdot\cos\theta\cdot\Gamma_{41} + \Gamma_{42}) = \sum_{n=1}^{3} r^{(3-n)}\cdot\cos^{(3-n)}[\theta]\cdot(\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21}\cdot r\cdot\cos\theta + \Gamma_{22}\cdot r\cdot\sin\theta + \Gamma_{23}\cdot r^2\cdot\sin\theta\cdot\cos\theta + \Gamma_{24}$$

$$\Gamma_{41}\cdot r\cdot r'\cdot\cos^2\theta + \Gamma_{42}\cdot r'\cdot\cos\theta - r^2\cdot\theta'\cdot\Gamma_{41}\cdot\sin\theta\cdot\cos\theta - \Gamma_{42}\cdot r\cdot\theta'\cdot\sin\theta = \sum_{n=1}^{3} r^{(3-n)}\cdot\cos^{(3-n)}[\theta]\cdot(\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21}\cdot r\cdot\cos\theta + \Gamma_{22}\cdot r\cdot\sin\theta + \Gamma_{23}\cdot r^2\cdot\sin\theta\cdot\cos\theta + \Gamma_{24}$$

$$\Gamma_{41} \cdot r \cdot r' \cdot \cos^2\theta + \Gamma_{42} \cdot r' \cdot \cos\theta - r^2 \cdot \theta' \cdot \Gamma_{41} \cdot \sin\theta \cdot \cos\theta - \Gamma_{42} \cdot r \cdot \theta' \cdot \sin\theta$$
$$= r^2 \cdot \cos^2\theta \cdot (\Gamma_{11} - \Gamma_{31}) + r \cdot \cos\theta \cdot (\Gamma_{12} - \Gamma_{32}) + \Gamma_{13} - \Gamma_{33} + \Gamma_{21} \cdot r \cdot \cos\theta$$
$$+ \Gamma_{22} \cdot r \cdot \sin\theta + \Gamma_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Gamma_{24}$$

$$r \cdot \cos^2\theta \cdot \{\Gamma_{41} \cdot r' - r \cdot (\Gamma_{11} - \Gamma_{31})\} + \cos\theta \cdot \{\Gamma_{42} \cdot r' - r \cdot (\Gamma_{12} - \Gamma_{32}) - \Gamma_{21} \cdot r\}$$
$$- r \cdot \sin\theta \cdot \{\Gamma_{42} \cdot \theta' + \Gamma_{22}\} - r^2 \cdot \sin\theta \cdot \cos\theta \cdot \{\theta' \cdot \Gamma_{41} + \Gamma_{23}\} + \{\Gamma_{33} - \Gamma_{13} - \Gamma_{24}\} = 0$$

$$h_1 = h_1(V_1, V_2)$$
$$= \frac{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}}{V_2 \cdot \Xi_{41} + \Xi_{42}}$$

$$h_1 = h_1(r, \theta)$$
$$= \frac{\sum_{n=1}^{3} r^{(3-n)} \cdot \sin^{(3-n)}[\theta] \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot r \cdot \sin\theta + \Xi_{22} \cdot r \cdot \cos\theta + \Xi_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Xi_{24}}{r \cdot \sin\theta \cdot \Xi_{41} + \Xi_{42}}$$

$$\frac{dV_2}{dt} = h_1(V_1, V_2) \Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$
$$= \frac{\sum_{n=1}^{3} r^{(3-n)} \cdot \sin^{(3-n)}[\theta] \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot r \cdot \sin\theta + \Xi_{22} \cdot r \cdot \cos\theta + \Xi_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Xi_{24}}{r \cdot \sin\theta \cdot \Xi_{41} + \Xi_{42}}$$

$$r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$
$$= \frac{\sum_{n=1}^{3} r^{(3-n)} \cdot \sin^{(3-n)}[\theta] \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot r \cdot \sin\theta + \Xi_{22} \cdot r \cdot \cos\theta + \Xi_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Xi_{24}}{r \cdot \sin\theta \cdot \Xi_{41} + \Xi_{42}}$$

$$\{r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta\} \cdot \{r \cdot \sin\theta \cdot \Xi_{41} + \Xi_{42}\}$$
$$= \sum_{n=1}^{3} r^{(3-n)} \cdot \sin^{(3-n)}[\theta] \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot r \cdot \sin\theta$$
$$+ \Xi_{22} \cdot r \cdot \cos\theta + \Xi_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Xi_{24}$$

$$r' \cdot r \cdot \Xi_{41} \cdot \sin^2\theta + \Xi_{42} \cdot r' \cdot \sin\theta + r^2 \cdot \theta' \cdot \Xi_{41} \cdot \cos\theta \cdot \sin\theta + \Xi_{42} \cdot r \cdot \theta' \cdot \cos\theta$$
$$= \sum_{n=1}^{3} r^{(3-n)} \cdot \sin^{(3-n)}[\theta] \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot r \cdot \sin\theta$$
$$+ \Xi_{22} \cdot r \cdot \cos\theta + \Xi_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Xi_{24}$$

$$r' \cdot r \cdot \Xi_{41} \cdot \sin^2\theta + \Xi_{42} \cdot r' \cdot \sin\theta + r^2 \cdot \theta' \cdot \Xi_{41} \cdot \cos\theta \cdot \sin\theta + \Xi_{42} \cdot r \cdot \theta' \cdot \cos\theta$$
$$= r^2 \cdot \sin^2\theta \cdot (\Xi_{11} - \Xi_{31}) + r \cdot \sin\theta \cdot (\Xi_{12} - \Xi_{32}) + \Xi_{13} - \Xi_{33}$$
$$+ \Xi_{21} \cdot r \cdot \sin\theta + \Xi_{22} \cdot r \cdot \cos\theta + \Xi_{23} \cdot r^2 \cdot \sin\theta \cdot \cos\theta + \Xi_{24}$$

$$r \cdot \sin^2\theta \cdot \{r' \cdot \Xi_{41} - r \cdot (\Xi_{11} - \Xi_{31})\} + \sin\theta \cdot \{\Xi_{42} \cdot r' - r \cdot (\Xi_{12} - \Xi_{32}) - \Xi_{21} \cdot r\}$$
$$+ r \cdot \cos\theta \cdot \{\Xi_{42} \cdot \theta' - \Xi_{22}\} + r^2 \cdot \cos\theta \cdot \sin\theta \cdot \{\theta' \cdot \Xi_{41} - \Xi_{23}\}$$
$$+ \{\Xi_{33} - \Xi_{13} - \Xi_{24}\} = 0$$

We can summarize our system equations in cylindrical (r, θ) coordinates:

$$(*)\quad \begin{array}{l} r \cdot \cos^2\theta \cdot \{\Gamma_{41} \cdot r' - r \cdot (\Gamma_{11} - \Gamma_{31})\} + \cos\theta \cdot \{\Gamma_{42} \cdot r' - r \cdot (\Gamma_{12} - \Gamma_{32}) - \Gamma_{21} \cdot r\} \\ -r \cdot \sin\theta \cdot \{\Gamma_{42} \cdot \theta' + \Gamma_{22}\} - r^2 \cdot \sin\theta \cdot \cos\theta \cdot \{\theta' \cdot \Gamma_{41} + \Gamma_{23}\} + \{\Gamma_{33} - \Gamma_{24} - \Gamma_{13}\} = 0 \end{array}$$

$$(**)\quad \begin{array}{l} r \cdot \sin^2\theta \cdot \{r' \cdot \Xi_{41} - r \cdot (\Xi_{11} - \Xi_{31})\} + \sin\theta \cdot \{\Xi_{42} \cdot r' - r \cdot (\Xi_{12} - \Xi_{32}) - \Xi_{21} \cdot r\} \\ + r \cdot \cos\theta \cdot \{\Xi_{42} \cdot \theta' - \Xi_{22}\} + r^2 \cdot \cos\theta \cdot \sin\theta \cdot \{\theta' \cdot \Xi_{41} - \Xi_{23}\} + \{\Xi_{33} - \Xi_{13} - \Xi_{24}\} = 0 \end{array}$$

For simplicity, we define the following system global parameters relations:

$$\Gamma_{41} = \Xi_{41};\ \Gamma_{11} - \Gamma_{31} = \Xi_{11} - \Xi_{31};\ \Gamma_{42} = \Xi_{42};\ \Gamma_{12} - \Gamma_{32} = \Xi_{12} - \Xi_{32};\ \Gamma_{21} = \Xi_{21}$$
$$\chi_{41} = \Gamma_{33} - \Gamma_{24} - \Gamma_{13};\ \chi_{42} = \Xi_{33} - \Xi_{13} - \Xi_{24}.$$

$$\chi_1 = \chi_1(r, r') = \Gamma_{41} \cdot r' - r \cdot (\Gamma_{11} - \Gamma_{31}) = r' \cdot \Xi_{41} - r \cdot (\Xi_{11} - \Xi_{31})$$

$$\begin{aligned} \chi_2 = \chi_2(r, r') &= \Gamma_{42} \cdot r' - r \cdot (\Gamma_{12} - \Gamma_{32}) - \Gamma_{21} \cdot r \\ &= \Xi_{42} \cdot r' - r \cdot (\Xi_{12} - \Xi_{32}) - \Xi_{21} \cdot r \end{aligned}$$

Based on the above assumptions we get the following two equations:

$$(*)\quad \begin{array}{l} r \cdot \cos^2\theta \cdot \chi_1 + \cos\theta \cdot \chi_2 - r \cdot \sin\theta \cdot \{\Gamma_{42} \cdot \theta' + \Gamma_{22}\} \\ -r^2 \cdot \sin\theta \cdot \cos\theta \cdot \{\theta' \cdot \Gamma_{41} + \Gamma_{23}\} + \chi_{41} = 0 \end{array}$$

$$(**)\quad \begin{array}{l} r \cdot \sin^2\theta \cdot \chi_1 + \sin\theta \cdot \chi_2 + r \cdot \cos\theta \cdot \{\Xi_{42} \cdot \theta' - \Xi_{22}\} \\ + r^2 \cdot \cos\theta \cdot \sin\theta \cdot \{\theta' \cdot \Xi_{41} - \Xi_{23}\} + \chi_{42} = 0 \end{array}$$

(*) + (**) → (***)

$$r \cdot \cos^2\theta \cdot \chi_1 + r \cdot \sin^2\theta \cdot \chi_1 + \cos\theta \cdot \chi_2 + \sin\theta \cdot \chi_2 - r \cdot \sin\theta \cdot \{\Gamma_{42} \cdot \theta' + \Gamma_{22}\}$$
$$+ r \cdot \cos\theta \cdot \{\Xi_{42} \cdot \theta' - \Xi_{22}\} - r^2 \cdot \sin\theta \cdot \cos\theta \cdot \{\theta' \cdot \Gamma_{41} + \Gamma_{23}\}$$
$$+ r^2 \cdot \cos\theta \cdot \sin\theta \cdot \{\theta' \cdot \Xi_{41} - \Xi_{23}\} + \chi_{41} + \chi_{42} = 0$$

We get one equation: $\chi_4 = \chi_{41} + \chi_{42}$

$$r \cdot \chi_1 + \chi_2 \cdot (\cos\theta + \sin\theta) - r \cdot \sin\theta \cdot \{\Gamma_{42} \cdot \theta' + \Gamma_{22}\} + r \cdot \cos\theta \cdot \{\Xi_{42} \cdot \theta' - \Xi_{22}\}$$
$$+ r^2 \cdot \cos\theta \cdot \sin\theta \cdot \{\theta' \cdot (\Xi_{41} - \Gamma_{41}) - (\Gamma_{23} + \Xi_{23})\} + \chi_4 = 0$$

We choose $d\theta/dt = \theta'$ in time such that $\theta' \cdot (\Xi_{41} - \Gamma_{41}) - (\Gamma_{23} + \Xi_{23}) = 0 \Rightarrow \theta' = \frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}}$

The last condition restricts our main result equation (***) to the below equation:

$$r \cdot \chi_1 + \chi_2 \cdot (\cos\theta + \sin\theta) - r \cdot \sin\theta \cdot \{\Gamma_{42} \cdot \theta' + \Gamma_{22}\} \\ + r \cdot \cos\theta \cdot \{\Xi_{42} \cdot \theta' - \Xi_{22}\} + \chi_4 = 0$$

$$r \cdot \chi_1 + \chi_2 \cdot (\cos\theta + \sin\theta) - \Gamma_{42} \cdot \theta' \cdot r \cdot \sin\theta - \Gamma_{22} \cdot r \cdot \sin\theta \\ + \Xi_{42} \cdot \theta' \cdot r \cdot \cos\theta - \Xi_{22} \cdot r \cdot \cos\theta + \chi_4 = 0$$

$$r \cdot \chi_1 + \chi_2 \cdot (\cos\theta + \sin\theta) - \Gamma_{22} \cdot r \cdot \sin\theta - \Xi_{22} \cdot r \cdot \cos\theta \\ + \Xi_{42} \cdot \theta' \cdot r \cdot \cos\theta - \Gamma_{42} \cdot \theta' \cdot r \cdot \sin\theta + \chi_4 = 0$$

We consider $\Xi_{42} = \Gamma_{42}; \Gamma_{22} = \Xi_{22}$ and we get the following expression:

$$r \cdot \chi_1 + \chi_2 \cdot (\cos\theta + \sin\theta) - \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) \\ + \Gamma_{42} \cdot \theta' \cdot r \cdot (\cos\theta - \sin\theta) + \chi_4 = 0$$

$$r \cdot \chi_1 + \chi_2 \cdot (\cos\theta + \sin\theta) - \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) \\ - \Gamma_{42} \cdot \theta' \cdot r \cdot (\sin\theta - \cos\theta) + \chi_4 = 0$$

$\chi_1(r, r') \to \chi_1$; $\chi_2(r, r') \to \chi_2$ yield the following expressions:

$$r \cdot \{\Gamma_{41} \cdot r' - r \cdot (\Gamma_{11} - \Gamma_{31})\} + \{\Gamma_{42} \cdot r' - r \cdot (\Gamma_{12} - \Gamma_{32}) - \Gamma_{21} \cdot r\} \cdot (\cos\theta + \sin\theta) \\ - \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) - \Gamma_{42} \cdot \theta' \cdot r \cdot (\sin\theta - \cos\theta) + \chi_{41} + \chi_{42} = 0$$

$$r \cdot \{\Gamma_{41} \cdot r' - r \cdot (\Gamma_{11} - \Gamma_{31})\} + \{\Gamma_{42} \cdot r' - r \cdot (\Gamma_{12} - \Gamma_{32}) - \Gamma_{21} \cdot r\} \cdot (\cos\theta + \sin\theta) \\ - \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) - \Gamma_{42} \cdot \theta' \cdot r \cdot (\sin\theta - \cos\theta) \\ + \{(\Gamma_{33} + \Xi_{33}) - (\Gamma_{24} + \Xi_{24}) - (\Gamma_{13} + \Xi_{13})\} = 0$$

$$\Gamma_{41} \cdot r \cdot r' - r^2 \cdot (\Gamma_{11} - \Gamma_{31}) + \Gamma_{42} \cdot r' \cdot (\cos\theta + \sin\theta) - r \cdot (\Gamma_{12} - \Gamma_{32}) \cdot (\cos\theta + \sin\theta) \\ - \Gamma_{21} \cdot r \cdot (\cos\theta + \sin\theta) - \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) - \Gamma_{42} \cdot \theta' \cdot r \cdot (\sin\theta - \cos\theta) \\ + \{(\Gamma_{33} + \Xi_{33}) - (\Gamma_{24} + \Xi_{24}) - (\Gamma_{13} + \Xi_{13})\} = 0$$

$$r' \cdot \{\Gamma_{41} \cdot r + \Gamma_{42} \cdot (\cos\theta + \sin\theta)\} - r^2 \cdot (\Gamma_{11} - \Gamma_{31}) \\ - r \cdot (\Gamma_{12} - \Gamma_{32}) \cdot (\cos\theta + \sin\theta) - \Gamma_{21} \cdot r \cdot (\cos\theta + \sin\theta) - \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) \\ - \Gamma_{42} \cdot \theta' \cdot r \cdot (\sin\theta - \cos\theta) + \{(\Gamma_{33} + \Xi_{33}) - (\Gamma_{24} + \Xi_{24}) - (\Gamma_{13} + \Xi_{13})\} = 0$$

$$r' \cdot \{\Gamma_{41} \cdot r + \Gamma_{42} \cdot (\cos\theta + \sin\theta)\} = r^2 \cdot (\Gamma_{11} - \Gamma_{31}) + r \cdot (\Gamma_{12} - \Gamma_{32}) \cdot (\cos\theta + \sin\theta) + \Gamma_{21} \cdot r \cdot (\cos\theta + \sin\theta) + \Gamma_{22} \cdot r \cdot (\sin\theta + \cos\theta) + \Gamma_{42} \cdot \theta' \cdot r \cdot (\sin\theta - \cos\theta) - \{(\Gamma_{33} + \Xi_{33}) - (\Gamma_{24} + \Xi_{24}) - (\Gamma_{13} + \Xi_{13})\}$$

$$r' \cdot \{\Gamma_{41} \cdot r + \Gamma_{42} \cdot (\cos\theta + \sin\theta)\} = r^2 \cdot (\Gamma_{11} - \Gamma_{31}) + r \cdot \{(\Gamma_{12} - \Gamma_{32}) \cdot (\cos\theta + \sin\theta) + \Gamma_{21} \cdot (\cos\theta + \sin\theta) + \Gamma_{22} \cdot (\sin\theta + \cos\theta) + \Gamma_{42} \cdot \theta' \cdot (\sin\theta - \cos\theta)\} - \{(\Gamma_{33} + \Xi_{33}) - (\Gamma_{24} + \Xi_{24}) - (\Gamma_{13} + \Xi_{13})\}$$

For simplicity, we define the following functions: $f_1(\theta) = \sin\theta + \cos\theta$

$$f_2(\theta) = \sin\theta - \cos\theta; \Delta_1 = \Gamma_{11} - \Gamma_{31}; \Delta_2 = \Gamma_{12} - \Gamma_{32};$$
$$\Delta_3 = (\Gamma_{33} + \Xi_{33}) - (\Gamma_{24} + \Xi_{24}) - (\Gamma_{13} + \Xi_{13})$$

And we get the following function:

$$r' \cdot \{\Gamma_{41} \cdot r + \Gamma_{42} \cdot f_1(\theta)\} = r^2 \cdot \Delta_1 + r \cdot \{\Delta_2 \cdot f_1(\theta) + \Gamma_{21} \cdot f_1(\theta) + \Gamma_{22} \cdot f_1(\theta) + \Gamma_{42} \cdot \theta' \cdot f_2(\theta)\} - \Delta_3$$

$$r' = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2 \cdot \Delta_1 + r \cdot \{\Delta_2 \cdot f_1(\theta) + \Gamma_{21} \cdot f_1(\theta) + \Gamma_{22} \cdot f_1(\theta) + \Gamma_{42} \cdot \theta' \cdot f_2(\theta)\} - \Delta_3}{\Gamma_{41} \cdot r + \Gamma_{42} \cdot f_1(\theta)}$$

$$\theta' = \frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2 \cdot \Delta_1 + r \cdot \left\{\Delta_2 \cdot f_1(\theta) + \Gamma_{21} \cdot f_1(\theta) + \Gamma_{22} \cdot f_1(\theta) + \Gamma_{42} \cdot \left\{\frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}}\right\} \cdot f_2(\theta)\right\} - \Delta_3}{\Gamma_{41} \cdot r + \Gamma_{42} \cdot f_1(\theta)}$$

We can summarize our last system expression cylindrical coordinates (r, θ):

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2 \cdot \Delta_1 + r \cdot \left\{\Delta_2 \cdot f_1(\theta) + \Gamma_{21} \cdot f_1(\theta) + \Gamma_{22} \cdot f_1(\theta) + \Gamma_{42} \cdot \left\{\frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}}\right\} \cdot f_2(\theta)\right\} - \Delta_3}{\Gamma_{41} \cdot r + \Gamma_{42} \cdot f_1(\theta)}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2 \cdot \Delta_1 + r \cdot \left\{f_1(\theta) \cdot [\Delta_2 + \Gamma_{21} + \Gamma_{22}] + \Gamma_{42} \cdot \left\{\frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}}\right\} \cdot f_2(\theta)\right\} - \Delta_3}{\Gamma_{41} \cdot r + \Gamma_{42} \cdot f_1(\theta)}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}} \Rightarrow \theta = \left\{\frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}}\right\} \cdot t + \text{Const}$$

$$\theta = \omega \cdot t = \frac{2 \cdot \pi}{T} \cdot t \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2 \cdot \pi}{T} \Rightarrow \frac{\Gamma_{23} + \Xi_{23}}{\Xi_{41} - \Gamma_{41}} = \frac{2 \cdot \pi}{T} \Rightarrow T = \frac{2 \cdot \pi \cdot (\Xi_{41} - \Gamma_{41})}{\Gamma_{23} + \Xi_{23}}$$

As a result, our solution has period $T = \frac{2\cdot\pi\cdot(\Xi_{41}-\Gamma_{41})}{\Gamma_{23}+\Xi_{23}}$.

To find our solution constant radius, we set $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$ which yields the following outcome:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0 \Rightarrow \frac{r^2\cdot\Delta_1 + r\cdot\left\{f_1(\theta)\cdot[\Delta_2+\Gamma_{21}+\Gamma_{22}] + \Gamma_{42}\cdot\left\{\frac{\Gamma_{23}+\Xi_{23}}{\Xi_{41}-\Gamma_{41}}\right\}\cdot f_2(\theta)\right\} - \Delta_3}{\Gamma_{41}\cdot r + \Gamma_{42}\cdot f_1(\theta)} = 0$$

$$\Gamma_{41}\cdot r + \Gamma_{42}\cdot f_1(\theta) \neq 0 \Rightarrow r \neq -\frac{\Gamma_{42}}{\Gamma_{41}}\cdot f_1(\theta) \Rightarrow r(t) \neq -\frac{\Gamma_{42}}{\Gamma_{41}}\cdot f_1(\theta = \left\{\frac{\Gamma_{23}+\Xi_{23}}{\Xi_{41}-\Gamma_{41}}\right\}\cdot t + \mathrm{Const})$$

We define for simplicity $\frac{\mathrm{d}r}{\mathrm{d}t} = 0 \Rightarrow \frac{r^2\cdot\Delta_1 + r\cdot g(\theta) - \Delta_3}{\Gamma_{41}\cdot r + \Gamma_{42}\cdot f_1(\theta)} = 0 \Rightarrow r^2\cdot\Delta_1 + r\cdot g(\theta) - \Delta_3 = 0$.

$$g = g(\theta) = f_1(\theta)\cdot[\Delta_2+\Gamma_{21}+\Gamma_{22}] + \Gamma_{42}\cdot\left\{\frac{\Gamma_{23}+\Xi_{23}}{\Xi_{41}-\Gamma_{41}}\right\}\cdot f_2(\theta)$$

$$g = g(\theta)|_{\Delta_2=\Gamma_{12}-\Gamma_{32}} = f_1(\theta)\cdot[\Gamma_{12}-\Gamma_{32}+\Gamma_{21}+\Gamma_{22}] + \Gamma_{42}\cdot\left\{\frac{\Gamma_{23}+\Xi_{23}}{\Xi_{41}-\Gamma_{41}}\right\}\cdot f_2(\theta)$$

$$r^2\cdot\Delta_1 + r\cdot g(\theta) - \Delta_3 = 0 \Rightarrow r_{1,2} = \frac{-g(\theta)\pm\sqrt{g^2(\theta)+4\cdot\Delta_1\cdot\Delta_3}}{2\cdot\Delta_1} > 0$$

We need to show the conditions that a close orbit still exists for a stable limit cycle. There is a stable limit cycle at $r = r_{1,2} = \frac{-g(\theta)\pm\sqrt{g^2(\theta)+4\cdot\Delta_1\cdot\Delta_3}}{2\cdot\Delta_1} > 0$.

We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$, such that $\frac{\mathrm{d}r}{\mathrm{d}t} < 0$ on the outer circle and $\frac{\mathrm{d}r}{\mathrm{d}t} > 0$ on the inner circle. Then the annulus $0 < r_{\min} \le r \le r_{\max}$ will be our desired trapping region. There are no fixed points in the annulus since we consider $\frac{\mathrm{d}\theta}{\mathrm{d}t} > 0$; hence if $r_{\min}$ and $r_{\max}$ can be found, the Poincare–Bendixson theorem will imply the existence of the closed orbit. To find $r_{\min}$, we require $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2\cdot\Delta_1 + r\cdot g(\theta) - \Delta_3}{\Gamma_{41}\cdot r + \Gamma_{42}\cdot f_1(\theta)} > 0$ for all θ. $\frac{r_{\min}^2\cdot\Delta_1 + r_{\min}\cdot g(\theta) - \Delta_3}{\Gamma_{41}\cdot r_{\min} + \Gamma_{42}\cdot f_1(\theta)} > 0$.

We have two subcases: (1) $r_{\min}^2\cdot\Delta_1 + r_{\min}\cdot g(\theta) - \Delta_3 > 0$ and $\Gamma_{41}\cdot r_{\min} + \Gamma_{42}\cdot f_1(\theta) > 0$. (2) $r_{\min}^2\cdot\Delta_1 + r_{\min}\cdot g(\theta) - \Delta_3 < 0$ and $\Gamma_{41}\cdot r_{\min} + \Gamma_{42}\cdot f_1(\theta) < 0$. To find $r_{\max}$, we require $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2\cdot\Delta_1 + r\cdot g(\theta) - \Delta_3}{\Gamma_{41}\cdot r + \Gamma_{42}\cdot f_1(\theta)} < 0$ for all θ. $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r^2\cdot\Delta_1 + r\cdot g(\theta) - \Delta_3}{\Gamma_{41}\cdot r + \Gamma_{42}\cdot f_1(\theta)} < 0$ and we have two subcases: (1) $r_{\max}^2\cdot\Delta_1 + r_{\max}\cdot g(\theta) - \Delta_3 > 0$ and $\Gamma_{41}\cdot r_{\max} + \Gamma_{42}\cdot f_1(\theta) < 0$.

(2) $r_{\max}^2\cdot\Delta_1 + r_{\max}\cdot g(\theta) - \Delta_3 < 0$ and $\Gamma_{41}\cdot r_{\max} + \Gamma_{42}\cdot f_1(\theta) > 0$.

System equations without periodic sources $X_1(t) = X_2(t) = 0$ are $\frac{dV_1}{dt} = g_1(V_1, V_2)\frac{dV_2}{dt} = h_1(V_1, V_2)$. Our system $dV/dt = f(V)$ with $V \in \mathbb{R}^2$ where there is a periodic solution $v(t) = \Phi(t)$ with period T. We must have $\rho_1 = 1$ then we get $\rho_1 \cdot \rho_2 = e^{\int_0^T tr(A(s))\cdot ds}|_{\rho_1=1} \Rightarrow \rho_2 = e^{\int_0^T tr(A(s))\cdot ds}$. The perturbation should be bounded and hence for the solution to be stable, $\rho_1 \leq 1; \rho_2 \leq 1$ and since we know $\rho 1 = 1$ and we wish ρ_1 and ρ_2 to be distinct, we must have $\rho_2 < 1;\ \rho_1 = 1$

$$\rho_2 < 1 \Rightarrow e^{\int_0^T tr(A(s))\cdot ds} < 1 \Rightarrow \int_0^T tr(A(s)) \cdot ds < 0$$

$$\Rightarrow \int_0^T tr(\frac{\partial g_1/\partial h_1}{\partial V_j}|_{\phi(s)}) \cdot ds < 0 \Rightarrow \int_0^T (\frac{\partial g_1}{\partial V_1} + \frac{\partial h_1}{\partial V_2})|_{\phi(s)} \cdot ds < 0$$

$\int_0^T \left(\frac{\partial g_1}{\partial V_1} + \frac{\partial h_1}{\partial V_2}\right)|_{V=\phi} \cdot ds < 0$ We get instability when $\int_0^T \left(\frac{\partial g_1}{\partial V_1} + \frac{\partial h_1}{\partial V_2}\right)|_{V=\phi} \cdot ds > 0$.

$$g_1(V_1, V_2) = \frac{\xi_1(V_1, V_2)}{\xi_2(V_1)} \Rightarrow g_1 = \frac{\xi_1}{\xi_2}; \frac{\partial g_1}{\partial V_1} = \frac{\frac{\partial \xi_1}{\partial V_1} \cdot \xi_2 - \frac{\partial \xi_2}{\partial V_1} \cdot \xi_1}{\xi_2^2}$$

We define the following functions: $\xi_1 = \xi_1(V_1, V_2);\ \xi_2 = \xi_2(V_1)$.

$$\xi_1 = \sum_{n=1}^{3} V_1^{3-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24};\ \xi_2 = V_1 \cdot \Gamma_{41} + \Gamma_{42}$$

$$\frac{\partial \xi_1}{\partial V_1} = \sum_{n=1}^{3} (3 - n) \cdot V_1^{2-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} + \Gamma_{23} \cdot V_2;\ \frac{\partial \xi_1}{\partial V_2} = \Gamma_{22} + \Gamma_{23} \cdot V_1$$

$\frac{\partial \xi_2}{\partial V_1} = \Gamma_{41};\ \frac{\partial \xi_2}{\partial V_2} = 0$. We get the following expression for $\frac{\partial g_1}{\partial V_1}$

$$\frac{\partial g_1}{\partial V_1} = \frac{\frac{\partial \xi_1}{\partial V_1} \cdot \xi_2 - \frac{\partial \xi_2}{\partial V_1} \cdot \xi_1}{\xi_2^2}$$

$$= \frac{\left\{\sum_{n=1}^{3} (3 - n) \cdot V_1^{2-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} + \Gamma_{23} \cdot V_2\right\} \cdot \left\{V_1 \cdot \Gamma_{41} + \Gamma_{42}\right\} - \Gamma_{41} \cdot \left\{\sum_{n=1}^{3} V_1^{3-n} \cdot (\Gamma_{1n} - \Gamma_{3n}) + \Gamma_{21} \cdot V_1 + \Gamma_{22} \cdot V_2 + \Gamma_{23} \cdot V_1 \cdot V_2 + \Gamma_{24}\right\}}{(V_1 \cdot \Gamma_{41} + \Gamma_{42})^2}$$

First, we take care of the numerator of the expression.

$$\left\{\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}+\Gamma_{23}\cdot V_2\right\}\cdot\{V_1\cdot\Gamma_{41}+\Gamma_{42}\}$$
$$-\Gamma_{41}\cdot\left\{\sum_{n=1}^{3}V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}\cdot V_1+\Gamma_{22}\cdot V_2+\Gamma_{23}\cdot V_1\cdot V_2+\Gamma_{24}\right\}$$
$$=\sum_{n=1}^{3}(3-n)\cdot V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})\cdot\Gamma_{41}+\Gamma_{42}\cdot\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})$$
$$+V_1\cdot\Gamma_{41}\cdot\Gamma_{21}+\Gamma_{42}\cdot\Gamma_{21}+V_1\cdot V_2\cdot\Gamma_{41}\cdot\Gamma_{23}+\Gamma_{42}\cdot\Gamma_{23}\cdot V_2$$
$$-\sum_{n=1}^{3}V_1^{3-n}\cdot\Gamma_{41}\cdot(\Gamma_{1n}-\Gamma_{3n})-\Gamma_{41}\cdot\Gamma_{21}\cdot V_1$$
$$-\Gamma_{41}\cdot\Gamma_{22}\cdot V_2-\Gamma_{41}\cdot\Gamma_{23}\cdot V_1\cdot V_2-\Gamma_{41}\cdot\Gamma_{24}$$

$$\sum_{n=1}^{3}(3-n)\cdot V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})\cdot\Gamma_{41}=2\cdot V_1^2\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{41}+V_1\cdot(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{41}$$

$$\mathit{\Gamma}_{42}\cdot\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})=2\cdot V_1\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{42}+(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{42}$$

$$\sum_{n=1}^{3}V_1^{3-n}\cdot\Gamma_{41}\cdot(\Gamma_{1n}-\Gamma_{3n})=V_1^2\cdot\Gamma_{41}\cdot(\Gamma_{11}-\Gamma_{31})+V_1\cdot\Gamma_{41}\cdot(\Gamma_{12}-\Gamma_{32})$$
$$+\Gamma_{41}\cdot(\Gamma_{13}-\Gamma_{33})$$

We get the following numerator expression:

$$\left\{\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}+\Gamma_{23}\cdot V_2\right\}\cdot\{V_1\cdot\Gamma_{41}+\Gamma_{42}\}$$
$$-\Gamma_{41}\cdot\left\{\sum_{n=1}^{3}V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}\cdot V_1+\Gamma_{22}\cdot V_2+\Gamma_{23}\cdot V_1\cdot V_2+\Gamma_{24}\right\}$$
$$=2\cdot V_1^2\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{41}+V_1\cdot(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{41}$$
$$+2\cdot V_1\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{42}+(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{42}+V_1\cdot\Gamma_{41}\cdot\Gamma_{21}+\Gamma_{42}\cdot\Gamma_{21}$$
$$+V_1\cdot V_2\cdot\Gamma_{41}\cdot\Gamma_{23}+\Gamma_{42}\cdot\Gamma_{23}\cdot V_2-V_1^2\cdot\Gamma_{41}\cdot(\Gamma_{11}-\Gamma_{31})$$
$$-V_1\cdot\Gamma_{41}\cdot(\Gamma_{12}-\Gamma_{32})-\Gamma_{41}\cdot(\Gamma_{13}-\Gamma_{33})$$
$$-\Gamma_{41}\cdot\Gamma_{21}\cdot V_1-\Gamma_{41}\cdot\Gamma_{22}\cdot V_2-\Gamma_{41}\cdot\Gamma_{23}\cdot V_1\cdot V_2-\Gamma_{41}\cdot\Gamma_{24}$$

$$\left\{\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}+\Gamma_{23}\cdot V_2\right\}\cdot\{V_1\cdot\Gamma_{41}+\Gamma_{42}\}$$
$$-\Gamma_{41}\cdot\left\{\sum_{n=1}^{3}V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}\cdot V_1+\Gamma_{22}\cdot V_2+\Gamma_{23}\cdot V_1\cdot V_2+\Gamma_{24}\right\}$$
$$=2\cdot V_1^2\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{41}-V_1^2\cdot\Gamma_{41}\cdot(\Gamma_{11}-\Gamma_{31})+V_1\cdot(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{41}$$
$$+2\cdot V_1\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{42}+V_1\cdot\Gamma_{41}\cdot\Gamma_{21}-V_1\cdot\Gamma_{41}\cdot(\Gamma_{12}-\Gamma_{32})-\Gamma_{41}\cdot\Gamma_{21}\cdot V_1$$
$$+\mathit{\Gamma}_{42}\cdot\mathit{\Gamma}_{23}\cdot V_2-\mathit{\Gamma}_{41}\cdot\mathit{\Gamma}_{22}\cdot V_2+V_1\cdot V_2\cdot\mathit{\Gamma}_{41}\cdot\mathit{\Gamma}_{23}-\mathit{\Gamma}_{41}\cdot\mathit{\Gamma}_{23}\cdot V_1\cdot V_2$$
$$-\Gamma_{41}\cdot\Gamma_{24}+\Gamma_{42}\cdot\Gamma_{21}+(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{42}-\Gamma_{41}\cdot(\Gamma_{13}-\Gamma_{33})$$

$$\left\{\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}+\Gamma_{23}\cdot V_2\right\}\cdot\{V_1\cdot\Gamma_{41}+\Gamma_{42}\}$$
$$-\Gamma_{41}\cdot\left\{\sum_{n=1}^{3}V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}\cdot V_1+\Gamma_{22}\cdot V_2+\Gamma_{23}\cdot V_1\cdot V_2+\Gamma_{24}\right\}$$
$$=V_1^2\cdot\Gamma_{41}\cdot\{2\cdot(\Gamma_{11}-\Gamma_{31})-(\Gamma_{11}-\Gamma_{31})\}+V_1\cdot\{(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{41}$$
$$+2\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{42}+\Gamma_{41}\cdot\Gamma_{21}-\Gamma_{41}\cdot(\Gamma_{12}-\Gamma_{32})-\Gamma_{41}\cdot\Gamma_{21}\}$$
$$+V_2\cdot\{\Gamma_{42}\cdot\Gamma_{23}-\Gamma_{41}\cdot\Gamma_{22}\}+V_1\cdot V_2\cdot\{\Gamma_{41}\cdot\Gamma_{23}-\Gamma_{41}\cdot\Gamma_{23}\}$$
$$-\Gamma_{41}\cdot\Gamma_{24}+\Gamma_{42}\cdot\Gamma_{21}+\Gamma_{12}\cdot\Gamma_{42}-\Gamma_{32}\cdot\Gamma_{42}-\Gamma_{41}\cdot\Gamma_{13}+\Gamma_{41}\cdot\Gamma_{33}$$

We define the following parameters: $\Omega_1=\Gamma_{41}\cdot\{2\cdot(\Gamma_{11}-\Gamma_{31})-(\Gamma_{11}-\Gamma_{31})\}$,

$$\Omega_2=(\Gamma_{12}-\Gamma_{32})\cdot\Gamma_{41}+2\cdot(\Gamma_{11}-\Gamma_{31})\cdot\Gamma_{42}$$
$$+\Gamma_{41}\cdot\Gamma_{21}-\Gamma_{41}\cdot(\Gamma_{12}-\Gamma_{32})-\Gamma_{41}\cdot\Gamma_{21}$$
$$\Omega_3=\Gamma_{42}\cdot\Gamma_{23}-\Gamma_{41}\cdot\Gamma_{22};\Omega_4=\Gamma_{41}\cdot\Gamma_{23}-\Gamma_{41}\cdot\Gamma_{23}$$
$$\Omega_5=-\Gamma_{41}\cdot\Gamma_{24}+\Gamma_{42}\cdot\Gamma_{21}+\Gamma_{12}\cdot\Gamma_{42}-\Gamma_{32}\cdot\Gamma_{42}-\Gamma_{41}\cdot\Gamma_{13}+\Gamma_{41}\cdot\Gamma_{33}$$

We get the following numerator expression:

$$\left\{\sum_{n=1}^{3}(3-n)\cdot V_1^{2-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}+\Gamma_{23}\cdot V_2\right\}\cdot\{V_1\cdot\Gamma_{41}+\Gamma_{42}\}$$
$$-\Gamma_{41}\cdot\left\{\sum_{n=1}^{3}V_1^{3-n}\cdot(\Gamma_{1n}-\Gamma_{3n})+\Gamma_{21}\cdot V_1+\Gamma_{22}\cdot V_2+\Gamma_{23}\cdot V_1\cdot V_2+\Gamma_{24}\right\}$$
$$=V_1^2\cdot\Omega_1+V_1\cdot\Omega_2+V_2\cdot\Omega_3+V_1\cdot V_2\cdot\Omega_4+\Omega_5$$

$$\frac{\partial \mathrm{g}_1}{\partial V_1}=\frac{\frac{\partial\xi_1}{\partial V_1}\cdot\xi_2-\frac{\partial\xi_2}{\partial V_1}\cdot\xi_1}{\xi_2^2}=\frac{V_1^2\cdot\Omega_1+V_1\cdot\Omega_2+V_2\cdot\Omega_3+V_1\cdot V_2\cdot\Omega_4+\Omega_5}{V_1^2\cdot\Gamma_{41}^2+2\cdot V_1\cdot\Gamma_{41}\cdot\Gamma_{42}+\Gamma_{42}^2}$$

$$\frac{dV_2}{dt} = h_1(V_1, V_2);\ \mathrm{h}_1(V_1, V_2) = \frac{\xi_4(V_1, V_2)}{\xi_5(V_2)} \Rightarrow h_1 = \frac{\xi_4}{\xi_5};\ \frac{\partial h_1}{\partial V_2} = \frac{\frac{\partial \xi_4}{\partial V_2} \cdot \xi_5 - \xi_4 \cdot \frac{\partial \xi_5}{\partial V_2}}{\xi_5^2}$$

We define the following functions: $\xi_4 = \xi_4(V_1, V_2);\ \xi_5 = \xi_5(V_2)$

$$\xi_4 = \sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24};\ \xi_5 = V_2 \cdot \Xi_{41} + \Xi_{42}$$

$$\frac{\partial \xi_4}{\partial V_2} = \sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1;\ \frac{\partial \xi_4}{\partial V_1} = \Xi_{22} + \Xi_{23} \cdot V_2$$

$\frac{\partial \xi_5}{\partial V_2} = \Xi_{41};\ \frac{\partial \xi_5}{\partial V_1} = 0$. We get the following expression for $\frac{\partial h_1}{\partial V_2}$

$$\frac{\partial h_1}{\partial V_2} = \frac{\frac{\partial \xi_4}{\partial V_2} \cdot \xi_5 - \xi_4 \cdot \frac{\partial \xi_5}{\partial V_2}}{\xi_5^2}$$

$$= \frac{\left\{\sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1\right\} \cdot \{V_2 \cdot \Xi_{41} + \Xi_{42}\} - \left\{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}\right\} \cdot \Xi_{41}}{(V_2 \cdot \Xi_{41} + \Xi_{42})^2}$$

First, we take care of the numerator of the expression.

$$\begin{aligned}
&\left\{\sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1\right\} \cdot \{V_2 \cdot \Xi_{41} + \Xi_{42}\} \\
&- \left\{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}\right\} \cdot \Xi_{41} \\
&= \sum_{n=1}^{3} (3-n) \cdot V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) \cdot \Xi_{41} + \sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) \cdot \Xi_{42} \\
&\quad + V_2 \cdot \Xi_{41} \cdot \Xi_{21} + \Xi_{42} \cdot \Xi_{21} + \Xi_{23} \cdot \Xi_{41} \cdot V_1 \cdot V_2 + \Xi_{23} \cdot \Xi_{42} \cdot V_1 - \sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) \cdot \Xi_{41} \\
&\quad - \Xi_{21} \cdot \Xi_{41} \cdot V_2 - \Xi_{22} \cdot \Xi_{41} \cdot V_1 - \Xi_{23} \cdot \Xi_{41} \cdot V_1 \cdot V_2 - \Xi_{24} \cdot \Xi_{41}
\end{aligned}$$

$$\sum_{n=1}^{3} (3-n) \cdot V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) \cdot \Xi_{41} = 2 \cdot V_2^2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{41} + V_2 \cdot (\Xi_{12} - \Xi_{32}) \cdot \Xi_{41}$$

$$\sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) \cdot \Xi_{42} = 2 \cdot V_2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{42} + (\Xi_{12} - \Xi_{32}) \cdot \Xi_{42}$$

$$\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) \cdot \Xi_{41} = V_2^2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{41} + V_2 \cdot (\Xi_{12} - \Xi_{32}) \cdot \Xi_{41} + (\Xi_{13} - \Xi_{33}) \cdot \Xi_{41}$$

We get the following numerator expression:

$$\begin{aligned}&\left\{\sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1\right\} \cdot \{V_2 \cdot \Xi_{41} + \Xi_{42}\} \\ &- \left\{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}\right\} \cdot \Xi_{41} \\ &= 2 \cdot V_2^2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{41} + V_2 \cdot (\Xi_{12} - \Xi_{32}) \cdot \Xi_{41} + 2 \cdot V_2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{42} \\ &+ (\Xi_{12} - \Xi_{32}) \cdot \Xi_{42} + V_2 \cdot \Xi_{41} \cdot \Xi_{21} + \Xi_{42} \cdot \Xi_{21} + \Xi_{23} \cdot \Xi_{41} \cdot V_1 \cdot V_2 + \Xi_{23} \cdot \Xi_{42} \cdot V_1 \\ &- V_2^2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{41} - V_2 \cdot (\Xi_{12} - \Xi_{32}) \cdot \Xi_{41} - (\Xi_{13} - \Xi_{33}) \cdot \Xi_{41} - \Xi_{21} \cdot \Xi_{41} \cdot V_2 \\ &- \Xi_{22} \cdot \Xi_{41} \cdot V_1 - \Xi_{23} \cdot \Xi_{41} \cdot V_1 \cdot V_2 - \Xi_{24} \cdot \Xi_{41}\end{aligned}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1\right\} \cdot \{V_2 \cdot \Xi_{41} + \Xi_{42}\} \\ &- \left\{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}\right\} \cdot \Xi_{41} \\ &= V_2^2 \cdot \Xi_{41} \cdot \{2 \cdot (\Xi_{11} - \Xi_{31}) - (\Xi_{11} - \Xi_{31})\} \\ &+ V_2 \cdot \{(\Xi_{12} - \Xi_{32}) \cdot \Xi_{41} + 2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{42} + \Xi_{41} \cdot \Xi_{21} \\ &- (\Xi_{12} - \Xi_{32}) \cdot \Xi_{41} - \Xi_{21} \cdot \Xi_{41}\} \\ &+ V_1 \cdot V_2 \cdot \{\Xi_{23} \cdot \Xi_{41} - \Xi_{23} \cdot \Xi_{41}\} + V_1 \cdot \{\Xi_{23} \cdot \Xi_{42} - \Xi_{22} \cdot \Xi_{41}\} \\ &- \Xi_{24} \cdot \Xi_{41} + (\Xi_{12} - \Xi_{32}) \cdot \Xi_{42} + \Xi_{42} \cdot \Xi_{21} - (\Xi_{13} - \Xi_{33}) \cdot \Xi_{41}\end{aligned}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1\right\} \cdot \{V_2 \cdot \Xi_{41} + \Xi_{42}\} \\ &- \left\{\sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24}\right\} \cdot \Xi_{41} \\ &= V_2^2 \cdot \Xi_{41} \cdot \{\Xi_{11} - \Xi_{31}\} + V_2 \cdot \{2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{42}\} + V_1 \cdot \{\Xi_{23} \cdot \Xi_{42} - \Xi_{22} \cdot \Xi_{41}\} \\ &- \Xi_{24} \cdot \Xi_{41} + (\Xi_{12} - \Xi_{32}) \cdot \Xi_{42} + \Xi_{42} \cdot \Xi_{21} - (\Xi_{13} - \Xi_{33}) \cdot \Xi_{41}\end{aligned}$$

We define the following parameters: $\Upsilon_1 = \Xi_{41} \cdot \{\Xi_{11} - \Xi_{31}\}; \Upsilon_2 = 2 \cdot (\Xi_{11} - \Xi_{31}) \cdot \Xi_{42}$

$$\Upsilon_3 = \Xi_{23} \cdot \Xi_{42} - \Xi_{22} \cdot \Xi_{41}; \Upsilon_4 = -\Xi_{24} \cdot \Xi_{41} + (\Xi_{12} - \Xi_{32}) \cdot \Xi_{42} + \Xi_{42} \cdot \Xi_{21} - (\Xi_{13} - \Xi_{33}) \cdot \Xi_{41}$$

We get the following numerator expression:

$$\left\{ \sum_{n=1}^{3} (3-n) \cdot V_2^{2-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} + \Xi_{23} \cdot V_1 \right\} \cdot \{V_2 \cdot \Xi_{41} + \Xi_{42}\}$$
$$- \left\{ \sum_{n=1}^{3} V_2^{3-n} \cdot (\Xi_{1n} - \Xi_{3n}) + \Xi_{21} \cdot V_2 + \Xi_{22} \cdot V_1 + \Xi_{23} \cdot V_1 \cdot V_2 + \Xi_{24} \right\} \cdot \Xi_{41}$$
$$= V_2^2 \cdot \Upsilon_1 + V_2 \cdot \Upsilon_2 + V_1 \cdot \Upsilon_3 + \Upsilon_4$$

$$\frac{\partial h_1}{\partial V_2} = \frac{\frac{\partial \xi_4}{\partial V_2} \cdot \xi_5 - \xi_4 \cdot \frac{\partial \xi_5}{\partial V_2}}{\xi_5^2} = \frac{V_2^2 \cdot \Upsilon_1 + V_2 \cdot \Upsilon_2 + V_1 \cdot \Upsilon_3 + \Upsilon_4}{V_2^2 \cdot \Xi_{41}^2 + 2 \cdot V_2 \cdot \Xi_{41} \cdot \Xi_{42} + \Xi_{42}^2}$$

We can summarize our last results for $\frac{\partial g_1}{\partial V_1}$ and $\frac{\partial h_1}{\partial V_2}$

$$\frac{\partial g_1}{\partial V_1} = \frac{\frac{\partial \xi_1}{\partial V_1} \cdot \xi_2 - \frac{\partial \xi_2}{\partial V_1} \cdot \xi_1}{\xi_2^2} = \frac{V_1^2 \cdot \Omega_1 + V_1 \cdot \Omega_2 + V_2 \cdot \Omega_3 + V_1 \cdot V_2 \cdot \Omega_4 + \Omega_5}{V_1^2 \cdot \Gamma_{41}^2 + 2 \cdot V_1 \cdot \Gamma_{41} \cdot \Gamma_{42} + \Gamma_{42}^2}$$
$$\frac{\partial h_1}{\partial V_2} = \frac{\frac{\partial \xi_4}{\partial V_2} \cdot \xi_5 - \xi_4 \cdot \frac{\partial \xi_5}{\partial V_2}}{\xi_5^2} = \frac{V_2^2 \cdot \Upsilon_1 + V_2 \cdot \Upsilon_2 + V_1 \cdot \Upsilon_3 + \Upsilon_4}{V_2^2 \cdot \Xi_{41}^2 + 2 \cdot V_2 \cdot \Xi_{41} \cdot \Xi_{42} + \Xi_{42}^2}$$

We need to find $\left(\frac{\partial g_1}{\partial V_1} + \frac{\partial h_1}{\partial V_2}\right)|_{V=\phi}$ at a solution with constant radius ($\mathrm{d}r/\mathrm{d}t = 0$). We already found the solution with constant radius:

$$r_{\text{const}} = r_{1,2} = \frac{-g(\theta) \pm \sqrt{g^2(\theta) + 4 \cdot \Delta_1 \cdot \Delta_3}}{2 \cdot \Delta_1} > 0 \text{ and find } \left(\frac{\partial g_1}{\partial V_1} + \frac{\partial h_1}{\partial V_2}\right)|_{r=r_{\text{const}}}$$
$$\rho_1 \cdot \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot \mathrm{d}s}|_{\rho_1=1} \Rightarrow \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot \mathrm{d}s} = \mathrm{e}^{\int_0^{T=\frac{2 \cdot \pi \cdot (\Xi_{41} - \Gamma_{41})}{\Gamma_{23} + \Xi_{23}}} \left(\frac{\partial g_1}{\partial V_1} + \frac{\partial h_1}{\partial V_2}\right)|_{r=r_{\text{const}}} \cdot \mathrm{d}s}$$

As a result, the limit cycle with radius $r_{\text{const}} = r_{1,2} = \frac{-g(\theta) \pm \sqrt{g^2(\theta) + 4 \cdot \Delta_1 \cdot \Delta_3}}{2 \cdot \Delta_1} > 0$ is stable if $\int_0^T tr(A(s)) \cdot \mathrm{d}s < 0$ or $\rho_2 < 1$ and unstable if $\int_0^T tr(A(s)) \cdot \mathrm{d}s > 0$ or $\rho_2 > 1$.

Remark In the last analysis, we consider system without periodic sources $X_1(t) = X_2(t) = 0$. It is reader's exercise to do the limit cycle stability analysis for all other cases $X_1(t) \neq X_2(t) \neq 0$ by using Floquet theory [46–48].

4.4 Optoisolation Circuit Second-Order ODE with Periodic Source

We have second-order ODE system with periodic source $a(t)$, where $a(t)$ is periodic with period T. The system has two main variables $X_1(t)$ and $V(t)$ which is the circuit output voltage in time. Our system's second-order ODE equations are

$$\frac{dV(t)}{dt} = \Gamma_1 \cdot \frac{d^2X_1(t)}{dt^2} + \Gamma_2 \cdot X_1(t); \; \frac{dX_1(t)}{dt} = \psi\left(V(t), \frac{dV(t)}{dt}, a(t), \frac{\partial a(t)}{\partial t}, \ldots\right)$$

First, we present our optoisolation circuit second-order ODE with periodic source. We have variable capacitor ($C1$) and variable inductor ($L1$) in series at the output circuit. The feedback output voltage is multiple by input periodic source $a(t)$, with period T and feed to input LED $D1$ circuit (Fig. 4.2).

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$$I_{C1} = I_{L1}; \; I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt}; \; V_{L1} = L_1 \cdot \frac{dI_{L1}}{dt}; \; V(t) = V_{C1} + V_{L1}$$
$$= L_1 \cdot \frac{dI_{C1}}{dt} + \frac{1}{C_1} \cdot \int I_{C1} \cdot dt$$

$$V_{CEQ1} = V(t) = V; \; I_{RCQ1} = I_{CQ1} + I_{C1} = \frac{V_{cc} - V}{R_{CQ1}} \Rightarrow I_{CQ1} = \frac{V_{cc} - V}{R_{CQ1}} - I_{C1}$$

The multiplication element ([]*[]) is implemented by using op-amps, resistors, capacitors, diodes, etc. Multiplication element's input current is zero since its input impedance is infinite $I_{in\{[]*[]\}} \to \varepsilon$; $R_{in\{[]*[]\}} \to \infty$; $I_{D1} = I_{R1}$.

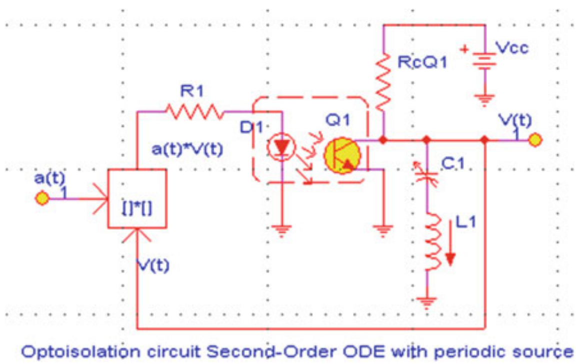

Fig. 4.2 Optoisolation circuit second-order ODE with periodic source

$$V(t)\cdot a(t) = V_{R1} + V_{D1} = I_{D1}\cdot R_1 + V_{D1};\ V_{D1} = V_t\cdot \ln\left\{\frac{I_{D1}}{I_0}+1\right\} \Rightarrow V(t)\cdot a(t) = I_{D1}\cdot R_1 + V_t\cdot \ln\left\{\frac{I_{D1}}{I_0}+1\right\}$$

By using Taylor series approximation $\ln\left\{\frac{I_{D1}}{I_0}+1\right\} \approx \frac{I_{D1}}{I_0} \Rightarrow V(t)\cdot a(t) = I_{D1}\cdot R_1 + V_t\cdot \frac{I_{D1}}{I_0}$.

$$V(t)\cdot a(t) = I_{D1}\cdot R_1 + V_t\cdot\frac{I_{D1}}{I_0} = I_{D1}\cdot\left\{R_1+\frac{V_t}{I_0}\right\} \Rightarrow I_{D1} = \frac{V(t)\cdot a(t)}{R_1+\frac{V_t}{I_0}}\Big|_{V=V(t)} = \frac{V\cdot a(t)}{R_1+\frac{V_t}{I_0}}$$

$$I_{BQ1} = k_1\cdot I_{D1} = \frac{k_1\cdot V(t)\cdot a(t)}{R_1+\frac{V_t}{I_0}}\Big|_{V=V(t)} = \frac{k_1\cdot V\cdot a(t)}{R_1+\frac{V_t}{I_0}};\ I_{EQ1} = I_{BQ1}+I_{CQ1} = \frac{k_1\cdot V\cdot a(t)}{R_1+\frac{V_t}{I_0}} + I_{CQ1}$$

$$I_{EQ1} = I_{BQ1}+I_{CQ1} = \frac{k_1\cdot V\cdot a(t)}{R_1+\frac{V_t}{I_0}} + I_{CQ1} = \frac{k_1\cdot V\cdot a(t)}{R_1+\frac{V_t}{I_0}} + \frac{V_{cc}-V}{R_{CQ1}} - I_{C1};$$
$$I_{CQ1} = \frac{V_{cc}-V}{R_{CQ1}} - I_{C1}$$

$$V(t) = V = V_{CEQ1} = V_t\cdot\ln\left\{\frac{(\alpha_{r1}\cdot I_{CQ1}-I_{EQ1})+I_{se}\cdot(\alpha_{r1}\cdot\alpha_{f1}-1)}{(I_{CQ1}-I_{EQ1}\cdot\alpha_{f1})+I_{sc}\cdot(\alpha_{r1}\cdot\alpha_{f1}-1)}\right\} + V_t\cdot\ \ln\left(\frac{I_{sc}}{I_{se}}\right);$$
$$\frac{I_{sc}}{I_{se}} \to 1 \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$\ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon \Rightarrow V \simeq V_t\cdot\ln\left\{\frac{(\alpha_{r1}\cdot I_{CQ1}-I_{EQ1})+I_{se}\cdot(\alpha_{r1}\cdot\alpha_{f1}-1)}{(I_{CQ1}-I_{EQ1}\cdot\alpha_{f1})+I_{sc}\cdot(\alpha_{r1}\cdot\alpha_{f1}-1)}\right\};$$
$$V_{L1} = L_1\cdot\frac{\mathrm{d}I_{C1}}{\mathrm{d}t};\ V_{C1} = \frac{1}{C_1}\cdot\int I_{C1}\cdot\mathrm{d}t$$

$$I_{C1} = I_{L1};\ \frac{\mathrm{d}V}{\mathrm{d}t} = L_1\cdot\frac{\mathrm{d}^2 I_{C1}}{\mathrm{d}t^2} + \frac{1}{C_1}\cdot I_{C1}$$

$$\alpha_{r1}\cdot I_{CQ1} - I_{EQ1} = \alpha_{r1}\cdot\left\{\frac{V_{cc}-V}{R_{CQ1}} - I_{C1}\right\} - \left\{\frac{k_1\cdot V\cdot a(t)}{R_1+\frac{V_t}{I_0}} + \frac{V_{cc}-V}{R_{CQ1}} - I_{C1}\right\}$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \frac{\alpha_{r1} \cdot (V_{cc} - V)}{R_{CQ1}} - \alpha_{r1} \cdot I_{C1} - \frac{k_1 \cdot V \cdot a(t)}{R_1 + \frac{V_t}{I_0}} - \frac{V_{cc} - V}{R_{CQ1}} + I_{C1}$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \frac{(\alpha_{r1} - 1) \cdot (V_{cc} - V)}{R_{CQ1}} + I_{C1} \cdot (1 - \alpha_{r1}) - \frac{k_1 \cdot V \cdot a(t)}{R_1 + \frac{V_t}{I_0}}$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \frac{(\alpha_{r1} - 1) \cdot V_{cc}}{R_{CQ1}} - \frac{(\alpha_{r1} - 1) \cdot V}{R_{CQ1}} + I_{C1} \cdot (1 - \alpha_{r1}) - \frac{k_1 \cdot V \cdot a(t)}{R_1 + \frac{V_t}{I_0}}$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \frac{(\alpha_{r1} - 1) \cdot V_{cc}}{R_{CQ1}} - V \cdot \left\{ \frac{(\alpha_{r1} - 1)}{R_{CQ1}} + \frac{k_1 \cdot a(t)}{R_1 + \frac{V_t}{I_0}} \right\} + I_{C1} \cdot (1 - \alpha_{r1})$$

We define the following function: $\xi_1(V, a(t)) = \frac{(\alpha_{r1} - 1) \cdot V_{cc}}{R_{CQ1}} - V \cdot \left\{ \frac{(\alpha_{r1} - 1)}{R_{CQ1}} + \frac{k_1 \cdot a(t)}{R_1 + \frac{V_t}{I_0}} \right\}$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \xi_1(V, a(t)) + I_{C1} \cdot (1 - \alpha_{r1}) = \xi_1 + I_{C1} \cdot (1 - \alpha_{r1});$$
$$\xi_1 = \xi_1(V, a(t))$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{V_{cc} - V}{R_{CQ1}} - I_{C1} - \left\{ \frac{k_1 \cdot V \cdot a(t)}{R_1 + \frac{V_t}{I_0}} + \frac{V_{cc} - V}{R_{CQ1}} - I_{C1} \right\} \cdot \alpha_{f1}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{V_{cc} - V}{R_{CQ1}} + I_{C1} \cdot (\alpha_{f1} - 1) - \frac{k_1 \cdot V \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}} - \frac{(V_{cc} - V) \cdot \alpha_{f1}}{R_{CQ1}}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{(V_{cc} - V)}{R_{CQ1}} \cdot (1 - \alpha_{f1}) + I_{C1} \cdot (\alpha_{f1} - 1) - \frac{k_1 \cdot V \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} - \frac{V \cdot (1 - \alpha_{f1})}{R_{CQ1}} - I_{C1} \cdot (1 - \alpha_{f1}) - \frac{k_1 \cdot V \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} - V \cdot \left\{ \frac{(1 - \alpha_{f1})}{R_{CQ1}} + \frac{k_1 \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}} \right\} - I_{C1} \cdot (1 - \alpha_{f1})$$

$$\xi_2(V, a(t)) = \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} - V \cdot \left\{ \frac{(1 - \alpha_{f1})}{R_{CQ1}} + \frac{k_1 \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}} \right\};$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \xi_2(V, a(t)) - I_{C1} \cdot (1 - \alpha_{f1})$$

$$\xi_2 = \xi_2(V, a(t)) \Rightarrow I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \xi_2 - I_{C1} \cdot (1 - \alpha_{f1})$$

$$V \simeq V_t \cdot \ln\left\{ \frac{\xi_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\xi_2 - I_{C1} \cdot (1 - \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \right\}$$
$$\Rightarrow e^{\left[\frac{V}{V_t}\right]} = \frac{\xi_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\xi_2 - I_{C1} \cdot (1 - \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}$$

$$\xi_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)$$
$$= \left\{ \xi_2 - I_{C1} \cdot (1 - \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \right\} \cdot e^{\left[\frac{V}{V_t}\right]}$$

$$\xi_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) = \xi_2 \cdot e^{\left[\frac{V}{V_t}\right]} - I_{C1} \cdot (1 - \alpha_{f1}) \cdot e^{\left[\frac{V}{V_t}\right]}$$
$$+ I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot e^{\left[\frac{V}{V_t}\right]}$$

$$I_{C1} \cdot (1 - \alpha_{r1}) + I_{C1} \cdot (1 - \alpha_{f1}) \cdot e^{\left[\frac{V}{V_t}\right]} = \xi_2 \cdot e^{\left[\frac{V}{V_t}\right]} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot e^{\left[\frac{V}{V_t}\right]}$$
$$- I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) - \xi_1$$

$$I_{C1} \cdot \left\{ (1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot e^{\left[\frac{V}{V_t}\right]} \right\} = \xi_2 \cdot e^{\left[\frac{V}{V_t}\right]} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot e^{\left[\frac{V}{V_t}\right]} - I_{se}$$
$$\cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) - \xi_1$$

$$I_{C1} = \frac{\xi_2 \cdot e^{\left[\frac{V}{V_t}\right]} + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot e^{\left[\frac{V}{V_t}\right]} - I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) - \xi_1}{(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot e^{\left[\frac{V}{V_t}\right]}}$$

$$I_{C1} = \frac{\xi_2 \cdot e^{\left[\frac{V}{V_t}\right]} - \xi_1 + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{sc} \cdot e^{\left[\frac{V}{V_t}\right]} - I_{se} \right\}}{(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot e^{\left[\frac{V}{V_t}\right]}}$$

Taylor series approximation: $e^{\left[\frac{V}{V_t}\right]} \approx \frac{V}{V_t} + 1; \Delta = \xi_2 - \xi_1$

$$I_{C1} = \frac{\xi_2 \cdot \left(\frac{V}{V_t} + 1\right) - \xi_1 + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{sc} \cdot \left(\frac{V}{V_t} + 1\right) - I_{se} \right\}}{(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot \left(\frac{V}{V_t} + 1\right)}; \Delta = \xi_2 - \xi_1$$

$$I_{C1} = \frac{\xi_2 - \xi_1 + \xi_2 \cdot \frac{V}{V_t} + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se} \right\}}{2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}}$$
$$= \frac{\Delta + \xi_2 \cdot \frac{V}{V_t} + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se} \right\}}{2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}}$$

$$\Delta = \xi_2 - \xi_1 = \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} - V \cdot \left\{ \frac{(1 - \alpha_{f1})}{R_{CQ1}} + \frac{k_1 \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}} \right\}$$
$$- \left\{ \frac{(\alpha_{r1} - 1) \cdot V_{cc}}{R_{CQ1}} - V \cdot \left\{ \frac{(\alpha_{r1} - 1)}{R_{CQ1}} + \frac{k_1 \cdot a(t)}{R_1 + \frac{V_t}{I_0}} \right\} \right\}$$

$$\Delta = \xi_2 - \xi_1$$
$$= \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} - \frac{(\alpha_{r1} - 1) \cdot V_{cc}}{R_{CQ1}} + V$$
$$\cdot \left\{ \frac{(\alpha_{r1} - 1)}{R_{CQ1}} + \frac{k_1 \cdot a(t)}{R_1 + \frac{V_t}{I_0}} - \frac{(1 - \alpha_{f1})}{R_{CQ1}} - \frac{k_1 \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}} \right\}$$

$$\Delta = \xi_2 - \xi_1$$
$$= \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} + \frac{(1 - \alpha_{r1}) \cdot V_{cc}}{R_{CQ1}} + V$$
$$\cdot \left\{ \frac{(\alpha_{r1} - 1)}{R_{CQ1}} + \frac{k_1 \cdot a(t)}{R_1 + \frac{V_t}{I_0}} - \frac{(1 - \alpha_{f1})}{R_{CQ1}} - \frac{k_1 \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}} \right\}$$

$$\Delta = \xi_2 - \xi_1$$
$$= \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} + \frac{(1 - \alpha_{r1}) \cdot V_{cc}}{R_{CQ1}} + V$$
$$\cdot \left\{ \frac{(\alpha_{r1} + \alpha_{f1} - 2)}{R_{CQ1}} + \frac{k_1 \cdot a(t) \cdot (1 - \alpha_{f1})}{R_1 + \frac{V_t}{I_0}} \right\}$$

$$\Delta = \xi_2 - \xi_1$$
$$= \frac{(2 - \alpha_{f1} - \alpha_{r1}) \cdot V_{cc}}{R_{CQ1}} + V \cdot \frac{k_1 \cdot a(t) \cdot (1 - \alpha_{f1})}{R_1 + \frac{V_t}{I_0}} - V \cdot \frac{(2 - \alpha_{r1} - \alpha_{f1})}{R_{CQ1}}$$

We define the following parameters for simplicity:

$$\Delta = \Delta(V, a(t)) = \xi_2 - \xi_1 = A_1 - V \cdot A_2 + V \cdot a(t) \cdot A_3$$

$$A_1 = \frac{(2 - \alpha_{f1} - \alpha_{r1}) \cdot V_{cc}}{R_{CQ1}};\ A_3 = \frac{k_1 \cdot (1 - \alpha_{f1})}{R_1 + \frac{V_t}{I_0}};\ A_2 = \frac{(2 - \alpha_{r1} - \alpha_{f1})}{R_{CQ1}}$$

$$\xi_2 = \xi_2(V, a(t)) = \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}} - V \cdot \frac{(1 - \alpha_{f1})}{R_{CQ1}} - V \cdot \frac{k_1 \cdot a(t) \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}};$$

$$A_4 = \frac{V_{cc} \cdot (1 - \alpha_{f1})}{R_{CQ1}}$$

$$A_5 = -\frac{(1 - \alpha_{f1})}{R_{CQ1}};\ A_6 = -\frac{k_1 \cdot \alpha_{f1}}{R_1 + \frac{V_t}{I_0}};\ \xi_2 = \xi_2(V, a(t)) = A_4 + V \cdot A_5 + V \cdot a(t) \cdot A_6$$

$$I_{C1} = \frac{\begin{array}{c} A_1 - V \cdot A_2 + V \cdot a(t) \cdot A_3 + \{A_4 + V \cdot A_5 + V \cdot a(t) \cdot A_6\} \cdot \frac{V}{V_t} \\ + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \left\{ I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se} \right\} \end{array}}{2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}}$$

$$I_{C1} = \frac{\left\{ \begin{array}{c} A_1 - V \cdot A_2 + V \cdot a(t) \cdot A_3 + A_4 \cdot \frac{V}{V_t} + \frac{V^2}{V_t} \cdot A_5 + \frac{V^2}{V_t} \cdot a(t) \cdot A_6 \\ + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot I_{sc} \cdot \frac{V}{V_t} + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot (I_{sc} - I_{se}) \end{array} \right\}}{2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}}$$

$$I_{C1} = \frac{\left[\begin{array}{l} \frac{V^2}{V_t} \cdot (A_5 + a(t) \cdot A_6) + V \cdot \left\{ (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t} + a(t) \cdot A_3 \right\} \\ \quad + A_1 + (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot (I_{sc} - I_{se}) \end{array} \right]}{2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}}$$

$$I_{C1} = \frac{\left[\begin{array}{l} \frac{V^2}{V_t} \cdot (A_5 + a(t) \cdot A_6) + V \cdot \left\{ (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t} + a(t) \cdot A_3 \right\} \\ \quad + A_1 - (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se}) \end{array} \right]}{2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}}$$

We define the following system parameters and parameters in time for simplicity:

$$B_1 = 2 - \alpha_{r1} - \alpha_{f1};\ B_2 = \frac{(1 - \alpha_{f1})}{V_t};\ B_3(t) = \frac{1}{V_t} \cdot [A_5 + a(t) \cdot A_6]$$

$$B_4(t) = (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t} + a(t) \cdot A_3; \quad \frac{\partial B_3(t)}{\partial t} = \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t};$$

$$B_5 = A_1 - (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se}) \frac{\partial B_4(t)}{\partial t} = \frac{\partial a(t)}{\partial t} \cdot A_3; \quad I_{C1} = \frac{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5}{B_1 + B_2 \cdot V}$$

We can summarize our results for $B_i(t), \frac{\partial B_i(t)}{\partial t} (i = 1, \ldots, 5)$ expressions in Table 4.5:

$$\frac{dI_{C1}}{dt} = \frac{d}{dt}\left\{\frac{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5}{B_1 + B_2 \cdot V}\right\}$$

$$= \frac{\left\{2 \cdot V \cdot \frac{dV}{dt} \cdot B_3(t) + V^2 \cdot \frac{\partial B_3(t)}{\partial t} + \frac{dV}{dt} \cdot B_4(t) + V \cdot \frac{\partial B_4(t)}{\partial t}\right\} \cdot \{B_1 + B_2 \cdot V\} - \{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5\} \cdot B_2 \cdot \frac{dV}{dt}}{[B_1 + B_2 \cdot V]^2}$$

$$f\left(V, \frac{dV}{dt}, t\right) = \frac{\left\{2 \cdot V \cdot \frac{dV}{dt} \cdot B_3(t) + V^2 \cdot \frac{\partial B_3(t)}{\partial t} + \frac{dV}{dt} \cdot B_4(t) + V \cdot \frac{\partial B_4(t)}{\partial t}\right\} \cdot \{B_1 + B_2 \cdot V\} - \{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5\} \cdot B_2 \cdot \frac{dV}{dt}}{[B_1 + B_2 \cdot V]^2}$$

$$\frac{dI_{C1}}{dt} = \frac{d}{dt}\left\{\frac{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5}{B_1 + B_2 \cdot V}\right\} \Rightarrow \frac{dI_{C1}}{dt} = f\left(V, \frac{dV}{dt}, t\right)$$

Summary We can represent our system by two main differential equations: $\frac{dV}{dt} = L_1 \cdot \frac{d^2 I_{C1}}{dt^2} + \frac{1}{C_1} \cdot I_{C1}$; $\frac{dI_{C1}}{dt} = f\left(V, \frac{dV}{dt}, t\right)$. We define new system variables in time $X_1 = X_1(t)$; $X_2 = X_2(t)$.

$$X_2 = I_{C1};\ X_1 = \frac{dX_2}{dt} = \frac{dI_{C1}}{dt};\ \frac{dX_1}{dt} = \frac{d^2 X_2}{dt^2} = \frac{d^2 I_{C1}}{dt^2} \Rightarrow \frac{dV}{dt} = L_1 \cdot \frac{dX_1}{dt} + \frac{1}{C_1} \cdot X_2$$

$$\frac{dV}{dt} = L_1 \cdot \frac{dX_1}{dt} + \frac{1}{C_1} \cdot X_2 \Rightarrow \frac{dX_1}{dt} = \frac{1}{L_1} \cdot \left\{\frac{dV}{dt} - \frac{1}{C_1} \cdot X_2\right\}$$

We get the following system differential equations:

$$\frac{dX_1}{dt} = \frac{1}{L_1} \cdot \left\{\frac{dV}{dt} - \frac{1}{C_1} \cdot X_2\right\};\ \frac{dX_2}{dt} = X_1;\ X_2 = I_{C1} = \frac{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5}{B_1 + B_2 \cdot V}$$

$$X_2 = \frac{V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5}{B_1 + B_2 \cdot V} \Rightarrow B_1 \cdot X_2 + B_2 \cdot V \cdot X_2 = V^2 \cdot B_3(t) + V \cdot B_4(t) + B_5$$

$$V^2 \cdot B_3(t) + V \cdot [B_4(t) - B_2 \cdot X_2] + (B_5 - B_1 \cdot X_2) = 0$$

$$\Rightarrow V_{\#,\#\#} = \frac{B_2 \cdot X_2 - B_4(t) \pm \sqrt{[B_4(t) - B_2 \cdot X_2]^2 - 4 \cdot B_3(t) \cdot (B_5 - B_1 \cdot X_2)}}{2 \cdot B_3(t)}$$

Table 4.5 System parameters and parameters in time

$B_i(t)\ (i = 1,\ldots,5)$	$\frac{\partial B_i(t)}{\partial t}\ (i = 1,\ldots,5)$
$B_1 = 2 - \alpha_{r1} - \alpha_{f1}$	$\frac{\partial B_1}{\partial t} = 0$
$B_2 = \frac{(1-\alpha_{f1})}{V_t}$	$\frac{\partial B_2}{\partial t} = 0$
$B_3(t) = \frac{1}{V_t} \cdot [A_5 + a(t) \cdot A_6]$	$\frac{\partial B_3(t)}{\partial t} = \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t}$
$B_4(t) = (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t} + a(t) \cdot A_3$	$\frac{\partial B_4(t)}{\partial t} = \frac{\partial a(t)}{\partial t} \cdot A_3$
$B_5 = A_1 - (1 - \alpha_{r1} \cdot \alpha_{f1}) \cdot (I_{sc} - I_{se})$	$\frac{\partial B_5}{\partial t} = 0$

We define the following functions: $\chi_1(t) = B_4(t) - B_2 \cdot X_2$; $\chi_2(t) = 4 \cdot B_3(t) \cdot (B_5 - B_1 \cdot X_2)$.

$$\chi_3(t) = 2 \cdot B_3(t);\ V_{\#,\#\#} = \frac{-\chi_1(t) \pm \sqrt{[\chi_1(t)]^2 - \chi_2(t)}}{\chi_3(t)}$$

$$V_{\#,\#\#} = \frac{-\chi_1(t) \pm \sqrt{[\chi_1(t)]^2 - \chi_2(t)}}{\chi_3(t)} = \frac{-\chi_1(t) \pm \{[\chi_1(t)]^2 - \chi_2(t)\}^{\frac{1}{2}}}{\chi_3(t)}$$

$$\begin{aligned}
\dot{\chi}_1(t) &= \frac{\partial B_4(t)}{\partial t} - B_2 \cdot \frac{\mathrm{d}X_2}{\mathrm{d}t} = \frac{\partial B_4(t)}{\partial t} - B_2 \cdot X_1; \\
\dot{\chi}_2(t) &= 4 \cdot \frac{\partial B_3(t)}{\partial t} \cdot (B_5 - B_1 \cdot X_2) - 4 \cdot B_3(t) \cdot B_1 \cdot \frac{\mathrm{d}X_2}{\mathrm{d}t} \dot{\chi}_2(t) \\
&= 4 \cdot \frac{\partial B_3(t)}{\partial t} \cdot (B_5 - B_1 \cdot X_2) - 4 \cdot B_3(t) \cdot B_1 \cdot X_1; \\
\dot{\chi}_3(t) &= 2 \cdot \frac{\partial B_3(t)}{\partial t} = 2 \cdot \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\dot{\chi}_1(t) &= \frac{\partial B_4(t)}{\partial t} - B_2 \cdot \frac{\mathrm{d}X_2}{\mathrm{d}t} = \frac{\partial a(t)}{\partial t} \cdot A_3 - B_2 \cdot X_1; \dot{\chi}_2(t) \\
&= 4 \cdot \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t} \cdot (B_5 - B_1 \cdot X_2) - 4 \cdot B_3(t) \cdot B_1 \cdot X_1
\end{aligned}$$

We can summarize our $\chi_i, \dot{\chi}_i\ (i = 1, 2, 3)$ expressions in the Table 4.6:

$$\begin{aligned}
\frac{\mathrm{d}V_{\#,\#\#}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{-\chi_1(t) \pm \{[\chi_1(t)]^2 - \chi_2(t)\}^{\frac{1}{2}}}{\chi_3(t)}\right\} \\
&= \frac{\left\{-\dot{\chi}_1 \pm \frac{1}{2} \cdot ([\chi_1]^2 - \chi_2)^{-\frac{1}{2}} \cdot (2 \cdot \chi_1 \cdot \dot{\chi}_1 - \dot{\chi}_2)\right\} \cdot \chi_3 - \dot{\chi}_3 \cdot \left\{-\chi_1 \pm \{[\chi_1]^2 - \chi_2\}^{\frac{1}{2}}\right\}}{[\chi_3]^2}
\end{aligned}$$

We define the following function: $\frac{\mathrm{d}v_{\#,\#\#}}{\mathrm{d}t} = g\left(X_1, X_2, a(t), \frac{\partial a(t)}{\partial t}\right)$

Table 4.6 Summary χ_i and $\dot{\chi}_i$ expressions

$\chi_i(i=1,2,3)$	$\dot{\chi}_i(i=1,2,3)$
$\chi_1(t)=B_4(t)-B_2\cdot X_2$	$\dot{\chi}_1(t)=\frac{\partial B_4(t)}{\partial t}-B_2\cdot\frac{\mathrm{d}X_2}{\mathrm{d}t}=\frac{\partial a(t)}{\partial t}\cdot A_3-B_2\cdot X_1$ $\dot{\chi}_1(t)=\frac{\partial B_4(t)}{\partial t}-B_2\cdot\frac{\mathrm{d}X_2}{\mathrm{d}t}=\frac{\partial a(t)}{\partial t}\cdot A_3-B_2\cdot X_1$
$\chi_2(t)=4\cdot B_3(t)\cdot(B_5-B_1\cdot X_2)$	$\dot{\chi}_2(t)=4\cdot\frac{\partial B_3(t)}{\partial t}\cdot(B_5-B_1\cdot X_2)-4\cdot B_3(t)\cdot B_1\cdot X_1$ $\dot{\chi}_2(t)=4\cdot\frac{A_6}{V_t}\cdot\frac{\partial a(t)}{\partial t}\cdot(B_5-B_1\cdot X_2)-4\cdot B_3(t)\cdot B_1\cdot X_1$
$\chi_3(t)=2\cdot B_3(t)$	$\dot{\chi}_3(t)=2\cdot\frac{\partial B_3(t)}{\partial t}=2\cdot\frac{A_6}{V_t}\cdot\frac{\partial a(t)}{\partial t}$

$$g=g\left(X_1,X_2,a(t),\frac{\partial a(t)}{\partial t}\right)$$

$$g\left(X_1,X_2,a(t),\frac{\partial a(t)}{\partial t}\right)=\frac{\left\{-\dot{\chi}_1\pm\frac{1}{2}\cdot([\chi_1]^2-\chi_2)^{-\frac{1}{2}}\cdot(2\cdot\chi_1\cdot\dot{\chi}_1-\dot{\chi}_2)\right\}\cdot\chi_3-\dot{\chi}_3\cdot\left\{-\chi_1(t)\pm\left\{[\chi_1(t)]^2-\chi_2(t)\right\}^{\frac{1}{2}}\right\}}{[\chi_3]^2}$$

$$\frac{\mathrm{d}X_1}{\mathrm{d}t}=\frac{1}{L_1}\cdot\left\{\frac{\mathrm{d}V_{\#,\#\#}}{\mathrm{d}t}-\frac{1}{C_1}\cdot X_2\right\};\ \frac{\mathrm{d}X_2}{\mathrm{d}t}=X_1\Rightarrow\frac{\mathrm{d}X_2}{\mathrm{d}t}=X_1;\ \frac{\mathrm{d}X_1}{\mathrm{d}t}=\frac{1}{L_1}\cdot\left\{g(X_1,X_2,a(t),\ldots)-\frac{1}{C_1}\cdot X_2\right\}$$

4.5 Optoisolation Circuit Second-Order ODE with Periodic Source Stability of a Limit Cycle

We discuss in this subchapter, optoisolation circuit second order with periodic source, limit cycle stability [85]. First, we need to prove that the system has periodic orbits and it is done by changing system cartesian coordinates ($X_1(t)$, $X_2(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next, we show that the cylinder is invariant. We approve that there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers. One of them is unity. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z-axis. In our system, we refer to Cartesian X_1–X_2 plane (with equation $X_3 = 0$). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X_1, X_2) are the same as for polar coordinates, namely $X_1(t) = r(t)\cdot\cos[\theta(t)]$; $X_2(t) = r$

$(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X_1^2 + X_2^2}$. $\theta(t) = 0$ if $X_1 = 0$ and $X_2 = 0$. $\theta(t) = \arcsin(X_2/r)$ if $X_1 \geq 0$.

$\theta(t) = -\arcsin(X_2/r) + \pi$ if $X_1 < 0$. We represent our system equation: $\frac{dX_1}{dt} = \frac{1}{L_1} \cdot \left\{ g(X_1, X_2, a(t), \ldots) - \frac{1}{C_1} \cdot X_1 \right\}$; $\frac{dX_2}{dt} = X_1$ by using cylindrical coordinates ($r(t)$, $\theta(t)$) [78–80].

$$X_1(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{dX_1(t)}{dt} = \frac{dr(t)}{dt} \cdot \cos[\theta(t)] - r(t) \cdot \frac{d\theta(t)}{dt} \cdot \sin[\theta(t)]$$
$$X_2(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{dX_2(t)}{dt} = \frac{dr(t)}{dt} \cdot \sin[\theta(t)] + r(t) \cdot \frac{d\theta(t)}{dt} \cdot \cos[\theta(t)]$$
$$\frac{dX_1(t)}{dt} = \frac{dX_1}{dt}; \frac{dr(t)}{dt} = r'; \frac{d\theta(t)}{dt} = \theta'; \theta(t) = \theta; r(t) = r$$

We get the equations: $\frac{dX_1}{dt} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta$; $\frac{dX_2}{dt} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$

$$\frac{dX_2}{dt} = X_1 \Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = r \cdot \cos\theta \Rightarrow r' \cdot \sin\theta = -r \cdot \theta' \cdot \cos\theta + r \cdot \cos\theta$$
$$r' \cdot \sin\theta = -r \cdot \theta' \cdot \cos\theta + r \cdot \cos\theta \Rightarrow r' \cdot \sin\theta = r \cdot \cos\theta \cdot (1 - \theta') \Rightarrow tg\theta = \frac{r \cdot (1 - \theta')}{r'}$$
$$tg\theta = \frac{r \cdot (1 - \theta')}{r'} \Rightarrow \theta = arctg\{\frac{r \cdot (1 - \theta')}{r'}\}; X_1 = r \cdot \cos\theta; X_2 = r \cdot \sin\theta$$

We can summarize our $\chi_i, \dot{\chi}_i (i = 1, 2, 3)$ cylindrical coordinates ($r(t)$, $\theta(t)$) expressions in Table 4.7:

We carry out our analysis first for system with constant source: $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$.

We can summarize our results for $B_i(t), \frac{\partial B_i(t)}{\partial t}$ $(i = 1, \ldots, 5)$ expressions with constant source: $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$ in Table 4.8:.

We can summarize our $\chi_i, \dot{\chi}_i (i = 1, 2, 3)$ cylindrical coordinates ($r(t)$, $\theta(t)$) expressions with constant source: $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$ in Table 4.9:.

To find our solution constant radius for system with constant source: $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$ we set $\frac{dr}{dt} = 0$ which yield the following outcome:

$$\frac{dX_2}{dt} = X_1 \Rightarrow tg\theta = \frac{r \cdot (1 - \theta')}{r'} \Rightarrow r' = \frac{r \cdot (1 - \theta')}{tg\theta}; \frac{dr}{dt} = 0 \Leftrightarrow \frac{r \cdot (1 - \theta')}{tg\theta} = 0$$

Since $r \neq 0$, we get two possible (1) $tg\theta \to \infty \Rightarrow \theta = \frac{\pi}{2} + \pi \cdot n = \pi \cdot (\frac{1}{2} + n) \, \forall \, n = 0, 1, 2, \ldots$.

Table 4.7 Cylindrical coordinates ($r(t)$, $\theta(t)$) expressions

$\chi_i (i = 1, 2, 3)$	$\dot{\chi}_i (i = 1, 2, 3)$
$\chi_1(t) = B_4(t) - B_2 \cdot X_2$ $\chi_1(t) = B_4(t) - B_2 \cdot r \cdot \sin\theta$ $\chi_1(t) = \{(\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t}$ $+ a(t) \cdot A_3\} - B_2 \cdot r \cdot \sin\theta$	$\dot{\chi}_1(t) = \frac{\partial a(t)}{\partial t} \cdot A_3 - B_2 \cdot X_1 = \frac{\partial a(t)}{\partial t} \cdot A_3 - B_2 \cdot r \cdot \cos\theta$
$\chi_2(t) = 4 \cdot B_3(t) \cdot (B_5 - B_1 \cdot X_2)$ $\chi_2(t) = 4 \cdot B_3(t) \cdot (B_5 - B_1 \cdot r \cdot \sin\theta)$ $\chi_2(t) = 4 \cdot \frac{1}{V_t} \cdot [A_5 + a(t) \cdot A_6] \cdot (B_5 - B_1 \cdot r \cdot \sin\theta)$	$\dot{\chi}_2(t) = 4 \cdot \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t} \cdot (B_5 - B_1 \cdot X_2) - 4 \cdot B_3(t) \cdot B_1 \cdot X_1$ $\dot{\chi}_2(t) = 4 \cdot \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t} \cdot (B_5 - B_1 \cdot r \cdot \sin\theta)$ $- 4 \cdot B_3(t) \cdot B_1 \cdot r \cdot \cos\theta$
$\chi_3(t) = 2 \cdot \frac{1}{V_t} \cdot [A_5 + a(t) \cdot A_6]$	$\dot{\chi}_3(t) = 2 \cdot \frac{\partial B_3(t)}{\partial t} = 2 \cdot \frac{A_6}{V_t} \cdot \frac{\partial a(t)}{\partial t}$

Table 4.8 Analysis first for system with constant source

$B_i(t)(i=1,\ldots,5);\ a(t)=1;\ \frac{\partial a(t)}{\partial t}=0$	$\frac{\partial B_i(t)}{\partial t}\ (i=1,\ldots,5);\ a(t)=1;\ \frac{\partial a(t)}{\partial t}=0$
$B_1=2-\alpha_{r1}-\alpha_{f1}$	$\frac{\partial B_1}{\partial t}=0$
$B_2=\frac{(1-\alpha_{f1})}{V_t}$	$\frac{\partial B_2}{\partial t}=0$
$B_3(t)=\frac{A_5+A_6}{V_t}$	$\frac{\partial B_3(t)}{\partial t}=0$
$B_4(t)=(\alpha_{r1}\cdot\alpha_{f1}-1)\cdot\frac{I_{sc}}{V_t}-A_2+\frac{A_4}{V_t}+A_3$	$\frac{\partial B_4(t)}{\partial t}=0$
$B_5=A_1-(1-\alpha_{r1}\cdot\alpha_{f1})\cdot(I_{sc}-I_{se})$	$\frac{\partial B_5}{\partial t}=0$

(2) $1-\theta'=0\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}t}=1\Rightarrow\theta=t+\text{const}\ ;\ \frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{2\cdot\pi}{T}=1\Rightarrow T=2\cdot\pi.$

We need to find $g(X_1,X_2,a(t),\ldots)$ in cylindrical coordinates ($r(t)$, $\theta(t)$) for system with constant source: $a(t)=1;\frac{\partial a(t)}{\partial t}=0$. The transformation: $g(X_1,X_2,a(t)=1,\ldots)\rightarrow g(r(t),\theta(t),a(t)=1,\ldots)$. We already got $g(X_1, X_2,\ldots)$ expression:

$$g\left(X_1,X_2,a(t),\frac{\partial a(t)}{\partial t}\right)=\frac{\left\{-\dot{\chi}_1\pm\frac{1}{2}\cdot([\chi_1]^2-\chi_2)^{-\frac{1}{2}}\cdot(2\cdot\chi_1\cdot\dot{\chi}_1-\dot{\chi}_2)\right\}\cdot\chi_3-\dot{\chi}_3\cdot\left\{-\chi_1(t)\pm\left\{[\chi_1(t)]^2-\chi_2(t)\right\}^{\frac{1}{2}}\right\}}{[\chi_3]^2}$$

We define new parameters for system with constant source $a(t)=1;\ \frac{\partial a(t)}{\partial t}=0$

$$\Omega_i\{a(t)=0;\frac{\partial a(t)}{\partial t}=0\}\,\forall\, i=1,2,3;\ \Omega_1=(\alpha_{r1}\cdot\alpha_{f1}-1)\cdot\frac{I_{sc}}{V_t}-A_2+\frac{A_4}{V_t}+A_3;$$
$$\Omega_2=4\cdot\frac{1}{V_t}\cdot[A_5+A_6]\cdot B_5$$

$$([\chi_1]^2-\chi_2)^{-\frac{1}{2}}|_{\substack{X_1=r\cdot\cos\theta\\X_2=r\cdot\sin\theta}}=(r^2\cdot B_2^2\cdot\sin^2\theta+r\cdot\sin\theta\cdot\{\Omega_3-2\cdot\Omega_1\cdot B_2\}+\Omega_1^2-\Omega_2)^{-\frac{1}{2}}$$

$$2\cdot\chi_1\cdot\dot{\chi}_1-\dot{\chi}_2|_{\substack{X_1=r\cdot\cos\theta\\X_2=r\cdot\sin\theta}}=-2\cdot(\Omega_1-B_2\cdot r\cdot\sin\theta)\cdot B_2\cdot r\cdot\cos\theta+\Omega_3\cdot r\cdot\cos\theta$$

$$2\cdot\chi_1\cdot\dot{\chi}_1-\dot{\chi}_2|_{\substack{X_1=r\cdot\cos\theta\\X_2=r\cdot\sin\theta}}=r^2\cdot B_2^2\cdot\ \sin 2\theta+r\cdot\ \cos\theta\cdot\{\Omega_3-2\cdot\Omega_1\cdot B_2\}$$
$$=r^2\cdot B_2^2\cdot\ \sin 2\theta+r\cdot\imath_1\cdot\ \cos\theta$$

Table 4.9 Cylindrical coordinates ($r(t)$, $\theta(t)$) expressions with constant source

$\chi_i\,(i = 1,2,3);\ a(t) = 1;\ \frac{\partial a(t)}{\partial t} = 0$	$\dot{\chi}_i\,(i = 1,2,3);\ a(t) = 1;\ \frac{\partial a(t)}{\partial t} = 0$
$\chi_1(t) = \left\{(\alpha_{r1}\cdot\alpha_{f1} - 1)\cdot\frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t}\right\} - B_2\cdot X_2$ $\chi_1(t) = \left\{(\alpha_{r1}\cdot\alpha_{f1} - 1)\cdot\frac{I_{sc}}{V_t} - A_2 + \frac{A_4}{V_t} + A_3\right\}$ $- B_2\cdot r\cdot \sin\theta$	$\dot{\chi}_1(t) = \frac{\partial a(t)}{\partial t}\cdot A_3 - B_2\cdot X_1 = -B_2\cdot r\cdot \cos\theta$
$\chi_2(t) = 4\cdot B_3(t)\cdot(B_5 - B_1\cdot X_2)$ $\chi_2(t) = 4\cdot\frac{1}{V_t}\cdot[A_5 + A_6]\cdot(B_5 - B_1\cdot r\cdot\sin\theta)$ $= 4\cdot\frac{1}{V_t}\cdot[A_5 + A_6]\cdot B_5 - 4\cdot\frac{1}{V_t}\cdot[A_5 + A_6]\cdot B_1\cdot r\cdot\sin\theta$	$\dot{\chi}_2(t) = 4\cdot\frac{A_6}{V_t}\cdot\frac{\partial a(t)}{\partial t}\cdot(B_5 - B_1\cdot r\cdot\sin\theta)$ $- 4\cdot B_3(t)\cdot B_1\cdot r\cdot\cos\theta$ $\dot{\chi}_2(t) = -4\cdot\left[\frac{A_5 + A_6}{V_t}\right]\cdot B_1\cdot r\cdot\cos\theta$
$\chi_3(t) = 2\cdot\frac{1}{V_t}\cdot[A_5 + A_6]$	$\dot{\chi}_3(t) = 2\cdot\frac{\partial B_3(t)}{\partial t} = 0$

$$\left\{[\chi_1(t)]^2 - \chi_2(t)\right\}^{\frac{1}{2}}\Big|_{\substack{X_1 = r\cdot\cos\theta \\ X_2 = r\cdot\sin\theta}} = (r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2)^{\frac{1}{2}}$$
$$= \sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}$$

Finally we get $g\left(X_1, X_2, a(t) = 1, \frac{\partial a(t)}{\partial t} = 0\right)$ expression.

$$g\left(X_1, X_2, a(t) = 1, \frac{\partial a(t)}{\partial t} = 0\right)$$
$$= \frac{\left\{-\dot{\chi}_1 \pm \frac{1}{2}\cdot([\chi_1]^2 - \chi_2)^{-\frac{1}{2}}\cdot(2\cdot\chi_1\cdot\dot{\chi}_1 - \dot{\chi}_2)\right\}\cdot\chi_3 - \dot{\chi}_3\cdot\left\{-\chi_1(t) \pm \left\{[\chi_1(t)]^2 - \chi_2(t)\right\}^{\frac{1}{2}}\right\}}{[\chi_3]^2}$$

$$g\left(X_1, X_2, a(t) = 1, \frac{\partial a(t)}{\partial t} = 0\right)$$
$$= \frac{\left\{B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2}\cdot\frac{1}{\sqrt{r^2\cdot B_2^2\cdot\sin^2\theta + r\cdot\imath_1\cdot\sin\theta + \imath_2}}\cdot(r^2 \cdot B_2^2 \cdot \sin 2\theta + r\cdot\imath_1\cdot\cos\theta)\right\}\cdot\Omega_4}{\Omega_4^2}$$

$$g\left(X_1, X_2, a(t) = 1, \frac{\partial a(t)}{\partial t} = 0\right) = \frac{\left\{B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2}\cdot\left(\frac{r^2\cdot B_2^2\cdot\sin 2\theta + r\cdot\imath_1\cdot\cos\theta}{\sqrt{r^2\cdot B_2^2\cdot\sin^2\theta + r\cdot\imath_1\cdot\sin\theta + \imath_2}}\right)\right\}\cdot\Omega_4}{\Omega_4^2}$$

$$g\left(X_1, X_2, a(t) = 1, \frac{\partial a(t)}{\partial t} = 0\right) = \frac{B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2}\cdot\left(\frac{r^2\cdot B_2^2\cdot\sin 2\theta + r\cdot\imath_1\cdot\cos\theta}{\sqrt{r^2\cdot B_2^2\cdot\sin^2\theta + r\cdot\imath_1\cdot\sin\theta + \imath_2}}\right)}{\Omega_4}$$

$$g\left(r, \theta, a(t) = 1, \frac{\partial a(t)}{\partial t} = 0\right) = \frac{B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2}\cdot\left(\frac{r^2\cdot B_2^2\cdot\sin 2\theta + r\cdot\imath_1\cdot\cos\theta}{\sqrt{r^2\cdot B_2^2\cdot\sin^2\theta + r\cdot\imath_1\cdot\sin\theta + \imath_2}}\right)}{\Omega_4}$$

We need to represent system first differential equation by using cylindrical coordinates ($r(t)$, $\theta(t)$) for system with constant source: $a(t) = 1$; $\frac{\partial a(t)}{\partial t} = 0$

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1}\cdot\left\{g(X_1, X_2, a(t) = 1, \ldots) - \frac{1}{C_1}\cdot X_1\right\} \rightarrow \frac{\mathrm{d}X_1(r,\theta)}{\mathrm{d}t}$$
$$= \frac{1}{L_1}\cdot\left\{g(r, \theta, a(t) = 1, \ldots) - \frac{1}{C_1}\cdot X_1(r,\theta)\right\}$$

$$\frac{dX_1}{dt} = r' \cdot \cos\theta - r\cdot\theta'\cdot\sin\theta; X_1 = r\cdot\cos\theta \Rightarrow$$

$$r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = \frac{1}{L_1} \cdot \left[\left\{ \frac{B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right)}{\Omega_4} \right\} - \frac{1}{C_1} \cdot r \cdot \cos\theta \right]$$

To find our solution constant radius for system with constant source $a(t) = 1$; $\frac{\partial a(t)}{\partial t} = 0$, we set $\frac{dr}{dt} = 0$ which yields the following outcome:

$$-r \cdot \theta' \cdot L_1 \cdot \sin\theta = \frac{1}{\Omega_4} \cdot \left\{ B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right) \right\} - \frac{1}{C_1} \cdot r \cdot \cos\theta$$

$$\frac{1}{C_1} \cdot r \cdot \cos\theta - r \cdot \theta' \cdot L_1 \cdot \sin\theta = \frac{1}{\Omega_4} \cdot \left\{ B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right) \right\}$$

$$\frac{\Omega_4}{C_1} \cdot r \cdot \cos\theta - r \cdot \theta' \cdot L_1 \cdot \Omega_4 \cdot \sin\theta = B_2 \cdot r \cdot \cos\theta \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right)$$

$$\frac{\Omega_4}{C_1} \cdot r \cdot \cos\theta - r \cdot \theta' \cdot L_1 \cdot \Omega_4 \cdot \sin\theta - B_2 \cdot r \cdot \cos\theta = \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right)$$

$$r \cdot \left\{ \left(\frac{\Omega_4}{C_1} - B_2 \right) \cdot \cos\theta - \theta' \cdot L_1 \cdot \Omega_4 \cdot \sin\theta \right\} = \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right)$$

We already got for our system's second differential equation constant radius solution $\frac{dr}{dt} = 0$; $\frac{dr}{dt} = 0 \Leftrightarrow \frac{r\cdot(1-\theta')}{tg\theta} = 0 \Rightarrow \frac{d\theta}{dt} = 1 \Rightarrow \theta = t + \text{const}$

$$r \cdot \left\{ \left(\frac{\Omega_4}{C_1} - B_2 \right) \cdot \cos\theta - L_1 \cdot \Omega_4 \cdot \sin\theta \right\} = \pm \frac{1}{2} \cdot \left(\frac{r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta}{\sqrt{r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2}} \right)$$

Squaring two sides of above equation yields:

$$r^2 \cdot \left\{ \left(\frac{\Omega_4}{C_1} - B_2 \right) \cdot \cos\theta - L_1 \cdot \Omega_4 \cdot \sin\theta \right\}^2 = \frac{1}{4} \cdot \frac{[r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta]^2}{[r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2]}$$

For simplicity, we define the following functions:

$$\mathrm{H}(\theta) = \left\{ \left(\frac{\Omega_4}{C_1} - B_2 \right) \cdot \cos\theta - L_1 \cdot \Omega_4 \cdot \sin\theta \right\}^2 \Rightarrow r^2 \cdot \mathrm{H}(\theta) = \frac{1}{4} \cdot \frac{[r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta]^2}{[r^2 \cdot B_2^2 \cdot \sin^2\theta + r \cdot \imath_1 \cdot \sin\theta + \imath_2]}$$

$$r^4 \cdot 4 \cdot B_2^2 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta + 4 \cdot r^3 \cdot \imath_1 \cdot \mathrm{H}(\theta) \cdot \sin\theta + r^2 \cdot \mathrm{H}(\theta) \cdot 4 \cdot \imath_2 = [r^2 \cdot B_2^2 \cdot \sin 2\theta + r \cdot \imath_1 \cdot \cos\theta]^2$$

$$r^4 \cdot 4 \cdot B_2^2 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta + 4 \cdot r^3 \cdot \imath_1 \cdot \mathrm{H}(\theta) \cdot \sin\theta + r^2 \cdot \mathrm{H}(\theta) \cdot 4 \cdot \imath_2 = r^4 \cdot B_2^4 \cdot \sin^2 2\theta + 2 \cdot r^3 \cdot B_2^2 \cdot \imath_1 \cdot \sin 2\theta \cdot \cos\theta + r^2 \cdot \imath_1^2 \cdot \cos^2\theta$$

$$r^4 \cdot 4 \cdot B_2^2 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta - r^4 \cdot B_2^4 \cdot \sin^2 2\theta + 4 \cdot r^3 \cdot \imath_1 \cdot \mathrm{H}(\theta) \cdot \sin\theta - 2 \cdot r^3 \cdot B_2^2 \cdot \imath_1 \cdot \sin 2\theta \cdot \cos\theta + r^2 \cdot \mathrm{H}(\theta) \cdot 4 \cdot \imath_2 - r^2 \cdot \imath_1^2 \cdot \cos^2\theta = 0$$

$$r^4 \cdot B_2^2 \cdot \{4 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta - B_2^2 \cdot \sin^2 2\theta\} + r^3 \cdot 2 \cdot \imath_1 \cdot \{2 \cdot \mathrm{H}(\theta) \cdot \sin\theta - B_2^2 \cdot \sin 2\theta \cdot \cos\theta\} + r^2 \cdot \{\mathrm{H}(\theta) \cdot 4 \cdot \imath_2 - \imath_1^2 \cdot \cos^2\theta\} = 0$$

$$r^2 \cdot [r^2 \cdot B_2^2 \cdot \{4 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta - B_2^2 \cdot \sin^2 2\theta\} + r \cdot 2 \cdot \imath_1 \cdot \{2 \cdot \mathrm{H}(\theta) \cdot \sin\theta - B_2^2 \cdot \sin 2\theta \cdot \cos\theta\} + \mathrm{H}(\theta) \cdot 4 \cdot \imath_2 - \imath_1^2 \cdot \cos^2\theta] = 0$$

We get two possible solutions: (1) $r^2 = 0 \rightarrow r = 0$ impossible since $r > 0$ for constant radius solution. (2) $r^2 \cdot B_2^2 \cdot \{4 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta - B_2^2 \cdot \sin^2 2\theta\} + r \cdot 2 \cdot \imath_1 \cdot \{2 \cdot \mathrm{H}(\theta) \cdot \sin\theta - B_2^2 \cdot \sin 2\theta \cdot \cos\theta\} + \mathrm{H}(\theta) \cdot 4 \cdot \imath_2 - \imath_1^2 \cdot \cos^2\theta = 0$

We define for simplicity three new functions of θ:

$$\Xi_1(\theta) = B_2^2 \cdot \{4 \cdot \mathrm{H}(\theta) \cdot \sin^2\theta - B_2^2 \cdot \sin^2 2\theta\}; \Xi_2(\theta) = 2 \cdot \imath_1 \cdot \{2 \cdot \mathrm{H}(\theta) \cdot \sin\theta - B_2^2 \cdot \sin 2\theta \cdot \cos\theta\}$$

$$\Xi_3(\theta) = \mathrm{H}(\theta) \cdot 4 \cdot \imath_2 - \imath_1^2 \cdot \cos^2\theta; \; r^2 \cdot \Xi_1(\theta) + r \cdot \Xi_2(\theta) + \Xi_3(\theta) = 0$$

$$r^2 \cdot \Xi_1(\theta) + r \cdot \Xi_2(\theta) + \Xi_3(\theta) = 0 \Rightarrow r_{1,2} = \frac{-\Xi_2(\theta) \pm \sqrt{\Xi_2^2(\theta) - 4 \cdot \Xi_1(\theta) \cdot \Xi_3(\theta)}}{2 \cdot \Xi_1(\theta)}$$

Constant radius solution $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$ must be for $r > 0$ then we have conditions for the above expression: $\Xi_2^2(\theta) - 4 \cdot \Xi_1(\theta) \cdot \Xi_3(\theta) \geq 0$ & if $\Xi_2^2(\theta) - 4 \cdot \Xi_1(\theta) \cdot \Xi_3(\theta) = 0$ then $r_{1,2} = \frac{-\Xi_2(\theta)}{2 \cdot \Xi_1(\theta)} \Rightarrow_{r_{1,2} > 0} \frac{\Xi_2(\theta)}{\Xi_1(\theta)} < 0$. Back to our system differential equation: $\frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1} \cdot \left\{g(X_1, X_2, a(t), \ldots) - \frac{1}{C_1} \cdot X_1\right\}$; $\frac{\mathrm{d}X_2}{\mathrm{d}t} = X_1; \; \Upsilon_1(X_1, X_2) = \frac{1}{L_1} \cdot \left\{g(X_1, X_2, a(t) = 1, \ldots) - \frac{1}{C_1} \cdot X_1\right\}$

$\Upsilon_2(X_1, X_2) = X_1; \; \Upsilon_1 = \Upsilon_1(X_1, X_2); \; \Upsilon_2 = \Upsilon_2(X_1, X_2)$. We get our system equations: $\frac{\mathrm{d}X_1}{\mathrm{d}t} = \Upsilon_1(X_1, X_2)$; $\frac{\mathrm{d}X_2}{\mathrm{d}t} = \Upsilon_2(X_1, X_2)$. We need to find $\left(\frac{\partial \Upsilon_1}{\partial X_1} + \frac{\partial \Upsilon_2}{\partial X_2}\right)|_{r=r_{1,2}}$ at a solution with constant radius ($\mathrm{d}r/\mathrm{d}t = 0$).

We already found the solution with constant radius:

$$r_{\mathrm{const}} = r_{1,2} = \frac{-\Xi_2(\theta) \pm \sqrt{\Xi_2^2(\theta) - 4 \cdot \Xi_1(\theta) \cdot \Xi_3(\theta)}}{2 \cdot \Xi_1(\theta)} > 0 \text{ \& find } \left(\frac{\partial \Upsilon_1}{\partial X_1} + \frac{\partial \Upsilon_2}{\partial X_2}\right)|_{r=r_{\mathrm{const}}}$$

$$\rho_1 \cdot \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot \mathrm{d}s}|_{\rho_1 = 1} \Rightarrow \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot \mathrm{d}s} = \mathrm{e}^{\int_0^{T = 2 \cdot \pi} \left(\frac{\partial \Upsilon_1}{\partial X_1} + \frac{\partial \Upsilon_2}{\partial X_2}\right)|_{r=r_{\mathrm{const}}} \cdot \mathrm{d}s}$$

As a result, the limit cycle with radius $r_{\mathrm{const}} = r_{1,2} = \frac{-\Xi_2(\theta) \pm \sqrt{\Xi_2^2(\theta) - 4 \cdot \Xi_1(\theta) \cdot \Xi_3(\theta)}}{2 \cdot \Xi_1(\theta)} > 0$.

Is stable if $\int_0^T tr(A(s)) \cdot \mathrm{d}s < 0$ or $\rho_2 < 1$ and unstable if $\int_0^T tr(A(s)) \cdot \mathrm{d}s > 0$ or $\rho_2 > 1$.

Remark In the last analysis we consider system with constant source $a(t) = 1$; $\partial a(t)/\partial t = 0$. It is reader's exercise to do the limit cycle stability analysis for all other cases $a(t) \neq 1$; $\partial a(t)/\partial t \neq 0$, periodic source by using Floquet theory [46, 47].

4.6 Optoisolation Circuit Hills Equations

We discuss in this subchapter Hill's equation system which is implemented by optoisolation circuit. We have dynamic system with one main variable $U(t)$. System Hill's equation $\frac{d^2U(t)}{dt^2} + [\delta + a(t)] \cdot U(t) = 0$, δ is a constant and $a(t)$ is a π-periodic function. We can write this equation in a different way, $x = \begin{pmatrix} U(t) \\ \frac{dU(t)}{dt} \end{pmatrix}$,

$\dot{x} = \begin{pmatrix} \frac{dU(t)}{dt} \\ \frac{d^2U(t)}{dt^2} \end{pmatrix}$. We get system $\dot{x} = A(t) \cdot x$; $A(t) = \begin{pmatrix} 0 & 1 \\ -[\delta + a(t)] & 0 \end{pmatrix}$.

If $A \in C^0(\mathbb{R}, \mathbb{R}^{dxd})$ or $A \in C^0(\mathbb{R}, \mathbb{C}^{dxd})$ is a $d \times d$ matrix function and there is $T > 0$ such that $A(t + T) = A(t)$ for all t, then differential equation is a periodic system or a T-periodic system. T is not the minimal period. If X(t) is a $d \times d$ matrix solution of a T-periodic system, then $X(t + T)$ also is a solution. If $X(t)$ is a fundamental matrix solution, then there is $d \times d$ constant matrix M such that $X(t + T) = X(t) \cdot M$ for all t and M is nonsingular. The M matrix is called a monodromy matrix and the eigenvalues ρ are called the Floquet multipliers. Since each Floquet multiplier ρ is different from zero, there is complex number λ, called a Floquet exponent, such that $\rho = e^{\lambda \cdot T}$. The Floquet multipliers and the real parts of the characteristic exponents are uniquely defined, but the imaginary parts of the characteristic exponents are not. We can always take $X(t = 0) = I$ in defining the monodromy matrix. The Floquet multipliers do depend upon the period T. A complex number $\rho = e^{\lambda \cdot T}$ is a Floquet multiplier of a T-period system if and only if there is a nontrivial solution of the form $e^{\lambda \cdot t} \cdot q(t)$, where $q(t)$ is T periodic. In particular, there is a periodic solution of period T if and only if there is a multiplier equal to +1. If ρ_j, $j = 1, 2, 3, \ldots, d$ are the Floquet multipliers of the T-periodic system then $\prod_{j=1}^{d} \rho_j = e^{\int_0^T TrA(s) \cdot ds}$. If M is a nonsingular $d \times d$ matrix, then there is $d \times d$ matrix B such that $M = e^B$. If $X(t)$ is a fundamental matrix for the T-periodic system, then there exist a $d \times d$ constant matrix B and nonsingular $d \times d$ T-periodic matrix $P(t)$ such that $X(t) = P(t) \cdot e^{B \cdot t}$. There are matrix B and a $2T$ periodic nonsingular matrix $P(t)$ such that $X(t) = P(t) \cdot e^{B \cdot t}$ holds. If M is the monodromy matrix for the fundamental solution $X(t)$, $X(t = 0) = I$, then the Floquet multipliers ρ_1, ρ_2 of $\frac{d^2U(t)}{dt^2} + [\delta + a(t)] \cdot U(t) = 0$ are the solutions of the equation: $\det[M - \rho \cdot I] = \rho^2 + (TrM) \cdot \rho + 1 = 0$ since $\det M = 1$. From the following theorem and last equation, we can determine the stability properties of zero solution of $\frac{d^2U(t)}{dt^2} + [\delta + a(t)] \cdot U(t) = 0$ from the magnitude of TrM. The theorem is for T-periodic, if M is a monodromy matrix, then the solution $x = 0$ is uniformly stable if and only if each $\rho \in \sigma(M)$ satisfy $|\rho| \leq 1$ and the ones with $|\rho| = 1$ have simple elementary divisors. It is uniformly asymptotically stable if and only if each $\rho \in \sigma(M)$ satisfies $|\rho| < 1$. There are some possibilities for our case [48, 85, 86].

(i) If $|TrM| < 2$, then the Floquet multipliers are complex and simple $|\rho_1| = |\rho_2| = 1$, $\rho_1 \neq \rho_2$, and the zero solution of $\frac{d^2U(t)}{dt^2} + \cdots = 0$ is uniformly stable.

(ii) If $|TrM| > 2$, then either $0 < \rho_1 < 1 < \rho_2$ or $\rho_2 < -1 < \rho_1 < 0$ and the zero solution of $\frac{d^2U(t)}{dt^2} + \cdots = 0$ is stable.

(iii) If $|TrM| = 2$, then $\rho_1 = \rho_2 = 1$ or $\rho_1 = \rho_2 = -1$. In the first situation, there must be a π-periodic solution of $\frac{d^2U(t)}{dt^2} + \cdots = 0$ and, in the latter situation, there must be a 2π-periodic solution of $\frac{d^2U(t)}{dt^2} + \cdots = 0$. The solutions of $\frac{d^2U(t)}{dt^2} + \cdots = 0$ are either all periodic or periodic functions times a linear function of t depending upon whether or not the monodromy matrix is diagonalizable.

If we consider Hill's equation with δ constant equal to zero ($\delta = 0$), we get the following reduced Hill's equation: $\frac{d^2U(t)}{dt^2} + a(t) \cdot U(t) = 0; a(t+T) = a(t)$.

We can write this equation in a different way, $x = \begin{pmatrix} U(t) \\ \frac{dU(t)}{dt} \end{pmatrix}, \dot{x} = \begin{pmatrix} \frac{dU(t)}{dt} \\ \frac{d^2U(t)}{dt^2} \end{pmatrix}$.

We get system $\dot{x} = A(t) \cdot x$; $A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}$.

Linear system theory can be applied to Floquet theory to analyze the stability of the zero solution of this T-periodic system. The first step in the stability analysis is an application of Liouville's formula. The Liouville's formula proposition is suppose that $t \to \Phi(t)$ is a matrix solution of the homogeneous linear system $\dot{x} = A(t) \cdot x \, \forall x \in \mathbb{R}^n$, $t \to A(t)$ is a smooth function from some open interval $J \subseteq \mathbb{R}$ to the space of $n \times n$ matrices, on the open interval J. If $t_0 \in J$, then $\det \Phi(t) = \det \Phi(t_0) \cdot e^{\int_{t_0}^{t} trA(s) \cdot ds}$, where det denotes determinant and tr denotes trace. In particular, $\Phi(t)$ is a fundamental matrix solution if and only if the columns of $\Phi(t_0)$ are linearly independent. The exponential of a matrix is defined as an infinite series and used this definition to prove that the homogeneous linear system $\dot{x} = A(t) \cdot x \, \forall x \in \mathbb{R}^n$ has a fundamental matrix solution, namely, $t \to e^{t \cdot A}$. We may recall that $\ddot{U} + p(t) \cdot \dot{U} + q(t) \cdot U = 0$ and the Wronskian of the two solutions U_1 and U_2 is defined by $W(t) = \det\begin{pmatrix} U_1(t) & U_2(t) \\ \dot{U}_1(t) & \dot{U}_2(t) \end{pmatrix}$ then $W(t) = W(t=0) \cdot e^{-\int_0^t p(s) \cdot ds}$. The equivalent first-order system $dx/dt = B(t) \cdot x$

$\dot{x} = B(t) \cdot x = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \cdot x$; $x = \begin{pmatrix} U(t) \\ \dot{U}(t) \end{pmatrix}$; $\dot{x} = \begin{pmatrix} \dot{U}(t) \\ \ddot{U}(t) \end{pmatrix}$ with fundamental matrix $\Psi(t)$, formula $W(t) = W(t=0) \cdot e^{-\int_0^t p(s) \cdot ds}$ is a special case of Liouville's formuladet $\Psi(t) = \det \Psi(t=0) \cdot e^{\int_{t_0}^{t} trB(s) \cdot ds}$. At any rate, let us apply Liouville's formula to the principal fundamental matrix $\Phi(t)$ at $t = 0$ for Hill's system to obtain the identity $\det\Phi(t) \equiv 1$. Since the determinant of a matrix is the product of the eigenvalues of the matrix, there is an important fact, the product of the characteristic multipliers of the monodromy matrix, $\Phi(T)$, is 1. The characteristic multipliers for Hill's equation are denoted by λ_1 and λ_2 and note that they are roots of the characteristic equation $\lambda^2 - [tr\Phi(T)] \cdot \lambda + \det \Phi(T) = 0$. For notational convenience let us set $2 \cdot \phi = tr\Phi(T)$ to obtain the equivalent characteristic equation: $\lambda^2 - 2 \cdot \phi \cdot \lambda + 1 = 0 \Rightarrow \lambda = \phi \pm \sqrt{\phi^2 - 1}$. There are several cases to consider depending on the value of ϕ [48, 78, 79] (Table 4.10).

Theorem *if* $a : \mathbb{R} \to \mathbb{R}$ *is a positive T-periodic function such that* $T \cdot \int_0^T a(t) \cdot dt \leq 4$ *then all solutions of the Hill's equation* $\ddot{x} + a(t) \cdot x = 0$ *are bounded. In particular, the trivial solution is stable.*

The next optoisolation circuit implements Hill's equation system. Delta input constant signal level (δ). $a(t)$ is a periodic source (Fig. 4.3).

$$U(t) = V_{CEQ1} + V_{CEQ2}; I_{RcQ1} = I_{CQ1};\ I_{EQ1} = I_{CQ2};\ I_{RcQ1} = I_{CQ1} = \frac{V_{DD} - U(t)}{RcQ1};$$
$$U = U(t)$$

We consider two identical optocouplers and the two photo transistors $Q1$, $Q2$ have the same parameters $(\alpha_{r1} = \alpha_{r2} = \alpha_r; \alpha_{f1} = \alpha_{f2} = \alpha_f)$ except coupling coefficients $k1$ and $k2$ [16, 25, 26].

$$V_{CEQ1} = V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\} + V_t \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right); \frac{I_{sc}}{I_{se}} \to 1$$
$$\Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$V_{CEQ2} = V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\} + V_t \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right); \frac{I_{sc}}{I_{se}} \to 1$$
$$\Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$V_{CEQ1} \simeq V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\};$$
$$V_{CEQ2} \simeq V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\}$$

$\frac{d^2U}{dt^2} = \ddot{U} = R_2 \cdot I_{D2} + V_{D2} = R_2 \cdot I_{D2} + V_t \cdot \ln\left(\frac{I_{D2}}{I_0} + 1\right)$; $\ln\left(\frac{I_{D2}}{I_0} + 1\right) \approx \frac{I_{D2}}{I_0}$ using Taylor series approximation.

$$\frac{d^2U}{dt^2} = \ddot{U} = R_2 \cdot I_{D2} + V_t \cdot \frac{I_{D2}}{I_0} = I_{D2} \cdot \left(R_2 + \frac{V_t}{I_0}\right)$$
$$\Rightarrow I_{D2} = \frac{\ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)}; I_{BQ2} = k_2 \cdot I_{D2} = \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)}$$
$$I_{EQ2} = I_{BQ2} + I_{CQ2} = \frac{k_2 \cdot \ddot{U}}{(R_2 + \frac{V_t}{I_0})} + I_{CQ2}; a(t)$$
$$+ \delta = R_1 \cdot I_{D1} + V_{D1} = R_1 \cdot I_{D1} + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right)$$

By using Taylor series approximation: $\ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx \frac{I_{D1}}{I_0}$; $a(t) + \delta = I_{D1} \cdot \left(R_1 + \frac{V_t}{I_0}\right)$

$$a(t) + \delta = I_{D1} \cdot \left(R_1 + \frac{V_t}{I_0}\right) \Rightarrow I_{D1} = \frac{a(t) + \delta}{\left(R_1 + \frac{V_t}{I_0}\right)}; I_{BQ1} = k_1 \cdot I_{D1} = \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1} = \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + I_{CQ1}; I_{CQ1} = \frac{V_{DD} - U}{RcQ1} \Rightarrow I_{EQ1}$$
$$= \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{V_{DD} - U}{RcQ1}$$

$$I_{CQ2} = I_{EQ1} = \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{V_{DD} - U}{RcQ1}; I_{EQ2} = I_{BQ2} + I_{CQ2}$$
$$= \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} + \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{V_{DD} - U}{RcQ1}$$

Table 4.10 Case to consider depending on the value of ϕ

Case to consider depending on the value of ϕ	Case description $\lambda = \phi \pm \sqrt{\phi^2 - 1}$
$\phi > 1$	λ_1 and λ_2 are distinct positive real numbers such $\lambda_1 \cdot \lambda_2 = 1$. We assume that $0 < \lambda_1 < 1$, $\lambda_2 > 1$ with $\lambda_1 = 1/\lambda_2$ and there is a real number $\mu > 0$ (characteristic exponent) such that $\lambda_1 = e^{-T \cdot \mu}$ and $\lambda_2 = e^{T \cdot \mu}$. There is a fundamental set of solutions of the form $e^{-\mu \cdot t} \cdot p_1(t)$; $e^{\mu \cdot t} \cdot p_2(t)$, where the real functions p_1 and p_2 are T periodic. In this case the zero solution is unstable
$\phi < -1$	λ_1 and λ_2 are both real and both negative. Since $\lambda_1 \cdot \lambda_2 = 1$, we may assume that $\lambda_1 < -1$; $-1 < \lambda_2 < 0$ with $\lambda_1 = 1/\lambda_2$. There is a real number $\mu > 0$ (characteristic exponent) such that $e^{2 \cdot T \cdot \mu} = \lambda_1^2$; $e^{-2 \cdot T \cdot \mu} = \lambda_2^2$. There is a fundamental set of solutions of the form, $e^{-\mu \cdot t} \cdot q_2(t) e^{-\mu \cdot t} \cdot q_2(t)$ where the real functions q_1 and q_2 are $2 \cdot T$ periodic. The zero solution is unstable
$-1 < \phi < 1$	λ_1 and λ_2 are complex conjugates each with nonzero imaginary part. Since $\lambda_1 \cdot \bar{\lambda}_1 = 1$, we have that $\lvert\lambda_1\rvert = 1$, and therefore both characteristic multipliers lie on the unit circle in the complex plane. Because both λ_1 and λ_2 have nonzero imaginary parts, one of these characteristic multipliers, say λ_1, lies in the upper half plane. Thus, there is a real number θ with $0 < \theta \cdot T < \pi$ and $e^{i \cdot \theta \cdot T} = \lambda_1$. There is a solution of the form $e^{i \cdot \theta \cdot t} \cdot [r(t) + i \cdot s(t)]$ with r and s both T-periodic functions. Hence there is a fundamental set of solutions of the form $r(t) \cdot \cos[\theta \cdot t] - s(t) \cdot \sin[\theta \cdot t]$ and $r(t) \cdot \sin[\theta \cdot t] + s(t) \cdot \cos[\theta \cdot t]$. The zero solution is stable but not asymptotically stable. The solutions are periodic if and only if there are relatively prime positive integers m and n such that $2 \cdot \pi \cdot m/\theta = n \cdot T$. If such integers exist, all solutions have period $n \cdot T$. if not, and then these solutions are quasiperiodic. *Remark*: suppose that $\Phi(t)$ is the principal fundamental matrix solution of Hill's equation at $t = 0$. If $\lvert tr\Phi(T)\rvert < 2$, then the zero solution is stable. If $\lvert tr\Phi(T)\rvert > 2$, then the zero solution is unstable
$\phi = 1$	$\lambda_1 = \lambda_2 = 1$ and the nature of the solutions depends on the canonical form of $\Phi(T)$. if $\Phi(T)$ is the identity, then $e^0 = \Phi(T)$ and there is a Floquet normal form $\Phi(t) = p(t)$ where $p(t)$ is T periodic and invertible. There is a fundamental set of periodic solutions and the zero solution is stable. If $\Phi(T)$ is not the identity, then there is nonsingular matrix C such that $C \cdot \Phi(T) \cdot C^{-1} = I + N = e^N$ where $N \neq 0$ is nilpotent. Thus, $\Phi(t)$ has a Floquet normal form $\Phi(t) = p(t) \cdot e^{t \cdot B}$ where $B = C^{-1} \cdot (N/T) \cdot C$. Because $e^{t \cdot B} = C^{-1} \cdot (I + \frac{t}{T} \cdot N) \cdot C$, the matrix $t \to e^{t \cdot B}$ is unbounded, and therefore the zero solution is unstable

(continued)

Table 4.10 (continued)

Case to consider depending on the value of ϕ	Case description $\lambda = \phi \pm \sqrt{\phi^2 - 1}$
$\phi = -1$	The situation is similar to case $\phi = 1$, expect the fundamental matrix is represented by $Q(t) \cdot e^{t \cdot B}$ where $Q(t)$ is a $2 \cdot T$-periodic matrix function

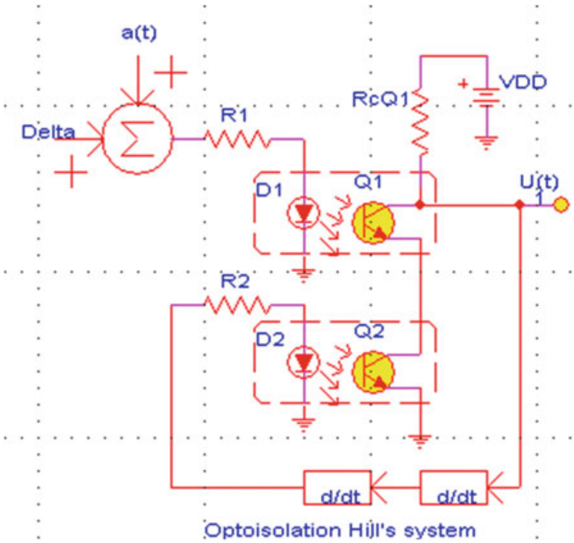

Fig. 4.3 Optoisolation Hill's system

$$U(t) = V_{CEQ1} + V_{CEQ2} = V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\} + V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\}$$

$$U(t) = V_t \cdot \ln\left[\begin{array}{c}\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\} \\ \cdot \left\{\frac{(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\}\end{array}\right]$$

$$e^{\left[\frac{U(t)}{V_t}\right]} = \frac{[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)]}{[(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)]}$$

By using Taylor series approximation: $e^{\left[\frac{U(t)}{V_t}\right]} = e^{\left[\frac{U}{V_t}\right]} \approx \frac{U}{V_t} + 1$

$$\left(\frac{U}{V_t}+1\right)=\frac{[(\alpha_r\cdot I_{CQ1}-I_{EQ1})+I_{se}\cdot(\alpha_r\cdot\alpha_f-1)]\cdot[(\alpha_r\cdot I_{CQ2}-I_{EQ2})+I_{se}\cdot(\alpha_r\cdot\alpha_f-1)]}{[(I_{CQ1}-I_{EQ1}\cdot\alpha_f)+I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)]\cdot[(I_{CQ2}-I_{EQ2}\cdot\alpha_f)+I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)]}$$

$$\begin{aligned}\alpha_r\cdot I_{CQ1}-I_{EQ1}&=\frac{\alpha_r\cdot(V_{DD}-U)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}-\frac{V_{DD}-U}{RcQ1}\\&=\frac{\alpha_r\cdot V_{DD}}{RcQ1}-\frac{\alpha_r\cdot U}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}-\frac{V_{DD}}{RcQ1}+\frac{U}{RcQ1}\end{aligned}$$

$$\begin{aligned}\alpha_r\cdot I_{CQ1}-I_{EQ1}&=\frac{(\alpha_r-1)\cdot V_{DD}}{RcQ1}+\frac{U\cdot(1-\alpha_r)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}\\&=-\frac{(1-\alpha_r)\cdot V_{DD}}{RcQ1}+\frac{U\cdot(1-\alpha_r)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}\end{aligned}$$

$$\begin{aligned}\alpha_r\cdot I_{CQ1}-I_{EQ1}&=-\frac{(1-\alpha_r)\cdot V_{DD}}{RcQ1}+\frac{U\cdot(1-\alpha_r)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}\\&=\frac{(U-V_{DD})\cdot(1-\alpha_r)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}\end{aligned}$$

$$\begin{aligned}I_{CQ1}-I_{EQ1}\cdot\alpha_f&=\frac{V_{DD}-U}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1\cdot\alpha_f}{\left(R_1+\frac{V_t}{I_0}\right)}-\frac{(V_{DD}-U)\cdot\alpha_f}{RcQ1}\\&=\frac{V_{DD}}{RcQ1}-\frac{U}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1\cdot\alpha_f}{\left(R_1+\frac{V_t}{I_0}\right)}-\frac{V_{DD}\cdot\alpha_f}{RcQ1}+\frac{U\cdot\alpha_f}{RcQ1}\end{aligned}$$

$$\begin{aligned}I_{CQ1}-I_{EQ1}\cdot\alpha_f&=\frac{V_{DD}\cdot(1-\alpha_f)}{RcQ1}+\frac{U\cdot(\alpha_f-1)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1\cdot\alpha_f}{\left(R_1+\frac{V_t}{I_0}\right)}\\&=\frac{V_{DD}\cdot(1-\alpha_f)}{RcQ1}-\frac{U\cdot(1-\alpha_f)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1\cdot\alpha_f}{\left(R_1+\frac{V_t}{I_0}\right)}\end{aligned}$$

$$I_{CQ1}-I_{EQ1}\cdot\alpha_f=\frac{(V_{DD}-U)\cdot(1-\alpha_f)}{RcQ1}-\frac{[a(t)+\delta]\cdot k_1\cdot\alpha_f}{\left(R_1+\frac{V_t}{I_0}\right)}$$

$$\begin{aligned}\alpha_r\cdot I_{CQ2}-I_{EQ2}=&\frac{\alpha_r\cdot[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}+\frac{\alpha_r\cdot(V_{DD}-U)}{RcQ1}-\frac{k_2\cdot\ddot{U}}{\left(R_2+\frac{V_t}{I_0}\right)}\\&-\frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}-\frac{V_{DD}-U}{RcQ1}\end{aligned}$$

$$\alpha_r \cdot I_{CQ2} - I_{EQ2} = \frac{(\alpha_r - 1) \cdot [a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(\alpha_r - 1) \cdot (V_{DD} - U)}{RcQ1} - \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)}$$

$$I_{CQ2} - I_{EQ2} \cdot \alpha_f = \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{V_{DD} - U}{RcQ1} - \frac{k_2 \cdot \ddot{U} \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)} - \frac{[a(t) + \delta] \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{(V_{DD} - U) \cdot \alpha_f}{RcQ1}$$

$$I_{CQ2} - I_{EQ2} \cdot \alpha_f = \frac{[a(t) + \delta] \cdot k_1 \cdot (1 - \alpha_f)}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} - \frac{k_2 \cdot \ddot{U} \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)}$$

First we find numerator expression:

$$[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)]$$

$$\begin{aligned}
&[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \\
&\quad = (\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) + (\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \\
&\quad\quad + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se}^2 \cdot (\alpha_r \cdot \alpha_f - 1)^2
\end{aligned}$$

$$\begin{aligned}
&[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \\
&\quad = (\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \\
&\quad\quad \cdot \{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} + I_{se}^2 \cdot (\alpha_r \cdot \alpha_f - 1)^2
\end{aligned}$$

(1) $(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2})$

$$(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) = \left\{\frac{(U - V_{DD}) \cdot (1 - \alpha_r)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1}{(R_1 + \frac{V_t}{I_0})}\right\} \cdot \left\{\frac{(\alpha_r - 1) \cdot [a(t) + \delta] \cdot k_1}{(R_1 + \frac{V_t}{I_0})} + \frac{(\alpha_r - 1) \cdot (V_{DD} - U)}{RcQ1} - \frac{k_2 \cdot \ddot{U}}{(R_2 + \frac{V_t}{I_0})}\right\}$$

$$(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) = \left\{ \frac{(U - V_{DD}) \cdot (1 - \alpha_r)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \right\} \cdot \left\{ \begin{array}{r} \frac{(\alpha_r - 1) \cdot [a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} \\ - \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \end{array} \right\}$$

$$(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) = \left\{ \frac{(U - V_{DD}) \cdot (1 - \alpha_r)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \right\} \cdot \left\{ \begin{array}{r} \frac{(1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} - \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \\ - \frac{(1 - \alpha_r) \cdot [a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \end{array} \right\}$$

$$\begin{aligned} &(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) \\ &\quad = \frac{(U - V_{DD})^2 \cdot (1 - \alpha_r)^2}{[RcQ1]^2} \\ &\qquad - \frac{(U - V_{DD}) \cdot (1 - \alpha_r) \cdot k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right) \cdot RcQ1} - \frac{(U - V_{DD}) \cdot (1 - \alpha_r)^2 \cdot [a(t) + \delta] \cdot k_1}{RcQ1 \cdot \left(R_1 + \frac{V_t}{I_0}\right)} \\ &\qquad - \frac{[a(t) + \delta] \cdot k_1 \cdot (1 - \alpha_r) \cdot (U - V_{DD})}{\left(R_1 + \frac{V_t}{I_0}\right) \cdot RcQ1} + \frac{[a(t) + \delta] \cdot k_1 \cdot k_2 \cdot \ddot{U}}{\left(R_1 + \frac{V_t}{I_0}\right) \cdot \left(R_2 + \frac{V_t}{I_0}\right)} \\ &\qquad + \frac{(1 - \alpha_r) \cdot [a(t) + \delta]^2 \cdot k_1^2}{\left(R_1 + \frac{V_t}{I_0}\right)^2} \end{aligned}$$

$$\begin{aligned} &(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) \\ &\quad = \frac{k_2 \cdot \ddot{U}}{(R_2 + \frac{V_t}{I_0})} \cdot \left\{ \frac{[a(t) + \delta] \cdot k_1}{(R_1 + \frac{V_t}{I_0})} - \frac{(U - V_{DD}) \cdot (1 - \alpha_r)}{RcQ1} \right\} \\ &\qquad + \frac{(U - V_{DD})^2 \cdot (1 - \alpha_r)^2}{[RcQ1]^2} - \frac{(U - V_{DD}) \cdot [a(t) + \delta] \cdot k_1 \cdot (1 - \alpha_r)}{RcQ1 \cdot (R_1 + \frac{V_t}{I_0})} \\ &\qquad \cdot \{2 - \alpha_r\} + \frac{(1 - \alpha_r) \cdot [a(t) + \delta]^2 \cdot k_1^2}{(R_1 + \frac{V_t}{I_0})^2} \end{aligned}$$

We define for simplicity the following functions:

$$\xi_1(U, a(t), \delta, \ldots) = \frac{k_2}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot \left\{ \frac{[a(t)+\delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{(U - V_{DD}) \cdot (1-\alpha_r)}{RcQ1} \right\}$$

$$\begin{aligned} &\xi_2(U, a(t), \delta, \ldots) \\ &= \frac{(U - V_{DD})^2 \cdot (1-\alpha_r)^2}{[RcQ1]^2} - \frac{(U - V_{DD}) \cdot [a(t)+\delta] \cdot k_1 \cdot (1-\alpha_r)}{RcQ1 \cdot \left(R_1 + \frac{V_t}{I_0}\right)} \cdot \{2 - \alpha_r\} \\ &\quad + \frac{(1-\alpha_r) \cdot [a(t)+\delta]^2 \cdot k_1^2}{\left(R_1 + \frac{V_t}{I_0}\right)^2} \end{aligned}$$

Then

$$(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) = \ddot{U} \cdot \xi_1(U, a(t), \delta, \ldots) + \xi_2(U, a(t), \delta, \ldots)$$

$$\begin{aligned} &\xi_1 = \xi_1(U, a(t), \delta, \ldots); \xi_2 = \xi_2(U, a(t), \delta, \ldots) \\ &\quad \Rightarrow (\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) = \ddot{U} \cdot \xi_1 + \xi_2 \end{aligned}$$

(2) $(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2})$

$$\begin{aligned} &(\alpha_r \cdot I_{CQ1} - I_{EQ1}) \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot (\alpha_r \cdot I_{CQ2} - I_{EQ2}) \\ &\quad = \{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \end{aligned}$$

$$\begin{aligned} &\{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \\ &= \left\{ \begin{aligned} &\frac{(U - V_{DD}) \cdot (1-\alpha_r)}{RcQ1} - \frac{[a(t)+\delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \\ &+ \frac{(\alpha_r - 1) \cdot [a(t)+\delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(\alpha_r - 1) \cdot (V_{DD} - U)}{RcQ1} \\ &- \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \end{aligned} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) \end{aligned}$$

$$\{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = \left\{ \begin{array}{r} \frac{(U - V_{DD}) \cdot (1 - \alpha_r)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(\alpha_r - 1) \cdot [a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \\ + \frac{(1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} - \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \end{array} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = -\frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + \left\{ \begin{array}{r} \frac{(U - V_{DD}) \cdot (1 - \alpha_r)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \\ + \frac{(\alpha_r - 1) \cdot [a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} \end{array} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = -\frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + \left\{ \begin{array}{r} -\frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(\alpha_r - 1) \cdot [a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \\ + \frac{2 \cdot (1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} \end{array} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = -\frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + \left\{ \begin{array}{r} \frac{[a(t) + \delta] \cdot k_1 \cdot (\alpha_r - 2)}{\left(R_1 + \frac{V_t}{I_0}\right)} \\ + \frac{2 \cdot (1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} \end{array} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = \frac{k_2 \cdot \ddot{U}}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) + \left\{ \begin{array}{c} \frac{2 \cdot (1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} \\ -\frac{[a(t) + \delta] \cdot k_1 \cdot (2 - \alpha_r)}{\left(R_1 + \frac{V_t}{I_0}\right)} \end{array} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

We define for simplicity the following functions:

$$\xi_3 = \frac{k_2}{(R_2 + \frac{V_t}{I_0})} \cdot I_{se} \cdot (1 - \alpha_r \cdot \alpha_f); \xi_4(U, a(t), \delta, \ldots) = \left\{ \begin{array}{c} \frac{2 \cdot (1 - \alpha_r) \cdot (U - V_{DD})}{RcQ1} \\ -\frac{[a(t) + \delta] \cdot k_1 \cdot (2 - \alpha_r)}{\left(R_1 + \frac{V_t}{I_0}\right)} \end{array} \right\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\xi_4 = \xi_4(U, a(t), \delta, \ldots);\ \{\alpha_r \cdot I_{CQ1} - I_{EQ1} + \alpha_r \cdot I_{CQ2} - I_{EQ2}\} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = \ddot{U} \cdot \xi_3 + \xi_4$$

(3) $\xi_5 = I_{se}^2 \cdot (\alpha_r \cdot \alpha_f - 1)^2$

We can summarize our numerator expression:

$$[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] = (1) + (2) + (3) = \ddot{U} \cdot \xi_1 + \xi_2 + \ddot{U} \cdot \xi_3 + \xi_4 + \xi_5 = \ddot{U} \cdot (\xi_1 + \xi_3) + \xi_2 + \xi_4 + \xi_5$$

$$\Delta_1 = \xi_1 + \xi_3; \Delta_2 = \xi_2 + \xi_4 + \xi_5;$$
$$[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] = \ddot{U} \cdot \Delta_1 + \Delta_2$$

ξ_3, ξ_5 are constant functions and they are not dependent on $U, a(t), \delta$.
Second, we find denominator expression:

$$[(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] = (I_{CQ1} - I_{EQ1} \cdot \alpha_f) \cdot (I_{CQ2} - I_{EQ2} \cdot \alpha_f) + \{I_{CQ1} - I_{EQ1} \cdot \alpha_f + I_{CQ2} - I_{EQ2} \cdot \alpha_f\} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + I_{sc}^2 \cdot (\alpha_r \cdot \alpha_f - 1)^2$$

(4) $(I_{CQ1} - I_{EQ1} \cdot \alpha_f) \cdot (I_{CQ2} - I_{EQ2} \cdot \alpha_f)$

$$
\begin{aligned}
&(I_{CQ1} - I_{EQ1} \cdot \alpha_f) \cdot (I_{CQ2} - I_{EQ2} \cdot \alpha_f) \\
&\quad = \left\{ \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} \right\} \\
&\qquad \cdot \left\{ \frac{[a(t) + \delta] \cdot k_1 \cdot (1 - \alpha_f)}{(R_1 + \frac{V_t}{I_0})} + \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} - \frac{k_2 \cdot \ddot{U} \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&(I_{CQ1} - I_{EQ1} \cdot \alpha_f) \cdot (I_{CQ2} - I_{EQ2} \cdot \alpha_f) = \left\{ \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} - \frac{[a(t) + \delta] \cdot k_1 \cdot \alpha_f}{(R_1 + \frac{V_t}{I_0})} \right\} \\
&\cdot \left\{ \frac{[a(t) + \delta] \cdot k_1 \cdot (1 - \alpha_f)}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} - \frac{k_2 \cdot \ddot{U} \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&(I_{CQ1} - I_{EQ1} \cdot \alpha_f) \cdot (I_{CQ2} - I_{EQ2} \cdot \alpha_f) \\
&\quad = \frac{(V_{DD} - U) \cdot (1 - \alpha_f)^2 \cdot [a(t) + \delta] \cdot k_1}{RcQ1 \cdot \left(R_1 + \frac{V_t}{I_0}\right)} + \frac{(V_{DD} - U)^2 \cdot (1 - \alpha_f)^2}{[RcQ1]^2} \\
&\qquad - \ddot{U} \cdot \frac{(V_{DD} - U) \cdot (1 - \alpha_f) \cdot k_2 \cdot \alpha_f}{RcQ1 \cdot \left(R_2 + \frac{V_t}{I_0}\right)} - \frac{[a(t) + \delta]^2 \cdot k_1^2 \cdot \alpha_f \cdot (1 - \alpha_f)}{\left(R_1 + \frac{V_t}{I_0}\right)^2} \\
&\qquad - \frac{[a(t) + \delta] \cdot k_1 \cdot \alpha_f \cdot (V_{DD} - U) \cdot (1 - \alpha_f)}{\left(R_1 + \frac{V_t}{I_0}\right) \cdot RcQ1} + \ddot{U} \cdot \frac{[a(t) + \delta] \cdot k_1 \cdot k_2 \cdot \alpha_f^2}{\left(R_1 + \frac{V_t}{I_0}\right) \cdot \left(R_2 + \frac{V_t}{I_0}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&(I_{CQ1} - I_{EQ1} \cdot \alpha_f) \cdot (I_{CQ2} - I_{EQ2} \cdot \alpha_f) \\
&\quad = \ddot{U} \cdot \frac{k_2 \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot \left\{ \frac{[a(t) + \delta] \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} \right\} \\
&\qquad + \frac{[a(t) + \delta] \cdot k_1 \cdot (V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1 \cdot \left(R_1 + \frac{V_t}{I_0}\right)} \cdot (1 - 2 \cdot \alpha_f) \\
&\qquad - \frac{[a(t) + \delta]^2 \cdot k_1^2 \cdot \alpha_f \cdot (1 - \alpha_f)}{\left(R_1 + \frac{V_t}{I_0}\right)^2} + \frac{(V_{DD} - U)^2 \cdot (1 - \alpha_f)^2}{[RcQ1]^2}
\end{aligned}
$$

We define for simplicity the following functions: $\xi_6 = \xi_6(U, a(t), \delta, \ldots)$

$$
\xi_6(U, a(t), \delta, \ldots) = \frac{k_2 \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot \left\{ \frac{[a(t) + \delta] \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{(V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} \right\}
$$

$$\xi_7(U, a(t), \delta, \ldots) = \frac{[a(t)+\delta]\cdot k_1 \cdot (V_{DD}-U)\cdot(1-\alpha_f)}{RcQ1\cdot\left(R_1+\frac{V_t}{I_0}\right)}\cdot(1-2\cdot\alpha_f) - \frac{[a(t)+\delta]^2\cdot k_1^2\cdot\alpha_f\cdot(1-\alpha_f)}{\left(R_1+\frac{V_t}{I_0}\right)^2} + \frac{(V_{DD}-U)^2\cdot(1-\alpha_f)^2}{[RcQ1]^2}$$

$$(I_{CQ1}-I_{EQ1}\cdot\alpha_f)\cdot(I_{CQ2}-I_{EQ2}\cdot\alpha_f) = \ddot{U}\cdot\xi_6(U,a(t),\delta,\ldots)+\xi_7(U,a(t),\delta,\ldots) = \ddot{U}\cdot\xi_6+\xi_7$$

(5) $\{I_{CQ1}-I_{EQ1}\cdot\alpha_f+I_{CQ2}-I_{EQ2}\cdot\alpha_f\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)$

$$\{I_{CQ1}-I_{EQ1}\cdot\alpha_f+I_{CQ2}-I_{EQ2}\cdot\alpha_f\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1) = \left\{\frac{(V_{DD}-U)\cdot(1-\alpha_f)}{RcQ1} - \frac{[a(t)+\delta]\cdot k_1\cdot\alpha_f}{\left(R_1+\frac{V_t}{I_0}\right)} + \frac{[a(t)+\delta]\cdot k_1\cdot(1-\alpha_f)}{\left(R_1+\frac{V_t}{I_0}\right)} + \frac{(V_{DD}-U)\cdot(1-\alpha_f)}{RcQ1} - \frac{k_2\cdot\ddot{U}\cdot\alpha_f}{\left(R_2+\frac{V_t}{I_0}\right)}\right\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)$$

$$\{I_{CQ1}-I_{EQ1}\cdot\alpha_f+I_{CQ2}-I_{EQ2}\cdot\alpha_f\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1) = \left\{\frac{2\cdot(V_{DD}-U)\cdot(1-\alpha_f)}{RcQ1} + \frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot(1-2\cdot\alpha_f)\right\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1) - \ddot{U}\cdot\frac{k_2\cdot\alpha_f}{\left(R_2+\frac{V_t}{I_0}\right)}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)$$

$$\{I_{CQ1}-I_{EQ1}\cdot\alpha_f+I_{CQ2}-I_{EQ2}\cdot\alpha_f\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1) = \ddot{U}\cdot\frac{k_2\cdot\alpha_f}{\left(R_2+\frac{V_t}{I_0}\right)}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f) + \left\{\frac{2\cdot(V_{DD}-U)\cdot(1-\alpha_f)}{RcQ1} + \frac{[a(t)+\delta]\cdot k_1}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot(1-2\cdot\alpha_f)\right\}\cdot I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)$$

We define for simplicity the following functions: $\xi_9 = \xi_9(U, a(t), \delta, \ldots)$

$$\xi_8 = \frac{k_2 \cdot \alpha_f}{\left(R_2 + \frac{V_t}{I_0}\right)} \cdot I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)$$

$$\xi_9 = \xi_9(U, a(t), \delta, \ldots) = \left\{ \frac{2 \cdot (V_{DD} - U) \cdot (1 - \alpha_f)}{RcQ1} + \frac{[a(t) + \delta] \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot (1 - 2 \cdot \alpha_f) \right\} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$\{I_{CQ1} - I_{EQ1} \cdot \alpha_f + I_{CQ2} - I_{EQ2} \cdot \alpha_f\} \cdot I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) = \ddot{U} \cdot \xi_8 + \xi_9$$

(6) $\xi_{10} = I_{sc}^2 \cdot (\alpha_r \cdot \alpha_f - 1)^2$

ξ_8, ξ_{10} are constant functions and they are not dependent on $U, a(t), \delta$.
The denominator expression:

$$[(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] = (4) + (5) + (6)$$

$$[(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] = \ddot{U} \cdot \xi_6 + \xi_7 + \ddot{U} \cdot \xi_8 + \xi_9 + \xi_{10} = \ddot{U} \cdot (\xi_6 + \xi_8) + \xi_7 + \xi_9 + \xi_{10}$$

We can summarize our result in Table 4.11:

$$\left(\frac{U}{V_t} + 1\right) = \frac{[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)]}{[(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)]}$$

$$\left(\frac{U}{V_t} + 1\right) = \frac{\ddot{U} \cdot (\xi_1 + \xi_3) + \xi_2 + \xi_4 + \xi_5}{\ddot{U} \cdot (\xi_6 + \xi_8) + \xi_7 + \xi_9 + \xi_{10}} \Rightarrow \left(\frac{U}{V_t} + 1\right) \cdot \{\ddot{U} \cdot (\xi_6 + \xi_8) + \xi_7 + \xi_9 + \xi_{10}\} = \ddot{U} \cdot (\xi_1 + \xi_3) + \xi_2 + \xi_4 + \xi_5$$

$$\ddot{U} \cdot \left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) + \left(\frac{U}{V_t} + 1\right) \cdot \{\xi_7 + \xi_9 + \xi_{10}\} = \ddot{U} \cdot (\xi_1 + \xi_3) + \xi_2 + \xi_4 + \xi_5$$

Table 4.11 Numerator expression and denominator expression

$\frac{[(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(\alpha_r \cdot I_{CQ2} - I_{EQ2}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)]}{[(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)] \cdot [(I_{CQ2} - I_{EQ2} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)]}$	
Numerator expression	Denominator expression
$\ddot{U} \cdot (\xi_1 + \xi_3) + \xi_2 + \xi_4 + \xi_5$	$\ddot{U} \cdot (\xi_6 + \xi_8) + \xi_7 + \xi_9 + \xi_{10}$

$$\ddot{U} \cdot \left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) - \ddot{U} \cdot (\xi_1 + \xi_3) + \left(\frac{U}{V_t} + 1\right) \cdot \{\xi_7 + \xi_9 + \xi_{10}\} - \{\xi_2 + \xi_4 + \xi_5\} = 0$$

$$\ddot{U} \cdot \left\{\left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) - (\xi_1 + \xi_3)\right\} + \left(\frac{U}{V_t} + 1\right) \cdot \{\xi_7 + \xi_9 + \xi_{10}\} - \{\xi_2 + \xi_4 + \xi_5\} = 0$$

$$\ddot{U} + \frac{\left(\frac{U}{V_t} + 1\right) \cdot \{\xi_7 + \xi_9 + \xi_{10}\} - \{\xi_2 + \xi_4 + \xi_5\}}{\left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) - (\xi_1 + \xi_3)} = 0;$$

$$\left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) - (\xi_1 + \xi_3) \neq 0$$

$$\frac{d^2U(t)}{dt^2} + [\delta + a(t)] \cdot U(t) = 0 \Leftrightarrow \ddot{U} + \frac{\left(\frac{U}{V_t} + 1\right) \cdot \{\xi_7 + \xi_9 + \xi_{10}\} - \{\xi_2 + \xi_4 + \xi_5\}}{\left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) - (\xi_1 + \xi_3)} = 0$$

For the above Hill's equation equivalent expressions must fulfill the below notation:

$$[\delta + a(t)] \cdot U = \frac{\left(\frac{U}{V_t} + 1\right) \cdot \{\xi_7 + \xi_9 + \xi_{10}\} - \{\xi_2 + \xi_4 + \xi_5\}}{\left(\frac{U}{V_t} + 1\right) \cdot (\xi_6 + \xi_8) - (\xi_1 + \xi_3)}$$

Remark It is the reader's task to find which restrictions need to be fulfilled by $U(t)$ expression that makes our optoisolation circuit represent Hill's system.

4.7 Exercises

1. In subchapter 4.3, we analyze optoisolation circuit limit cycle stability by using Floquet theory for system without periodic sources $X_i(t) = 0$, $i = 1{,}2$ and analyze optoisolation circuit limit cycle stability for the following cases:

   1.1 $X_1(t) = \sin(2 \cdot \pi \cdot t)$; $X_2(t) = 0$. 1.2 $X_1(t) = 0$; $X_2(t) = -\sin(2 \cdot \pi \cdot t)$.
   1.2 $X_1(t) = \sin(2 \cdot \pi \cdot t)$; $X_2(t) = -\sin(2 \cdot \pi \cdot t)$.
   1.3 $X_1(t) = \sin(2 \cdot \pi \cdot t) + \cos(2 \cdot \pi \cdot t)$; $X_2(t) = |X_1(t)|$.
   1.4 $X_1(t) = \sin(2 \cdot \pi \cdot t) + \cos(2 \cdot \pi \cdot t)$; $X_2(t) = -\sin(2 \cdot \pi \cdot t) - \cos(2 \cdot \pi \cdot t)$.

2. We have an optoisolation circuits which are characterized by two main variables $V_1(t)$ and $V_2(t)$. The first variable $V_1(t)$ characterizes the output voltage in time for the first optocoupler's output voltage (V_{CEQ1}) and the second variable $V_2(t)$ characterizes the output voltage in time for the second optocoupler's output voltage (V_{CEQ2}). The first and the second optoisolation circuits have input periodic sources $X_1(t)$ and $X_2(t)$ which are coupled to the other circuit by optocouplers. These periodic sources $X_i(t)$ for $i = 1, 2$ with $X_1(t) = A_0 \cdot \sin(2 \cdot \pi \cdot t)$ and $X_2(t) = -A_0 \cdot \sin(2 \cdot \pi \cdot t)$; $A_0 \gg V_{D1ON}$, $A_0 \gg V_{D2ON}$.

   2.1 You need to inspect system stability, limit cycle, and limit cycle stability by using Floquet theory for the cases $X_i(t) = 0$ for $i = 1{,}2$ and $X_i(t) = (-1)^{i+1} \cdot A_0 \cdot \sin(2 \cdot \pi \cdot t)$ for $i = 1{,}2$. $A_0 \gg V_{D1ON}$, $A_0 \gg V_{D2ON}$.
   2.2 Additionally we consider two special cases. If the two periodic sources are completely uncoupled ($k_1 = 0$; $k_2 = 0$) or coupled ($k_1 \neq 0$; $k_2 \neq 0$). Analyze circuits limit cycle stability for these cases.
   2.3 Analyze system limit cycle stability by using Floquet theory for $X_i(t) = (-1)^{i+1} \cdot A_0 \cdot \sin(2 \cdot \pi \cdot t) + (-1)^{i+1} \cdot A_0 \cdot \cos(2 \cdot \pi \cdot t)$ for $i = 1{,}2$. $A_0 \gg V_{D1ON}$, $A_0 \gg V_{D2ON}$.
   2.3 How optoisolation circuits limit cycle stability depends on Ω parameter, $k_1 = k$; $k_2 = k \cdot \Omega$. Differentiate two subcases $\Omega > 0$ and $\Omega < 0$.
   2.5 Analyze system limit cycle stability by using Floquet theory for $X_i(t) = (-1)^{i+1} \cdot |V_i(t)| \cdot A_0 \cdot \sin(2 \cdot \pi \cdot t)$ for $i = 1, 2$. $A_0 \gg V_{D1ON}$, $A_0 \gg V_{D2ON}$.
   2.6 Analyze system limit cycle stability by using Floquet theory for $X_i(t) = (-1)^{i+1} \cdot \text{sgn}\{V_i(t)\} \cdot A_0 \cdot \sin(2 \cdot \pi \cdot t)$ for $i = 1, 2$. $A_0 \gg V_{D1ON}$, $A_0 \gg V_{D2ON}$. (Fig. 4.4)

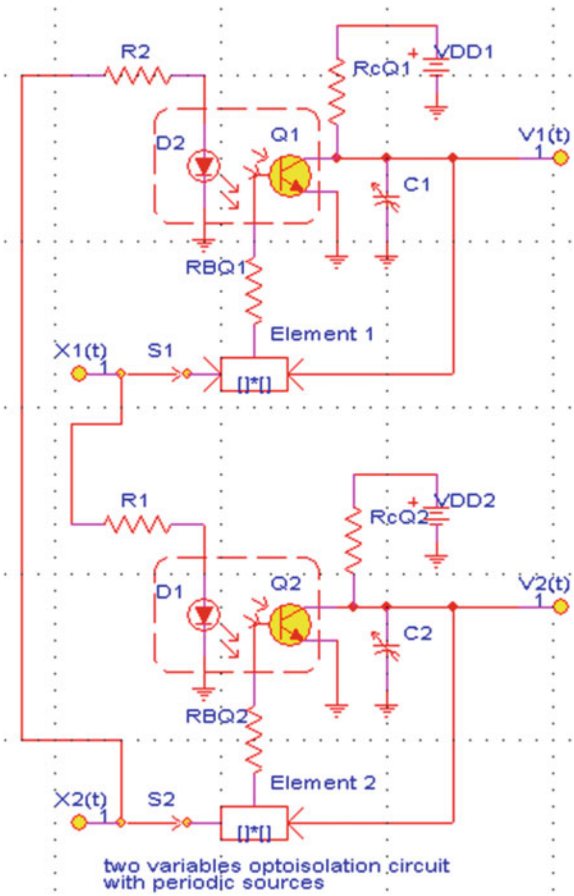

Fig. 4.4 Two variables optoisolation circuit with periodic source

3. We have an optoisolation system which is characterized by three nonlinear differential equations (X, Y, Z main variables), a_i, b_i, d_i (i = 1,2) system parameters and $\frac{dX}{dt} = f_1; \frac{dY}{dt} = f_2; \frac{dZ}{dt} = f_3$; $\frac{dX}{dt} = X \cdot (1 - X) - \frac{a_1 \cdot X}{(1 + b_1 \cdot X)} \cdot Y$; $\frac{dY}{dt} = \frac{a_1 \cdot X}{(1 + b_1 \cdot X)} \cdot Y - \frac{a_2 \cdot Y}{(1 + b_2 \cdot Y)} \cdot Z - d_1 \cdot Y$; $\frac{dZ}{dt} = \frac{a_2 \cdot Y}{(1 + b_2 \cdot Y)} \cdot Z - d_2 \cdot Z$.

3.1 Find system fixed points and plot phase space, $X(t)$, $Y(t)$, $Z(t)$.
3.2 Find system limit cycle and discuss stability of limit cycle by using Floquet theory. We test its stability by asking if small perturbations away from the limit cycle grow or shrink over a complete period.
3.3 Show that as nondimensional parameter b_1 increases, the system undergoes the period doubling route to chaos.
3.4 Implement the system by using optoisolation circuits, op-amps, resistors, capacitors, etc. How circuit optoisolation coupling coefficients (k_1, k_2,…) influence system limit cycle stability.

3.5 We have the case $a_1 = a_2 = a$; $b_1 = b_2 = b$; $d_1 \neq d_2$. Analyze how system limit cycle stability is influenced by parameters a, b values variation. Implement it by using optoisolation circuits.

4. Consider the system of two coupled oscillators with periodic parametric excitation. $X = X(t)$, $Y = Y(t)$ are system variables, a is system parameter. $\ddot{X} + (1 + a \cdot \cos[2 \cdot \pi \cdot t]) \cdot X = Y - X; \ddot{Y} + (1 + a \cdot \cos[2 \cdot \pi \cdot t]) \cdot Y = X - Y$

4.1 Find system fixed points for $a = 0$ and discuss stability.
4.2 How the system dynamic changes in the (ω, a)-space?
4.3 Find system limit cycle and discuss stability of limit cycle by using Floquet theory. We test its stability by asking if small perturbations away from the limit cycle grow or shrink over a complete period.
4.4 Implement the system by using optoisolation circuits, Op-Amps, resistors, capacitors, etc. How circuit optoisolation coupling coefficients (k_1, k_2,...) influence system limit cycle stability.
4.5 System parameter a is now dependent on system main variables X, Y; $a = a(X,Y)$. $a = |X \cdot Y|$. Analyze system dynamic, fixed points, stability, limit cycle and limit cycle stability by using Floquet theory.
4.6 System parameter a is now dependent on system main variables X, Y; $a = a(X,Y)$. $a = \mathrm{sign}\{X \cdot Y\}$. Analyze system dynamic, fixed points, stability, limit cycle and limit cycle stability by using Floquet theory.

5. We have system of three variables X, Y, Z which is characterized by the following differential equations:

$$\frac{dX}{dt} = X - Y - X \cdot (X^2 + Y^2); \ \frac{dY}{dt} = X + Y - Y \cdot (X^2 + Y^2);$$
$$\frac{dZ}{dt} = Z + X \cdot Z - Z^3$$

5.1 Find system fixed points and discuss stability.
5.2 Prove that the system has periodic orbits. Hint: change to cylindrical coordinates, show that the cylinder (with radius one whose axis of symmetry is the z-axis) is invariant.
5.3 prove that there is a stable periodic orbit. The stable periodic orbit has three Floquet multipliers, one of them is unity.
5.4 Implement the system by using optoisolation circuits, discuss stability, limit cycle, and limit cycle stability.
5.5 How system dynamic changes when one of optoisolation element's coupling coefficient is zero ($k_i = 0$) for $i \in 1, 2, \ldots$?

6. We have the following 2D system with variables U_1, U_2. μ is a bifurcation control parameter and n, m parameters of the system $n, m \in \mathbb{R}$; $n, m \in \mathbb{N}$. System differential equations are

$$\frac{dU_1}{dt} = m \cdot \mu \cdot U_1 - n \cdot \mu \cdot U_2 - U_1^2 + U_2^2; \ \frac{dU_2}{dt} = \mu \cdot U_2 + U_1 \cdot U_2$$

6.1 Find system fixed points and discuss stability for $\mu > 0$ and $\mu < 0$. m, n parameters have the following options: (6.1a) $m \neq n$, (6.1b) $m = n$, (6.1c) $m \neq 0$, $n = 0$, (6.1d) $m = 0$, $n \neq 0$. Discuss every case.

6.2 Represent the system by using cylinder coordinates (r,θ), find limit cycle and discuss limit cycle stability by using Floquet theory.

6.3 Implement the system by using optoisolation circuits and discrete components (Op-Amps, resistors, capacitors, diodes, etc.). How optoisolation elements coupling coefficients (k_1, $k_{2,\ldots}$) influence our system limit cycle stability.

6.4 Some of our system variables are taking only absolute values, system differential equation are

$$\frac{dU_1}{dt} = m \cdot \mu \cdot U_1 - n \cdot \mu \cdot |U_2| - U_1^2 + U_2^2; \ \frac{dU_2}{dt} = \mu \cdot U_2 + |U_1| \cdot U_2$$

Discuss all cases of system dynamic ($U_1 > 0$, $U_2 > 0$; $U_1 > 0$, $U_2 < 0$; $U_1 < 0$, $U_2 > 0$; $U_1 < 0$, $U_2 < 0$). How system dynamic changes in each case. Find for each case system limit cycle and discuss system limit cycle stability by using Floquet theory.

6.5 We exchange between $U_1^2 \rightleftharpoons U_2^2$ in our system equations. We get the following system equations:

$$\frac{dU_1}{dt} = m \cdot \mu \cdot U_1 - n \cdot \mu \cdot U_2 - U_2^2 + U_1^2; \ \frac{dU_2}{dt} = \mu \cdot U_2 + U_1 \cdot U_2$$

How system dynamic and limit cycle stability change?

7. Consider the following differential system with X_1, X_2 variables, and μ, σ system parameters.

$$\frac{dX_1}{dt} = \mu \cdot X_1 - X_1^2 - 2 \cdot X_1 \cdot X_2; \ \frac{dX_2}{dt} = (\mu - \sigma) \cdot X_2 + X_1 \cdot X_2 + X_2^2,$$

where σ is a fixed parameter and μ is the bifurcation parameter.

7.1 Differentiate the following cases: $\mu > 0$, $\sigma > 0$; $\mu > 0$, $\sigma < 0$; $\mu < 0$, $\sigma > 0$; $\mu < 0$, $\sigma < 0$. Find fixed points and discuss stability.

7.2 Represent the system by using cylinder coordinates (r,θ), find limit cycle and discuss limit cycle stability by using Floquet theory.

7.3 $\mu = \sigma$, How system dynamic changes? Discuss limit cycle stability by using Floquet theory for $\mu = \sigma$.

7.4 Implement the system by using optoisolation circuits and discrete components (Op-amps, resistors, capacitors, diodes, etc.) for $\mu = \sigma$.

7.4 Discuss system limit cycle stability for $\mu = n \cdot \sigma;\ n \in \mathbb{N}$.

8. We have second-order ODE optoisolation system with periodic source $a(t)$, see Sect. (4.4) circuit. The periodic source is $a(t) = \sin[2 \cdot \pi \cdot t]$, $\partial a(t)/\partial t = 2 \cdot \pi \cdot \cos[2 \cdot \pi \cdot t]$, $T = 1$ s.

   8.1 Find system equations and represent it as a system of two first-order ODE equations.
   8.2 Find system solution constant radius ($\mathrm{d}r/\mathrm{d}t = 0$) as a function of optoisolation circuit's parameters.
   8.3 Find system limit cycle and discuss limit cycle stability by using Floquet theory.
   8.4 The circuit periodic source $a(t) = \sin[2 \cdot \pi \cdot t] + \cos[\pi \cdot t]$. How does the system dynamic change? Discuss limit cycle stability by using Floquet theory.

9. We have linearized Hill's equations which describe optoisolation system, where n is a system parameter and x, y, z are system variables.

$$\ddot{x} - 2 \cdot n \cdot \dot{y} - 3 \cdot n^2 \cdot x = 0;\ \ddot{y} + 2 \cdot n \cdot \dot{x} = 0;\ \ddot{z} + n^2 \cdot z = 0$$

   9.1 Find system fixed points and discuss stability.
   9.2 How system stability depends on "n" parameters.
   9.3 There is a five-dimensional manifold in the phase space corresponding to periodic orbits. Find them.
   9.4 Implement the Hill's system by using optoisolation elements and discrete components (op-amps, resistors, capacitors, diodes, etc.).
   9.5 Bring system equations to a set of typical Hill's equations: $\mathrm{d}^2 U_i(t)/\mathrm{d}t^2 + [\delta_i + a_i(t)] \cdot U_i(t) = 0;\ a_i(t) \to \varepsilon$. U_i—represents new variable. Ex# $\ddot{x} - 2 \cdot n \cdot \dot{y} - 3 \cdot n^2 \cdot x = 0 \Rightarrow \dot{y} = \frac{1}{2 \cdot n} \cdot \ddot{x} - \frac{3}{2} \cdot n \cdot x \ddot{y} = \frac{1}{2 \cdot n} \cdot \dddot{x} - \frac{3}{2} \cdot n \cdot \dot{x} \Rightarrow \ddot{y} + \quad 2 \cdot n \cdot \dot{x} = 0 \Rightarrow \frac{1}{2 \cdot n} \cdot \dddot{x} - \frac{3}{2} \cdot n \cdot \dot{x} + 2 \cdot n \cdot \dot{x} = 0 \frac{1}{2 \cdot n} \cdot \dddot{x} - \quad \frac{3}{2} \cdot n \cdot \dot{x} + 2 \cdot n \cdot \dot{x} = 0 \Rightarrow \frac{1}{2 \cdot n} \cdot \dddot{x} + \frac{1}{2} \cdot n \cdot \dot{x} = 0 \Rightarrow \dddot{x} + n^2 \cdot \dot{x} = 0$
   We define new variable $\dot{x} = \xi;\ \ddot{x} = \dot{\xi};\ \dddot{x} = \ddot{\xi} \Rightarrow \ddot{\xi} + n^2 \cdot \xi = 0$
   $\ddot{\xi} + n^2 \cdot \xi = 0$ it typical Hill's equation when $a_i(t) \to \varepsilon;\ \delta = n^2$.
   Do the same for all system original second-order differential equations. Find Floquet multipliers and discuss all possible options.

10. We have the following system Hill's equation $\frac{\mathrm{d}^2 U(t)}{\mathrm{d}t^2} + [\delta \cdot sign\{a(t)\} + a(t)] \cdot U(t) = 0$, δ is a constant and $a(t)$ is a π-periodic function.

   10.1 Find system $A(t)$ matrix, monodromy matrix, Floquet exponents, and Floquet multipliers. Differentiate all possible cases.
   10.2 Represent the system by using optoisolation elements and discrete components (op-amps, resistors, capacitors, diodes, etc.).

10.3 Investigate system behavior for $a(t) = \sin(\pi \cdot t) + \cos(2 \cdot \pi \cdot t)$.

10.4 Investigate system behavior for $a(t) = \text{sign}\{\sin(\pi \cdot t)\} \cdot \cos(2 \cdot \pi \cdot t)$.

10.5 Represent $\delta \cdot sign\{a(t)\} + a(t)$ expression in the way that our optoisolation circuit represents Hill's equation.

Chapter 5
Optoisolation NDR Circuits Behavior Investigation by Using Floquet Theory

Floquet theory is the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points. Another way to describe Floquet theory, it is the study of linear systems of differential equations with periodic coefficients. Floquet theory can use for variety of OptoNDR circuits, linear stability analysis for, when dealing with an optoisolation periodic system. Although Floquet theory is a linear theory, OptoNDR circuits' nonlinear models can be linearized near limit cycle solutions to enable the use of Floquet theory. Floquet theory deals with continuous-time systems. OptoNDR circuits' periodic study has many engineering implementations like oscillators, amplifiers, logic and memory. Electrical negative resistance is often used to design oscillators. Many topologies are possible, such as the Colpitts oscillator, Hartley oscillator, Wien bridge oscillator, and some types of relaxation oscillators. Another engineering implementation is to use OptoNDR as a Chua's element in Chua's circuit. Autonomous Chua's circuit is containing three energy storage elements, a linear resistor and a nonlinear resistor N_R (Chua's element), and its discrete circuitry design and implementations. Since Chua's circuit is an extremely simple system, and yet it exhibits a rich variety of bifurcations and chaos among the chaos-producing mechanisms, it has a special significance. The term Chua's element is a general description for a two-terminal nonlinear resistor with piecewise-linear characteristic. There are two forms of Chua's element, the first type is a voltage-controlled nonlinear element characterized by $i_R = f(v_R)$, and the other type is a current-controlled nonlinear element characterized by $v_R = g(i_R)$. Chaotic *o*scillators designed with Chua's element are generally based on a single, three-segment, odd-symmetric, voltage-controlled piecewise-linear nonlinear resistor structure. OptoNDR systems can be periodic and continuous in time therefore Floquet theory is an ideal way for behavior and stability analysis [46–48, 81].

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5.1 OptoNDR Circuit Floquet Theory Analysis

We consider an OptoNDR circuit with storage elements (variable capacitor and variable Inductance) in the output port (see Fig. 5.1). At $t = 0$ switch $S1$ moves his position from OFF state to ON state. Circuit dynamic starts and $V(t)$ is the main system variable. We consider that initially capacitor $C1$ is charged to $V_{C1}(t = 0)$ and inductance $L1$ is charged to $I_{L1}(t = 0)$. $V_{DD} \gg V_{Q1(\text{sustain})}$ and photo transistor $Q1$ reaches his sustaining voltage (breakover voltage) after we move switch $S1$ to ON state. After we move $S1$ to ON state capacitor $C1$ starts to charge and the current through inductor $L1$ rise up. Circuit output voltage $V_{CEQ1} = V(t)$ is growing up.

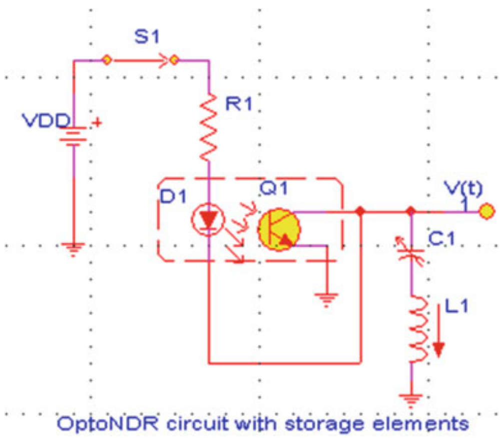
$$V(t) = V = V_{CEQ1} = V_{C1} + V_{L1}; V = V_{CEQ1};\ V_{DD} - V = I_{R1} \cdot R_1 + V_{D1}; I_{R1} = I_{D1}$$

Taylor series approximation: $\ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx \frac{I_{D1}}{I_0}$

$$V_{D1} = \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx \frac{I_{D1}}{I_0};\ V_{DD} - V = I_{D1} \cdot R_1 + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx I_{D1} \cdot R_1 + \frac{V_t}{I_0} \cdot I_{D1}$$

$$V_{DD} - V = I_{D1} \cdot \left(R_1 + \frac{V_t}{I_0}\right) \Rightarrow I_{D1} = \frac{V_{DD} - V}{\left(R_1 + \frac{V_t}{I_0}\right)};\ I_{BQ1} = k_1 \cdot I_{D1} = \frac{(V_{DD} - V) \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

Fig. 5.1 OptoNDR circuit with storage elements



$$I_{C1} = I_{L1};\ I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt} \Rightarrow V_{C1} = \frac{1}{C_1} \cdot \int I_{C1} \cdot dt;$$
$$V_{L1} = L_1 \cdot \frac{dI_{L1}}{dt};\ V = V_{C1} + V_{L1} = \frac{1}{C_1} \cdot \int I_{C1} \cdot dt + L_1 \cdot \frac{dI_{L1}}{dt}$$

$$\frac{d}{dt}\left\{V = \frac{1}{C_1} \cdot \int I_{C1} \cdot dt + L_1 \cdot \frac{dI_{L1}}{dt}\right\} \Rightarrow \dot{V} = \frac{1}{C_1} \cdot I_{C1} + L_1 \cdot \ddot{I}_{L1};\ \dot{V} = \frac{dV}{dt};$$
$$\ddot{I}_{L1} = \frac{d^2 I_{L1}}{dt^2}$$

$$I_{C1} = I_{L1} \Rightarrow \dot{V} = \frac{1}{C_1} \cdot I_{C1} + L_1 \cdot \ddot{I}_{C1};\ I_{D1} = I_{CQ1} + I_{C1} \Rightarrow I_{C1} = I_{D1} - I_{CQ1} = \frac{V_{DD} - V}{\left(R_1 + \frac{V_t}{I_0}\right)} - I_{CQ1}$$

$$I_{C1} = I_{D1} - I_{CQ1} \Rightarrow \dot{I}_{C1} = \dot{I}_{D1} - \dot{I}_{CQ1} \Rightarrow \ddot{I}_{C1} = \ddot{I}_{D1} - \ddot{I}_{CQ1};\ I_{D1} = \frac{V_{DD} - V}{\left(R_1 + \frac{V_t}{I_0}\right)} \Rightarrow \dot{I}_{D1} = -\frac{\dot{V}}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$I_{D1} = \frac{V_{DD} - V}{\left(R_1 + \frac{V_t}{I_0}\right)} \Rightarrow \ddot{I}_{D1} = -\frac{\ddot{V}}{R_1 + \frac{V_t}{I_0}};\ I_{EQ1} = I_{BQ1} + I_{CQ1} = \frac{(V_{DD} - V) \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + I_{CQ1}$$
$$V = V_{CEQ1} = V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\}$$
$$+ V_t \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right);\ \frac{I_{sc}}{I_{se}} \to 1 \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$\ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon \Rightarrow V = V_{CEQ1} \approx V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\}$$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = \alpha_r \cdot I_{CQ1} - \frac{(V_{DD} - V) \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)}$$
$$- I_{CQ1} = I_{CQ1} \cdot (\alpha_r - 1) - \frac{(V_{DD} - V) \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)}$$
$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = I_{CQ1} - \frac{(V_{DD} - V) \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)}$$
$$- I_{CQ1} \cdot \alpha_f = I_{CQ1} \cdot (1 - \alpha_f) - \frac{(V_{DD} - V) \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$V \approx V_t \cdot \ln\left\{\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right\} \Rightarrow e^{\left[\frac{V}{V_t}\right]}$$
$$\approx \frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

Taylor series approximation:

$$e^{\left[\frac{V}{V_t}\right]} \approx \frac{V}{V_t} + 1;\ \frac{V}{V_t} + 1 \approx \frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

$$\frac{V}{V_t} + 1 \simeq \frac{I_{CQ1} \cdot (\alpha_r - 1) - \frac{(V_{DD} - V) \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{I_{CQ1} \cdot (1 - \alpha_f) - \frac{(V_{DD} - V) \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)};$$

$$\ddot{I}_{C1} = -\left\{\frac{\ddot{V}}{\left(R_1 + \frac{V_t}{I_0}\right)} + \ddot{I}_{CQ1}\right\}$$

$$\dot{V} = \frac{1}{C_1} \cdot I_{C1} + L_1 \cdot \ddot{I}_{C1} \Rightarrow \dot{V}$$
$$= \frac{1}{C_1} \cdot \left\{\frac{V_{DD} - V}{\left(R_1 + \frac{V_t}{I_0}\right)} - I_{CQ1}\right\} - L_1 \cdot \left\{\frac{\ddot{V}}{\left(R_1 + \frac{V_t}{I_0}\right)} + \ddot{I}_{CQ1}\right\}$$

$$\dot{V} = \frac{1}{C_1} \cdot I_{C1} + L_1 \cdot \ddot{I}_{C1} \Rightarrow \dot{V}$$
$$= \frac{1}{C_1} \cdot \frac{(V_{DD} - V)}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{1}{C_1} \cdot I_{CQ1} - \ddot{V} \cdot \frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} - L_1 \cdot \ddot{I}_{CQ1}$$

We define new system variables:

$$X = \dot{V};\ \dot{X} = \ddot{V};\ Y = V \Rightarrow \dot{Y} = \dot{V}; \dot{Y} = X$$

$$\frac{V}{V_t} + 1 \simeq \frac{I_{CQ1} \cdot (\alpha_r - 1) - \frac{V_{DD} \cdot k_1}{(R_1 + \frac{V_t}{I_0})} + \frac{V \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{I_{CQ1} \cdot (1 - \alpha_f) - \frac{V_{DD} \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} + \frac{V \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}$$

$$\frac{V}{V_t} + 1 \simeq \frac{I_{CQ1} \cdot (\alpha_r - 1) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - \frac{V_{DD} \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + V \cdot \frac{k_1}{\left(R_1 + \frac{V_t}{I_0}\right)}}{I_{CQ1} \cdot (1 - \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) - \frac{V_{DD} \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)} + V \cdot \frac{k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)}}$$

For simplicity we define the following parameters:

$$A_1 = \alpha_r - 1; A_2 = I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - \frac{V_{DD} \cdot k_1}{\left(R_1 + \frac{V_t}{I_0}\right)};$$

$$A_3 = \frac{k_1}{\left(R_1 + \frac{V_t}{I_0}\right)}; A_4 = 1 - \alpha_f$$

$$A_5 = I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) - \frac{V_{DD} \cdot k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)}; A_6 = \frac{k_1 \cdot \alpha_f}{\left(R_1 + \frac{V_t}{I_0}\right)};$$

$$A_6 = A_3 \cdot \alpha_f; \frac{V}{V_t} + 1 \simeq \frac{I_{CQ1} \cdot A_1 + A_2 + V \cdot A_3}{I_{CQ1} \cdot A_4 + A_5 + V \cdot A_6}$$

$$\frac{V}{V_t} + 1 \simeq \frac{I_{CQ1} \cdot A_1 + A_2 + V \cdot A_3}{I_{CQ1} \cdot A_4 + A_5 + V \cdot A_6} \Rightarrow I_{CQ1} \cdot A_4 \cdot \left(\frac{V}{V_t} + 1\right) + A_5 \cdot \left(\frac{V}{V_t} + 1\right) + V \cdot A_6 \cdot \left(\frac{V}{V_t} + 1\right) = I_{CQ1} \cdot A_1 + A_2 + V \cdot A_3$$

$$A_5 \cdot \left(\frac{V}{V_t} + 1\right) + V \cdot A_6 \cdot \left(\frac{V}{V_t} + 1\right) - A_2 - V \cdot A_3 = I_{CQ1} \cdot A_1 - I_{CQ1} \cdot A_4 \cdot \left(\frac{V}{V_t} + 1\right)$$

$$I_{CQ1} \cdot \left\{A_1 - A_4 \cdot \left(\frac{V}{V_t} + 1\right)\right\} = A_5 \cdot \left(\frac{V}{V_t} + 1\right) + V \cdot A_6 \cdot \left(\frac{V}{V_t} + 1\right) - A_2 - V \cdot A_3$$

$$I_{CQ1} \cdot \left\{A_1 - \frac{A_4}{V_t} \cdot V - A_4\right\} = \frac{A_5}{V_t} \cdot V + A_5 + \frac{A_6}{V_t} \cdot V^2 + V \cdot A_6 - A_2 - V \cdot A_3$$

$$I_{CQ1} \cdot \left\{-\frac{A_4}{V_t} \cdot V + A_1 - A_4\right\} = \frac{A_6}{V_t} \cdot V^2 + V \cdot \left[\frac{A_5}{V_t} + A_6 - A_3\right] + [A_5 - A_2]$$

$$\Rightarrow I_{CQ1} = \frac{\frac{A_6}{V_t} \cdot V^2 + V \cdot \left[\frac{A_5}{V_t} + A_6 - A_3\right] + [A_5 - A_2]}{-\frac{A_4}{V_t} \cdot V + A_1 - A_4}$$

$$\eta_1 = \frac{A_6}{V_t}; \eta_2 = \frac{A_5}{V_t} + A_6 - A_3; \eta_3 = A_5 - A_2; \eta_4 = -\frac{A_4}{V_t};$$

$$\eta_5 = A_1 - A_4;\ I_{CQ1} = \frac{V^2 \cdot \eta_1 + V \cdot \eta_2 + \eta_3}{V \cdot \eta_4 + \eta_5}$$

$$\frac{dI_{CQ1}}{dt} = \frac{[2 \cdot V \cdot \dot{V} \cdot \eta_1 + \dot{V} \cdot \eta_2] \cdot [V \cdot \eta_4 + \eta_5] - [V^2 \cdot \eta_1 + V \cdot \eta_2 + \eta_3] \cdot \dot{V} \cdot \eta_4}{[V \cdot \eta_4 + \eta_5]^2}$$

$$\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} = \frac{\begin{array}{l} 2\cdot V^2\cdot \dot{V}\cdot \eta_1\cdot \eta_4 + 2\cdot V\cdot \dot{V}\cdot \eta_1\cdot \eta_5 + \dot{V}\cdot V\cdot \eta_4\cdot \eta_2 + \dot{V}\cdot \eta_2\cdot \eta_5 \\ -V^2\cdot \dot{V}\cdot \eta_4\cdot \eta_1 - V\cdot \dot{V}\cdot \eta_4\cdot \eta_2 - \dot{V}\cdot \eta_4\cdot \eta_3 \end{array}}{[V\cdot \eta_4 + \eta_5]^2}$$

$$\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} = \frac{V^2\cdot \dot{V}\cdot \eta_1\cdot \eta_4 + V\cdot \dot{V}\cdot 2\cdot \eta_1\cdot \eta_5 + \dot{V}\cdot (\eta_2\cdot \eta_5 - \eta_4\cdot \eta_3)}{[V\cdot \eta_4 + \eta_5]^2};$$

$$\xi_1 = \eta_1\cdot \eta_4; \xi_2 = 2\cdot \eta_1\cdot \eta_5; \xi_3 = \eta_2\cdot \eta_5 - \eta_4\cdot \eta_3$$

$$\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} = \frac{V^2\cdot \dot{V}\cdot \xi_1 + V\cdot \dot{V}\cdot \xi_2 + \dot{V}\cdot \xi_3}{[V\cdot \eta_4 + \eta_5]^2}; \frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{V^2\cdot \dot{V}\cdot \xi_1 + V\cdot \dot{V}\cdot \xi_2 + \dot{V}\cdot \xi_3}{[V\cdot \eta_4 + \eta_5]^2}\right\}$$

$$\frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2} = \frac{\begin{array}{l} \{(2\cdot V\cdot [\dot{V}]^2 + V^2\cdot \ddot{V})\cdot \xi_1 + ([\dot{V}]^2 + V\cdot \ddot{V})\cdot \xi_2 + \ddot{V}\cdot \xi_3\}\cdot [V\cdot \eta_4 + \eta_5]^2 \\ -\{V^2\cdot \dot{V}\cdot \xi_1 + V\cdot \dot{V}\cdot \xi_2 + \dot{V}\cdot \xi_3\}\cdot 2\cdot (V\cdot \eta_4 + \eta_5)\cdot \dot{V}\cdot \eta_4 \end{array}}{[V\cdot \eta_4 + \eta_5]^4}$$

$$\phi_1(V, \dot{V}, \ddot{V}) = \{(2\cdot V\cdot [\dot{V}]^2 + V^2\cdot \ddot{V})\cdot \xi_1 + ([\dot{V}]^2 + V\cdot \ddot{V})\cdot \xi_2 + \ddot{V}\cdot \xi_3\}$$

$$\phi_1(V, \dot{V}, \ddot{V})|_{V=Y,\dot{V}=X,\ddot{V}=\dot{X}} = \{(2\cdot Y\cdot X^2 + Y^2\cdot \dot{X})\cdot \xi_1 + (X^2 + Y\cdot \dot{X})\cdot \xi_2 + \dot{X}\cdot \xi_3\}$$

$$\begin{aligned}\phi_1(V, \dot{V}, \ddot{V})|_{V=Y,\dot{V}=X,\ddot{V}=\dot{X}} &= 2\cdot Y\cdot X^2\cdot \xi_1 + Y^2\cdot \dot{X}\cdot \xi_1 + X^2\cdot \xi_2 \\ &\quad + Y\cdot \dot{X}\cdot \xi_2 + \dot{X}\cdot \xi_3\end{aligned}$$

$$\phi_1(V, \dot{V}, \ddot{V})|_{V=Y,\dot{V}=X,\ddot{V}=\dot{X}} = \dot{X}\cdot [Y^2\cdot \xi_1 + Y\cdot \xi_2 + \xi_3] + X^2\cdot [2\cdot Y\cdot \xi_1 + \xi_2]$$

$$f_1(Y) = Y^2\cdot \xi_1 + Y\cdot \xi_2 + \xi_3; f_2(X, Y) = X^2\cdot [2\cdot Y\cdot \xi_1 + \xi_2]$$

$$\phi_1(Y, X, \dot{X}) = \dot{X}\cdot [Y^2\cdot \xi_1 + Y\cdot \xi_2 + \xi_3] + X^2\cdot [2\cdot Y\cdot \xi_1 + \xi_2]$$

$$\begin{aligned}\phi_1(Y, X, \dot{X}) &= \dot{X}\cdot [Y^2\cdot \xi_1 + Y\cdot \xi_2 + \xi_3] + X^2\cdot [2\cdot Y\cdot \xi_1 + \xi_2] \Rightarrow \phi_1(Y, X, \dot{X}) \\ &= \dot{X}\cdot f_1(Y) + f_2(X, Y)\end{aligned}$$

$$\begin{aligned}\phi_2(V) &= [V\cdot \eta_4 + \eta_5]^2 = V^2\cdot \eta_4^2 + 2\cdot V\cdot \eta_4\cdot \eta_5 + \eta_5^2; \\ \phi_2(V)|_{V=Y} &= Y^2\cdot \eta_4^2 + 2\cdot Y\cdot \eta_4\cdot \eta_5 + \eta_5^2\end{aligned}$$

$$\begin{aligned}\phi_3(V, \dot{V}) &= \{V^2\cdot \dot{V}\cdot \xi_1 + V\cdot \dot{V}\cdot \xi_2 + \dot{V}\cdot \xi_3\}|_{V=Y,\dot{V}=X} \\ &= Y^2\cdot X\cdot \xi_1 + Y\cdot X\cdot \xi_2 + X\cdot \xi_3\end{aligned}$$

$$\begin{aligned}\phi_4(V, \dot{V}) &= 2\cdot (V\cdot \eta_4 + \eta_5)\cdot \dot{V}\cdot \eta_4|_{V=Y,\dot{V}=X} = 2\cdot Y\cdot X\cdot \eta_4^2 + 2\cdot X\cdot \eta_4\cdot \eta_5 \\ &= 2\cdot \eta_4\cdot [Y\cdot X\cdot \eta_4 + X\cdot \eta_5]\end{aligned}$$

$$\frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2} = \frac{\begin{array}{l}\{(2\cdot V\cdot[\dot{V}]^2 + V^2\cdot\ddot{V})\cdot\xi_1 + ([\dot{V}]^2 + V\cdot\ddot{V})\cdot\xi_2 + \ddot{V}\cdot\xi_3\}\cdot[V\cdot\eta_4+\eta_5]^2\\ -\{V^2\cdot\dot{V}\cdot\xi_1 + V\cdot\dot{V}\cdot\xi_2 + \dot{V}\cdot\xi_3\}\cdot 2\cdot(V\cdot\eta_4+\eta_5)\cdot\dot{V}\cdot\eta_4\end{array}}{[V\cdot\eta_4+\eta_5]^4} \Rightarrow$$

$$\begin{aligned}\frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2} &= \frac{\phi_1(Y,X,\dot{X})\cdot\phi_2(Y) - \phi_3(Y,X)\cdot\phi_4(Y,X)}{[\phi_2(Y)]^2}\\ &= \frac{\phi_1(Y,X,\dot{X})}{\phi_2(Y)} - \frac{\phi_3(Y,X)\cdot\phi_4(Y,X)}{[\phi_2(Y)]^2}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2} &= \frac{\dot{X}\cdot f_1(Y) + f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(Y,X)\cdot\phi_4(Y,X)}{[\phi_2(Y)]^2}\\ &= \dot{X}\cdot\frac{f_1(Y)}{\phi_2(Y)} + \left\{\frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(Y,X)\cdot\phi_4(Y,X)}{[\phi_2(Y)]^2}\right\}\end{aligned}$$

For simplicity we define the following functions:

$$f_3(Y) = \frac{f_1(Y)}{\phi_2(Y)}; f_4(X,Y) = \frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y)\cdot\phi_4(X,Y)}{[\phi_2(Y)]^2};$$

$$\frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2} = \dot{X}\cdot f_3(Y) + f_4(X,Y)$$

$$I_{CQ1} = \left.\frac{V^2\cdot\eta_1 + V\cdot\eta_2 + \eta_3}{V\cdot\eta_4+\eta_5}\right|_{V=Y} = \frac{Y^2\cdot\eta_1 + Y\cdot\eta_2 + \eta_3}{Y\cdot\eta_4+\eta_5};$$

$$I_{CQ1} = f_5(Y) = \frac{Y^2\cdot\eta_1 + Y\cdot\eta_2 + \eta_3}{Y\cdot\eta_4+\eta_5}$$

$$\dot{V} = \frac{1}{C_1}\cdot\frac{(V_{DD}-V)}{\left(R_1+\frac{V_t}{I_0}\right)} - \frac{1}{C_1}\cdot I_{CQ1} - \ddot{V}\cdot\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} - L_1\cdot\ddot{I}_{CQ1}$$

$$X = \frac{1}{C_1}\cdot\frac{(V_{DD}-Y)}{\left(R_1+\frac{V_t}{I_0}\right)} - \frac{1}{C_1}\cdot f_5(Y) - \dot{X}\cdot\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} - L_1\cdot[\dot{X}\cdot f_3(Y) + f_4(X,Y)]$$

$$\begin{aligned}X = \frac{1}{C_1}\cdot\frac{(V_{DD}-Y)}{\left(R_1+\frac{V_t}{I_0}\right)} - \frac{1}{C_1}\cdot f_5(Y) - \dot{X}\cdot\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} - \dot{X}\cdot L_1\cdot f_3(Y)\\ - L_1\cdot f_4(X,Y)\end{aligned}$$

$$X - \frac{1}{C_1}\cdot\frac{(V_{DD}-Y)}{\left(R_1+\frac{V_t}{I_0}\right)} + \frac{1}{C_1}\cdot f_5(Y) + L_1\cdot f_4(X,Y) = -\dot{X}\cdot\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} - \dot{X}\cdot L_1\cdot f_3(Y)$$

$$\dot{X} \cdot \frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + \dot{X} \cdot L_1 \cdot f_3(Y) = -X + \frac{1}{C_1} \cdot \frac{(V_{DD} - Y)}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X, Y)$$

$$\dot{X} \cdot \left\{ \frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y) \right\} = \frac{1}{C_1} \cdot \frac{(V_{DD} - Y)}{\left(R_1 + \frac{V_t}{I_0}\right)} - X - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X, Y)$$

$$\dot{X} = \frac{\frac{1}{C_1} \cdot \frac{(V_{DD} - Y)}{\left(R_1 + \frac{V_t}{I_0}\right)} - X - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X, Y)}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)}$$

We can summarize our system differential equations new two variables (X, Y):

$$\dot{X} = \frac{\frac{1}{C_1} \cdot \frac{(V_{DD} - Y)}{\left(R_1 + \frac{V_t}{I_0}\right)} - X - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X, Y)}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)}$$

$$\dot{Y} = X$$

We need to check in which conditions the system has periodic orbits and it is done by changing system Cartesian coordinates ($X(t)$, $Y(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. We approve that there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers. One of them is unity. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z axis. Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X, Y) are the same as for polar coordinates, namely $X(t) = r(t) \cdot \cos[\theta(t)]$; $Y(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X^2 + Y^2}$. $\theta(t) = 0$ if $X = 0$ and $Y = 0$. $\theta(t) = \arcsin(Y/r)$ if $X >= 0$. $\theta(t) = -\arcsin(Y/r) + \pi$ if $X < 0$. We represent our system equation by using cylindrical coordinates ($r(t)$, $\theta(t)$).

$$X(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t} \cdot \cos[\theta(t)] - r(t) \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \cdot \sin[\theta(t)]$$

$$Y(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t} \cdot \sin[\theta(t)] + r(t) \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \cdot \cos[\theta(t)]$$

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t}; \quad \frac{\mathrm{d}r(t)}{\mathrm{d}t} = r'; \quad \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \theta'; \theta(t) = \theta; r(t) = r$$

We get the equations:

$$\frac{dX}{dt} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta; \frac{dY}{dt} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$

From the second system differential equation (5.2) we get $\dot{r} = \frac{r \cdot [1-\dot{\theta}]}{tg\theta}$.

$$\dot{Y} = X \Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = r \cdot \cos\theta \Rightarrow r' \cdot \sin\theta = r \cdot \cos\theta \cdot [1 - \theta']$$
$$\Rightarrow \dot{r} = \frac{r \cdot [1 - \dot{\theta}]}{tg\theta}$$

To find our system solution constant radius we set $\frac{dr}{dt} = 0$ which yield the following outcome:

$$\frac{dY}{dt} = X \Rightarrow tg\theta = \frac{r \cdot (1-\theta')}{r'} \Rightarrow r' = \frac{r \cdot (1-\theta')}{tg\theta}; \frac{dr}{dt} = 0 \Leftrightarrow \frac{r \cdot (1-\theta')}{tg\theta} = 0$$

Since $r \neq 0$, we get two possible equations as follows:

1. $tg\theta \to \infty \Rightarrow \theta = \frac{\pi}{2} + \pi \cdot n = \pi \cdot \left(\frac{1}{2} + n\right) \quad \forall n = 0, 1, 2, \ldots$
2. $1 - \theta' = 0 \Rightarrow \frac{d\theta}{dt} = 1 \Rightarrow \theta = t + \text{const}; \frac{d\theta}{dt} = \frac{2 \cdot \pi}{T} = 1 \Rightarrow T = 2 \cdot \pi.$

We define

$$\Upsilon_1(X,Y) = \frac{\frac{1}{C_1} \cdot \frac{(V_{DD}-Y)}{\left(R_1 + \frac{V_t}{I_0}\right)} - X - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X,Y)}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)}; \Upsilon_2(X,Y) = X$$

We get system differential equations:

$$\frac{dX}{dt} = \Upsilon_1(X,Y); \frac{dY}{dt} = \Upsilon_2(X,Y)$$

We need to find $\Upsilon_1(X,Y)$ and $\Upsilon_2(X,Y)$ in cylindrical coordinates ($r(t)$, $\theta(t)$) for our system. The transformation $\Upsilon_1(X,Y) \to \Upsilon_1(r(t),\theta(t)); \Upsilon_2(X,Y) \to \Upsilon_2(r(t),\theta(t))$.

$$f_3(r(t),\theta(t)); f_4(X,Y) \to f_4(r(t),\theta(t)); f_5(X,Y) \to f_5(r(t),\theta(t)).$$

$$f_3(Y) = \frac{f_1(Y)}{\phi_2(Y)} = \left.\frac{Y^2 \cdot \xi_1 + Y \cdot \xi_2 + \xi_3}{Y^2 \cdot \eta_4^2 + 2 \cdot Y \cdot \eta_4 \cdot \eta_5 + \eta_5^2}\right| \begin{matrix} X = r \cdot \cos\theta \\ Y = r \cdot \sin\theta \end{matrix}$$

$$= \frac{r^2 \cdot \sin^2\theta \cdot \xi_1 + r \cdot \sin\theta \cdot \xi_2 + \xi_3}{r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2}$$

$$f_4(X,Y) = \frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y) \cdot \phi_4(X,Y)}{[\phi_2(Y)]^2} = \frac{X^2 \cdot [2 \cdot Y \cdot \xi_1 + \xi_2]}{Y^2 \cdot \eta_4^2 + 2 \cdot Y \cdot \eta_4 \cdot \eta_5 + \eta_5^2}$$
$$- \frac{[Y^2 \cdot X \cdot \xi_1 + Y \cdot X \cdot \xi_2 + X \cdot \xi_3] \cdot \{2 \cdot \eta_4 \cdot [Y \cdot X \cdot \eta_4 + X \cdot \eta_5]\}}{[Y^2 \cdot \eta_4^2 + 2 \cdot Y \cdot \eta_4 \cdot \eta_5 + \eta_5^2]^2}$$

$$X^2 = r^2 \cdot \cos^2\theta; Y^2 = r^2 \cdot \sin^2\theta; X = r \cdot \cos\theta; Y = r \cdot \sin\theta$$

$$f_4(X,Y) = \left.\frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y) \cdot \phi_4(X,Y)}{[\phi_2(Y)]^2}\right|_{\substack{X = r \cdot \cos\theta \\ Y = r \cdot \sin\theta}}$$
$$= \frac{r^2 \cdot \cos^2\theta \cdot [2 \cdot r \cdot \sin\theta \cdot \xi_1 + \xi_2]}{r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2}$$
$$- \frac{[r^2 \cdot \sin^2\theta \cdot r \cdot \cos\theta \cdot \xi_1 + r \cdot \sin\theta \cdot r \cdot \cos\theta \cdot \xi_2 + r \cdot \cos\theta \cdot \xi_3] \cdot \{2 \cdot \eta_4 \cdot [r \cdot \sin\theta \cdot r \cdot \cos\theta \cdot \eta_4 + r \cdot \cos\theta \cdot \eta_5]\}}{[r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2]^2}$$

$$f_4(X,Y) = \left.\frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y) \cdot \phi_4(X,Y)}{[\phi_2(Y)]^2}\right|_{\substack{X = r \cdot \cos\theta \\ Y = r \cdot \sin\theta}}$$
$$= \frac{r^2 \cdot \cos^2\theta \cdot [2 \cdot r \cdot \sin\theta \cdot \xi_1 + \xi_2]}{r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2}$$
$$- \frac{[r^2 \cdot \sin^2\theta \cdot r \cdot \cos\theta \cdot \xi_1 + r \cdot \sin\theta \cdot r \cdot \cos\theta \cdot \xi_2 + r \cdot \cos\theta \cdot \xi_3] \cdot \{2 \cdot \eta_4 \cdot [r \cdot \sin\theta \cdot r \cdot \cos\theta \cdot \eta_4 + r \cdot \cos\theta \cdot \eta_5]\}}{[r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2]^2}$$

$$f_4(X,Y) = \left.\frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y) \cdot \phi_4(X,Y)}{[\phi_2(Y)]^2}\right|_{\substack{X = r \cdot \cos\theta \\ Y = r \cdot \sin\theta}}$$
$$= \frac{2 \cdot r^3 \cdot \sin\theta \cdot \cos^2\theta \cdot \xi_1 + r^2 \cdot \cos^2\theta \cdot \xi_2}{r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2}$$
$$- \frac{[r^3 \cdot \sin^2\theta \cdot \cos\theta \cdot \xi_1 + r^2 \cdot \sin\theta \cdot \cos\theta \cdot \xi_2 + r \cdot \cos\theta \cdot \xi_3] \cdot \{2 \cdot \eta_4 \cdot [r^2 \cdot \sin\theta \cdot \cos\theta \cdot \eta_4 + r \cdot \cos\theta \cdot \eta_5]\}}{[r^2 \cdot \sin^2\theta \cdot \eta_4^2 + 2 \cdot r \cdot \sin\theta \cdot \eta_4 \cdot \eta_5 + \eta_5^2]^2}$$

$$f_4(X,Y) = \frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y)\cdot\phi_4(X,Y)}{[\phi_2(Y)]^2}\Bigg|_{\substack{X=r\cdot\cos\theta \\ Y=r\cdot\sin\theta}}$$

$$= \frac{r^3\cdot\sin 2\theta\cdot\cos\theta\cdot\xi_1 + r^2\cdot\cos^2\theta\cdot\xi_2}{r^2\cdot\sin^2\theta\cdot\eta_4^2 + 2\cdot r\cdot\sin\theta\cdot\eta_4\cdot\eta_5 + \eta_5^2}$$

$$- \frac{[r^3\cdot\frac{1}{2}\sin 2\theta\cdot\sin\theta\cdot\xi_1 + r^2\cdot\frac{1}{2}\sin 2\theta\cdot\xi_2 + r\cdot\cos\theta\cdot\xi_3]\cdot\{2\cdot\eta_4\cdot[r^2\cdot\frac{1}{2}\sin 2\theta\cdot\eta_4 + r\cdot\cos\theta\cdot\eta_5]\}}{[r^2\cdot\sin^2\theta\cdot\eta_4^2 + 2\cdot r\cdot\sin\theta\cdot\eta_4\cdot\eta_5 + \eta_5^2]^2}$$

$$f_5(Y) = \frac{Y^2\cdot\eta_1 + Y\cdot\eta_2 + \eta_3}{Y\cdot\eta_4 + \eta_5}\Bigg|_{\substack{X=r\cdot\cos\theta \\ Y=r\cdot\sin\theta}} = \frac{r^2\cdot\sin^2\theta\cdot\eta_1 + r\cdot\sin\theta\cdot\eta_2 + \eta_3}{r\cdot\sin\theta\cdot\eta_4 + \eta_5}$$

To find our system solution constant radius we set $\frac{dr}{dt} = 0$ which yield to:

$$\frac{dX}{dt} = \Upsilon_1(X,Y) \Rightarrow r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta = \Upsilon_1(r,\theta); \quad \frac{dr}{dt} = 0 \Rightarrow -r\cdot\theta'\cdot\sin\theta = \Upsilon_1(r,\theta)$$

$$-r\cdot\theta'\cdot\sin\theta = \Upsilon_1(r,\theta) \Rightarrow -r\cdot\theta'\cdot\sin\theta = \frac{\frac{1}{C_1}\cdot\frac{(V_{DD}-r\cdot\sin\theta)}{\left(R_1+\frac{V_t}{I_0}\right)} - r\cdot\cos\theta - \frac{1}{C_1}\cdot f_5(r,\theta) - L_1\cdot f_4(r,\theta)}{\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} + L_1\cdot f_3(r,\theta)}$$

We already got from system second differential equation constant radius solution

$$\frac{dr}{dt} = 0 \Rightarrow \frac{d\theta}{dt} = 1 \Rightarrow \theta = t + \text{const}; \frac{d\theta}{dt} = \frac{2\cdot\pi}{T} = 1 \Rightarrow T = 2\cdot\pi$$

$$-r\cdot\sin\theta = \Upsilon_1(r,\theta) \Rightarrow -r\cdot\sin\theta = \frac{\frac{1}{C_1}\cdot\frac{(V_{DD}-r\cdot\sin\theta)}{\left(R_1+\frac{V_t}{I_0}\right)} - r\cdot\cos\theta - \frac{1}{C_1}\cdot f_5(r,\theta) - L_1\cdot f_4(r,\theta)}{\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} + L_1\cdot f_3(r,\theta)}$$

$$-r\cdot\sin\theta\cdot\left\{\frac{L_1}{\left(R_1+\frac{V_t}{I_0}\right)} + L_1\cdot f_3(r,\theta)\right\} = \frac{1}{C_1}\cdot\frac{(V_{DD}-r\cdot\sin\theta)}{\left(R_1+\frac{V_t}{I_0}\right)} - r\cdot\cos\theta - \frac{1}{C_1}\cdot f_5(r,\theta) - L_1\cdot f_4(r,\theta)$$

$$-r \cdot \sin\theta \cdot \left\{ \frac{C_1 \cdot L_1}{(R_1 + \frac{V_t}{I_0})} + C_1 \cdot L_1 \cdot f_3(r,\theta) \right\} = \frac{(V_{DD} - r \cdot \sin\theta)}{\left(R_1 + \frac{V_t}{I_0}\right)} - C_1 \cdot r \cdot \cos\theta - f_5(r,\theta) - C_1 \cdot L_1 \cdot f_4(r,\theta)$$

$$r \cdot \sin\theta \cdot \left\{ \frac{C_1 \cdot L_1}{(R_1 + \frac{V_t}{I_0})} + C_1 \cdot L_1 \cdot f_3(r,\theta) \right\} = C_1 \cdot r \cdot \cos\theta + f_5(r,\theta) + C_1 \cdot L_1 \cdot f_4(r,\theta) - \frac{(V_{DD} - r \cdot \sin\theta)}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$r \cdot \sin\theta \cdot \left\{ \frac{C_1 \cdot L_1}{(R_1 + \frac{V_t}{I_0})} + C_1 \cdot L_1 \cdot f_3(r,\theta) \right\} - C_1 \cdot r \cdot \cos\theta - \frac{r \cdot \sin\theta}{\left(R_1 + \frac{V_t}{I_0}\right)} = f_5(r,\theta) + C_1 \cdot L_1 \cdot f_4(r,\theta) - \frac{V_{DD}}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$r \cdot \sin\theta \cdot \left\{ \frac{C_1 \cdot L_1}{(R_1 + \frac{V_t}{I_0})} + C_1 \cdot L_1 \cdot f_3(r,\theta) - C_1 \cdot \frac{1}{tg\theta} - \frac{1}{(R_1 + \frac{V_t}{I_0})} \right\} = f_5(r,\theta) + C_1 \cdot L_1 \cdot f_4(r,\theta) - \frac{V_{DD}}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

From the above equation we need to get constant radius $r = r_{\text{const–radius}}$.

$$r_{\text{const-radius}} = \frac{f_5(r_{\text{const-radius}},\theta) + C_1 \cdot L_1 \cdot f_4(r_{\text{const-radius}},\theta) - \frac{V_{DD}}{\left(R_1 + \frac{V_t}{I_0}\right)}}{\sin\theta \cdot \left\{ \frac{C_1 \cdot L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + C_1 \cdot L_1 \cdot f_3(r_{\text{const-radius}},\theta) - C_1 \cdot \frac{1}{tg\theta} - \frac{1}{\left(R_1 + \frac{V_t}{I_0}\right)} \right\}}$$

Remark We need to do algebraic manipulation in the above expression and get expression without θ variable in time and find $r_{\text{const-radius}}$ as a function of circuit parameters (C_1, L_1, α_f, α_r, I_{se}, I_{sc}, etc.).

We need to find $\left(\frac{\partial \Upsilon_1}{\partial X} + \frac{\partial \Upsilon_2}{\partial Y}\right)\Big|_{r = r_{\text{const-radius}}}$ at a solution with constant radius ($\mathrm{d}r/\mathrm{d}t = 0$).

$$\rho_1 \cdot \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot ds}\Bigg|_{\rho_1 = 1} \Rightarrow \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot ds} = \mathrm{e}^{\int_0^{T = 2\cdot\pi} \left(\frac{\partial \Upsilon_1}{\partial X} + \frac{\partial \Upsilon_2}{\partial Y}\right)\big|_{r = r_{\text{const}}} \cdot ds}$$

As a result, the limit cycle with radius $r_{\text{const-radius}} > 0$ is stable if $\int_0^T tr(A(s)) \cdot ds < 0$ or $\rho_2 < 1$, and unstable if $\int_0^T tr(A(s)) \cdot ds > 0$ or $\rho_2 > 1$. We need to find the expressions for $\frac{\partial \Upsilon_1}{\partial X}, \frac{\partial \Upsilon_2}{\partial Y}$. $\Upsilon_2(X,Y) = X \Rightarrow \frac{\partial \Upsilon_2(X,Y)}{\partial Y} = 0$

$$\Upsilon_1(X,Y) = \frac{\frac{1}{C_1} \cdot \frac{(V_{DD}-Y)}{\left(R_1 + \frac{V_t}{I_0}\right)} - X - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X,Y)}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)}$$

$$\Upsilon_1(X,Y) = \frac{1}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)} \cdot \left\{ \frac{1}{C_1} \cdot \frac{(V_{DD}-Y)}{(R_1 + \frac{V_t}{I_0})} - X - \frac{1}{C_1} \cdot f_5(Y) - L_1 \cdot f_4(X,Y) \right\}$$

$$\frac{\partial \Upsilon_1(X,Y)}{\partial X} = -\frac{1}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)} \cdot \left\{ 1 + L_1 \cdot \frac{\partial f_4(X,Y)}{\partial X} \right\}$$

$$f_4(X,Y) = \frac{f_2(X,Y)}{\phi_2(Y)} - \frac{\phi_3(X,Y) \cdot \phi_4(X,Y)}{[\phi_2(Y)]^2}$$
$$\Rightarrow \frac{\partial f_4(X,Y)}{\partial X} = \frac{1}{\phi_2(Y)} \cdot \frac{\partial f_2(X,Y)}{\partial X} - \frac{1}{[\phi_2(Y)]^2} \cdot \left\{ \frac{\partial \phi_3(X,Y)}{\partial X} \cdot \phi_4(X,Y) + \phi_3(X,Y) \cdot \frac{\partial \phi_4(X,Y)}{\partial X} \right\}$$

$$f_2(X,Y) = X^2 \cdot [2 \cdot Y \cdot \xi_1 + \xi_2] \Rightarrow \frac{\partial f_2(X,Y)}{\partial X} = 2 \cdot X \cdot [2 \cdot Y \cdot \xi_1 + \xi_2]$$

$$\phi_3(X,Y) = Y^2 \cdot X \cdot \xi_1 + Y \cdot X \cdot \xi_2 + X \cdot \xi_3 \Rightarrow \frac{\partial \phi_3(X,Y)}{\partial X} = Y^2 \cdot \xi_1 + Y \cdot \xi_2 + \xi_3$$

$$\phi_4(X,Y) = 2 \cdot \eta_4 \cdot [Y \cdot X \cdot \eta_4 + X \cdot \eta_5] \Rightarrow \frac{\partial \phi_4(X,Y)}{\partial X} = 2 \cdot \eta_4 \cdot [Y \cdot \eta_4 + \eta_5]$$

$$\frac{\partial f_4(X,Y)}{\partial X} = \frac{1}{[Y^2 \cdot \eta_4^2 + 2 \cdot Y \cdot \eta_4 \cdot \eta_5 + \eta_5^2]} \cdot 2 \cdot X \cdot [2 \cdot Y \cdot \xi_1 + \xi_2] - \frac{1}{[Y^2 \cdot \eta_4^2 + 2 \cdot Y \cdot \eta_4 \cdot \eta_5 + \eta_5^2]^2} \cdot \{[Y^2 \cdot \xi_1 + Y \cdot \xi_2 + \xi_3] \cdot 2 \cdot \eta_4 \cdot X \cdot [Y \cdot \eta_4 + \eta_5] + 2 \cdot \eta_4 \cdot X \cdot [Y^2 \cdot \xi_1 + Y \cdot \xi_2 + \xi_3] \cdot [Y \cdot \eta_4 + \eta_5]\}$$

$$\frac{\partial\Upsilon_1}{\partial X} + \frac{\partial\Upsilon_2}{\partial Y} = -\frac{1}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)} \cdot \left\{1 + L_1 \cdot \frac{\partial f_4(X,Y)}{\partial X}\right\}$$

$$\left(\frac{\partial\Upsilon_1}{\partial X} + \frac{\partial\Upsilon_2}{\partial Y}\right)\Bigg|_{r=r_{\text{const-radius}}} = \left(-\frac{1}{\frac{L_1}{(R_1 + \frac{V_t}{I_0})} + L_1 \cdot f_3(Y)} \cdot \left\{1 + L_1 \cdot \frac{\partial f_4(X,Y)}{\partial X}\right\}\right)\Bigg|_{r=r_{\text{const-radius}}}$$

$$\rho_1 \cdot \rho_2 = e^{\int_0^T tr(A(s))\cdot ds}\Bigg|_{\rho_1=1} \Rightarrow \rho_2 = e^{\int_0^T tr(A(s))\cdot ds} = e^{\int_0^{T=2\cdot\pi}\left(\frac{\partial\Upsilon_1}{\partial X} + \frac{\partial\Upsilon_2}{\partial Y}\right)\Big|_{r=r_{\text{const}}}\cdot ds}$$

$$tr(A(s)) = \left(\frac{\partial\Upsilon_1}{\partial X} + \frac{\partial\Upsilon_2}{\partial Y}\right)\Bigg|_{r=r_{\text{const}}}$$
$$= \left(-\frac{1}{\frac{L_1}{\left(R_1 + \frac{V_t}{I_0}\right)} + L_1 \cdot f_3(Y)} \cdot \left\{1 + L_1 \cdot \frac{\partial f_4(X,Y)}{\partial X}\right\}\right)\Bigg|_{r=r_{\text{const-radius}}}$$

$$\rho_2 = e^{\int_0^T tr(A(s))\cdot ds} = e^{\int_0^{T=2\cdot\pi}\left(\frac{\partial\Upsilon_1}{\partial X} + \frac{\partial\Upsilon_2}{\partial Y}\right)\Big|_{r=r_{\text{const}}}\cdot ds}$$
$$= \exp\left\{\int_0^{T=2\cdot\pi}\left(-\frac{1}{\frac{L_1}{(R_1 + \frac{V_t}{I_0})} + L_1 \cdot f_3(Y)} \cdot \left\{1 + L_1 \cdot \frac{\partial f_4(X,Y)}{\partial X}\right\}\right)\Bigg|_{r=r_{\text{const-radius}}} \cdot ds\right\}$$

5.2 Chua's Circuit Fixed Points and Stability Analysis

Autonomous Chua's circuit has many implementations in various electrical circuits. Chua's circuit contains three energy storage elements (two capacitors and one inductor), a linear resistor and nonlinear resistor N_R, and its discrete circuitry design and implementations. Chua's circuit is an extremely simple system, but it exhibits a rich variety of bifurcations and chaos among the chaos-producing mechanism. Chua's circuit fixed points and stability analysis is a crucial step toward understanding circuits and systems behavior which include Chua's circuits. Several realizations of Chua's circuit are possible. The methodologies used in these realizations can be divided into two basic categories. In the first approach, a variety of circuit topologies have been considered for realizing the nonlinear resistor N_R in

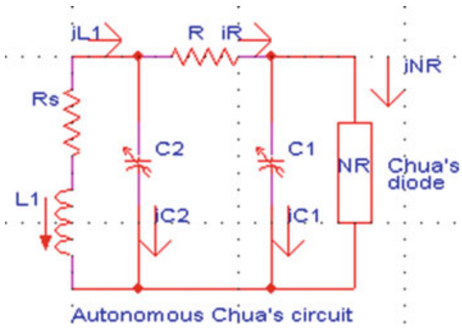


Fig. 5.2 Autonomous Chua's circuit

Chua's circuit. The main idea in the second approach related to the implementation of Chua's circuit is an inductor less realization of Chua's circuit [81, 116] (Fig. 5.2).

The nonlinear resistor concept and Chua's diode are important elements in system concept analysis. The term Chua's diode is a general description for a two-terminal nonlinear resistor with piecewise-linear characteristic. Chua's diode is defined in two forms. The first type of Chua's diode is a voltage controlled nonlinear element characterized by $i_R = f(v_R)$, and the other type is a current-controlled nonlinear element characterized by $v_R = g(i_R)$. Chaotic oscillators designed with Chua's diode are generally based on a single, three-segment, voltage-controlled piecewise-linear nonlinear resistor structure. We refer in our analysis to such a voltage-controlled characteristic of Chua's diode. Figure 5.3 describes three-segment odd-symmetric voltage-controlled piecewise-linear characteristic of Chua's diode [81].

In the above $i_R = f(v_R)$ graph, α_a and α_b are the inner and outer slopes, respectively, and $\pm\Gamma(\Gamma > 0)$ denote the breakpoints. The concave resistor N_R is a piecewise-linear voltage-controlled resistor uniquely specified by $(\alpha_a, \alpha_b, \Gamma)$ parameters. $i_R = f(v_R) = \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{|v_R + \Gamma| - |v_R - \Gamma|\}$; $\Gamma > 0$.

We differentiate three graph regions $i_R = f(v_R)_{I/II/III}$ equations.

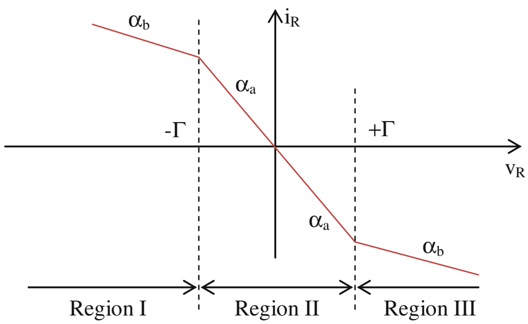


Fig. 5.3 Describes three-segment odd-symmetric voltage-controlled piecewise-linear characteristic of Chua's diode

Region I

$$v_R < -\Gamma \Rightarrow v_R + \Gamma < 0 \Rightarrow |v_R + \Gamma| = -(v_R + \Gamma)\ \&\ |v_R - \Gamma| = -(v_R - \Gamma)$$

$$v_R < -\Gamma \Rightarrow i_R = f(v_R) = \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{-v_R - \Gamma + v_R - \Gamma\}$$
$$= \alpha_b \cdot v_R - (\alpha_a - \alpha_b) \cdot \Gamma$$

Region II

$$-\Gamma \le v_R \le +\Gamma$$

$$-\Gamma \le v_R \le +\Gamma \Rightarrow v_R - \Gamma \le 0\ \&\ v_R + \Gamma \ge 0 \Rightarrow |v_R + \Gamma| = v_R + \Gamma\ \&\ |v_R - \Gamma| = -v_R + \Gamma$$
$$i_R = f(v_R) = \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{|v_R + \Gamma| - |v_R - \Gamma|\}$$
$$= \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{v_R + \Gamma + v_R - \Gamma\}$$
$$i_R = f(v_R) = \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{|v_R + \Gamma| - |v_R - \Gamma|\} = \alpha_a \cdot v_R$$

Region III

$$v_R > \Gamma \Rightarrow v_R - \Gamma > 0 \Rightarrow |v_R + \Gamma| = v_R + \Gamma \& |v_R - \Gamma| = v_R - \Gamma$$

$$i_R = f(v_R) = \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{|v_R + \Gamma| - |v_R - \Gamma|\}$$
$$= \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{v_R + \Gamma - v_R + \Gamma\}$$
$$i_R = f(v_R) = \alpha_b \cdot v_R + \frac{1}{2} \cdot (\alpha_a - \alpha_b) \cdot \{|v_R + \Gamma| - |v_R - \Gamma|\} = \alpha_b \cdot v_R + (\alpha_a - \alpha_b) \cdot \Gamma$$

We can summery our results in Table 5.1.
We can write system differential equations.

$$i_{L1}(t=0) = Z_0; V_{C1}(t=0) = V_R(t=0) = X_0; V_{C2}(t=0) = Y_0; V_R \to X; V_{C2} \to Y; i_{L1} \to Z$$

$$C_1 \cdot \frac{dV_R}{dt} = \frac{1}{R} \cdot (V_{C2} - V_R) - f(V_R); C_2 \cdot \frac{dV_{C2}}{dt} = i_{L1} - \frac{1}{R} \cdot (V_{C2} - V_R); L_1 \cdot \frac{di_{L1}}{dt}$$
$$= -V_{C2} - i_{L1} \cdot R_S$$

Table 5.1 Summary of our results

$v_R < -\Gamma \Rightarrow v_R + \Gamma < 0$	$-\Gamma \le v_R \le +\Gamma$	$v_R > \Gamma \Rightarrow v_R - \Gamma > 0$
$i_R = f(v_R) = \alpha_b \cdot v_R - (\alpha_a - \alpha_b) \cdot \Gamma$	$i_R = f(v_R) = \alpha_a \cdot v_R$	$i_R = f(v_R) = \alpha_b \cdot v_R + (\alpha_a - \alpha_b) \cdot \Gamma$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot (Y - X) - \frac{1}{C_1} \cdot f(X); \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - X); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

We have three cases which we analyze system fixed points and stability.

Case I

$$v_R < -\Gamma \Rightarrow v_R + \Gamma < 0 \Rightarrow i_R = f(X) = \alpha_b \cdot X - (\alpha_a - \alpha_b) \cdot \Gamma$$
$$= \alpha_b \cdot X + (\alpha_b - \alpha_a) \cdot \Gamma$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot (Y - X) - \frac{1}{C_1} \cdot \{\alpha_b \cdot X + (\alpha_b - \alpha_a) \cdot \Gamma\}; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - X)$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot Y - \frac{1}{C_1 \cdot R} \cdot X - \frac{1}{C_1} \cdot \alpha_b \cdot X - \frac{1}{C_1} \cdot (\alpha_b - \alpha_a) \cdot \Gamma;$$
$$\dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - X)$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot Y - \frac{1}{C_1} \cdot X \cdot \left(\frac{1}{R} + \alpha_b\right) - \frac{1}{C_1} \cdot (\alpha_b - \alpha_a) \cdot \Gamma; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - X)$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

We find system fixed points by setting $\dot{X} = 0; \dot{Y} = 0; \dot{Z} = 0$.

$$\dot{X} = 0 \Rightarrow \frac{1}{C_1 \cdot R} \cdot Y^* - \frac{1}{C_1} \cdot X^* \cdot \left(\frac{1}{R} + \alpha_b\right) - \frac{1}{C_1} \cdot (\alpha_b - \alpha_a) \cdot \Gamma = 0;$$
$$\dot{Y} = 0 \Rightarrow \frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot (Y^* - X^*) = 0$$
$$\dot{Z} = 0 \Rightarrow -\frac{1}{L_1} \cdot Y^* - \frac{R_S}{L_1} \cdot Z^* = 0$$

$$-\frac{1}{L_1} \cdot Y^* - \frac{R_S}{L_1} \cdot Z^* = 0 \Rightarrow Z^* = -\frac{1}{R_S} \cdot Y^*; \frac{1}{C_1 \cdot R} \cdot Y^* - \frac{1}{C_1} \cdot X^* \cdot \left(\frac{1}{R} + \alpha_b\right)$$
$$-\frac{1}{C_1} \cdot (\alpha_b - \alpha_a) \cdot \Gamma = 0$$

$$\frac{1}{C_1 \cdot R} \cdot Y^* - \frac{1}{C_1} \cdot X^* \cdot \left(\frac{1}{R} + \alpha_b\right) - \frac{1}{C_1} \cdot (\alpha_b - \alpha_a) \cdot \Gamma = 0$$
$$\Rightarrow \frac{1}{C_1} \cdot X^* \cdot \left(\frac{1}{R} + \alpha_b\right) = \frac{1}{C_1 \cdot R} \cdot Y^* - \frac{1}{C_1} \cdot (\alpha_b - \alpha_a) \cdot \Gamma$$

$$X^* \cdot \left(\frac{1}{R} + \alpha_b\right) = \frac{1}{R} \cdot Y^* - (\alpha_b - \alpha_a) \cdot \Gamma \Rightarrow X^* = \frac{1}{R \cdot \left(\frac{1}{R} + \alpha_b\right)} \cdot Y^* - \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma$$

$$X^* = \frac{1}{R \cdot \left(\frac{1}{R} + \alpha_b\right)} \cdot Y^* - \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma \Rightarrow X^* = \frac{1}{(1 + R \cdot \alpha_b)} \cdot Y^* - \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma$$

$$\frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot (Y^* - X^*) = 0 \Rightarrow \frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot Y^* + \frac{1}{R \cdot C_2} \cdot X^* = 0$$

$$\frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot Y^* + \frac{1}{R \cdot C_2} \cdot X^* = 0 \Rightarrow R \cdot Z^* - Y^* + X^* = 0$$

$$R \cdot Z^* - Y^* + X^* = 0 \Rightarrow R \cdot \left(-\frac{1}{R_S} \cdot Y^*\right) - Y^* + \frac{1}{(1 + R \cdot \alpha_b)} \cdot Y^* - \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma = 0$$

$$-\frac{R}{R_S} \cdot Y^* - Y^* + \frac{1}{(1 + R \cdot \alpha_b)} \cdot Y^* - \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma = 0$$
$$\Rightarrow Y^* \cdot \left\{\frac{1}{(1 + R \cdot \alpha_b)} - \frac{R}{R_S} - 1\right\} = \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma$$

$$Y^* \cdot \left\{\frac{1}{(1 + R \cdot \alpha_b)} - \frac{R}{R_S} - 1\right\} = \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma \Rightarrow Y^* = \frac{1}{\left\{\frac{1}{(1 + R \cdot \alpha_b)} - \frac{R}{R_S} - 1\right\}} \cdot \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma$$

$$Z^* = -\frac{1}{R_S} \cdot Y^* \Rightarrow Z^* = -\frac{1}{R_S} \cdot \frac{1}{\left\{\frac{1}{(1 + R \cdot \alpha_b)} - \frac{R}{R_S} - 1\right\}} \cdot \frac{(\alpha_b - \alpha_a)}{\left(\frac{1}{R} + \alpha_b\right)} \cdot \Gamma$$

$$X^* = \frac{1}{(1+R\cdot\alpha_b)}\cdot Y^* - \frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma \Rightarrow X^*$$
$$= \frac{1}{(1+R\cdot\alpha_b)}\cdot\frac{1}{\left\{\frac{1}{(1+R\cdot\alpha_b)}-\frac{R}{R_S}-1\right\}}\cdot\frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma - \frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma$$

$$X^* = \frac{1}{\left\{1-(1+R\cdot\alpha_b)\cdot\left(\frac{R}{R_S}+1\right)\right\}}\cdot\frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma - \frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma$$

$$X^* = \frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma\cdot\left\{\frac{1}{\left\{1-(1+R\cdot\alpha_b)\cdot\left(\frac{R}{R_S}+1\right)\right\}}-1\right\} \Rightarrow X^*$$
$$= \frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma\cdot\frac{(1+R\cdot\alpha_b)\cdot\left(\frac{R}{R_S}+1\right)}{\left\{1-(1+R\cdot\alpha_b)\cdot\left(\frac{R}{R_S}+1\right)\right\}}$$

We can summarize our fixed point's results for the first case $v_R < -\Gamma \Rightarrow v_R + \Gamma < 0$ in Table 5.2.

Results Our system fixed points for the first case are independent on circuit energy storage elements (C_1, C_2, and L_1). They depend only on circuit's dissipative elements (R, R_s) and Chua's diode parameters [81].

The next step is to discuss system stability for the first case. We can characterize our circuit's equations by the following notation.

$$\dot{X} = g_1(X,Y,Z);\ \dot{Y} = g_2(X,Y,Z);\ \dot{Z} = g_3(X,Y,Z)$$
$$g_1(X,Y,Z) = \frac{1}{C_1\cdot R}\cdot Y - \frac{1}{C_1}\cdot X\cdot\left(\frac{1}{R}+\alpha_b\right) - \frac{1}{C_1}\cdot(\alpha_b-\alpha_a)\cdot\Gamma;$$
$$g_2(X,Y,Z) = \frac{1}{C_2}\cdot Z - \frac{1}{R\cdot C_2}\cdot(Y-X)$$
$$g_3(X,Y,Z) = -\frac{1}{L_1}\cdot Y - \frac{R_S}{L_1}\cdot Z$$

Table 5.2 Fixed point's results for the first case $v_R < -\Gamma \Rightarrow v_R + \Gamma < 0$

Case I $v_R < -\Gamma \Rightarrow v_R+\Gamma<0 \Rightarrow i_R = f(X) = \alpha_b\cdot X - (\alpha_a-\alpha_b)\cdot\Gamma = \alpha_b\cdot X + (\alpha_b-\alpha_a)\cdot\Gamma$
$X^* = \frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma\cdot\frac{(1+R\cdot\alpha_b)\cdot\left(\frac{R}{R_S}+1\right)}{\left\{1-(1+R\cdot\alpha_b)\cdot\left(\frac{R}{R_S}+1\right)\right\}}$
$Y^* = \frac{1}{\left\{\frac{1}{(1+R\cdot\alpha_b)}-\frac{R}{R_S}-1\right\}}\cdot\frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma$
$Z^* = -\frac{1}{R_S}\cdot\frac{1}{\left\{\frac{1}{(1+R\cdot\alpha_b)}-\frac{R}{R_S}-1\right\}}\cdot\frac{(\alpha_b-\alpha_a)}{\left(\frac{1}{R}+\alpha_b\right)}\cdot\Gamma$

We need to implement linearization technique. The below matrix A is called the Jacobian matrix at the fixed points (X^{*}, Y^{*}, Z^{*}).

$$\frac{\partial g_1(X,Y,Z)}{\partial X} = -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right); \frac{\partial g_1(X,Y,Z)}{\partial Y} = \frac{1}{C_1 \cdot R}; \frac{\partial g_1(X,Y,Z)}{\partial Z} = 0$$

$$\frac{\partial g_2(X,Y,Z)}{\partial X} = \frac{1}{R \cdot C_2}; \frac{\partial g_2(X,Y,Z)}{\partial Y} = -\frac{1}{R \cdot C_2}; \frac{\partial g_2(X,Y,Z)}{\partial Z} = \frac{1}{C_2}$$

$$\frac{\partial g_3(X,Y,Z)}{\partial X} = 0; \frac{\partial g_3(X,Y,Z)}{\partial Y} = -\frac{1}{L_1}; \frac{\partial g_3(X,Y,Z)}{\partial Z} = -\frac{R_S}{L_1}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} \Rightarrow A - \lambda \cdot I$$

$$= \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} \Rightarrow \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} -\lambda \cdot I$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) - \lambda & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} - \lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} - \lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \det \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) - \lambda & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} - \lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} - \lambda \end{pmatrix} = 0$$

$$\det\begin{pmatrix} -\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)-\lambda & \frac{1}{C_1\cdot R} & 0 \\ \frac{1}{R\cdot C_2} & -\frac{1}{R\cdot C_2}-\lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1}-\lambda \end{pmatrix}$$

$$= -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)+\lambda\right\}\cdot\begin{pmatrix} -\left(\frac{1}{R\cdot C_2}+\lambda\right) & \frac{1}{C_2} \\ -\frac{1}{L_1} & -\left(\frac{R_S}{L_1}+\lambda\right) \end{pmatrix}$$

$$-\frac{1}{C_1\cdot R}\cdot\begin{pmatrix} \frac{1}{R\cdot C_2} & \frac{1}{C_2} \\ 0 & -\left(\frac{R_S}{L_1}+\lambda\right) \end{pmatrix}$$

$$= -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)+\lambda\right\}\cdot\left\{\left(\frac{1}{R\cdot C_2}+\lambda\right)\cdot\left(\frac{R_S}{L_1}+\lambda\right)+\frac{1}{L_1\cdot C_2}\right\}$$

$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det(A-\lambda\cdot I) = -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)+\lambda\right\}$$
$$\cdot\left\{\left(\frac{1}{R\cdot C_2}+\lambda\right)\cdot\left(\frac{R_S}{L_1}+\lambda\right)+\frac{1}{L_1\cdot C_2}\right\}+\frac{1}{C_1\cdot C_2\cdot R^2}$$
$$\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det(A-\lambda\cdot I) = -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)+\lambda\right\}\cdot\left\{\lambda^2+\lambda\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)\right\}$$
$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\frac{R_S}{L_1}+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\lambda$$

$$\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)+\lambda\right\}\cdot\left\{\lambda^2+\lambda\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)\right\}$$
$$=\lambda^3+\lambda^2\cdot\left(\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)+\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)$$
$$+\lambda\cdot\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_b\right)\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)\right\}$$
$$+\frac{1}{L_1\cdot C_1\cdot C_2}\cdot\left(\frac{1}{R}+\alpha_b\right)\cdot\left(\frac{R_S}{R}+1\right)$$

For simplicity we define new parameters:

$$\eta_1 = \frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) + \frac{1}{R \cdot C_2} + \frac{R_S}{L_1}; \eta_2 = \frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) \cdot \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right) + \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right)$$

$$\eta_3 = \frac{1}{L_1 \cdot C_1 \cdot C_2} \cdot \left(\frac{1}{R} + \alpha_b\right) \cdot \left(\frac{R_S}{R} + 1\right)$$

$$\left\{\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) + \lambda\right\} \cdot \left\{\lambda^2 + \lambda \cdot \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right) + \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right)\right\} = \lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \eta_2 + \eta_3$$

$$\det(A - \lambda \cdot I) = -(\lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \eta_2 + \eta_3) + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \left(\frac{R_S}{L_1} + \lambda\right)$$

$$\det(A - \lambda \cdot I) = -(\lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \eta_2 + \eta_3) + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \frac{R_S}{L_1} + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \lambda$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot (\eta_2 - \frac{1}{C_1 \cdot C_2 \cdot R^2}) + \eta_3 - \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \frac{R_S}{L_1} = 0$$

We define

$$a = 1; b = \eta_1; c = \eta_2 - \frac{1}{C_1 \cdot C_2 \cdot R^2}; d = \eta_3 - \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \frac{R_S}{L_1}$$

Our cubic equation $a \cdot \lambda^3 + b \cdot \lambda^2 + c \cdot \lambda + d = 0$ with real coefficients has at least one solution λ among the real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant Δ,

$$\Delta = -4 \cdot b^3 \cdot d + b^2 \cdot c^2 - 4 \cdot a \cdot c^3 + 18 \cdot a \cdot b \cdot c \cdot d - 27 \cdot a^2 \cdot d^2.$$

If $\Delta > 0$, then the equation has three distinct real roots. If $\Delta < 0$, then the equation has one real root and a pair of complex conjugate roots. If $\Delta = 0$, then (at least) two roots coincide. It may be that the equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple real root. A possible way to decide between these subcases is to compute the resultant of the cubic and its second derivative: a triple root exists if and only if this resultant vanishes [7, 8].

Table 5.3 Eigenvalues of A (Jacobian) vs type of critical point (fixed point)

Eigenvalues of A (Jacobian)	Type of critical point (fixed point)
Real, unequal, same sign	Node (stable/unstable), driven away or back from steady-state value
Real, unequal, opposite sign	Saddle point and is unstable
Real and equal	Node (stable/unstable), driven away or back from steady-state value
Complex conjugate	Spiral point (stable/unstable), oscillates around steady-state value with decreasing/increasing amplitude
Pure imaginary	Center (oscillates around steady-state value with constant amplitude)

We get three degree polynomial in λ (cubic equation in λ) which can have three roots: λ_1, λ_2, λ_3. We classify system critical point (fixed point) according to Table 5.3.

Case II

$$-\Gamma \leq v_R \leq +\Gamma; i_R = f(v_R) = \alpha_a \cdot v_R; i_R = f(X) = \alpha_a \cdot X$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot (Y - X) - \frac{1}{C_1} \cdot \alpha_a \cdot X; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - X)$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot Y - X \cdot \frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_a\right); \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot Y + \frac{1}{R \cdot C_2} \cdot X$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

We find system fixed points by setting $\dot{X} = 0; \dot{Y} = 0; \dot{Z} = 0$.

$$\dot{X} = 0 \Rightarrow \frac{1}{C_1 \cdot R} \cdot Y^* - X^* \cdot \frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_a\right) = 0;$$
$$\dot{Y} = 0 \Rightarrow \frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot Y^* + \frac{1}{R \cdot C_2} \cdot X^* = 0$$
$$\dot{Z} = 0 \Rightarrow -\frac{1}{L_1} \cdot Y^* - \frac{R_S}{L_1} \cdot Z^* = 0$$

$$-\frac{1}{L_1}\cdot Y^* - \frac{R_S}{L_1}\cdot Z^* = 0 \Rightarrow Z^* = -\frac{1}{R_S}\cdot Y^*$$

$$\frac{1}{C_1\cdot R}\cdot Y^* - X^*\cdot\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right) = 0 \Rightarrow X^* = \frac{1}{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)}\cdot\frac{1}{C_1\cdot R}\cdot Y^* = \frac{1}{(1+\alpha_a\cdot R)}\cdot Y^*$$

$$\frac{1}{C_2}\cdot Z^* - \frac{1}{R\cdot C_2}\cdot Y^* + \frac{1}{R\cdot C_2}\cdot X^* = 0 \Rightarrow Y^*\cdot\frac{1}{C_2}\cdot\left\{\frac{1}{R}\cdot\left\{\frac{1}{(1+\alpha_a\cdot R)}-1\right\}-\frac{1}{R_S}\right\} = 0 \Rightarrow Y^* = 0$$

$$Y^* = 0 \Rightarrow Z^* = 0 \Rightarrow X^* = 0$$

The next step is to discuss system stability for the first case. We can characterize our circuit's equations by the following notation.

$$\dot{X} = g_1(X,Y,Z);\dot{Y} = g_2(X,Y,Z);\dot{Z} = g_3(X,Y,Z)$$

$$g_1(X,Y,Z) = \frac{1}{C_1\cdot R}\cdot Y - X\cdot\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right);$$
$$g_2(X,Y,Z) = \frac{1}{C_2}\cdot Z - \frac{1}{R\cdot C_2}\cdot Y + \frac{1}{R\cdot C_2}\cdot X$$
$$g_3(X,Y,Z) = -\frac{1}{L_1}\cdot Y - \frac{R_S}{L_1}\cdot Z$$

We need to implement linearization technique. The below matrix A is called the Jacobian matrix at the fixed points (X^*, Y^*, Z^*).

$$\frac{\partial g_1(X,Y,Z)}{\partial X} = -\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right);\frac{\partial g_1(X,Y,Z)}{\partial Y} = \frac{1}{C_1\cdot R};\frac{\partial g_1(X,Y,Z)}{\partial Z} = 0$$

$$\frac{\partial g_2(X,Y,Z)}{\partial X} = \frac{1}{R\cdot C_2};\frac{\partial g_2(X,Y,Z)}{\partial Y} = -\frac{1}{R\cdot C_2};\frac{\partial g_2(X,Y,Z)}{\partial Z} = \frac{1}{C_2}$$

$$\frac{\partial g_3(X,Y,Z)}{\partial X} = 0;\frac{\partial g_3(X,Y,Z)}{\partial Y} = -\frac{1}{L_1};\frac{\partial g_3(X,Y,Z)}{\partial Z} = -\frac{R_S}{L_1}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)}$$

$$\Rightarrow A - \lambda \cdot I = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)}$$

$$= \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_a\right) & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \det \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_a\right) - \lambda & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} - \lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} - \lambda \end{pmatrix} = 0$$

$$\det \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_a\right) - \lambda & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} - \lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} - \lambda \end{pmatrix}$$

$$= -\left\{\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_a\right) + \lambda\right\} \cdot \begin{pmatrix} -\left(\frac{1}{R \cdot C_2} + \lambda\right) & \frac{1}{C_2} \\ -\frac{1}{L_1} & -\left(\frac{R_S}{L_1} + \lambda\right) \end{pmatrix}$$

$$-\frac{1}{C_1 \cdot R} \cdot \begin{pmatrix} \frac{1}{R \cdot C_2} & \frac{1}{C_2} \\ 0 & -\left(\frac{R_S}{L_1} + \lambda\right) \end{pmatrix}$$

$$\det\begin{pmatrix} -\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)-\lambda & \frac{1}{C_1\cdot R} & 0 \\ \frac{1}{R\cdot C_2} & -\frac{1}{R\cdot C_2}-\lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1}-\lambda \end{pmatrix}$$
$$= -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\lambda\right\}\cdot\left\{\left(\frac{1}{R\cdot C_2}+\lambda\right)\cdot\left(\frac{R_S}{L_1}+\lambda\right)+\frac{1}{L_1\cdot C_2}\right\}$$
$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$
$$= -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\lambda\right\}\cdot\left\{\lambda^2+\lambda\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}\right\}$$
$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\frac{R_S}{L_1}+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\lambda$$

$$\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\lambda\right\}\cdot\left\{\lambda^2+\lambda\cdot(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1})+\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}\right\}$$
$$=\lambda^3+\lambda^2\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right\}$$
$$+\lambda\cdot\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}\right\}$$
$$+\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)\cdot\left(\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}\right)$$

For simplicity we define new parameters:

$$\Omega_1=\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\frac{1}{R\cdot C_2}+\frac{R_S}{L_1};$$
$$\Omega_2=\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}$$

$$\Omega_3=\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)\cdot\left(\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}\right)$$

$$\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\lambda\right\}\cdot\left\{\lambda^2+\lambda\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{R\cdot C_2}\cdot\frac{R_S}{L_1}+\frac{1}{L_1\cdot C_2}\right\}$$
$$=\lambda^3+\lambda^2\cdot\Omega_1+\lambda\cdot\Omega_2+\Omega_3$$

$$\det\begin{pmatrix} -\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)-\lambda & \frac{1}{C_1\cdot R} & 0 \\ \frac{1}{R\cdot C_2} & -\frac{1}{R\cdot C_2}-\lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1}-\lambda \end{pmatrix}$$

$$= -\left\{\frac{1}{C_1}\cdot\left(\frac{1}{R}+\alpha_a\right)+\lambda\right\}\cdot\left\{\left(\frac{1}{R\cdot C_2}+\lambda\right)\cdot\left(\frac{R_S}{L_1}+\lambda\right)+\frac{1}{L_1\cdot C_2}\right\}$$

$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$= -\{\lambda^3+\lambda^2\cdot\Omega_1+\lambda\cdot\Omega_2+\Omega_3\}+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det(A-\lambda\cdot I) = -\{\lambda^3+\lambda^2\cdot\Omega_1+\lambda\cdot\Omega_2+\Omega_3\}+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det(A-\lambda\cdot I)=0 \Rightarrow \lambda^3+\lambda^2\cdot\Omega_1+\lambda\cdot\Omega_2+\Omega_3-\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\frac{R_S}{L_1}$$

$$-\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\lambda=0$$

$$\det(A-\lambda\cdot I)=0 \Rightarrow \lambda^3+\lambda^2\cdot\Omega_1+\lambda\cdot\left[\Omega_2-\frac{1}{C_1\cdot C_2\cdot R^2}\right]$$

$$+\Omega_3-\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\frac{R_S}{L_1}=0$$

We define

$$a=1; b=\Omega_1; c=\Omega_2-\frac{1}{C_1\cdot C_2\cdot R^2}; d=\Omega_3-\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\frac{R_S}{L_1}$$

Our cubic equation $a\cdot\lambda^3+b\cdot\lambda^2+c\cdot\lambda+d=0$ with real coefficients has at least one solution λ among the real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant Δ, $\Delta=-4\cdot b^3\cdot d+b^2\cdot c^2-4\cdot a\cdot c^3+18\cdot a\cdot b\cdot c\cdot d-27\cdot a^2\cdot d^2$.

If $\Delta>0$, then the equation has three distinct real roots. If $\Delta<0$, then the equation has one real root and a pair of complex conjugate roots. If $\Delta=0$, then (at least) two roots coincide. It may be that the equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple real root. A possible way to decide between these subcases is to compute the resultant of the cubic and its second derivative: a triple root exists if and only if this resultant vanishes.

We get three degree polynomial in λ (cubic equation in λ) which can have three roots: λ_1, λ_2, λ_3. We classify system critical point (fixed point) according to Table 5.4 [5, 6].

Table 5.4 Eigenvalues of A (Jacobian) vs type of critical point (fixed point)

Eigenvalues of A (Jacobian)	Type of critical point (fixed point)
Real, unequal, same sign	Node (stable/unstable), driven away or back from steady-state value
Real, unequal, opposite sign	Saddle point and is unstable
Real and equal	Node (stable/unstable), driven away or back from steady-state value
Complex conjugate	Spiral point (stable/unstable), oscillates around steady-state value with decreasing/increasing amplitude
Pure imaginary	Center (oscillates around steady-state value with constant amplitude)

Case III

$$v_R > \Gamma \Rightarrow v_R - \Gamma > 0; i_R = f(X) = \alpha_b \cdot X + (\alpha_a - \alpha_b) \cdot \Gamma$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot (Y - X) - \frac{1}{C_1} \cdot \{\alpha_b \cdot X + (\alpha_a - \alpha_b) \cdot \Gamma\}; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - X)$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot Y - \frac{1}{C_1 \cdot R} \cdot X - \frac{1}{C_1} \cdot \alpha_b \cdot X - \frac{1}{C_1} \cdot (\alpha_a - \alpha_b) \cdot \Gamma;$$
$$\dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot Y + \frac{1}{R \cdot C_2} \cdot X$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{X} = \frac{1}{C_1 \cdot R} \cdot Y - \frac{1}{C_1} \cdot X \cdot (\frac{1}{R} + \alpha_b) - \frac{1}{C_1} \cdot (\alpha_a - \alpha_b) \cdot \Gamma;$$
$$\dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot Y + \frac{1}{R \cdot C_2} \cdot X$$
$$\dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

We find system fixed points by setting $\dot{X} = 0; \dot{Y} = 0; \dot{Z} = 0$.

$$\dot{X} = 0 \Rightarrow \frac{1}{C_1 \cdot R} \cdot Y^* - \frac{1}{C_1} \cdot X^* \cdot \left(\frac{1}{R} + \alpha_b\right) - \frac{1}{C_1} \cdot (\alpha_a - \alpha_b) \cdot \Gamma = 0;$$
$$\dot{Y} = 0 \Rightarrow \frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot Y^* + \frac{1}{R \cdot C_2} \cdot X^* = 0$$
$$\dot{Z} = 0 \Rightarrow -\frac{1}{L_1} \cdot Y^* - \frac{R_S}{L_1} \cdot Z^* = 0$$

$$-\frac{1}{L_1}\cdot Y^* - \frac{R_S}{L_1}\cdot Z^* = 0 \Rightarrow Z^* = -\frac{1}{R_S}\cdot Y^*; X^* = \frac{1}{(1+\alpha_b\cdot R)}\cdot Y^* - \frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma$$

$$\frac{1}{C_2}\cdot Z^* - \frac{1}{R\cdot C_2}\cdot Y^* + \frac{1}{R\cdot C_2}\cdot X^* = 0 \Rightarrow$$

$$-\frac{1}{R_S\cdot C_2}\cdot Y^* - \frac{1}{R\cdot C_2}\cdot Y^* + \frac{1}{R\cdot C_2}\cdot\left\{\frac{1}{(1+\alpha_b\cdot R)}\cdot Y^* - \frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma\right\} = 0$$

$$-\frac{1}{R_S\cdot C_2}\cdot Y^* - \frac{1}{R\cdot C_2}\cdot Y^* + \frac{1}{R\cdot C_2}\cdot\frac{1}{(1+\alpha_b\cdot R)}\cdot Y^* - \frac{1}{R\cdot C_2}\cdot\frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma = 0$$

$$Y^*\cdot\left\{\frac{1}{R}\cdot\frac{1}{(1+\alpha_b\cdot R)} - \frac{1}{R_S} - \frac{1}{R}\right\} = \frac{1}{(1+R\cdot\alpha_b)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma$$

$$\begin{aligned} Y^* &= \frac{1}{\left\{\frac{1}{R}\cdot\left[\frac{1}{(1+\alpha_b\cdot R)}-1\right]-\frac{1}{R_S}\right\}}\cdot\frac{1}{(1+R\cdot\alpha_b)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma \\ &= \frac{1}{\left\{\frac{1}{(1+\alpha_b\cdot R)}-1-\frac{R}{R_S}\right\}}\cdot\frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma \end{aligned}$$

$$Z^* = -\frac{1}{R_S}\cdot Y^* = -\frac{1}{R_S}\cdot\frac{1}{\left\{\frac{1}{(1+\alpha_b\cdot R)}-1-\frac{R}{R_S}\right\}}\cdot\frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma$$

$$X^* = \frac{1}{(1+\alpha_b\cdot R)}\cdot\frac{1}{\left\{\frac{1}{(1+\alpha_b\cdot R)}-1-\frac{R}{R_S}\right\}}\cdot\frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma - \frac{1}{\left(\frac{1}{R}+\alpha_b\right)}\cdot(\alpha_a-\alpha_b)\cdot\Gamma$$

$$X^* = \left\{\frac{1}{\left\{1-(1+\alpha_b\cdot R)\cdot\left(1+\frac{R}{R_S}\right)\right\}}-1\right\}\cdot\frac{(\alpha_a-\alpha_b)\cdot\Gamma}{\left(\frac{1}{R}+\alpha_b\right)}$$

We can summarize our fixed point's results for the third case $v_R > \Gamma \Rightarrow v_R - \Gamma > 0 i_R = f(X) = \alpha_b\cdot X + (\alpha_a-\alpha_b)\cdot\Gamma$ in Table 5.5.

Results Our system fixed points for the third case are independent on circuit energy storage elements (C_1, C_2, and L_1). They depend only on circuit's dissipative elements (R, R_s) and Chua's diode parameters.

The next step is to discuss system stability for the third case. We can characterize our circuit's equations by the following notation.

Table 5.5 Summary our system fixed point

Case III $v_R > \Gamma \Rightarrow v_R - \Gamma > 0; i_R = f(X) = \alpha_b \cdot X + (\alpha_a - \alpha_b) \cdot \Gamma$
$X^* = \left\{ \frac{1}{\left\{1-(1+\alpha_b \cdot R) \cdot \left(1+\frac{R}{R_S}\right)\right\}} - 1 \right\} \cdot \frac{(\alpha_a - \alpha_b) \cdot \Gamma}{\left(\frac{1}{R} + \alpha_b\right)}$
$Y^* = \frac{1}{\left\{\frac{1}{(1+\alpha_b \cdot R)} - 1 - \frac{R}{R_S}\right\}} \cdot \frac{1}{\left(\frac{1}{R} + \alpha_b\right)} \cdot (\alpha_a - \alpha_b) \cdot \Gamma$
$Z^* = -\frac{1}{R_S} \cdot Y^* = -\frac{1}{R_S} \cdot \frac{1}{\left\{\frac{1}{(1+\alpha_b \cdot R)} - 1 - \frac{R}{R_S}\right\}} \cdot \frac{1}{\left(\frac{1}{R} + \alpha_b\right)} \cdot (\alpha_a - \alpha_b) \cdot \Gamma$

$$\dot{X} = g_1(X,Y,Z); \dot{Y} = g_2(X,Y,Z); \dot{Z} = g_3(X,Y,Z)$$

$$g_1(X,Y,Z) = \frac{1}{C_1 \cdot R} \cdot Y - \frac{1}{C_1} \cdot X \cdot \left(\frac{1}{R} + \alpha_b\right) - \frac{1}{C_1} \cdot (\alpha_a - \alpha_b) \cdot \Gamma;$$
$$g_2(X,Y,Z) = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot Y + \frac{1}{R \cdot C_2} \cdot X$$
$$g_3(X,Y,Z) = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

We need to implement linearization technique. The below matrix A is called the Jacobian matrix at the fixed points (X^*, Y^*, Z^*).

$$\frac{\partial g_1(X,Y,Z)}{\partial X} = -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right); \frac{\partial g_1(X,Y,Z)}{\partial Y} = \frac{1}{C_1 \cdot R}; \frac{\partial g_1(X,Y,Z)}{\partial Z} = 0$$

$$\frac{\partial g_2(X,Y,Z)}{\partial X} = \frac{1}{R \cdot C_2}; \frac{\partial g_2(X,Y,Z)}{\partial Y} = -\frac{1}{R \cdot C_2}; \frac{\partial g_2(X,Y,Z)}{\partial Z} = \frac{1}{C_2}$$

$$\frac{\partial g_3(X,Y,Z)}{\partial X} = 0; \frac{\partial g_3(X,Y,Z)}{\partial Y} = -\frac{1}{L_1}; \frac{\partial g_3(X,Y,Z)}{\partial Z} = -\frac{R_S}{L_1}$$

The Jacobian is the same like in Case I and stability analysis is the same except system's fixed point it is related to.

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)}$$
$$\Rightarrow A - \lambda \cdot I = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)}$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} & \frac{\partial g_1}{\partial Z} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} & \frac{\partial g_2}{\partial Z} \\ \frac{\partial g_3}{\partial X} & \frac{\partial g_3}{\partial Y} & \frac{\partial g_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} \quad -\lambda \cdot I$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) - \lambda & \frac{1}{C_1 \cdot R} & 0 \\ \frac{1}{R \cdot C_2} & -\frac{1}{R \cdot C_2} - \lambda & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\frac{R_S}{L_1} - \lambda \end{pmatrix}$$

For simplicity we define new parameters:

$$\eta_1 = \frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) + \frac{1}{R \cdot C_2} + \frac{R_S}{L_1}; \eta_2$$
$$= \frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) \cdot \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right) + \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right)$$

$$\eta_3 = \frac{1}{L_1 \cdot C_1 \cdot C_2} \cdot \left(\frac{1}{R} + \alpha_b\right) \cdot \left(\frac{R_S}{R} + 1\right)$$

$$\left\{\frac{1}{C_1} \cdot \left(\frac{1}{R} + \alpha_b\right) + \lambda\right\} \cdot \left\{\lambda^2 + \lambda \cdot \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right) + \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right)\right\} = \lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \eta_2 + \eta_3$$

$$\det(A - \lambda \cdot I) = -(\lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \eta_2 + \eta_3) + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \left(\frac{R_S}{L_1} + \lambda\right)$$
$$\det(A - \lambda \cdot I) = -(\lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \eta_2 + \eta_3) + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \frac{R_S}{L_1} + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \lambda$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot \eta_1 + \lambda \cdot \left(\eta_2 - \frac{1}{C_1 \cdot C_2 \cdot R^2}\right) + \eta_3 - \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \frac{R_S}{L_1} = 0$$

We define $a = 1; b = \eta_1; c = \eta_2 - \frac{1}{C_1 \cdot C_2 \cdot R^2}; d = \eta_3 - \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \frac{R_S}{L_1}$

Our cubic equation $a \cdot \lambda^3 + b \cdot \lambda^2 + c \cdot \lambda + d = 0$ with real coefficients has at least one solution λ among the real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant Δ, $\Delta = -4 \cdot b^3 \cdot d + b^2 \cdot c^2 - 4 \cdot a \cdot c^3 + 18 \cdot a \cdot b \cdot c \cdot d - 27 \cdot a^2 \cdot d^2$.

If $\Delta > 0$, then the equation has three distinct real roots. If $\Delta < 0$, then the equation has one real root and a pair of complex conjugate roots. If $\Delta = 0$, then (at least) two roots coincide. It may be that the equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple real root. A possible way to decide between these subcases is to compute the resultant of the cubic and its second derivative: a triple root exists if and only if this resultant vanishes. We get three degree polynomial in λ (cubic equation in λ) which can have three roots: λ_1, λ_2, λ_3. We classify system critical point (fixed point) according to Table 5.6.

We need to plot system phase plane X–Y–Z, X–Y, X–Z, Y–Z, and $X(t)$, $Y(t)$, $Z(t)$ (Figs. 5.4, 5.5, 5.6 and 5.7).

$$C_1 = 1000\,\text{uF} = 1\,\text{mF}; C_2 = 1000\,\text{uF} = 1\,\text{mF}; R = 1\,k\Omega; L_1 = 10\,\text{mH}; R_s = 1\,\Omega$$

$$C_1 \cdot R = 1; C_2 \cdot R = 1; \frac{1}{L_1} = 100; \frac{R_s}{L_1} = 100; \frac{1}{C_1} = 1000; \frac{1}{C_2} = 1000$$

Matlab Script $\alpha_a \to a; \alpha_b \to b; \Gamma \to c$

Table 5.6 Eigenvalues of A (Jacobian) vs type of critical point (fixed point)

Eigenvalues of A (Jacobian)	Type of critical point (fixed point)
Real, unequal, same sign	Node (stable/unstable), driven away or back from steady-state value
Real, unequal, opposite sign	Saddle point and is unstable
Real and equal	Node (stable/unstable), driven away or back from steady-state value
Complex conjugate	Spiral point (stable/unstable), oscillates around steady-state value with decreasing/increasing amplitude
Pure imaginary	Center (oscillates around steady-state value with constant amplitude)

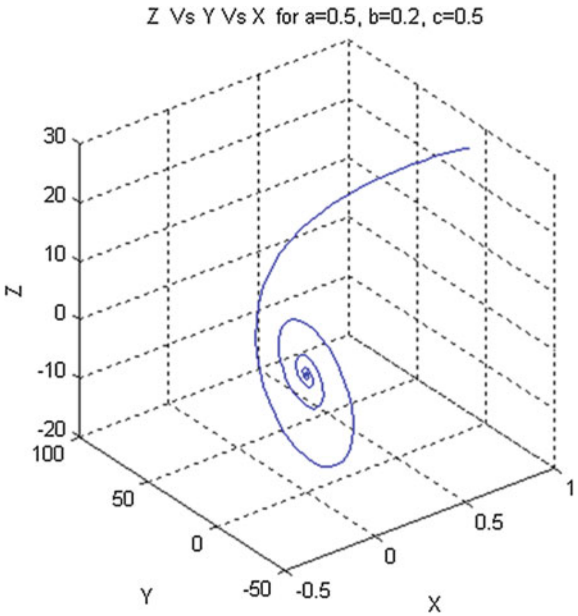


Fig. 5.4 *Z* versus *Y* versus *X* for $a = 0.5$, $b = 0.2$, $c = 0.5$

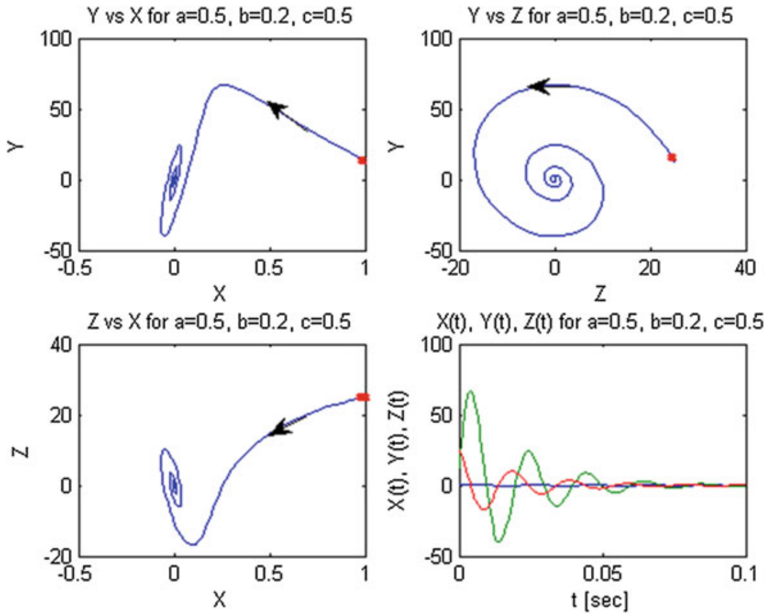


Fig. 5.5 *Y* versus *X* for $a = 0.5$; $b = 0.2$; $c = 0.5$, *Y* versus *Z* for $a = 0.5$; $b = 0.2$; $c0 = 0.5$, *Z* versus *X* for $a = 0.5$; $b = 0.2$; $c = 0.5$, $X(t)$, $Y(t)$, $Z(t)$ for $a = 0.5$; $Y = 0.2$; $Z = 0.5$

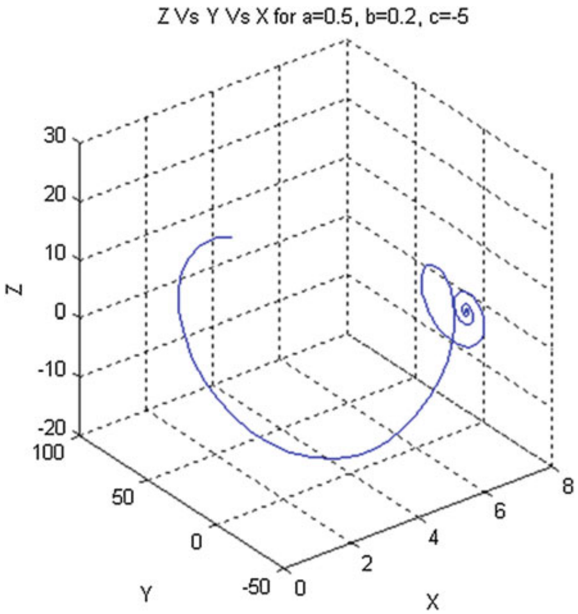


Fig. 5.6 Z versus Y versus X for a = 0.5, b = 0.2, c = −5

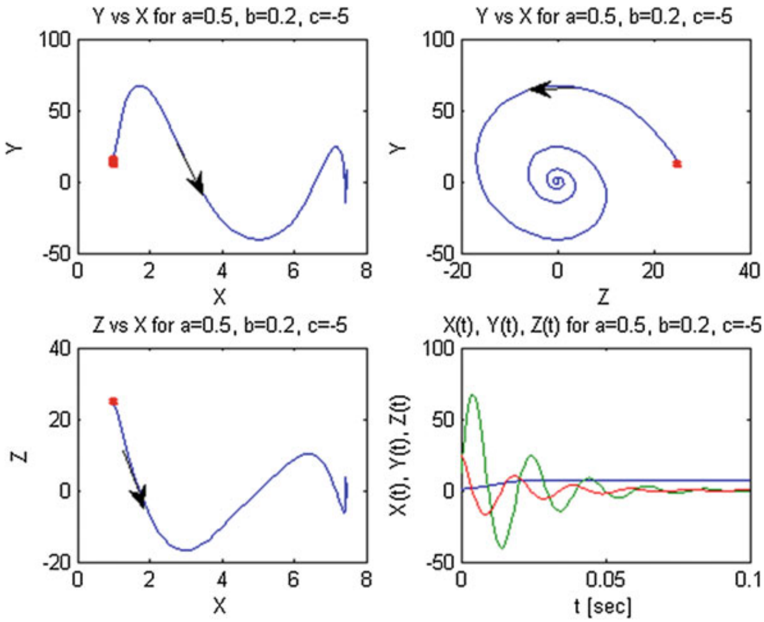


Fig. 5.7 Y versus X for a = 0.5, b = 0.2, c = −5; Y versus X for a = 0.5, b = 0.2, c = −5; Z versus X for a = 0.5, b = 0.2, c = −5; $X(t)$, $Y(t)$, $Z(t)$ for a = 0.5, b = 0.2, c = −5

```
function g=chua(t,x,a,b,c)
g=zeros(3,1);
g(1)=(x(2)-x(1))-1000.0*(b*x(1)+0.5*(a-b)*(abs(x(1)+c)-abs(x(1)-c)));
g(2)=1000.0*x(3)-(x(2)-x(1));
g(3)=-100*x(2)-100*x(3);

function h=chua1(a,b,c,x0,y0,z0)
[t,x]=ODE45 (@chua,[0,.1],[x0,y0,z0],[],a,b,c);
%plot3 (x(:,1),x(:,2),x(:,3));
%xlabel ('X')
%ylabel ('Y')
%zlabel ('Z')
%grid on
%axis square
subplot(2,2,1);plot(x(:,1),x(:,2));subplot(2,2,2);plot(x(:,3),x(:,2))
;subplot(2,2,3);plot(x(:,1),x(:,3));subplot(2,2,4);plot(t,x);
```

5.3 Chua's Circuit with OptoNDR Element Stability Analysis

We replace Chua's diode by OptoNDR element in Chua's circuit. The circuit contains three energy storage elements (Inductance $L1$, Capacitors $C1$ and $C2$). Chua's circuit exhibits different and unique variety of bifurcations and chaos among the chaos-producing mechanism with OptoNDR element. OptoNDR element is a current-controlled nonlinear element characterized by $v_{NDR} = g(i_{NDR})$. We divide circuit methodologies to two basic categories. The first category includes a variety of circuit topologies which considered for realizing the OptoNDR element in Chua's circuit. The second category approach related to the implementation of Chua's circuit, several alternative hybrid realizations of Chua's circuit combining circuit topologies proposed for the OptoNDR and the inductor element. The OptoNDR dynamical mechanism is described by i_{NDR} increasing which cause to v_{NDR} increase up to breakover voltage, then v_{NDR} decreases and photo transistor enters to saturation state.

Initially circuit's energy storage elements have some amount of energy. Capacitors $C1$ and $C2$ are charged to $V_{C1}(t = 0)$ and $V_{C2}(t = 0)$, respectively. We consider, Inductor $L1$ is charged to initial current $I_{L1}(t = 0)$ which is enough to cause Chua's circuit OptoNDR element gets breakover voltage [81, 85, 86, 116].

We can write system differential equations (Fig. 5.8).

$$I_{NDR} = I_{OptoNDR}; V_{NDR} = V_{OptoNDR}; i_{L1}(t=0) = Z_0; V_{C1}(t=0) \Rightarrow I_{NDR}(t=0) = X_0$$
$$V_{C2}(t=0) = Y_0; I_{NDR} \to X; V_{C2} \to Y; i_{L1} \to Z; V_{NDR} = g(I_{NDR}) = g(X)$$
$$C_1 \cdot \frac{dV_{NDR}}{dt} = \frac{1}{R} \cdot (V_{C2} - V_{NDR}) - I_{NDR}; C_2 \cdot \frac{dV_{C2}}{dt} = i_{L1} - \frac{1}{R} \cdot (V_{C2} - V_{NDR});$$
$$L_1 \cdot \frac{di_{L1}}{dt} = -V_{C2} - i_{L1} \cdot R_S$$

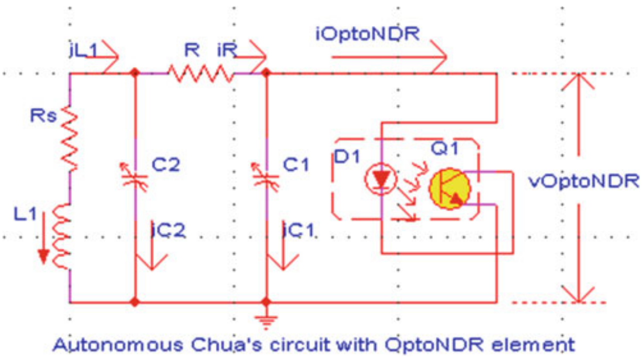


Fig. 5.8 Autonomous Chua's circuit with OptoNDR element

$$\dot{V}_{NDR} = \frac{1}{C_1 \cdot R} \cdot (Y - V_{NDR}) - \frac{1}{C_1} \cdot X; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - V_{NDR}); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{g}(X) = \frac{1}{C_1 \cdot R} \cdot (Y - g(X)) - \frac{1}{C_1} \cdot X; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - g(X)); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

We need to find $g()$ function $(V_{NDR} = g(I_{NDR}) = g(X))$ in our system. $I_{NDR} \rightleftarrows X$ $V_{NDR} \rightleftarrows g(X)$ $Ic = I_{CQ1}; Ie = I_{EQ1}; I_b = I_{BQ1}; V_{ce} = V_{CEQ1}$

We need to implement the Regular Ebers–Moll Model to the OptoNDR element and get a complete final expression for the Negative Differential Resistance (NDR) characteristics $V_{NDR} = g(I_{NDR}) = g(X)$ of that element [85, 86].

$Ide = \frac{\alpha r \cdot Ic - Ie}{\alpha r \cdot \alpha f - 1}$ and $Idc = \frac{Ic - Ie \cdot \alpha f}{\alpha r \cdot \alpha f - 1}$, now we can get the expression for Vbe and Vbc.

$$Vbe = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot Ic - Ie}{Ise \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] \text{ AND}$$

$$Vbc = Vt \cdot \ln\left[\left(\frac{Ic - Ie \cdot \alpha f}{Isc \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$Vce = Vcb + Vbe$, but $Vcb = -Vbc$, then $\boldsymbol{Vce = Vbe - Vbc}$

$$Vce = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot Ic - Ie}{Ise \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{Ic - Ie \cdot \alpha f}{Isc \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$Vce = Vt \cdot \ln\left[\frac{(\alpha r \cdot Ic - Ie) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{(Ic - Ie \cdot \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{Isc}{Ise}\right), \quad I_{D1} = I_{cQ1} = Ic$$

$$V_{D1} = Vt \cdot \ln\left[\frac{I_{D1}}{Io} + 1\right], \quad V_{D1} = Vt \cdot \ln\left[\frac{Ic}{Io} + 1\right],$$

$$Ib = k \cdot Ic; Ie = Ib + Ic; Ie = Ic \cdot (k + 1)$$

Now we proceed to the circuit model for analytical analysis. The optical coupling between the LED ($D1$) to Photo Transistor ($Q1$) is represented as Transistor dependent base current on LED ($D1$) current. $Ib = k \cdot Id_1$, $Id_1 = Ic_1$, $Ib = Ic \cdot k$ (k is coupling coefficient between LED $D1$ and photo transistor $Q1$).

Assumption $k > 1$, for keeping the saturation process after breakover occurs.

As long as the Photo transistor ($Q1$) is in Cut off region the current Ic, Ie, and Ib are very low. When the Photo transistor reaches the Breakover voltage it (the Photo Transistor) enters the saturation region (Vce decrease, Ic Increase). The region in which Vce deceases and Ic increases is the Negative Differential Resistance area of the $V{-}I$ Characteristics. The positive feedback in which the Transistor collector current increases and then, Ib increases ($Ib = Ic \bullet k$) is repeated in increasing cycles. The positive feedback ends when the photo transistor reaches saturation state. Finally' we arrive at an expression which is the voltage ($V_{NDR} \approx Vce$) as a function of the current ($I_{NDR} \approx Ic$) for the NDR circuit.

$$\begin{aligned} V_{NDR} &= V_{D1} + Vce \\ &= Vt \cdot \ln\left[\frac{Ic \cdot (\alpha r - k - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{Ic \cdot (1 - \alpha f \cdot (1 + k)) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{Isc}{Ise}\right) + Vt \\ &\quad \cdot \ln\left(\frac{Ic}{I_0} + 1\right) \end{aligned}$$

Assume $Vt \cdot \ln\left(\frac{Isc}{Ise}\right) \approx 0$, we get the expression for NDR value for that region. The value is changed as the process come closer to the photo transistor saturation region. $Ic = I_{NDR}$

$$\begin{aligned} V_{NDR} &= V_{D1} + Vce \\ &= Vt \cdot \ln\left[\frac{I_{NDR} \cdot (\alpha r - k - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{I_{NDR} \cdot (1 - \alpha f \cdot (1 + k)) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \\ &\quad \cdot \ln\left(\frac{Isc}{Ise}\right) + Vt \cdot \ln\left(\frac{I_{NDR}}{I_0} + 1\right) \end{aligned}$$

$$\ln\left(\frac{Isc}{Ise}\right) \to \varepsilon \Rightarrow V_{NDR} = V_{D1} + Vce$$
$$= Vt \cdot \ln\left[\frac{I_{NDR} \cdot (\alpha r - k - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{I_{NDR} \cdot (1 - \alpha f \cdot (1+k)) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt$$
$$\cdot \ln\left(\frac{I_{NDR}}{I_0} + 1\right)$$

$$V_{NDR} = g(I_{NDR}) = Vt \cdot \ln\left[\frac{I_{NDR} \cdot (\alpha r - k - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{I_{NDR} \cdot (1 - \alpha f \cdot (1+k)) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{I_{NDR}}{I_0} + 1\right)$$

For simplicity we define the following parameters:

$$A_1 = (\alpha r - k - 1); A_2 = Ise \cdot (\alpha r \cdot \alpha f - 1); A_3 = (1 - \alpha f \cdot (1+k));$$
$$A_4 = Isc \cdot (\alpha r \cdot \alpha f - 1)$$

$$V_{NDR} = g(I_{NDR}) = Vt \cdot \ln\left[\frac{I_{NDR} \cdot A_1 + A_2}{I_{NDR} \cdot A_3 + A_4}\right] + Vt \cdot \ln\left(\frac{I_{NDR}}{I_0} + 1\right);$$
$$\frac{\mathrm{d}V_{NDR}}{\mathrm{d}t} = \frac{\mathrm{d}g(I_{NDR})}{\mathrm{d}t} = \frac{\mathrm{d}g(I_{NDR})}{dI_{NDR}} \cdot \frac{\mathrm{d}I_{NDR}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}V_{NDR}}{\mathrm{d}t} = \frac{\mathrm{d}g(I_{NDR})}{\mathrm{d}t} = Vt \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left[\frac{I_{NDR} \cdot A_1 + A_2}{I_{NDR} \cdot A_3 + A_4}\right]\right\} + Vt \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left(\frac{I_{NDR}}{I_0} + 1\right)\right\}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left[\frac{I_{NDR} \cdot A_1 + A_2}{I_{NDR} \cdot A_3 + A_4}\right]\right\} = \frac{\dot{I}_{NDR} \cdot A_1 \cdot (I_{NDR} \cdot A_3 + A_4) - \dot{I}_{NDR} \cdot A_3 \cdot (I_{NDR} \cdot A_1 + A_2)}{[I_{NDR} \cdot A_1 + A_2] \cdot [I_{NDR} \cdot A_3 + A_4]}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left[\frac{I_{NDR} \cdot A_1 + A_2}{I_{NDR} \cdot A_3 + A_4}\right]\right\} = \dot{I}_{NDR} \cdot \left\{\frac{A_1 \cdot A_4 - A_3 \cdot A_2}{[I_{NDR} \cdot A_1 + A_2] \cdot [I_{NDR} \cdot A_3 + A_4]}\right\}$$

$$\xi_1(I_{NDR}) = \frac{A_1 \cdot A_4 - A_3 \cdot A_2}{[I_{NDR} \cdot A_1 + A_2] \cdot [I_{NDR} \cdot A_3 + A_4]} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left[\frac{I_{NDR} \cdot A_1 + A_2}{I_{NDR} \cdot A_3 + A_4}\right]\right\}$$
$$= \dot{I}_{NDR} \cdot \xi_1(I_{NDR})$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left(\frac{I_{NDR}}{I_0} + 1\right)\right\} = \frac{1}{\frac{I_{NDR}}{I_0} + 1} \cdot \frac{\dot{I}_{NDR}}{I_0} = \dot{I}_{NDR} \cdot \frac{1}{(I_{NDR} + I_0)}; \xi_2(I_{NDR}) = \frac{1}{(I_{NDR} + I_0)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left(\frac{I_{NDR}}{I_0} + 1\right)\right\} = \dot{I}_{NDR} \cdot \xi_2(I_{NDR}); \frac{\mathrm{d}V_{NDR}}{\mathrm{d}t} = \frac{\mathrm{d}g(I_{NDR})}{\mathrm{d}t}$$
$$= \dot{I}_{NDR} \cdot V_t \cdot \{\xi_1(I_{NDR}) + \xi_2(I_{NDR})\}$$

$$\psi(I_{NDR}) = V_t \cdot \{\xi_1(I_{NDR}) + \xi_2(I_{NDR})\} \Rightarrow \frac{\mathrm{d}V_{NDR}}{\mathrm{d}t} = \frac{\mathrm{d}g(I_{NDR})}{\mathrm{d}t} = \dot{I}_{NDR} \cdot \psi(I_{NDR})$$

Back to our system differential equations: $\dot{V}_{NDR} = \frac{\mathrm{d}V_{NDR}}{\mathrm{d}t} = \frac{\mathrm{d}g(I_{NDR})}{\mathrm{d}t} = \dot{I}_{NDR} \cdot \psi(I_{NDR})$

$$\dot{V}_{NDR} = \frac{1}{C_1 \cdot R} \cdot (Y - V_{NDR}) - \frac{1}{C_1} \cdot X; \dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - V_{NDR}); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$I_{NDR} \to X \Rightarrow \dot{V}_{NDR} = \frac{\mathrm{d}V_{NDR}}{\mathrm{d}t} = \frac{\mathrm{d}g(X)}{\mathrm{d}t} = \dot{X} \cdot \psi(X); V_{NDR} \to g(X)$$

$$\dot{X} \cdot \psi(X) = \frac{1}{C_1 \cdot R} \cdot (Y - g(X)) - \frac{1}{C_1} \cdot X;$$
$$\dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - g(X)); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\dot{X} = \frac{1}{C_1 \cdot R \cdot \psi(X)} \cdot [Y - g(X)] - \frac{1}{C_1 \cdot \psi(X)} \cdot X;$$
$$\dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - g(X)); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$V_{NDR} = g(X) = Vt \cdot \ln\left[\frac{X \cdot A_1 + A_2}{X \cdot A_3 + A_4}\right] + Vt \cdot \ln\left(\frac{X}{I_0} + 1\right)$$
$$= Vt \cdot \ln\left\{\left[\frac{X \cdot A_1 + A_2}{X \cdot A_3 + A_4}\right] \cdot \left(\frac{X}{I_0} + 1\right)\right\}$$

We find system fixed points by setting $\dot{X} = 0; \dot{Y} = 0; \dot{Z} = 0$.

$$\dot{X} = 0 \Rightarrow \frac{1}{C_1 \cdot R \cdot \psi(X^*)} \cdot [Y^* - g(X^*)] - \frac{1}{C_1 \cdot \psi(X^*)} \cdot X^* = 0;$$
$$\dot{Y} = 0 \Rightarrow \frac{1}{C_2} \cdot Z^* - \frac{1}{R \cdot C_2} \cdot [Y^* - g(X^*)] = 0$$
$$\dot{Z} = 0 \Rightarrow -\frac{1}{L_1} \cdot Y^* - \frac{R_S}{L_1} \cdot Z^* = 0 \Rightarrow Z^* = -\frac{1}{R_S} \cdot Y^*$$
$$\Rightarrow -\frac{1}{R_S \cdot C_2} \cdot Y^* - \frac{1}{R \cdot C_2} \cdot [Y^* - g(X^*)] = 0$$

We get two fixed points (X^*, Y^*) equations:

$$\frac{1}{C_1 \cdot R \cdot \psi(X^*)} \cdot [Y^* - g(X^*)] - \frac{1}{C_1 \cdot \psi(X^*)} \cdot X^* = 0; -\frac{1}{R_S \cdot C_2} \cdot Y^* - \frac{1}{R \cdot C_2} \cdot [Y^* - g(X^*)] = 0$$

$$-\frac{1}{R_S \cdot C_2} \cdot Y^* - \frac{1}{R \cdot C_2} \cdot [Y^* - g(X^*)] = 0 \Rightarrow Y^* \cdot \left(\frac{R}{R_S} + 1\right) = g(X^*) \Rightarrow Y^* = \frac{1}{\left(\frac{R}{R_S} + 1\right)} \cdot g(X^*)$$

$$Y^* = \frac{1}{\left(\frac{R}{R_S} + 1\right)} \cdot g(X^*) \Rightarrow \frac{1}{C_1 \cdot R \cdot \psi(X^*)} \cdot g(X^*) \cdot \left[\frac{1}{\left(\frac{R}{R_S} + 1\right)} - 1\right] - \frac{1}{C_1 \cdot \psi(X^*)} \cdot X^* = 0$$

$$\frac{g(X^*)}{\psi(X^*)} \cdot \left[\frac{1}{\left(\frac{R}{R_S} + 1\right)} - 1\right] \cdot \frac{1}{C_1 \cdot R} - \frac{1}{C_1} \cdot \frac{X^*}{\psi(X^*)} = 0 \Rightarrow \frac{1}{C_1 \cdot \psi(X^*)} \cdot \left\{g(X^*) \cdot \left[\frac{1}{\left(\frac{R}{R_S} + 1\right)} - 1\right] \cdot \frac{1}{R} - X^*\right\} = 0$$

We have two possible options as follows:

1. $\frac{1}{C_1 \cdot \psi(X^*)} = 0 \Rightarrow \frac{1}{\psi(X^*)} \to \varepsilon \Rightarrow \psi(X^*) \to \infty \Rightarrow \{\xi_1(X^*) + \xi_2(X^*)\} \to \infty$
2. $\left\{g(X^*) \cdot \left[\frac{1}{\left(\frac{R}{R_S}+1\right)} - 1\right] \cdot \frac{1}{R} - X^*\right\} = 0 \Rightarrow g(X^*) \cdot \left[\frac{1}{\left(\frac{R}{R_S}+1\right)} - 1\right] \cdot \frac{1}{R} = X^*$

$$g(X^*) = Vt \cdot \ln\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] + Vt \cdot \ln\left(\frac{X^*}{I_0} + 1\right) = Vt \cdot \ln\left\{\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)\right\}$$

$$g(X^*) \cdot \left[\frac{1}{\left(\frac{R}{R_S} + 1\right)} - 1\right] \cdot \frac{1}{R} = X^* \Rightarrow Vt \cdot \ln\left\{\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)\right\} \cdot \left[\frac{1}{\left(\frac{R}{R_S} + 1\right)} - 1\right] \cdot \frac{1}{R} = X^*$$

$$\ln\left\{\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)\right\} = \frac{1}{Vt \cdot \frac{1}{R} \cdot \left[\frac{1}{\left(\frac{R}{R_S}+1\right)} - 1\right]} \cdot X^*; \Xi(R, R_S) = \frac{1}{Vt \cdot \frac{1}{R} \cdot \left[\frac{1}{\left(\frac{R}{R_S}+1\right)} - 1\right]}$$

$$\Xi(R, R_S) \cdot X^* = \ln\left\{\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)\right\} \Rightarrow \mathrm{e}^{\Xi(R,R_S) \cdot X^*} = \left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)$$

Taylor series approximation: $e^{\Xi(R,R_S)\cdot X^*} \approx 1 + \Xi(R,R_S)\cdot X^*$

$$e^{\Xi(R,R_S)\cdot X^*} = \left[\frac{X^*\cdot A_1 + A_2}{X^*\cdot A_3 + A_4}\right]\cdot\left(\frac{X^*}{I_0}+1\right)$$
$$\Rightarrow 1 + \Xi(R,R_S)\cdot X^* = \left[\frac{X^*\cdot A_1 + A_2}{X^*\cdot A_3 + A_4}\right]\cdot\left(\frac{X^*}{I_0}+1\right)$$

$$1 + \Xi(R,R_S)\cdot X^* = \left[\frac{X^*\cdot A_1 + A_2}{X^*\cdot A_3 + A_4}\right]\cdot\left(\frac{X^*}{I_0}+1\right)$$
$$\Rightarrow [1 + \Xi(R,R_S)\cdot X^*]\cdot[X^*\cdot A_3 + A_4]$$
$$= [X^*\cdot A_1 + A_2]\cdot\left(\frac{X^*}{I_0}+1\right)$$

$$X^*\cdot A_3 + A_4 + \Xi(R,R_S)\cdot A_3\cdot[X^*]^2 + A_4\cdot\Xi(R,R_S)\cdot X^*$$
$$= \frac{[X^*]^2\cdot A_1}{I_0} + X^*\cdot A_1 + \frac{A_2\cdot X^*}{I_0} + A_2$$

$$\Xi(R,R_S)\cdot A_3\cdot[X^*]^2 - \frac{[X^*]^2\cdot A_1}{I_0} + X^*\cdot A_3 - X^*\cdot A_1 - \frac{A_2\cdot X^*}{I_0} + A_4\cdot\Xi(R,R_S)\cdot X^* + A_4 - A_2 = 0$$

$$[X^*]^2\cdot\left\{\Xi(R,R_S)\cdot A_3 - \frac{A_1}{I_0}\right\} + X^*\cdot\left\{A_3 - A_1 - \frac{A_2}{I_0} + A_4\cdot\Xi(R,R_S)\right\} + A_4 - A_2 = 0$$

We define the following parameters for simplicity: $\Xi_1;\Xi_2;\Xi_3$

$$\Xi_1 = \Xi(R,R_S)\cdot A_3 - \frac{A_1}{I_0};\Xi_2 = A_3 - A_1 - \frac{A_2}{I_0} + A_4\cdot\Xi(R,R_S);$$
$$\Xi_3 = A_4 - A_2 \Rightarrow [X^*]^2\cdot\Xi_1 + X^*\cdot\Xi_2 + \Xi_3 = 0$$

$$[X^*]^2\cdot\Xi_1 + X^*\cdot\Xi_2 + \Xi_3 = 0 \Rightarrow X^*_{1,2} = \frac{-\Xi_2 \pm\sqrt{\Xi_2^2 - 4\cdot\Xi_1\cdot\Xi_3}}{2\cdot\Xi_1}$$

Back to option (1):

$$\{\xi_1(X^*) + \xi_2(X^*)\}\to\infty;\xi_1(X^*) = \frac{A_1\cdot A_4 - A_3\cdot A_2}{[X^*\cdot A_1 + A_2]\cdot[X^*\cdot A_3 + A_4]}$$

$$\xi_2(X^*) = \frac{1}{(X^*+I_0)};\frac{A_1\cdot A_4 - A_3\cdot A_2}{[X^*\cdot A_1 + A_2]\cdot[X^*\cdot A_3 + A_4]} + \frac{1}{(X^*+I_0)}\to\infty$$

$$\frac{A_1 \cdot A_4 - A_3 \cdot A_2}{[X^* \cdot A_1 + A_2] \cdot [X^* \cdot A_3 + A_4]} + \frac{1}{(X^* + I_0)} \to \infty$$
$$\Rightarrow \frac{[A_1 \cdot A_4 - A_3 \cdot A_2] \cdot (X^* + I_0) + [X^* \cdot A_1 + A_2] \cdot [X^* \cdot A_3 + A_4]}{[X^* \cdot A_1 + A_2] \cdot [X^* \cdot A_3 + A_4] \cdot (X^* + I_0)} \to \infty$$

$$[X^* \cdot A_1 + A_2] \cdot [X^* \cdot A_3 + A_4] \cdot (X^* + I_0) \to 0 \Rightarrow X^* = -\frac{A_2}{A_1}; X^* = -\frac{A_4}{A_3}; X^* = -I_0; k < 1$$

$$A_1 = (\alpha r - k - 1) < 0; A_2 = Ise \cdot (\alpha r \cdot \alpha f - 1) < 0;$$
$$A_3 = (1 - \alpha f \cdot (1 + k)) > 0; A_4 = Isc \cdot (\alpha r \cdot \alpha f - 1) < 0$$

$$A_3 = (1 - \alpha f \cdot (1 + k)) > 0 \Rightarrow k < \frac{1 - \alpha f}{\alpha f}; 0 < \alpha f < 1 \Rightarrow k < \frac{1 - \alpha f}{\alpha f} > 0$$
$$\Rightarrow 0 < k < \frac{1 - \alpha f}{\alpha f}; I_0 = 10^{-6}$$

$$\alpha r = 0.5; \alpha f = 0.98 \Rightarrow \frac{1 - \alpha f}{\alpha f} = 0.02 \Rightarrow 0 < k < \frac{1 - \alpha f}{\alpha f} \Rightarrow 0 < k < 0.02; Ise = 1uA; Isc = 2uA$$

$$A_1 = (\alpha r - k - 1) < 0 \Rightarrow 0.5 - k - 1 < 0 \Rightarrow k > -0.5;$$
$$A_2 = Ise \cdot (\alpha r \cdot \alpha f - 1) < 0 \Rightarrow A_2 = -0.51 \cdot 10^{-6}$$

$$A_4 = Isc \cdot (\alpha r \cdot \alpha f - 1) < 0 \Rightarrow -1.02 \cdot 10^{-6}; X^* = -\frac{A_2}{A_1}\bigg|_{\substack{A_2 < 0 \\ A_1 < 0}} < 0; X^* = -\frac{A_4}{A_3}\bigg|_{\substack{A_4 < 0 \\ A_3 > 0}} > 0$$

Since $I_{NDR} > 0 \Rightarrow I_{NDR}^* > 0; I_{NDR}^* \to X^* \Rightarrow X^* > 0$. We choose only X coordinate fixed point which is positive $X^* = -\frac{A_4}{A_3}\Big|_{\substack{A_4 < 0 \\ A_3 > 0}} > 0 \Rightarrow X^* = -\frac{Isc \cdot (\alpha r \cdot \alpha f - 1)}{(1 - \alpha f \cdot (1 + k))} > 0$

$$A_1 \cdot A_4 - A_3 \cdot A_2|_{\substack{A_1 \cdot A_4 > 0 \\ A_3 \cdot A_2 < 0}} > 0; X^* = -\frac{Isc \cdot (\alpha r \cdot \alpha f - 1)}{(1 - \alpha f \cdot (1 + k))} > 0 \Rightarrow X^*$$
$$= \frac{Isc \cdot (1 - \alpha r \cdot \alpha f)}{(1 - \alpha f \cdot (1 + k))}$$

We can summery our system fixed points in Table 5.7.

We need to discuss system stability. $I_{NDR} \to X; V_{C2} \to Y; i_{L1} \to Z$

Table 5.7 Summary of our system fixed points

Fixed point coordinate	Option (1)	Option (2)
X^*	$X^* = \frac{Isc \cdot (1-\alpha r \cdot \alpha f)}{(1-\alpha f \cdot (1+k))}$	$X^*_{1,2} = \frac{-\Xi_2 \pm \sqrt{\Xi_2^2 - 4 \cdot \Xi_1 \cdot \Xi_3}}{2 \cdot \Xi_1}$
Y^*	$g(X^*) = Vt \cdot \ln\left\{\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)\right\}$ $Y^* = \frac{1}{\left(\frac{R}{R_S} + 1\right)} \cdot g(X^*)$	$g(X^*) = Vt \cdot \ln\left\{\left[\frac{X^* \cdot A_1 + A_2}{X^* \cdot A_3 + A_4}\right] \cdot \left(\frac{X^*}{I_0} + 1\right)\right\}$ $Y^* = \frac{1}{\left(\frac{R}{R_S} + 1\right)} \cdot g(X^*)$
Z^*	$Z^* = -\frac{1}{R_S} \cdot Y^* = -\frac{1}{(R+R_S)} \cdot g(X^*)$	$Z^* = -\frac{1}{R_S} \cdot Y^* = -\frac{1}{(R+R_S)} \cdot g(X^*)$

$$\dot{X} = \frac{1}{C_1 \cdot R \cdot \psi(X)} \cdot [Y - g(X)] - \frac{1}{C_1 \cdot \psi(X)} \cdot X;$$
$$\dot{Y} = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - g(X)); \dot{Z} = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z$$

$$\Upsilon_1(X, Y, Z) = \frac{1}{C_1 \cdot R \cdot \psi(X)} \cdot [Y - g(X)] - \frac{1}{C_1 \cdot \psi(X)} \cdot X;$$
$$\Upsilon_2(X, Y, Z) = \frac{1}{C_2} \cdot Z - \frac{1}{R \cdot C_2} \cdot (Y - g(X))$$

$$\Upsilon_3(X, Y, Z) = -\frac{1}{L_1} \cdot Y - \frac{R_S}{L_1} \cdot Z; \dot{X} = \Upsilon_1(X, Y, Z); \dot{Y} = \Upsilon_2(X, Y, Z); \dot{Z} = \Upsilon_3(X, Y, Z)$$

$$\Upsilon_1(X, Y, Z) = \frac{1}{C_1 \cdot R \cdot \psi(X)} \cdot [Y - g(X)] - \frac{1}{C_1 \cdot \psi(X)} \cdot X$$
$$= \frac{1}{C_1 \cdot \psi(X)} \cdot \left\{\frac{1}{R} \cdot [Y - g(X)] - X\right\}$$

We need to calculate our Chua's circuit with OptoNDR Jacobian elements.

$$\frac{\partial \Upsilon_1}{\partial X}, \frac{\partial \Upsilon_1}{\partial Y}, \frac{\partial \Upsilon_1}{\partial Z}, \frac{\partial \Upsilon_2}{\partial X}, \frac{\partial \Upsilon_2}{\partial Y}, \frac{\partial \Upsilon_2}{\partial Z}, \frac{\partial \Upsilon_3}{\partial X}, \frac{\partial \Upsilon_3}{\partial Y}, \frac{\partial \Upsilon_3}{\partial Z},$$
$$\Upsilon_1 = \Upsilon_1(X, Y, Z), \Upsilon_2 = \Upsilon_2(X, Y, Z), \Upsilon_3 = \Upsilon_3(X, Y, Z)$$

$$\frac{d}{dX}\left[\frac{1}{\psi(X)}\right] = -\frac{\mathrm{d}\psi(X)/dX}{\psi^2(X)}; \frac{d}{dX} \leftrightarrow \frac{\partial}{\partial X}; \frac{\partial \Upsilon_1}{\partial Y} = \frac{1}{C_1 \cdot \psi(X) \cdot R}, \frac{\partial \Upsilon_1}{\partial Z} = 0$$

$$\frac{\partial \Upsilon_1}{\partial X} = -\frac{1}{C_1 \cdot \psi(X)} \cdot \left[\frac{1}{\psi(X)} \cdot \frac{\partial \psi(X)}{\partial X} \cdot \left\{\frac{1}{R} \cdot [Y - g(X)] - X\right\} + \frac{1}{R} \cdot \frac{\partial g(X)}{\partial X} + 1\right]$$

$$\frac{\partial \Upsilon_2}{\partial X} = \frac{1}{R \cdot C_2} \cdot \frac{\partial g(X)}{\partial X}; \frac{\partial \Upsilon_2}{\partial Y} = -\frac{1}{R \cdot C_2}; \frac{\partial \Upsilon_2}{\partial Z} = \frac{1}{C_2};$$
$$\frac{\partial \Upsilon_3}{\partial X} = 0; \frac{\partial \Upsilon_3}{\partial Y} = -\frac{1}{L_1}; \frac{\partial \Upsilon_3}{\partial Z} = -\frac{R_S}{L_1}$$

We classify our system fixed point as stable (Spiral, Node) or unstable (Spiral, Node), which is done by inspection of the system characteristic equation.

$$A - \lambda \cdot I = \begin{pmatrix} \frac{\partial \Upsilon_1}{\partial X} & \frac{\partial \Upsilon_1}{\partial Y} & \frac{\partial \Upsilon_1}{\partial Z} \\ \frac{\partial \Upsilon_2}{\partial X} & \frac{\partial \Upsilon_2}{\partial Y} & \frac{\partial \Upsilon_2}{\partial Z} \\ \frac{\partial \Upsilon_3}{\partial X} & \frac{\partial \Upsilon_3}{\partial Y} & \frac{\partial \Upsilon_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \lambda$$
$$= \begin{pmatrix} \frac{\partial \Upsilon_1}{\partial X} - \lambda & \frac{\partial \Upsilon_1}{\partial Y} & \frac{\partial \Upsilon_1}{\partial Z} \\ \frac{\partial \Upsilon_2}{\partial X} & \frac{\partial \Upsilon_2}{\partial Y} - \lambda & \frac{\partial \Upsilon_2}{\partial Z} \\ \frac{\partial \Upsilon_3}{\partial X} & \frac{\partial \Upsilon_3}{\partial Y} & \frac{\partial \Upsilon_3}{\partial Z} - \lambda \end{pmatrix}_{(X^*,Y^*,Z^*)}$$

$$A - \lambda \cdot I = \begin{pmatrix} \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} - \lambda & \left.\frac{1}{C_1 \cdot \psi(X) \cdot R}\right|_{X^*,Y^*,Z^*} & 0 \\ \frac{1}{R \cdot C_2} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*} & -\left(\frac{1}{R \cdot C_2} + \lambda\right) & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\left(\frac{R_S}{L_1} + \lambda\right) \end{pmatrix}$$

Let $|A - \lambda \cdot I| = 0$ then

$$\det|A - \lambda \cdot I| = \det\left|\begin{pmatrix} \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} - \lambda & \left.\frac{1}{C_1 \cdot \psi(X) \cdot R}\right|_{X^*,Y^*,Z^*} & 0 \\ \frac{1}{R \cdot C_2} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*} & -\left(\frac{1}{R \cdot C_2} + \lambda\right) & \frac{1}{C_2} \\ 0 & -\frac{1}{L_1} & -\left(\frac{R_S}{L_1} + \lambda\right) \end{pmatrix}\right| = 0$$

$$\det|A - \lambda \cdot I| = \left(\left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} - \lambda\right) \cdot \det\begin{pmatrix} -\left(\frac{1}{R \cdot C_2} + \lambda\right) & \frac{1}{C_2} \\ -\frac{1}{L_1} & -\left(\frac{R_S}{L_1} + \lambda\right) \end{pmatrix}$$
$$- \left.\frac{1}{C_1 \cdot \psi(X) \cdot R}\right|_{X^*,Y^*,Z^*} \cdot \det\begin{pmatrix} \frac{1}{R \cdot C_2} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*} & \frac{1}{C_2} \\ 0 & -\left(\frac{R_S}{L_1} + \lambda\right) \end{pmatrix}$$

$$\det|A-\lambda\cdot I|=\left(\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}-\lambda\right)\cdot\left\{\left(\frac{1}{R\cdot C_2}+\lambda\right)\cdot\left(\frac{R_S}{L_1}+\lambda\right)+\frac{1}{L_1\cdot C_2}\right\}$$
$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*}\cdot\left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det|A-\lambda\cdot I|=\left(\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}-\lambda\right)\cdot\left\{\lambda^2+\lambda\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)+\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)\right\}$$
$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*}\cdot\left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det|A-\lambda\cdot I|=-\lambda^3+\lambda^2\cdot\left\{\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}-\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)\right\}$$
$$+\lambda\cdot\left\{\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)-\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)\right\}$$
$$+\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)$$
$$+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*}\cdot\left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\left(\frac{R_S}{L_1}+\lambda\right)$$

$$\det|A-\lambda\cdot I|=-\lambda^3+\lambda^2\cdot\left\{\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}-\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)\right\}$$
$$+\lambda\cdot\left\{\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\left(\frac{1}{R\cdot C_2}+\frac{R_S}{L_1}\right)-\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)\right.$$
$$\left.+\frac{1}{C_1\cdot C_2\cdot R^2}\cdot\left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*}\cdot\left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}\right\}$$
$$+\left.\frac{\partial\Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*}\cdot\frac{1}{L_1\cdot C_2}\cdot\left(\frac{R_S}{R}+1\right)$$
$$+\frac{R_S}{C_1\cdot C_2\cdot R^2\cdot L_1}\cdot\left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*}\cdot\left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}$$

We define the following parameters for simplicity: Γ_0, Γ_1, Γ_2, Γ_3

$$\Gamma_0 = -1; \Gamma_1 = \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} - \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right)$$

$$\Gamma_2 = \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} \cdot \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right) - \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right) + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}$$

$$\Gamma_3 = \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} \cdot \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right) + \frac{R_S}{C_1 \cdot C_2 \cdot R^2 \cdot L_1} \cdot \left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}$$

$$\det|A - \lambda \cdot I| = 0 \Rightarrow \Gamma_0 \cdot \lambda^3 + \Gamma_1 \cdot \lambda^2 + \Gamma_2 \cdot \lambda + \Gamma_3 = 0$$

We get cubic function of the form $\Gamma_0 \cdot \lambda^3 + \Gamma_1 \cdot \lambda^2 + \Gamma_2 \cdot \lambda + \Gamma_3$ where Γ_0 is nonzero, a polynomial of degree three. The derivative of a cubic function is a quadratic function. The integral of a cubic function is a quartic function. Setting $\det|A - \lambda \cdot I| = 0$ and assuming $\Gamma_0 \neq 0$; $\Gamma_0 = -1$ produces a cubic equation of the form: $\Gamma_0 \cdot \lambda^3 + \Gamma_1 \cdot \lambda^2 + \Gamma_2 \cdot \lambda + \Gamma_3 = 0$. The coefficients Γ_0, Γ_1, Γ_2, Γ_3 are real numbers. Every cubic equation with real coefficients (Γ_0, Γ_1, Γ_2, Γ_3) has at least one solution λ among the real numbers. This is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant $\Delta = -4 \cdot \Gamma_1^3 \cdot \Gamma_3 + \Gamma_1^2 \cdot \Gamma_2^2 + 4 \cdot \Gamma_2^3 - 18 \cdot \Gamma_1 \cdot \Gamma_2 \cdot \Gamma_3 - 27 \cdot \Gamma_3^2$. The following cases need to be considered. If $\Delta > 0$, then the equation has three distinct real root. If $\Delta < 0$, then the equation has one real root and a pair of complex conjugate roots. If $\Delta = 0$, then at least two roots coincide.

We classify system critical point (fixed point) according to Table 5.8 [5].

We need to implement our system/circuit stability analysis to specific circuit parameters. Table 5.9 describes typical circuit parameters.

Table 5.8 Eigenvalues of *A* (Jacobian) vs type of critical point (fixed point)

Eigenvalues of *A* (Jacobian)	Type of critical point (fixed point)
Real, unequal, same sign	Node (stable/unstable), driven away or back from steady-state value
Real, unequal, opposite sign	Saddle point and is unstable
Real and equal	Node (stable/unstable), driven away or back from steady-state value
Complex conjugate	Spiral point (Stable/Unstable), oscillates around steady-state value with decreasing/increasing amplitude
Pure imaginary	Center (oscillates around steady-state value with constant amplitude)

Table 5.9 Typical circuit parameters

I_{se}	1 uA	I_{sc}	2 uA
I_0	1 uA	α_f	0.98
α_r	0.5	k	0.01
L_1	10 mH	R_S	1 Ω
R	1 kΩ	C_1	1000 uF
C_2	1000 uF	V_t	0.026 V = 26 mV

Since I_{NDR} can get only positive values $I_{NDR} >= 0$, we get possible fixed point values $X^* \rightarrow I_{NDR}$ which are positive.

$$A_1 = (\alpha r - k - 1) = -0.51; A_2 = Ise \cdot (\alpha r \cdot \alpha f - 1) = -0.51 \cdot 10^{-6}$$
$$A_3 = (1 - \alpha f \cdot (1 + k)) = 0.0102; A_4 = Isc \cdot (\alpha r \cdot \alpha f - 1) = -1.02 \cdot 10^{-6}$$

$$\Xi(R = 1k\Omega, R_S = 1\Omega) = \frac{1}{Vt \cdot \frac{1}{R} \cdot \left[\frac{1}{\left(\frac{R}{R_S} + 1\right)} - 1 \right]} = -38850;$$

$$\Xi_1 = \Xi(R, R_S) \cdot A_3 - \frac{A_1}{I_0} = 509603.8$$

$$\Xi_2 = A_3 - A_1 - \frac{A_2}{I_0} + A_4 \cdot \Xi(R, R_S) = 1.069; \Xi_3 = A_4 - A_2 = -0.51 \cdot 10^{-6}$$

We can summery our system fixed points values in Table 5.10.

The next step is to find system characteristic equations for option (1) and option (2) which summery in Table 5.10.

Option (1) $X^* = 92.72$ uA, $Y^* = 0.28$ mV, $Z^* = -0.28$ mA, $g(X^*) = 0.2834$

Table 5.10 Summary of our system fixed points values

Fixed point coordinate	Option (1)	Option (2)
$X^* \rightarrow I_{NDR}{*}$	$X^* = \frac{Isc \cdot (1 - \alpha r \cdot \alpha f)}{(1 - \alpha f \cdot (1 + k))} = 92.72\,\text{uA}$	$X^*_{1,2} = \frac{-\Xi_2 + \sqrt{\Xi_2^2 - 4 \cdot \Xi_1 \cdot \Xi_3}}{2 \cdot \Xi_1} = 0.4\,\text{uA}$
$Y^* \rightarrow V_{C2}{*}$	$g(X^*) = 0.2834$ $Y^* = \frac{1}{\left(\frac{R}{R_S} + 1\right)} \cdot g(X^*) = 0.28\,\text{mV}$	$g(X^*) = -0.00042$ $Y^* = \frac{1}{\left(\frac{R}{R_S} + 1\right)} \cdot g(X^*) = -0.4\,\text{uA}$
$Z^* \rightarrow I_{L1}{*}$	$Z^* = -\frac{1}{R_S} \cdot Y^* = -0.28\,\text{mA}$	$Z^* = -\frac{1}{R_S} \cdot Y^* = 0.4\,\text{uA}$

$$\Gamma_0 = -1; \Gamma_1 = \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} - \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right)$$

$$\Gamma_2 = \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} \cdot \left(\frac{1}{R \cdot C_2} + \frac{R_S}{L_1}\right) - \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right) + \frac{1}{C_1 \cdot C_2 \cdot R^2} \cdot \left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}$$

$$\Gamma_3 = \left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} \cdot \frac{1}{L_1 \cdot C_2} \cdot \left(\frac{R_S}{R} + 1\right) + \frac{R_S}{C_1 \cdot C_2 \cdot R^2 \cdot L_1} \cdot \left.\frac{1}{\psi(X)}\right|_{X^*,Y^*,Z^*} \cdot \left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*}$$

$$\frac{\partial \Upsilon_1}{\partial X} = -\frac{1}{C_1 \cdot \psi(X)} \cdot \left[\frac{1}{\psi(X)} \cdot \frac{\partial \psi(X)}{\partial X} \cdot \left\{\frac{1}{R} \cdot [Y - g(X)] - X\right\} + \frac{1}{R} \cdot \frac{\partial g(X)}{\partial X} + 1\right]$$

$$\left.\frac{\partial g(X)}{\partial X}\right|_{X^*,Y^*,Z^*} = V_t \cdot \left[\frac{A_1 \cdot A_4 - A_3 \cdot A_2}{(X^* \cdot A_1 + A_2) \cdot (X^* \cdot A_3 + A_4)}\right] + V_t \cdot \left[\frac{1}{X^* + I_0}\right]; \psi(X)|_{X^*,Y^*,Z^*}$$

$$\left.\frac{\partial \psi(X)}{\partial X}\right|_{X^*,Y^*,Z^*}; \xi_1(X) = \frac{A_1 \cdot A_4 - A_3 \cdot A_2}{[X \cdot A_1 + A_2] \cdot [X \cdot A_3 + A_4]}; \xi_2(X) = \frac{1}{(X + I_0)}; \psi(X) = \frac{\partial g(X)}{\partial X}$$

$$\psi(X) = V_t \cdot \{\xi_1(X) + \xi_2(X)\} \Rightarrow \frac{\partial \psi(X)}{\partial X} = V_t \cdot \left\{\frac{\partial \xi_1(X)}{\partial X} + \frac{\partial \xi_2(X)}{\partial X}\right\}; \psi(X^* = 92.72uA) = 3808$$

$$\frac{\partial \xi_1(X)}{\partial X} = \frac{[A_3 \cdot A_2 - A_1 \cdot A_4] \cdot \{2 \cdot A_1 \cdot A_3 \cdot X + A_1 \cdot A_4 + A_3 \cdot A_2\}}{[X \cdot A_1 + A_2]^2 \cdot [X \cdot A_3 + A_4]^2}; \frac{\partial \xi_2(X)}{\partial X} = \frac{-1}{(X + I_0)^2}$$

$$\xi_1(X^* = 92.72uA) = 135710.5; \xi_2(X^* = 92.72uA) = 10752.68; \left.\frac{\partial \xi_1(X)}{\partial X}\right|_{X=X^*} = 12839.5 \cdot 10^6$$

$$\left.\frac{\partial \xi_2(X)}{\partial X}\right|_{X=X^*} = -115.6 \cdot 10^6; \left.\frac{\partial \psi(X)}{\partial X}\right|_{X=X^*} = V_t \cdot \left\{\left.\frac{\partial \xi_1(X)}{\partial X}\right|_{X=X^*} + \left.\frac{\partial \xi_2(X)}{\partial X}\right|_{X=X^*}\right\} = 330.82 \cdot 10^6$$

$$\left.\frac{\partial \Upsilon_1}{\partial X}\right|_{X^*,Y^*,Z^*} = 7.1657; \Gamma_0 = -1; \Gamma_1 = -93.83; \Gamma_2 = -99375.27; \Gamma_3 = 717386.57$$

$$\Gamma_0 \cdot \lambda^3 + \Gamma_1 \cdot \lambda^2 + \Gamma_2 \cdot \lambda + \Gamma_3 = 0 \Rightarrow \lambda_1 = -0.5050 + 3.1233\mathrm{i}$$
$$\lambda_2 = -0.5050 - 3.1233\mathrm{i}\ ; \lambda_3 = 0.0717$$

The above polynomial is represented in MATLAB software as [-1-93.83-99375.27 717386.57] and the roots of the characteristic polynomial are returned in a column vector by

EDU >> *p* = [-1-93.83-99375.27 717386.57]; *r* = roots (*p*)

r =

1.0e + 002 *

−0.5050 + 3.1233i

−0.5050-3.1233i

0.0717

Our fixed point in option (1) is a saddle point.

Remark It is reader exercise to discuss stability of system fixed point in option (2).

We run MATLAB script to find system phase portrait and behavior in time.

We need to plot system phase plane *X*–*Y*–*Z*, *X*–*Y*, *X*–*Z*, *Y*–*Z*, and *X*(*t*), *Y*(*t*), *Z*(*t*) (Figs. 5.9, 5.10, 5.11, and 5.12).

$\Psi(X) \to p$; $g(X) \to$ m; $C1 = C1 = 1000$ uF; Rs = 1 Ω; $R = 1$ kΩ; $L1 = 10$ mH

$X_0 \to I_{NDR}(t = 0)$; $Y_0 \to V_{C2}(t = 0)$; $Z_0 \to I_{L1}(t = 0)$

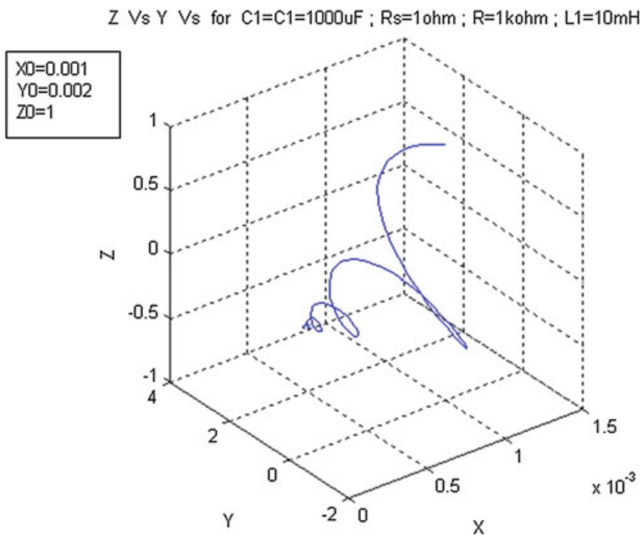


Fig. 5.9 *Z* versus *Y* versus *X* for *C*1 = *C*1 = 1000 μF; Rs = 1 Ω; *R* = 1 kΩ; *L*1 = 10 mH

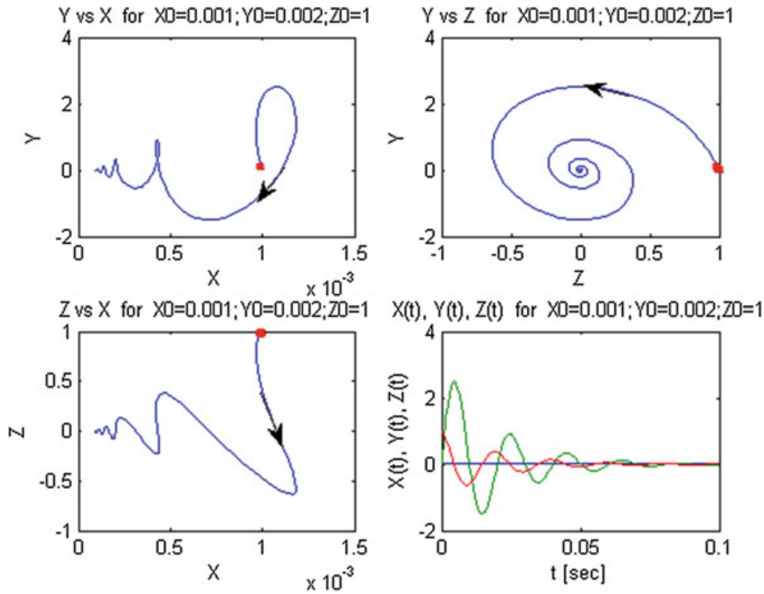


Fig. 5.10 *Y* versus *X* for *X*0 = 0.001, *Y*0 = 0.002, *Z*0 = 1; *Y* versus *Z* for *X*0 = 0.001, *Y*0 = 0.002, *Z*0 = 1; *Z* versus *X* for *X*0 = 0.001, *Y*0 = 0.002, *Z*0 = 1; *X*(*t*), *Y*(*t*), *Z*(*t*) for *X*0 = 0.001, *Y*0 = 0.002, *Z*0 = 1

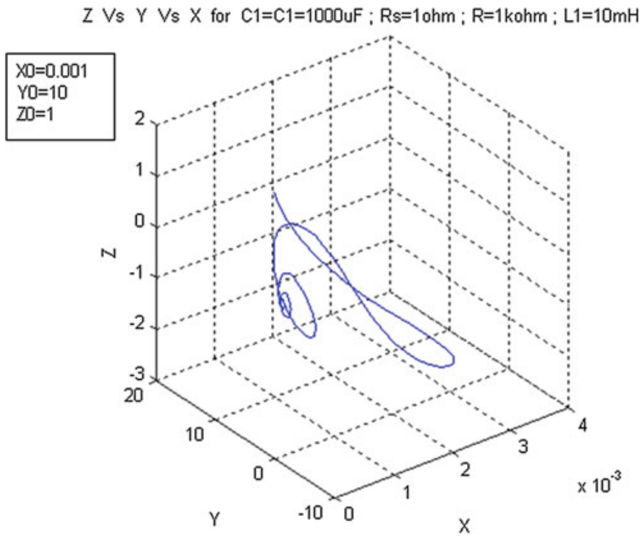


Fig. 5.11 *Z* versus *Y* versus *X* for *C*1 = *C*1 = 1000 μF; Rs = 1 Ω; *R* = 1 kΩ; *L*1 = 10 mH

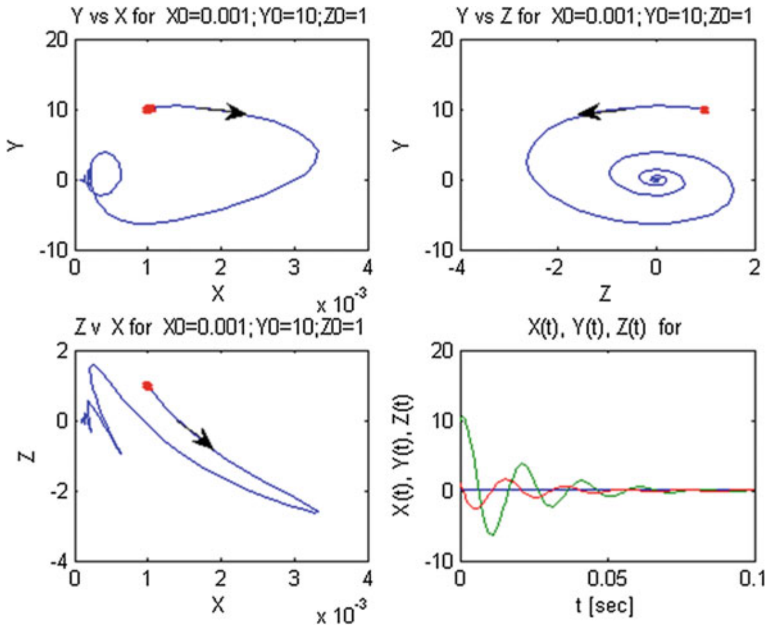


Fig. 5.12 *Y* versus *X* for *X*0 = 0.001, *Y*0 = 10, *Z*0 = 1; *Y* versus *Z* for *X*0 = 0.001, *Y*0 = 10, *Z*0 = 1; *Z* versus *X* for *X*0 = 0.001, *Y*0 = 10, *Z*0 = 1; *X*(*t*), *Y*(*t*), *Z*(*t*) for *X*0 = 0.001, *Y*0 = 10, *Z*0 = 1

```
function g=chuaopto(t,x)
g=zeros(3,1);
m=0.026*log(((-0.51*x(1)-0.51*0.000001)./(x(1)*0.0102-
1.02*0.000001)).*(x(1)/0.000001+1));
p=0.026*((0.52*0.000001+0.005*0.000001)./((-0.51*x(1)-
0.51*0.000001).*(x(1)*0.0102-1.02*0.000001))+1/(x(1)+0.000001));
g(1)=(1/p).*(x(2)-m)-(1000/p).*x(1);
g(2)=1000*x(3)-x(2)+m;
g(3)=-100*x(2)-100*x(3);
```

```
function h=chuaopto1(x0,y0,z0)
[t,x]=ODE45 (@chuaopto,[0,.1],[x0,y0,z0],[]);
%plot3 (x(:,1),x(:,2),x(:,3));
%xlabel ('X')
%ylabel ('Y')
%zlabel ('Z')
%grid on
%axis square
subplot(2,2,1);plot(x(:,1),x(:,2));subplot(2,2,2);plot(x(:,3),x(:,2))
;subplot(2,2,3);plot(x(:,1),x(:,3));subplot(2,2,4);plot(t,x);
```

5.4 OptoNDR Circuit's Two Variables Analysis

We consider an OptoNDR circuit with two storage elements (variable capacitor and variable Inductance see Fig. 5.13). At $t = 0$ switch $S1$ moves his position from OFF state to ON state. Circuit dynamic starts and $V(t)$ is the main system variable. We consider that initially capacitor $C1$ is charged to $VC1(t = 0)$ and inductance $L1$ is charged to $IL1(t = 0)$. $V_{BB} \gg VQ1$(sustain) and photo transistor $Q1$ can reach his sustaining voltage (breakover voltage) after we move switch $S1$ to ON state. After we move $S1$ to ON state capacitor $C1$ starts to charge and once V_{CEQ1} reaches sustain voltage the current through inductor $L1$ rise up [1, 2, 85].

Now we proceed to the circuit model for analytical analysis. The optical coupling between the LED ($D1$) to Photo transistor ($Q1$) is represented as transistor dependent base current on LED ($D1$) current. $I_{BQ1} = k_1 \cdot I_{D1} - \ldots$.

The (–) sign is for partial dependent current which flow through $L1$–$R2$ branch (Fig. 5.14).

$$I_{R1} = I_{D1}; I_{D1} = I_{CQ1} + I_{C1}; V_{CEQ1} = V_{C1};$$

$$I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt} = C_1 \cdot \frac{dV_{CEQ1}}{dt}; V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right]$$

$$I_{C1} = C_1 \cdot \frac{dV}{dt}; V_{BB} = I_{D1} \cdot R_1 + V_{D1} + V_{CEQ1};$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1}; k_1 \cdot I_{D1} = I_{BQ1} + I_{L1}$$

$$I_{L1} = I_{R2}; V_{BEQ1} = V_{L1} + V_{R2}; V_{L1} = L_1 \cdot \frac{dI_{L1}}{dt}; V_{BEQ1} = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2$$

Fig. 5.13 OptoNDR circuit with two energy stroage elements

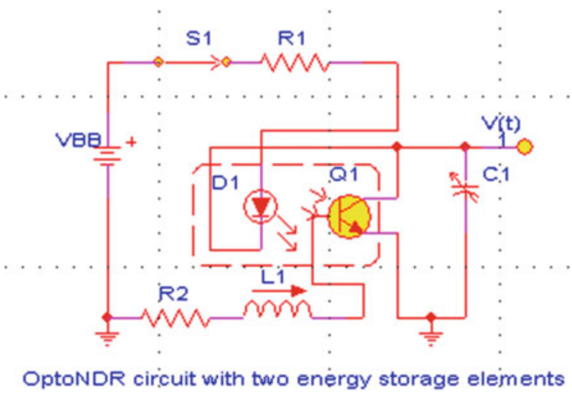
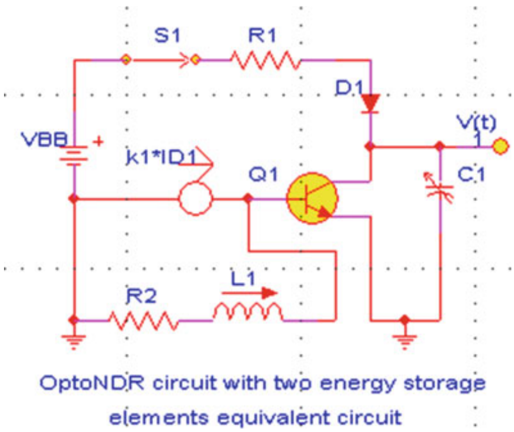


Fig. 5.14 OptoNDR circuit with two energy stroage elements equivalent circuit



$$V_{BEQ1} = V_t \cdot \ln\left[\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right];$$

$$V_{CEQ1} = V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

Assumption $\frac{I_{sc}}{I_{se}} \to 1 \Rightarrow \ln\left[\frac{I_{sc}}{I_{se}}\right] \to \varepsilon$

$$\frac{I_{sc}}{I_{se}} \to 1 \Rightarrow \ln\left[\frac{I_{sc}}{I_{se}}\right] \to \varepsilon \Rightarrow V = V_{CEQ1}$$
$$= V_t \cdot \ln\left[\frac{(\alpha_r \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_f) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}\right]$$

$$V_{BB} = I_{D1} \cdot R_1 + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) + V; V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx V_t \cdot \frac{I_{D1}}{I_0};$$
$$V_{BB} = I_{D1} \cdot R_1 + V_t \cdot \frac{I_{D1}}{I_0} + V$$

Taylor series approximation: $V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx V_t \cdot \frac{I_{D1}}{I_0}; I_{D1} = \frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}$

$$V_{BB} = I_{D1} \cdot R_1 + V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) + V; V_t \cdot \ln\left(\frac{I_{D1}}{I_0} + 1\right) \approx V_t \cdot \frac{I_{D1}}{I_0};$$
$$V_{BB} = I_{D1} \cdot R_1 + V_t \cdot \frac{I_{D1}}{I_0} + V$$

$$V_{BEQ1} = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2 \Rightarrow V_t \cdot \ln\left[\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2$$

$$k_1 \cdot I_{D1} = I_{BQ1} + I_{L1} \Rightarrow I_{L1} = k_1 \cdot I_{D1} - I_{BQ1} \Rightarrow I_{L1} = k_1 \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - I_{BQ1}$$

$$I_{D1} = I_{CQ1} + I_{C1}; k_1 \cdot I_{D1} = I_{BQ1} + I_{L1} \Rightarrow I_{BQ1} = k_1 \cdot I_{D1} - I_{L1} = k_1 \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - I_{L1}$$

$$I_{D1} = I_{CQ1} + I_{C1} \Rightarrow I_{CQ1} = I_{D1} - I_{C1} = \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt};$$

$$I_{EQ1} = (k_1 + 1) \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt} - I_{L1}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1} \Rightarrow I_{EQ1} = k_1 \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - I_{L1} + \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt} = (k_1 + 1) \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt} - I_{L1}$$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = \alpha_r \cdot \left\{\left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt}\right\} - \left\{(k_1 + 1) \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt} - I_{L1}\right\}$$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = [\alpha_r - (k_1 + 1)] \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] + (1 - \alpha_r) \cdot C_1 \cdot \frac{dV}{dt} + I_{L1};$$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = \xi_1\left(V, \frac{dV}{dt}\right) + I_{L1}$$

We define for simplicity

$$\xi_1\left(V, \frac{dV}{dt}\right) = [\alpha_r - (k_1 + 1)] \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] + (1 - \alpha_r) \cdot C_1 \cdot \frac{dV}{dt}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt} - \left\{(k_1 + 1) \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right] - C_1 \cdot \frac{dV}{dt} - I_{L1}\right\} \cdot \alpha_f$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = [1 - (k_1 + 1) \cdot \alpha_f] \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}} \right] - (1 - \alpha_f) \cdot C_1 \cdot \frac{dV}{dt} + I_{L1} \cdot \alpha_f;$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_f = \xi_2 \left(V, \frac{dV}{dt} \right) + I_{L1} \cdot \alpha_f$$

We define for simplicity $\xi_2\left(V, \frac{dV}{dt}\right) = [1 - (k_1 + 1) \cdot \alpha_f] \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}}\right]$ $-(1 - \alpha_f) \cdot C_1 \cdot \frac{dV}{dt}$

$$\alpha_r \cdot I_{CQ1} - I_{EQ1} = \xi_1 \left(V, \frac{dV}{dt} \right) + I_{L1}; I_{CQ1} - I_{EQ1} \cdot \alpha_f = \xi_2 \left(V, \frac{dV}{dt} \right) + I_{L1} \cdot \alpha_f$$

$$V_t \cdot \ln \left[\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1 \right] = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2$$

$$\Rightarrow V_t \cdot \ln \left[\frac{\xi_1 \left(V, \frac{dV}{dt} \right) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1 \right] = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2$$

$$V_t \cdot \ln \left[\frac{\xi_1 \left(V, \frac{dV}{dt} \right) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1 \right] = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2$$

$$\Rightarrow \frac{dI_{L1}}{dt} = \frac{V_t}{L_1} \cdot \ln \left[\frac{\xi_1 \left(V, \frac{dV}{dt} \right) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1 \right] - I_{L1} \cdot \frac{R_2}{L_1}$$

$$\xi_1 \left(V, \frac{dV}{dt} \right) = [\alpha_r - (k_1 + 1)] \cdot \left[\frac{V_{BB}}{R_1 + \frac{V_t}{I_0}} \right] - [\alpha_r - (k_1 + 1)] \cdot \left[\frac{V}{R_1 + \frac{V_t}{I_0}} \right] + (1 - \alpha_r) \cdot C_1 \cdot \frac{dV}{dt}$$

$$A_1 = [\alpha_r - (k_1 + 1)] \cdot \left[\frac{V_{BB}}{R_1 + \frac{V_t}{I_0}} \right]; A_2 = -\frac{[\alpha_r - (k_1 + 1)]}{R_1 + \frac{V_t}{I_0}};$$

$$A_3 = (1 - \alpha_r) \cdot C_1; A_1 = -A_2 \cdot V_{BB} \Rightarrow \frac{A_1}{A_2} = -V_{BB}$$

$$\xi_1 \left(V, \frac{dV}{dt} \right) = A_1 + A_2 \cdot V + A_3 \cdot \frac{dV}{dt};$$

$$\xi_2 \left(V, \frac{dV}{dt} \right) = [1 - (k_1 + 1) \cdot \alpha_f] \cdot \left[\frac{V_{BB} - V}{R_1 + \frac{V_t}{I_0}} \right] - (1 - \alpha_f) \cdot C_1 \cdot \frac{dV}{dt}$$

$$\xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) = [1-(k_1+1)\cdot\alpha_f]\cdot\left(\frac{V_{BB}}{R_1+\frac{V_t}{I_0}}\right) - \frac{[1-(k_1+1)\cdot\alpha_f]}{R_1+\frac{V_t}{I_0}}\cdot V - (1-\alpha_f)\cdot C_1\cdot\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$A_4 = [1-(k_1+1)\cdot\alpha_f]\cdot\left(\frac{V_{BB}}{R_1+\frac{V_t}{I_0}}\right); A_5 = -\frac{[1-(k_1+1)\cdot\alpha_f]}{R_1+\frac{V_t}{I_0}}; A_6 = -(1-\alpha_f)\cdot C_1$$

$$A_4 = -A_5\cdot V_{BB} \Rightarrow \frac{A_4}{A_5} = -V_{BB} \Rightarrow \frac{A_1}{A_2} = \frac{A_4}{A_5}; \xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) = A_4 + A_5\cdot V + A_6\cdot\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$V = V_t\cdot\ln\left[\frac{(\alpha_r\cdot I_{CQ1}-I_{EQ1}) + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)}{(I_{CQ1}-I_{EQ1}\cdot\alpha_f) + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)}\right]$$
$$\Rightarrow V = V_t\cdot\ln\left[\frac{\xi_1\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)}{\xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1}\cdot\alpha_f + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)}\right]$$

$$V = V_t\cdot\ln\left[\frac{\xi_1\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)}{\xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1}\cdot\alpha_f + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)}\right]$$
$$\Rightarrow e^{\left[\frac{V}{V_t}\right]} = \frac{\xi_1\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)}{\xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1}\cdot\alpha_f + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)}$$

Taylor series approximation: $e^{\left[\frac{V}{V_t}\right]} \approx \frac{V}{V_t}+1; \xi_1 = \xi_1\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right); \xi_2 = \xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right)$

$$\left(\frac{V}{V_t}+1\right) = \frac{\xi_1\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)}{\xi_2\left(V,\frac{\mathrm{d}V}{\mathrm{d}t}\right) + I_{L1}\cdot\alpha_f + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)};$$
$$\left(\frac{V}{V_t}+1\right) = \frac{\xi_1 + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)}{\xi_2 + I_{L1}\cdot\alpha_f + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)}$$

$$\xi_1 + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1) = \xi_2\cdot\left(\frac{V}{V_t}+1\right) + I_{L1}\cdot\alpha_f\cdot\left(\frac{V}{V_t}+1\right) + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)\cdot\left(\frac{V}{V_t}+1\right)$$

$$A_1 + A_2\cdot V + A_3\cdot\frac{\mathrm{d}V}{\mathrm{d}t} + I_{L1} + I_{se}\cdot(\alpha_r\cdot\alpha_f-1)$$
$$= \left(A_4 + A_5\cdot V + A_6\cdot\frac{\mathrm{d}V}{\mathrm{d}t}\right)\cdot\left(\frac{V}{V_t}+1\right)$$
$$+ I_{L1}\cdot\alpha_f\cdot\left(\frac{V}{V_t}+1\right) + I_{sc}\cdot(\alpha_r\cdot\alpha_f-1)\cdot\left(\frac{V}{V_t}+1\right)$$

$$A_1 + A_2 \cdot V + A_3 \cdot \frac{dV}{dt} + I_{L1} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = A_4 \cdot \left(\frac{V}{V_t} + 1\right) + A_5 \cdot V \cdot \left(\frac{V}{V_t} + 1\right) + A_6 \cdot \frac{dV}{dt} \cdot \left(\frac{V}{V_t} + 1\right) + I_{L1} \cdot \alpha_f \cdot \left(\frac{V}{V_t} + 1\right) + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \left(\frac{V}{V_t} + 1\right)$$

$$A_1 + A_2 \cdot V + A_3 \cdot \frac{dV}{dt} + I_{L1} + I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = A_4 \cdot \frac{V}{V_t} + A_4 + A_5 \cdot \frac{V^2}{V_t} + A_5 \cdot V + A_6 \cdot \frac{dV}{dt} \cdot \frac{V}{V_t} + A_6 \cdot \frac{dV}{dt} + I_{L1} \cdot \alpha_f \cdot \frac{V}{V_t} + I_{L1} \cdot \alpha_f + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{V}{V_t} + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)$$

$$A_3 \cdot \frac{dV}{dt} - A_6 \cdot \frac{dV}{dt} \cdot \frac{V}{V_t} - A_6 \cdot \frac{dV}{dt} = A_4 \cdot \frac{V}{V_t} + A_4 + A_5 \cdot \frac{V^2}{V_t} + A_5 \cdot V + I_{L1} \cdot \alpha_f \cdot \frac{V}{V_t} + I_{L1} \cdot \alpha_f + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) \cdot \frac{V}{V_t} + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) - A_1 - A_2 \cdot V - I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - I_{L1}$$

$$\left\{A_3 - A_6 \cdot \left(\frac{V}{V_t} + 1\right)\right\} \cdot \frac{dV}{dt} = -A_1 - I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + A_4 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1) + V \cdot \left(-A_2 + \frac{A_4}{V_t} + \frac{I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}{V_t} + A_5\right) + (\alpha_f - 1) \cdot I_{L1} + I_{L1} \cdot V \cdot \frac{\alpha_f}{V_t} + \frac{A_5}{V_t} \cdot V^2$$

For simplicity we define the following parameters: $B_3 = \alpha_f - 1$; $B_4 = \frac{\alpha_f}{V_t}$; $B_5 = \frac{A_5}{V_t}$

$$B_1 = -A_1 - I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) + A_4 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1);$$
$$B_2 = -A_2 + \frac{A_4}{V_t} + \frac{I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}{V_t} + A_5$$

$$\left\{A_3 - A_6 \cdot \left(\frac{V}{V_t} + 1\right)\right\} \cdot \frac{dV}{dt} = B_1 + V \cdot B_2 + B_3 \cdot I_{L1} + I_{L1} \cdot V \cdot B_4 + B_5 \cdot V^2$$

$$\frac{dV}{dt} = \frac{B_5 \cdot V^2 + V \cdot B_2 + B_1 + B_3 \cdot I_{L1} + I_{L1} \cdot V \cdot B_4}{[A_3 - A_6] - \frac{A_6}{V_t} \cdot V}; B_6 = A_3 - A_6; B_7 = -\frac{A_6}{V_t}$$

$$\frac{dV}{dt} = \frac{B_5 \cdot V^2 + V \cdot B_2 + B_1 + B_3 \cdot I_{L1} + I_{L1} \cdot V \cdot B_4}{B_6 + B_7 \cdot V};$$
$$\psi_1(V, I_{L1}) = \frac{B_5 \cdot V^2 + V \cdot B_2 + B_1 + B_3 \cdot I_{L1} + I_{L1} \cdot V \cdot B_4}{B_6 + B_7 \cdot V}$$

$$V_t \cdot \ln\left[\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] = L_1 \cdot \frac{dI_{L1}}{dt} + I_{L1} \cdot R_2$$
$$\Rightarrow \frac{dI_{L1}}{dt} = \frac{V_t}{L_1} \cdot \ln\left[\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$

$$\frac{dI_{L1}}{dt} = \frac{V_t}{L_1} \cdot \ln\left[\frac{\alpha_r \cdot I_{CQ1} - I_{EQ1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$
$$\Rightarrow \frac{dI_{L1}}{dt} = \frac{V_t}{L_1} \cdot \ln\left[\frac{\xi_1(V, \frac{dV}{dt}) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$

$$\frac{dI_{L1}}{dt} = \frac{V_t}{L_1} \cdot \ln\left[\frac{\xi_1(V, \frac{dV}{dt}) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$
$$\Rightarrow \frac{dI_{L1}}{dt} = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot V + A_3 \cdot \frac{dV}{dt} + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$

$$\frac{dV}{dt} = \psi_1(V, I_{L1}) \Rightarrow \frac{dI_{L1}}{dt}$$
$$= \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot V + A_3 \cdot \psi_1(V, I_{L1}) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$

$$\psi_2(V, I_{L1}) = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot V + A_3 \cdot \psi_1(V, I_{L1}) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1} \Rightarrow \frac{dI_{L1}}{dt} = \psi_2(V, I_{L1})$$

We can summarize our system differential equation: $\frac{dV}{dt} = \psi_1(V, I_{L1})$; $\frac{dI_{L1}}{dt} = \psi_2(V, I_{L1})$

$$\psi_1(V, I_{L1}) = \frac{B_5 \cdot V^2 + V \cdot B_2 + B_1 + B_3 \cdot I_{L1} + I_{L1} \cdot V \cdot B_4}{B_6 + B_7 \cdot V}$$
$$\psi_2(V, I_{L1}) = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot V + A_3 \cdot \psi_1(V, I_{L1}) + I_{L1}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - I_{L1} \cdot \frac{R_2}{L_1}$$

$$V \to X; I_{L1} \to Y \Rightarrow \frac{dX}{dt} = \psi_1(X, Y); \frac{dY}{dt} = \psi_2(X, Y)$$

$$\psi_1(X, Y) = \frac{B_5 \cdot X^2 + X \cdot B_2 + B_1 + B_3 \cdot Y + Y \cdot X \cdot B_4}{B_6 + B_7 \cdot X}$$

$$\psi_2(X, Y) = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X, Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - Y \cdot \frac{R_2}{L_1}$$

To find system fixed points we set $\frac{dX}{dt} = 0 \Rightarrow \psi_1(X^*, Y^*) = 0$; $\frac{dY}{dt} = 0 \Rightarrow \psi_2(X^*, Y^*) = 0$

$$\frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot X^* + Y^*}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - Y^* \cdot \frac{R_2}{L_1} = 0$$

$$\Rightarrow V_t \cdot \ln\left[\frac{A_1 + A_2 \cdot X^* + Y^*}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] = Y^* \cdot R_2$$

$$V_t \cdot \ln\left[\frac{A_1 + A_2 \cdot X^* + Y^*}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] = Y^* \cdot R_2 \Rightarrow e^{\frac{Y^* \cdot R_2}{V_t}} = \frac{A_1 + A_2 \cdot X^* + Y^*}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1$$

Taylor series approximation: $e^{\frac{Y^* \cdot R_2}{V_t}} \approx \frac{Y^* \cdot R_2}{V_t} + 1$

$$\frac{Y^* \cdot R_2}{V_t} = \frac{A_1 + A_2 \cdot X^* + Y^*}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} \Rightarrow \frac{Y^* \cdot R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = A_1 + A_2 \cdot X^* + Y^*$$

$$\frac{Y^* \cdot R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) = A_1 + A_2 \cdot X^* + Y^*$$

$$\Rightarrow Y^* \cdot \left\{\frac{R_2 \cdot I_{se}}{V_t} \cdot (\alpha_r \cdot \alpha_f - 1) - 1\right\} = A_2 \cdot X^* + A_1$$

$$Y^* \cdot \left\{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1\right\} = A_2 \cdot X^* - A_1$$

$$\Rightarrow Y^* = \frac{A_2}{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1} \cdot X^* + \frac{A_1}{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1}$$

$$Y^* = -\frac{\alpha_r - (k_1 + 1)}{\left\{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1\right\} \cdot \left(R_1 + \frac{V_t}{I_0}\right)} \cdot X^* + \frac{[\alpha_r - (k_1 + 1)] \cdot V_{BB}}{\left\{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1\right\} \cdot \left(R_1 + \frac{V_t}{I_0}\right)}$$

$$\Gamma_0 = -\frac{\alpha_r - (k_1+1)}{\left\{\frac{R_2}{V_t}\cdot I_{se}\cdot(\alpha_r\cdot\alpha_f - 1) - 1\right\}\cdot\left(R_1 + \frac{V_t}{I_0}\right)};$$

$$\Gamma_1 = \frac{[\alpha_r - (k_1+1)]\cdot V_{BB}}{\left\{\frac{R_2}{V_t}\cdot I_{se}\cdot(\alpha_r\cdot\alpha_f - 1) - 1\right\}\cdot\left(R_1 + \frac{V_t}{I_0}\right)};\Gamma_1 = -\Gamma_0\cdot V_{BB}$$

$$Y^* = \Gamma_0\cdot X^* + \Gamma_1;\frac{\mathrm{d}X}{\mathrm{d}t} = 0 \Rightarrow \psi_1(X^*, Y^*) = 0$$
$$\Rightarrow \frac{B_5\cdot[X^*]^2 + X^*\cdot B_2 + B_1 + B_3\cdot Y^* + Y^*\cdot X^*\cdot B_4}{B_6 + B_7\cdot X^*} = 0$$

$$B_6 + B_7\cdot X^* \neq 0 \Rightarrow X^* \neq -\frac{B_6}{B_7};B_5\cdot[X^*]^2 + X^*\cdot B_2 + B_1 + B_3\cdot Y^* + Y^*\cdot X^*\cdot B_4 = 0$$

$$B_5\cdot[X^*]^2 + X^*\cdot B_2 + B_1 + B_3\cdot(\Gamma_0\cdot X^* + \Gamma_1) + (\Gamma_0\cdot X^* + \Gamma_1)\cdot X^*\cdot B_4 = 0$$

$$B_5\cdot[X^*]^2 + X^*\cdot B_2 + B_1 + B_3\cdot\Gamma_0\cdot X^* + B_3\cdot\Gamma_1 + \Gamma_0\cdot B_4\cdot[X^*]^2 + \Gamma_1\cdot B_4\cdot X^* = 0$$
$$(B_5 + \Gamma_0\cdot B_4)\cdot[X^*]^2 + X^*\cdot(B_2 + B_3\cdot\Gamma_0 + \Gamma_1\cdot B_4) + B_1 + B_3\cdot\Gamma_1 = 0$$

$$\Omega_0 = B_5 + \Gamma_0\cdot B_4;\Omega_1 = B_2 + B_3\cdot\Gamma_0 + \Gamma_1\cdot B_4;$$
$$\Omega_2 = B_1 + B_3\cdot\Gamma_1;\Omega_0\cdot[X^*]^2 + \Omega_1\cdot X^* + \Omega_2 = 0$$

$$\Omega_0\cdot[X^*]^2 + \Omega_1\cdot X^* + \Omega_2 = 0 \Rightarrow X^* = \frac{-\Omega_1 \pm \sqrt{\Omega_1^2 - 4\cdot\Omega_0\cdot\Omega_2}}{2\cdot\Omega_0}$$

We consider X^* (V) fixed point is only real and has positive value

$$\Omega_1^2 - 4\cdot\Omega_0\cdot\Omega_2 > 0;\frac{-\Omega_1 \pm \sqrt{\Omega_1^2 - 4\cdot\Omega_0\cdot\Omega_2}}{2\cdot\Omega_0} > 0$$

We need to find system fixed points for specific circuit's parameters for two options: (1) V_{BB1} = 15 V and (2) V_{BB2} = 150 V (Table 5.11).

The next table gives as circuit's parameters values for two options: (1) V_{BB1} = 15 V and (2) V_{BB2} = 150 V (Table 5.12).

Table 5.11 Circuit's parameters values for two options: V_{BB1} = 15 V and V_{BB2} = 150 V

I_{se}	1 uA	I_{sc}	2 uA
I_0	1 uA	α_f	0.98
α_r	0.5	k_1	0.01
L_1	100 uH	R_2	1 kΩ
R_1	1 kΩ	C_1	0.1 uF
V_{BB}	15 V/150 V	V_t	0.026 V = 26 mV

Table 5.12 Circuit's parameters values for two options: *1* V_{BB1} = 15 V and *2* V_{BB2} = 150 V

Circuit parameter	V_{BB1} = 15 V	V_{BB2} = 150 V
$A_1 = [\alpha_r - (k_1+1)] \cdot \left[\frac{V_{BB}}{R_1 + \frac{V_t}{I_0}}\right]$	-0.28×10^{-3}	-2.8×10^{-3}
$A_2 = -\frac{[\alpha_r - (k_1+1)]}{R_1 + \frac{V_t}{I_0}}$ $A_1 = -A_2 \cdot V_{BB} \Rightarrow \frac{A_1}{A_2} = -V_{BB}$	18.8×10^{-6}	18.8×10^{-6}
$A_3 = (1-\alpha_r) \cdot C_1$	0.05×10^{-6}	0.05×10^{-6}
$A_4 = [1 - (k_1+1) \cdot \alpha_f] \cdot \left(\frac{V_{BB}}{R_1 + \frac{V_t}{I_0}}\right)$	5.5×10^{-6}	55×10^{-6}
$A_5 = -\frac{[1-(k_1+1)\cdot \alpha_f]}{R_1 + \frac{V_t}{I_0}}$ $A_4 = -A_5 \cdot V_{BB} \Rightarrow \frac{A_4}{A_5} = -V_{BB} \Rightarrow \frac{A_1}{A_2} = \frac{A_4}{A_5}$	-0.37×10^{-6}	-0.37×10^{-6}
$A_6 = -(1-\alpha_f) \cdot C_1$	-0.002×10^{-6}	-0.002×10^{-6}
$B_1 = -A_1 - I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)$ $+ A_4 + I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)$	0.284×10^{-3}	2.85×10^{-3}
$B_2 = -A_2 + \frac{A_4}{V_t} + \frac{I_{sc} \cdot (\alpha_r \cdot \alpha_f - 1)}{V_t} + A_5$	153.13×10^{-6}	2056.98×10^{-6}
$B_3 = \alpha_f - 1$	-0.02	-0.02
$B_4 = \frac{\alpha_f}{V_t}$	37.69	37.69
$B_5 = \frac{A_5}{V_t} = -\frac{[1-(k_1+1)\cdot\alpha_f]}{\left[R_1 + \frac{V_t}{I_0}\right] \cdot V_t}$	-14.23×10^{-6}	-14.23×10^{-6}
$B_6 = A_3 - A_6 = (1-\alpha_r) \cdot C_1 + (1-\alpha_f) \cdot C_1$	0.052×10^{-6}	0.052×10^{-6}
$B_7 = -\frac{A_6}{V_t} = \frac{(1-\alpha_f)\cdot C_1}{V_t}$	0.076×10^{-6}	0.076×10^{-6}
$\Gamma_0 = -\frac{\alpha_r - (k_1+1)}{\left\{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1\right\} \cdot \left(R_1 + \frac{V_t}{I_0}\right)}$	-18.5×10^{-6}	-18.5×10^{-6}
$\Gamma_1 = \frac{[\alpha_r - (k_1+1)] \cdot V_{BB}}{\left\{\frac{R_2}{V_t} \cdot I_{se} \cdot (\alpha_r \cdot \alpha_f - 1) - 1\right\} \cdot \left(R_1 + \frac{V_t}{I_0}\right)}$	0.2775×10^{-3}	2.775×10^{-3}
$\Omega_0 = B_5 + \Gamma_0 \cdot B_4$	-711.49×10^{-6}	-711.49×10^{-6}
$\Omega_1 = B_2 + B_3 \cdot \Gamma_0 + \Gamma_1 \cdot B_4$	10612.47×10^{-6}	106.646×10^{-3}
$\Omega_2 = B_1 + B_3 \cdot \Gamma_1$	0.2785×10^{-3}	2.794×10^{-3}
$\Delta = \Omega_1^2 - 4 \cdot \Omega_0 \cdot \Omega_2 > 0$	113.09×10^{-6}	11381.31×10^{-6}
$X^* = \frac{-\Omega_1 \pm \sqrt{\Omega_1^2 - 4 \cdot \Omega_0 \cdot \Omega_2}}{2 \cdot \Omega_0} > 0$ $X^* \to V^*$	$X^{(0)} = 14.928\, V$ $X^{(1)} = -0.0123\, V$	$X^{(0)} = 150\, V$ $X^{(1)} = -0.0239\, V$
$Y^* = \Gamma_0 \cdot X^* + \Gamma_1$ $Y^* \to I_{L1}^*$	$Y^{(0)} = 1.5\, uA$ $Y^{(1)} = 277\, uA$	$Y^{(0)} \to 0$ $Y^{(1)} = 2.77\, mA$

Remark We got two possible circuit's fixed points which based on initial conditions we start our circuit dynamic ($V(t = 0) \rightarrow X(t = 0)$, $I_{L1}(t = 0) \rightarrow Y(t = 0)$).

It is reader task to find Basin Of Attraction (BOA) for these fixed points.

We need to calculate our OptoNDR circuit's two variables Jacobian elements.

$$V \rightarrow X; I_{L1} \rightarrow Y \Rightarrow \frac{\mathrm{d}X}{\mathrm{d}t} = \psi_1(X,Y); \frac{\mathrm{d}Y}{\mathrm{d}t} = \psi_2(X,Y); \psi_1 = \psi_1(X,Y); \psi_2 = \psi_2(X,Y)$$

$$\psi_1(X,Y) = \frac{B_5 \cdot X^2 + X \cdot B_2 + B_1 + B_3 \cdot Y + Y \cdot X \cdot B_4}{B_6 + B_7 \cdot X};$$

$$\psi_2(X,Y) = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X,Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - Y \cdot \frac{R_2}{L_1}$$

$$\frac{\partial\psi_1}{\partial X}, \frac{\partial\psi_1}{\partial Y}, \frac{\partial\psi_2}{\partial X}, \frac{\partial\psi_2}{\partial Y};$$

$$\frac{\partial\psi_1}{\partial X} = \frac{B_7 \cdot B_5 \cdot X^2 + 2 \cdot B_5 \cdot B_6 \cdot X + Y \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) + B_2 \cdot B_6 - B_1 \cdot B_7}{(B_6 + B_7 \cdot X)^2}$$

$$\frac{\partial\psi_1}{\partial Y} = \frac{B_3 + X \cdot B_4}{B_6 + B_7 \cdot X}; \frac{\partial\psi_2}{\partial X} = \frac{V_t \cdot \left(A_2 + A_3 \cdot \frac{\partial\psi_1}{\partial X}\right)}{L_1 \cdot \{A_1 + A_2 \cdot X + A_3 \cdot \psi_1 + Y + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}}$$

$$\frac{\partial\psi_2}{\partial Y} = \frac{V_t \cdot \left(A_3 \cdot \frac{\partial\psi_1}{\partial Y} + 1\right)}{L_1 \cdot \{A_1 + A_2 \cdot X + A_3 \cdot \psi_1 + Y + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}} - \frac{R_2}{L_1}$$

We classify our system fixed points as stable (Spiral, Node) or unstable (Spiral, Node), which is done by inspection of the system characteristic equation.

$$A - \lambda \cdot I = \begin{pmatrix} \frac{\partial\psi_1}{\partial X} & \frac{\partial\psi_1}{\partial Y} \\ \frac{\partial\psi_2}{\partial X} & \frac{\partial\psi_2}{\partial Y} \end{pmatrix}_{(X^*,Y^*)} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \lambda = \begin{pmatrix} \frac{\partial\psi_1}{\partial X} - \lambda & \frac{\partial\psi_1}{\partial Y} \\ \frac{\partial\psi_2}{\partial X} & \frac{\partial\psi_2}{\partial Y} - \lambda \end{pmatrix}_{(X^*,Y^*)}$$

$$A - \lambda \cdot I = \begin{pmatrix} \left(\frac{\partial\psi_1}{\partial X} - \lambda\right)\Big|_{(X^*,Y^*)} & \frac{\partial\psi_1}{\partial Y}\Big|_{(X^*,Y^*)} \\ \frac{\partial\psi_2}{\partial X}\Big|_{(X^*,Y^*)} & \left(\frac{\partial\psi_2}{\partial Y} - \lambda\right)\Big|_{(X^*,Y^*)} \end{pmatrix}$$

$$A - \lambda \cdot I = \left(\begin{array}{cc} \left(\frac{\partial \psi_1}{\partial X} - \lambda\right)\Big|_{(X^*,Y^*)} & \left\{\frac{V_t \cdot (A_2 + A_3 \cdot \frac{\partial \psi_1}{\partial X})}{L_1 \cdot \{A_1 + A_2 \cdot X + A_3 \cdot \psi_1 + Y + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}}\right\}\Bigg|_{(X^*,Y^*)} \\ \frac{\partial \psi_1}{\partial Y}\Big|_{(X^*,Y^*)} & \left(\left\{\frac{V_t \cdot (A_3 \cdot \frac{\partial \psi_1}{\partial Y} + 1)}{L_1 \cdot \{A_1 + A_2 \cdot X + A_3 \cdot \psi_1 + Y + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}} - \frac{R_2}{L_1}\right\} - \lambda\right)\Bigg|_{(X^*,Y^*)} \end{array} \right\}$$

$$A - \lambda \cdot I = \left(\frac{\partial \psi_1}{\partial X} - \lambda\right)\Bigg|_{(X^*,Y^*)} \cdot \left(\left\{\frac{V_t \cdot \left(A_3 \cdot \frac{\partial \psi_1}{\partial Y} + 1\right)}{L_1 \cdot (A_1 + A_2 \cdot X + A_3 \cdot \psi_1 + Y + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1\}} - \frac{R_2}{L_1}\right\} - \lambda\right)\Bigg|_{(X^*,Y^*)} - \left\{\frac{V_t \cdot (A_2 + A_3 \cdot \frac{\partial \psi_1}{\partial X})}{L_1 \cdot \{A_1 + A_2 \cdot X + A_3 \cdot \psi_1 + Y + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}}\right\}\Bigg|_{(X^*,Y^*)} \cdot \frac{\partial \psi_1}{\partial Y}\Bigg|_{(X^*,Y^*)}$$

$$\frac{\partial \psi_1}{\partial X}\Bigg|_{(X^*,Y^*)} = \frac{B_7 \cdot B_5 \cdot [X^*]^2 + 2 \cdot B_5 \cdot B_6 \cdot X^* + Y^* \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) + B_2 \cdot B_6 - B_1 \cdot B_7}{(B_6 + B_7 \cdot X^*)^2}$$

$$\frac{\partial \psi_1}{\partial Y}\Bigg|_{(X^*,Y^*)} = \frac{B_3 + X^* \cdot B_4}{B_6 + B_7 \cdot X^*}; \frac{\partial \psi_2}{\partial X}\Bigg|_{(X^*,Y^*)} = \frac{V_t \cdot \left(A_2 + A_3 \cdot \frac{\partial \psi_1}{\partial X}\Big|_{(X^*,Y^*)}\right)}{L_1 \cdot \{A_1 + A_2 \cdot X^* + A_3 \cdot \psi_1 + Y^* + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1\}}$$

$$A - \lambda \cdot I|_{(X^*,Y^*)} = \frac{V_t \cdot \left(A_3 \cdot \frac{\partial \psi_1}{\partial Y}\Big|_{(X^*,Y^*)} + 1\right) \cdot \frac{\partial \psi_1}{\partial X}\Big|_{(X^*,Y^*)}}{L_1 \cdot \{A_1 + A_2 \cdot X^* + A_3 \cdot \psi_1(X^*,Y^*) + Y^* + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1\}} - \frac{R_2}{L_1} \cdot \frac{\partial \psi_1}{\partial X}\Bigg|_{(X^*,Y^*)} - \lambda \cdot \frac{\partial \psi_1}{\partial X}\Bigg|_{(X^*,Y^*)} - \frac{V_t \cdot \left(A_3 \cdot \frac{\partial \psi_1}{\partial Y}\Big|_{(X^*,Y^*)} + 1\right)}{L_1 \cdot \{A_1 + A_2 \cdot X^* + A_3 \cdot \psi_1(X^*,Y^*) + Y^* + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}} \cdot \lambda + \frac{R_2}{L_1} \cdot \lambda + \lambda^2 - \frac{V_t \cdot \left(A_2 + A_3 \cdot \frac{\partial \psi_1}{\partial X}\Big|_{(X^*,Y^*)}\right) \cdot \frac{\partial \psi_1}{\partial Y}\Big|_{(X^*,Y^*)}}{L_1 \cdot \{A_1 + A_2 \cdot X^* + A_3 \cdot \psi_1(X^*,Y^*) + Y^* + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}}$$

$$A-\lambda\cdot I|_{(X^*,Y^*)}=\lambda^2+\lambda \cdot\left\{\frac{R_2}{L_1}-\left.\frac{\partial\psi_1}{\partial X}\right|_{(X^*,Y^*)}-\frac{V_t\cdot\left(A_3\cdot\left.\frac{\partial\psi_1}{\partial Y}\right|_{(X^*,Y^*)}+1\right)}{L_1\cdot\{A_1+A_2\cdot X^*+A_3\cdot\psi_1(X^*,Y^*)+Y^*+I_{se}\cdot(\alpha_f\cdot\alpha_r-1)\}}\right\}$$

$$+\frac{V_t\cdot\left(A_3\cdot\left.\frac{\partial\psi_1}{\partial Y}\right|_{(X^*,Y^*)}+1\right)\cdot\left.\frac{\partial\psi_1}{\partial X}\right|_{(X^*,Y^*)}}{L_1\cdot\{A_1+A_2\cdot X^*+A_3\cdot\psi_1(X^*,Y^*)+Y^*+I_{se}\cdot(\alpha_f\cdot\alpha_r-1)\}}-\frac{R_2}{L_1}\cdot\left.\frac{\partial\psi_1}{\partial X}\right|_{(X^*,Y^*)}$$

For simplicity we define two parameters: $\Xi_1,\Xi_2;A-\lambda\cdot I|_{(X^*,Y^*)}=\lambda^2+\lambda\cdot\Xi_1+\Xi_2$

$$\det|A-\lambda\cdot I|_{(X^*,Y^*)}=0\Rightarrow\lambda^2+\Xi_1\cdot\lambda+\Xi_2=0\Rightarrow\lambda_1;\lambda_2$$

$$\Xi_1=\frac{R_2}{L_1}-\left.\frac{\partial\psi_1}{\partial X}\right|_{(X^*,Y^*)}-\frac{V_t\cdot\left(A_3\cdot\left.\frac{\partial\psi_1}{\partial Y}\right|_{(X^*,Y^*)}+1\right)}{L_1\cdot\{A_1+A_2\cdot X^*+A_3\cdot\psi_1(X^*,Y^*)+Y^*+I_{se}\cdot(\alpha_f\cdot\alpha_r-1)\}}$$

$$\Xi_2=\frac{V_t\cdot\left(A_3\cdot\left.\frac{\partial\psi_1}{\partial Y}\right|_{(X^*,Y^*)}+1\right)\cdot\left.\frac{\partial\psi_1}{\partial X}\right|_{(X^*,Y^*)}}{L_1\cdot\{A_1+A_2\cdot X^*+A_3\cdot\psi_1(X^*,Y^*)+Y^*+I_{se}\cdot(\alpha_f\cdot\alpha_r-1)\}}-\frac{R_2}{L_1}\cdot\left.\frac{\partial\psi_1}{\partial X}\right|_{(X^*,Y^*)}$$

We need to discuss circuit's stability for V_{BB} = 15 V (two fixed points) (Table 5.13).

Remark If two eigenvalues real, unequal, opposite sign then fixed point is a saddle point and is unstable.

5.5 OptoNDR Circuit's Two Variables Analysis by Using Floquet Theory

We need to check in which conditions the OptoNDR circuit's two variables system has periodic orbits and it is done by changing system Cartesian coordinates ($X(t)$, $Y(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. We approve that there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers [46–48]. One of them is unity. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation z = 0), and the cylindrical axis is the Cartesian z axis. Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X, Y) are the same as for polar coordinates, namely

Table 5.13 Discussed circuit's stability for VBB = 15 v

Functional parameter	V_{BB} = 15 V First fixed point $X^{(0)} = 14.92$, $Y^{(0)} = 1.5 \times 10^{-6}$	V_{BB} = 15 V Second fixed point $X^{(1)} = -0.0123$, $Y^{(1)} = 277 \times 10^{-6}$
$\psi_1(X^*, Y^*) = \frac{B_5 \cdot [X^*]^2 + X^* \cdot B_2 + B_1 + B_3 \cdot Y^* + Y^* \cdot X^* \cdot B_4}{B_6 + B_7 \cdot X^*}$	206.31	2909.14
$\left.\frac{\partial \psi_1}{\partial X}\right\vert_{(X^*, Y^*)} = \frac{\{B_7 \cdot B_5 \cdot [X^*]^2 + 2 \cdot B_5 \cdot B_6 \cdot X^* + Y^* \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) + B_2 \cdot B_6 - B_1 \cdot B_7\}}{(B_6 + B_7 \cdot X^*)^2}$	−181.48	211056.92
$\left.\frac{\partial \psi_1}{\partial Y}\right\vert_{(X^*, Y^*)} = \frac{B_3 + X^* \cdot B_4}{B_6 + B_7 \cdot X^*}$	474.52×10^6	-9.45×10^6
$\Xi_1 = \frac{R_2}{L_1} - \left.\frac{\partial \psi_1}{\partial X}\right\vert_{(X^*, Y^*)} - \frac{V_t \cdot \left(A_3 \cdot \left.\frac{\partial \psi_1}{\partial Y}\right\vert_{(X^*, Y^*)} + 1\right)}{L_1 \cdot \{A_1 + A_2 \cdot X^* + A_3 \cdot \psi_1(X^*, Y^*) + Y^* + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}}$	$\approx -53.45 \times 10^7$	8822943.1
$\Xi_2 = \frac{V_t \cdot \left(A_3 \cdot \left.\frac{\partial \psi_1}{\partial Y}\right\vert_{(X^*, Y^*)} + 1\right) \cdot \left.\frac{\partial \psi_1}{\partial X}\right\vert_{(X^*, Y^*)}}{L_1 \cdot \{A_1 + A_2 \cdot X^* + A_3 \cdot \psi_1(X^*, Y^*) + Y^* + I_{se} \cdot (\alpha_f \cdot \alpha_r - 1)\}} - \frac{R_2}{L_1} \cdot \left.\frac{\partial \psi_1}{\partial X}\right\vert_{(X^*, Y^*)}$	-9711.64×10^7	-154036.92×10^7
Circuit characteristic equation's fixed points (X^*, Y^*) eigenvalues λ_1, λ_2	$\lambda_1 = +5.34 \times 10^8$ $\lambda_2 = -\varepsilon$ ($\varepsilon \to 0$) Saddle point	$\lambda_1 = -8.99 \times 10^6$ $\lambda_2 = +171280$ Saddle point

$X(t) = r(t) \cdot \cos[\theta(t)]$; $Y(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X^2 + Y^2}$. $\theta(t) = 0$ if $X = 0$ and $Y = 0$. $\theta(t) = \arcsin(Y/r)$ if $X >= 0$. $\theta(t) = -\arcsin(Y/r) + \pi$ if $X < 0$. We represent our system equation by using cylindrical coordinates $(r(t), \theta(t))$.

$$X(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t} \cdot \cos[\theta(t)] - r(t) \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \cdot \sin[\theta(t)]$$
$$Y(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t} \cdot \sin[\theta(t)] + r(t) \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \cdot \cos[\theta(t)]$$
$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t}; \frac{\mathrm{d}r(t)}{\mathrm{d}t} = r'; \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \theta'; \theta(t) = \theta; r(t) = r$$

We get the equations:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta; \frac{\mathrm{d}Y}{\mathrm{d}t} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$

$$V \to X; I_{L1} \to Y \Rightarrow \frac{\mathrm{d}X}{\mathrm{d}t} = \psi_1(X, Y); \frac{\mathrm{d}Y}{\mathrm{d}t} = \psi_2(X, Y); \psi_1 = \psi_1(X, Y); \psi_2 = \psi_2(X, Y)$$

$$\psi_1(X, Y) = \frac{B_5 \cdot X^2 + X \cdot B_2 + B_1 + B_3 \cdot Y + Y \cdot X \cdot B_4}{B_6 + B_7 \cdot X};$$
$$\psi_2(X, Y) = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X, Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - Y \cdot \frac{R_2}{L_1}$$

We need to find $\psi_1(X, Y)$ and $\psi_2(X, Y)$ in cylindrical coordinates $(r(t), \theta(t))$ for our system. The transformation $\psi_1(X, Y) \to \psi_1(r(t), \theta(t)); \psi_2(X, Y) \to \psi_2(r(t), \theta(t))$.

$$\psi_1(X, Y) = \left.\frac{B_5 \cdot X^2 + X \cdot B_2 + B_1 + B_3 \cdot Y + Y \cdot X \cdot B_4}{B_6 + B_7 \cdot X}\right|_{\substack{X = r \cdot \cos\theta \\ Y = r \cdot \sin\theta}}$$

$$\psi_2(X, Y) = \left.\frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X, Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - Y \cdot \frac{R_2}{L_1}\right|_{\substack{X = r \cdot \cos\theta \\ Y = r \cdot \sin\theta}}$$

We get two equations, two variables $r(t), \theta(t)$:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \psi_1(X, Y) \Rightarrow r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = \frac{B_5 \cdot r^2 \cdot \cos^2\theta + r \cdot \cos\theta \cdot B_2 + B_1 + B_3 \cdot r \cdot \sin\theta + r^2 \cdot \sin 2\theta \cdot B_4}{B_6 + B_7 \cdot r \cdot \cos\theta}$$

$$\frac{dY}{dt} = \psi_2(X,Y) \Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$
$$= \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta + A_3 \cdot \psi_1(r(t),\theta(t)) + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - r \cdot \sin\theta \cdot \frac{R_2}{L_1}$$

To find our system solution constant radius we set $\frac{dr}{dt} = 0$ which yield to:

$$r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = \psi_1(r,\theta)$$

$$r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$
$$= \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta + A_3 \cdot \{r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta\} + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right]$$
$$- r \cdot \sin\theta \cdot \frac{R_2}{L_1}$$

$$r' + r \cdot \theta' \cdot \frac{1}{tg\theta} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1}$$
$$\cdot \ln[\frac{A_1 + A_2 \cdot r \cdot \cos\theta + A_3 \cdot \{r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta\} + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1]$$
$$- r \cdot \frac{R_2}{L_1}$$

$$r' = 0 \Rightarrow r \cdot \theta' \cdot \frac{1}{tg\theta} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \theta' \cdot \sin\theta + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - r \cdot \frac{R_2}{L_1}$$

$$r \cdot \theta' \cdot \frac{1}{tg\theta} + r \cdot \frac{R_2}{L_1} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \theta' \cdot \sin\theta + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right]$$

We consider periodic $2 \bullet \pi$, $\theta = \frac{2 \cdot \pi}{T} \cdot t \Rightarrow \theta' = \frac{2 \cdot \pi}{T}; \theta' = 1 \Rightarrow T = 2 \cdot \pi$

$$r \cdot \frac{1}{tg\theta} + r \cdot \frac{R_2}{L_1} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \sin\theta + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right]$$

Back to the first equation:

$$\frac{dX}{dt} = \psi_1(X,Y) \Rightarrow r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta$$
$$= \frac{B_5 \cdot r^2 \cdot \cos^2\theta + r \cdot \cos\theta \cdot B_2 + B_1 + B_3 \cdot r \cdot \sin\theta + r^2 \cdot \sin 2\theta \cdot B_4}{B_6 + B_7 \cdot r \cdot \cos\theta}$$

$$r' - r \cdot \theta' \cdot tg\theta = \frac{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + B_3 \cdot r \cdot tg\theta + r^2 \cdot 2 \cdot \sin\theta \cdot B_4}{B_6 + B_7 \cdot r \cdot \cos\theta}$$

$$r' = 0; \theta' = 1 \Rightarrow -r \cdot tg\theta = \frac{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + B_3 \cdot r \cdot tg\theta + r^2 \cdot 2 \cdot \sin\theta \cdot B_4}{B_6 + B_7 \cdot r \cdot \cos\theta}$$

We can summarize our two equations in r and θ:

$$(^{*})\ \ -r \cdot tg\theta = \frac{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + B_3 \cdot r \cdot tg\theta + r^2 \cdot 2 \cdot \sin\theta \cdot B_4}{B_6 + B_7 \cdot r \cdot \cos\theta}$$

$$(^{**})\ r \cdot \frac{1}{tg\theta} + r \cdot \frac{R_2}{L_1} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \sin\theta + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right]$$

From above equations we need to get constant radius $r = r_{\text{const-radius}}$.

$$-r_{\text{const-radius}} \cdot tg\theta = \frac{\left\{B_5 \cdot r_{\text{const-radius}}^2 \cdot \cos\theta + r_{\text{const-radius}} \cdot B_2 + \frac{B_1}{\cos\theta} + B_3 \cdot r_{\text{const-radius}} \cdot tg\theta + r_{\text{const-radius}}^2 \cdot 2 \cdot \sin\theta \cdot B_4\right\}}{B_6 + B_7 \cdot r_{\text{const-radius}} \cdot \cos\theta}$$

$$r_{\text{const-radius}} \cdot \frac{1}{tg\theta} + r_{\text{const-radius}} \cdot \frac{R_2}{L_1} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{\{A_1 + A_2 \cdot r_{\text{const-radius}} \cdot \cos\theta - A_3 \cdot r_{\text{const-radius}} \cdot \sin\theta + r_{\text{const-radius}} \cdot \sin\theta\}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right]$$

$$(^{*}) \Rightarrow -r \cdot tg\theta = \frac{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + B_3 \cdot r \cdot tg\theta + r^2 \cdot 2 \cdot \sin\theta \cdot B_4}{B_6 + B_7 \cdot r \cdot \cos\theta}$$

$$\Rightarrow -r \cdot [B_6 + B_7 \cdot r \cdot \cos\theta] \cdot tg\theta - B_3 \cdot r \cdot tg\theta = B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + r^2 \cdot 2 \cdot \sin\theta \cdot B_4$$

$$\Rightarrow -r \cdot [B_6 + B_7 \cdot r \cdot \cos\theta + B_3] \cdot tg\theta = B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + r^2 \cdot 2 \cdot \sin\theta \cdot B_4$$

$$tg\theta = \frac{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + r^2 \cdot 2 \cdot \sin\theta \cdot B_4}{-r \cdot [B_6 + B_7 \cdot r \cdot \cos\theta + B_3]}$$

$$\Rightarrow \frac{1}{tg\theta} = \frac{-r \cdot [B_6 + B_7 \cdot r \cdot \cos\theta + B_3]}{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + r^2 \cdot 2 \cdot \sin\theta \cdot B_4}$$

$$(^{**}) \Rightarrow r \cdot \frac{1}{tg\theta} + r \cdot \frac{R_2}{L_1} = \frac{1}{\sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \sin\theta + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right]$$

$$\Rightarrow \frac{1}{tg\theta} = \frac{1}{r \cdot \sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \sin\theta + r \cdot \sin\theta}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - \frac{R_2}{L_1}$$

We get the one expression:

$$\frac{-r \cdot [B_6 + B_7 \cdot r \cdot \cos\theta + B_3]}{B_5 \cdot r^2 \cdot \cos\theta + r \cdot B_2 + \frac{B_1}{\cos\theta} + r^2 \cdot 2 \cdot \sin\theta \cdot B_4} = \frac{1}{r \cdot \sin\theta} \cdot \frac{V_t}{L_1} \cdot \ln\left[\frac{\{A_1 + A_2 \cdot r \cdot \cos\theta - A_3 \cdot r \cdot \sin\theta + r \cdot \sin\theta\}}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - \frac{R_2}{L_1}$$

Remark We need to do algebraic manipulation in the above expression and get expression without θ variable in time and find $r_{\text{const-radius}}$ as a function of circuit parameters (C_1, L_1, α_f, α_r, I_{se}, I_{sc}, etc.).

We need to find $\left(\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y}\right)\Big|_{r=r_{\text{const-radius}}}$ at a solution with constant radius (dr/dt = 0).

$$\rho_1 \cdot \rho_2 = \left. e^{\int_0^T tr(A(s)) \cdot ds} \right|_{\rho_1 = 1} \Rightarrow \rho_2 = e^{\int_0^T tr(A(s)) \cdot ds} = e^{\int_0^{T=2\cdot\pi} \left(\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y}\right)\big|_{r=r_{\text{const}}} \cdot ds}$$

As a result, the limit cycle with radius $r_{\text{const-radius}} > 0$. Is stable if $\int_0^T tr(A(s)) \cdot ds < 0$ or $\rho_2 < 1$ and unstable if $\int_0^T tr(A(s)) \cdot ds > 0$ or $\rho_2 > 1$. We need to find the expressions for $\frac{\partial \psi_1(X,Y)}{\partial X}, \frac{\partial \psi_2(X,Y)}{\partial Y}$

$$\psi_1(X, Y) = \frac{B_5 \cdot X^2 + X \cdot B_2 + B_1 + B_3 \cdot Y + Y \cdot X \cdot B_4}{B_6 + B_7 \cdot X}$$

$$\frac{\partial \psi_1(X, Y)}{\partial X} = \frac{B_5 \cdot B_7 \cdot X^2 + Y \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) + 2 \cdot B_5 \cdot B_6 \cdot X + (B_2 \cdot B_6 - B_1 \cdot B_7)}{(B_6 + B_7 \cdot X)^2}$$

$$\psi_2(X, Y) = \frac{V_t}{L_1} \cdot \ln\left[\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X, Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right] - Y \cdot \frac{R_2}{L_1}$$

$$\frac{\partial \psi_2(X, Y)}{\partial Y} = \frac{V_t}{L_1} \cdot \frac{1}{\left(\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X,Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right)} \cdot \frac{A_3 \cdot \frac{\partial \psi_1(X,Y)}{\partial Y} + 1}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} - \frac{R_2}{L_1}$$

$$\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y} = \frac{B_5 \cdot B_7 \cdot X^2 + Y \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) + 2 \cdot B_5 \cdot B_6 \cdot X + (B_2 \cdot B_6 - B_1 \cdot B_7)}{(B_6 + B_7 \cdot X)^2} + \frac{V_t}{L_1} \cdot \frac{1}{\left\{\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X,Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right\}} \cdot \frac{A_3 \cdot \frac{\partial \psi_1(X,Y)}{\partial Y} + 1}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} - \frac{R_2}{L_1}$$

$$\left(\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y}\right)\bigg|_{r = r_{\text{const-radius}}} = \frac{\begin{array}{c}\{B_5 \cdot B_7 \cdot X^2 + Y \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) \\ + 2 \cdot B_5 \cdot B_6 \cdot X + (B_2 \cdot B_6 - B_1 \cdot B_7)\}\end{array}}{(B_6 + B_7 \cdot X)^2}\Bigg|_{r = r_{\text{const-radius}}} + \frac{V_t}{L_1} \cdot \frac{1}{\left\{\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X,Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right\}\Big|_{r = r_{\text{const-radius}}}} \cdot \frac{\left[A_3 \cdot \frac{\partial \psi_1(X,Y)}{\partial Y}\Big|_{r = r_{\text{const-radius}}} + 1\right]}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} - \frac{R_2}{L_1}$$

$$\rho_1 \cdot \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot ds}\Bigg|_{\rho_1 = 1} \Rightarrow \rho_2 = \mathrm{e}^{\int_0^T tr(A(s)) \cdot ds} = \mathrm{e}^{\int_0^{T = 2 \cdot \pi} \left(\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y}\right)\big|_{r = r_{\text{const}}} \cdot ds}$$

$$tr(A(s)) = \left(\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y}\right)\bigg|_{r = r_{\text{const}}} = \frac{\begin{array}{c}\{B_5 \cdot B_7 \cdot X^2 + Y \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) \\ + 2 \cdot B_5 \cdot B_6 \cdot X + (B_2 \cdot B_6 - B_1 \cdot B_7)\}\end{array}}{(B_6 + B_7 \cdot X)^2}\Bigg|_{r = r_{\text{const-radius}}} + \frac{V_t}{L_1} \cdot \frac{1}{\left\{\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X,Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right\}\Big|_{r = r_{\text{const-radius}}}} \cdot \frac{\left[A_3 \cdot \frac{\partial \psi_1(X,Y)}{\partial Y}\Big|_{r = r_{\text{const-radius}}} + 1\right]}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} - \frac{R_2}{L_1}$$

$$\rho_2 = e^{\int_0^T tr(A(s)) \cdot ds} = e^{\int_0^{T = 2 \cdot \pi} \left(\frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_2}{\partial Y}\right)\big|_{r = r_{\text{const}}} \cdot ds}$$
$$= \exp\left\{\int_0^{T = 2 \cdot \pi} \left[\frac{\begin{array}{c}\{B_5 \cdot B_7 \cdot X^2 + Y \cdot (B_4 \cdot B_6 - B_3 \cdot B_7) \\ + 2 \cdot B_5 \cdot B_6 \cdot X + (B_2 \cdot B_6 - B_1 \cdot B_7)\}\end{array}}{(B_6 + B_7 \cdot X)^2}\Bigg|_{r = r_{\text{const-radius}}} + \frac{V_t}{L_1} \cdot \frac{1}{\left\{\frac{A_1 + A_2 \cdot X + A_3 \cdot \psi_1(X,Y) + Y}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} + 1\right\}\Big|_{r = r_{\text{const-radius}}}} \cdot \frac{\left[A_3 \cdot \frac{\partial \psi_1(X,Y)}{\partial Y}\Big|_{r = r_{\text{const-radius}}} + 1\right]}{I_{se} \cdot (\alpha_r \cdot \alpha_f - 1)} - \frac{R_2}{L_1}\right] \cdot ds\right\}$$

5.6 OptoNDR Circuit's Second-Order ODE with Periodic Source Stability of a Limit Cycle

We can use Floquet theory to test the stability of optoisolation circuits limit cycle solution. It helps us to understand how dynamics depend on circuit parameter values. We have OptoNDR circuit which characterized by set of differential equations:$\frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n), \ldots, \frac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n)$. We study the stability of the limit cycle solution, first we have to locate it [5–7]. One point on the limit cycles, as well as its period T, such that $x_1(t+T) = x_1(t), \ldots, x_n(t+T) = x_n(t)$. Once we know the limit cycle and test its stability by checking if small perturbations away from the limit cycle grow or shrink over a complete period. It is done by finding the stability of the periodic system, obtained by linearizing the circuit model around the limit cycle. We define the Jacobian matrix of our system differential equations:

$$A(t) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdot s & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{(x_1(t),\ldots,x_n(t))\text{limitcycle}}$$

The Floquet multipliers of $A(t)$ determine the stability of the limit cycle. Floquet multiplier of $A(t)$ as some circuit parameters in increase or decrease, the dominant Floquet multiplier passes through −1, signifying a period doubling bifurcation of the limit cycle. We present our OptoNDR circuit second-order ODE with periodic source. We have variable capacitor ($C1$) and variable Inductance ($L1$) in series at the output circuit. The feedback output voltage is multiplied by input periodic source $a(t)$, with period T and feed to input LED $D1$ circuit. $V_{cc} \gg V_{Q1\ \text{break}}$, at $t = 0$ switch $S1$ move from OFF state to ON state [85, 86] (Fig. 5.15).

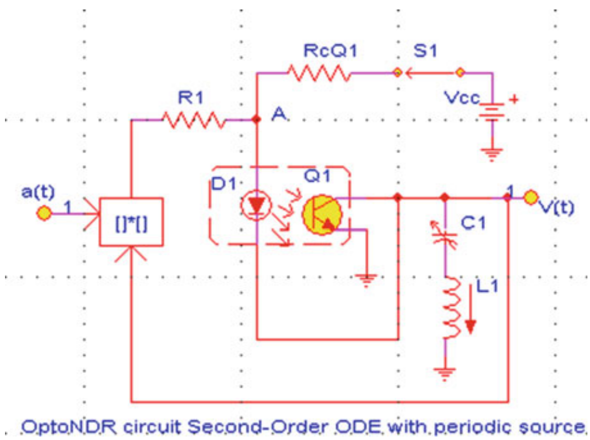


Fig. 5.15 OptoNDR circuit second-order ODE with periodic source

$$I_{C1} = I_{L1}; I_{C1} = C_1 \cdot \frac{\mathrm{d}V_{C1}}{\mathrm{d}t}; V_{L1} = L_1 \cdot \frac{\mathrm{d}I_{L1}}{\mathrm{d}t}; V(t) = V_{C1} + V_{L1} = L_1 \cdot \frac{\mathrm{d}I_{C1}}{\mathrm{d}t} + \frac{1}{C_1} \cdot \int I_{C1} \cdot \mathrm{d}t$$

$$V_{CEQ1} = V(t) = V; I_{RCQ1} = \frac{V_{cc} - V_A}{R_{CQ1}}; I_{D1} = I_{R1} + I_{RCQ1}; I_{D1} = I_{CQ1} + I_{C1} \Rightarrow I_{CQ1} = I_{D1} - I_{C1}$$

The multiplication element ([]*[]) is implemented by using op-amps, resistors, capacitors, diodes, etc. Multiplication element's input current is zero since his input impedance is infinite $I_{in\{[]*[]\}} \to \varepsilon; R_{in\{[]*[]\}} \to \infty$.

$$V(t) \cdot a(t) = V_{R1} + V_{D1} + V(t) = I_{R1} \cdot R_1 + V_{D1} + V(t);$$
$$I_{D1} = I_{R1} + I_{RCQ1} \Rightarrow I_{R1} = I_{D1} - I_{RCQ1}$$
$$V(t) \cdot a(t) = I_{R1} \cdot R_1 + V_{D1} + V(t);$$
$$V_{D1} = V_t \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\} \Rightarrow V(t) \cdot a(t) = I_{R1} \cdot R_1 + V_t \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\} + V(t)$$

$$V(t) \cdot a(t) = (I_{D1} - I_{RCQ1}) \cdot R_1 + V_t \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\} + V(t) \Rightarrow V(t) \cdot [a(t) - 1] = (I_{D1} - I_{RCQ1}) \cdot R_1 + V_t \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\}$$

$$V(t) \cdot [a(t) - 1] = (I_{D1} - I_{RCQ1}) \cdot R_1 + V_t \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\} \Rightarrow V(t) = \frac{(I_{D1} - I_{RCQ1})}{[a(t) - 1]} \cdot R_1 + \frac{V_t}{[a(t) - 1]} \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\}$$

By using Taylor series approximation $\ln\left\{\frac{I_{D1}}{I_0} + 1\right\} \approx \frac{I_{D1}}{I_0}$

$$V(t) = \frac{(I_{D1} - I_{RCQ1})}{[a(t) - 1]} \cdot R_1 + \frac{V_t}{[a(t) - 1]} \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\} \Rightarrow V(t) = \frac{(I_{D1} - I_{RCQ1})}{[a(t) - 1]} \cdot R_1 + \frac{V_t}{[a(t) - 1]} \cdot \frac{I_{D1}}{I_0}$$

$$V(t) = \frac{(I_{D1} - I_{RCQ1})}{[a(t) - 1]} \cdot R_1 + \frac{V_t}{[a(t) - 1]} \cdot \frac{I_{D1}}{I_0} \Rightarrow V(t) = \frac{I_{D1}}{[a(t) - 1]} \cdot \left(R_1 + \frac{V_t}{I_0}\right) - \frac{I_{RCQ1} \cdot R_1}{[a(t) - 1]}$$

$$V = V(t); V = \frac{I_{D1}}{[a(t)-1]} \cdot \left(R_1 + \frac{V_t}{I_0}\right) - \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}$$
$$\Rightarrow I_{D1} = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right)$$
$$I_{BQ1} = k_1 \cdot I_{D1} = \frac{k_1 \cdot [a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right);$$
$$I_{EQ1} = I_{BQ1} + I_{CQ1} = \frac{k_1 \cdot [a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) + I_{CQ1}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1} = \frac{k_1 \cdot [a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) + I_{D1} - I_{C1}$$
$$= \frac{(k_1+1) \cdot [a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) - I_{C1}$$
$$V(t) = V = V_{CEQ1} = V_t \cdot \ln\left\{\frac{(\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}$$
$$+ V_t \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right); \frac{I_{sc}}{I_{se}} \to 1 \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$\ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon \Rightarrow V \simeq V_t \cdot \ln\left\{\frac{(\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\};$$
$$I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt}; V_{L1} = L_1 \cdot \frac{dI_{L1}}{dt}$$

$$I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt} \Rightarrow V_{C1} = \frac{1}{C_1} \cdot \int I_{C1};$$
$$V_{L1} = L_1 \cdot \frac{dI_{L1}}{dt} \Rightarrow I_{L1} = \frac{1}{L_1} \cdot \int V_{L1} \cdot dt; V = V_{C1} + V_{L1} = \frac{1}{C_1} \cdot \int I_{C1} + L_1 \cdot \frac{dI_{L1}}{dt}$$
$$I_{C1} = I_{L1} \Rightarrow V = V_{C1} + V_{L1} = \frac{1}{C_1} \cdot \int I_{C1} + L_1 \cdot \frac{dI_{C1}}{dt}; \frac{dV}{dt} = \frac{1}{C_1} \cdot I_{C1} + L_1 \cdot \frac{d^2 I_{C1}}{dt^2}$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \alpha_{r1} \cdot I_{CQ1} - \left[\frac{(k_1+1) \cdot [a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) - I_{C1}\right]$$

$$I_{CQ1} = I_{D1} - I_{C1} = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) - I_{C1}$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \alpha_{r1} \cdot \left[\frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) - I_{C1}\right] - \left[\frac{(k_1+1) \cdot [a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) - I_{C1}\right]$$

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) \cdot (\alpha_{r1} - k_1 - 1) + I_{C1} \cdot (1 - \alpha_{r1})$$

$$I_{RCQ1} = \frac{V_{cc} - V_A}{R_{CQ1}} = \frac{V_{cc}}{R_{CQ1}} - \frac{V_A}{R_{CQ1}}; V_A = V_{D1} + V = \left(V_t \cdot \ln\left\{\frac{I_{D1}}{I_0} + 1\right\} + V\right)\Bigg|_{\ln\left\{\frac{I_{D1}}{I_0}+1\right\} \approx \frac{I_{D1}}{I_0}} = V_t \cdot \frac{I_{D1}}{I_0} + V$$

$$V_A = V_t \cdot \frac{I_{D1}}{I_0} + V = \frac{V_t}{I_0} \cdot \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) + V$$

$$I_{RCQ1} = \frac{V_{cc} - \left\{\frac{V_t}{I_0} \cdot \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right) + V\right\}}{R_{CQ1}} = \frac{(V_{cc} - V)}{R_{CQ1}} - \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}\right)$$

$$I_{RCQ1} = \frac{(V_{cc} - V)}{R_{CQ1}} - \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{[a(t)-1] \cdot V}{\left(R_1 + \frac{V_t}{I_0}\right)} - \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}$$

$$I_{RCQ1} + \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]} = \frac{(V_{cc} - V)}{R_{CQ1}} - \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{[a(t)-1] \cdot V}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$I_{RCQ1}\cdot\left[1+\frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{R_1}{\left(R_1+\frac{V_t}{I_0}\right)}\right]=\frac{(V_{cc}-V)}{R_{CQ1}}-\frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{[a(t)-1]\cdot V}{\left(R_1+\frac{V_t}{I_0}\right)}$$

$$I_{RCQ1}\cdot\left[1+\frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{R_1}{\left(R_1+\frac{V_t}{I_0}\right)}\right]=\frac{1}{R_{CQ1}}\cdot\left\{V_{cc}-V\cdot\left(1+\frac{V_t}{I_0}\cdot\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\right)\right\}$$

$$I_{RCQ1}=\frac{1}{R_{CQ1}\cdot\left[1+\frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{R_1}{\left(R_1+\frac{V_t}{I_0}\right)}\right]}\cdot\left\{V_{cc}-V\cdot\left(1+\frac{V_t}{I_0}\cdot\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\right)\right\}$$

We define:

$$I_{RCQ1}=\xi_1(V,a(t))=\frac{1}{R_{CQ1}\cdot\left[1+\frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{R_1}{(R_1+\frac{V_t}{I_0})}\right]}\cdot\left\{V_{cc}-V\cdot\left(1+\frac{V_t}{I_0}\cdot\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\right)\right\}$$

$$\alpha_{r1}\cdot I_{CQ1}-I_{EQ1}=\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left(V+\frac{\xi_1(V,a(t))\cdot R_1}{[a(t)-1]}\right)\cdot(\alpha_{r1}-k_1-1)+I_{C1}\cdot(1-\alpha_{r1})$$

We define:

$$\eta_1(V,a(t))=\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left(V+\frac{\xi_1(V,a(t))\cdot R_1}{[a(t)-1]}\right)\cdot(\alpha_{r1}-k_1-1);$$

$$\alpha_{r1}\cdot I_{CQ1}-I_{EQ1}=\eta_1(V,a(t))+I_{C1}\cdot(1-\alpha_{r1})$$

$$I_{CQ1}-I_{EQ1}\cdot\alpha_{f1}=[I_{D1}-I_{C1}]-I_{EQ1}\cdot\alpha_{f1}=\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left(V+\frac{I_{RCQ1}\cdot R_1}{[a(t)-1]}\right)-I_{C1}-\left[\frac{(k_1+1)\cdot[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot(V+\frac{I_{RCQ1}\cdot R_1}{[a(t)-1]})-I_{C1}\right]\cdot\alpha_{f1}$$

$$I_{CQ1}-I_{EQ1}\cdot\alpha_{f1}=\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left(V+\frac{I_{RCQ1}\cdot R_1}{[a(t)-1]}\right)-\frac{(k_1+1)\cdot[a(t)-1]\cdot\alpha_{f1}}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left(V+\frac{I_{RCQ1}\cdot R_1}{[a(t)-1]}\right)+I_{C1}\cdot(\alpha_{f1}-1)$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{[a(t)-1]}{(R_1 + \frac{V_t}{I_0})} \cdot (V + \frac{I_{RCQ1} \cdot R_1}{[a(t)-1]}) \cdot [1 - (k_1+1) \cdot \alpha_{f1}] + I_{C1} \cdot (\alpha_{f1} - 1); I_{RCQ1} = \xi_1(V, a(t))$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t)-1]}\right) \cdot [1 - (k_1+1) \cdot \alpha_{f1}] + I_{C1} \cdot (\alpha_{f1} - 1); I_{RCQ1} = \xi_1(V, a(t))$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t)-1]}\right) \cdot [1 - (k_1+1) \cdot \alpha_{f1}] - I_{C1} \cdot (1 - \alpha_{f1}); I_{RCQ1} = \xi_1(V, a(t))$$

We define: $\eta_2(V, a(t)) = \frac{[a(t)-1]}{(R_1 + \frac{V_t}{I_0})} \cdot (V + \frac{\xi_1(V,a(t)) \cdot R_1}{[a(t)-1]}) \cdot [1 - (k_1+1) \cdot \alpha_{f1}]$

$$I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \eta_2(V, a(t)) - I_{C1} \cdot (1 - \alpha_{f1}); \eta_1 = \eta_1(V, a(t)); \eta_2 = \eta_2(V, a(t))$$

We get two expressions:

$$\alpha_{r1} \cdot I_{CQ1} - I_{EQ1} = \eta_1 + I_{C1} \cdot (1 - \alpha_{r1}); I_{CQ1} - I_{EQ1} \cdot \alpha_{f1} = \eta_2 - I_{C1} \cdot (1 - \alpha_{f1})$$

$$V \simeq V_t \cdot \ln\left\{\frac{(\alpha_{r1} \cdot I_{CQ1} - I_{EQ1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\} \Rightarrow V \simeq V_t \cdot \ln\left\{\frac{\eta_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_2 - I_{C1} \cdot (1 - \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}\right\}$$

$$e^{\frac{V}{V_t}} \simeq \frac{\eta_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\eta_2 - I_{C1} \cdot (1 - \alpha_{f1}) + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)} \Rightarrow \eta_2 \cdot e^{\frac{V}{V_t}} - I_{C1} \cdot (1 - \alpha_{f1}) \cdot e^{\frac{V}{V_t}} + I_{sc} \cdot e^{\frac{V}{V_t}} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) = \eta_1 + I_{C1} \cdot (1 - \alpha_{r1}) + I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)$$

$$I_{C1} = \frac{\eta_2 \cdot e^{\frac{V}{V_t}} - \eta_1 + I_{sc} \cdot e^{\frac{V}{V_t}} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) - I_{se} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{[(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot e^{\frac{V}{V_t}}]}$$

$$I_{C1} = \frac{\eta_2 \cdot e^{\frac{V}{V_t}} - \eta_1 + [I_{sc} \cdot e^{\frac{V}{V_t}} - I_{se}] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{[(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot e^{\frac{V}{V_t}}]}$$

Taylor series approximation: $e^{\left[\frac{V}{V_t}\right]} \approx \frac{V}{V_t} + 1; \Delta = \eta_2 - \eta_1$

$$I_{C1} = \frac{\eta_2 \cdot \left(\frac{V}{V_t} + 1\right) - \eta_1 + \left[I_{sc} \cdot \left(\frac{V}{V_t} + 1\right) - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot \left(\frac{V}{V_t} + 1\right)\right]}$$
$$= \frac{\eta_2 \cdot \frac{V}{V_t} + \eta_2 - \eta_1 + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot \left(\frac{V}{V_t} + 1\right)\right]}$$

$$I_{C1} = \frac{\eta_2 \cdot \left(\frac{V}{V_t} + 1\right) - \eta_1 + \left[I_{sc} \cdot \left(\frac{V}{V_t} + 1\right) - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[(1 - \alpha_{r1}) + (1 - \alpha_{f1}) \cdot \left(\frac{V}{V_t} + 1\right)\right]}$$
$$= \frac{\eta_2 \cdot \frac{V}{V_t} + \Delta + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]}$$

$$\Delta = \eta_2 - \eta_1 = \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}\right) \cdot [1 - (k_1 + 1) \cdot \alpha_{f1}]$$
$$- \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}\right) \cdot (\alpha_{r1} - k_1 - 1)$$

$$\Delta = \eta_2 - \eta_1 = \frac{[a(t) - 1]}{(R_1 + \frac{V_t}{I_0})} \cdot (V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}) \cdot \{[1 - (k_1 + 1) \cdot \alpha_{f1}] - (\alpha_{r1} - k_1 - 1)\}$$

$$[1 - (k_1 + 1) \cdot \alpha_{f1}] - (\alpha_{r1} - k_1 - 1) = 2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}$$

$$\Delta = \eta_2 - \eta_1 = \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}\right) \cdot [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]$$

$$I_{C1} = \frac{\eta_2 \cdot \frac{V}{V_t} + \Delta + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]}$$

$$\frac{\mathrm{d}I_{C1}}{\mathrm{d}t} = \frac{\begin{array}{l} \left[\frac{\mathrm{d}\eta_2}{\mathrm{d}t} \cdot \frac{V}{V_t} + \frac{\eta_2}{V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{I_{sc}}{V_t} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)\right] \cdot \left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right] \\ -\left[\eta_2 \cdot \frac{V}{V_t} + \Delta + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)\right] \cdot \frac{(1 - \alpha_{f1})}{V_t} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \end{array}}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]^2}$$

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \left[\frac{\mathrm{d}a(t)/\mathrm{d}t}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}\right) + \frac{[a(t) - 1]}{(R_1 + \frac{V_t}{I_0})} \cdot \left(\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\frac{\mathrm{d}\xi_1(V,a(t))}{\mathrm{d}t} \cdot R_1 \cdot [a(t) - 1] - \frac{\mathrm{d}a(t)}{\mathrm{d}t} \cdot \xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]^2}\right)\right] \cdot [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]$$

$$I_{RCQ1} = \xi_1(V, a(t)) = \frac{1}{R_{CQ1} \cdot \left[1 + \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{R_1}{\left(R_1 + \frac{V_t}{I_0}\right)}\right]} \cdot \left\{V_{cc} - V \cdot \left(1 + \frac{V_t}{I_0} \cdot \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)}\right)\right\}$$

$$\xi_1 = \xi_1(V, a(t)); \frac{\mathrm{d}\xi_1}{\mathrm{d}t} = -\frac{1}{R_{CQ1} \cdot \left[1 + \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{R_1}{\left(R_1 + \frac{V_t}{I_0}\right)}\right]} \cdot \left\{\frac{\mathrm{d}V}{\mathrm{d}t} \cdot \left(1 + \frac{V_t}{I_0} \cdot \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)}\right) + V \cdot \frac{V_t}{I_0} \cdot \frac{\mathrm{d}a(t)/\mathrm{d}t}{\left(R_1 + \frac{V_t}{I_0}\right)})\right\}$$

$$\eta_2(V, a(t)) = \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}\right) \cdot [1 - (k_1 + 1) \cdot \alpha_{f1}]$$

$$\eta_2 = \eta_2(V, a(t)); \frac{\mathrm{d}\eta_2}{\mathrm{d}t} = \left\{\frac{\mathrm{d}a(t)/\mathrm{d}t}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1 \cdot R_1}{[a(t) - 1]}\right) + \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\frac{\mathrm{d}\xi_1}{\mathrm{d}t} \cdot R_1 \cdot [a(t) - 1] - \xi_1 \cdot R_1 \cdot \frac{\mathrm{d}a(t)}{\mathrm{d}t}}{[a(t) - 1]^2}\right)\right\} \cdot [1 - (k_1 + 1) \cdot \alpha_{f1}]$$

We define function $f\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, t\right) : \frac{\mathrm{d}I_{C1}}{\mathrm{d}t} = f\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, t\right)$

$$f\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, t\right) = \frac{\left[\frac{\mathrm{d}\eta_2}{\mathrm{d}t} \cdot \frac{V}{V_t} + \frac{\eta_2}{V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}\Delta}{\mathrm{d}t} + \frac{I_{sc}}{V_t} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)\right] \cdot \left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right] - \left[\eta_2 \cdot \frac{V}{V_t} + \Delta + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)\right] \cdot \frac{(1-\alpha_{f1})}{V_t} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]^2}$$

Summary We can represent our system by two main differential equations: $\frac{\mathrm{d}V}{\mathrm{d}t} = L_1 \cdot \frac{\mathrm{d}^2 I_{C1}}{\mathrm{d}t^2} + \frac{1}{C_1} \cdot I_{C1}; \frac{\mathrm{d}I_{C1}}{\mathrm{d}t} = f\left(V, \frac{\mathrm{d}V}{\mathrm{d}t}, t\right)$. We define new system variables in time $X_1 = X_1(t)$; $X_2 = X_2(t)$.

$$X_2 = I_{C1}; X_1 = \frac{\mathrm{d}X_2}{\mathrm{d}t} = \frac{\mathrm{d}I_{C1}}{\mathrm{d}t}; \frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{\mathrm{d}^2 X_2}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 I_{C1}}{\mathrm{d}t^2} \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = L_1 \cdot \frac{\mathrm{d}X_1}{\mathrm{d}t} + \frac{1}{C_1} \cdot X_2$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = L_1 \cdot \frac{\mathrm{d}X_1}{\mathrm{d}t} + \frac{1}{C_1} \cdot X_2 \Rightarrow \frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1} \cdot \left\{\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{1}{C_1} \cdot X_2\right\}.$$

We get the following system differential equations:

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1} \cdot \left\{\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{1}{C_1} \cdot X_2\right\}; \frac{\mathrm{d}X_2}{\mathrm{d}t} = X_1;$$

$$X_2 = I_{C1} = \frac{\eta_2 \cdot \frac{V}{V_t} + \Delta + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]}$$

$$X_2 = I_{C1} = \frac{\left\{\frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1 \cdot R_1}{[a(t)-1]}\right) \cdot [1 - (k_1 + 1) \cdot \alpha_{f1}]\right\} \cdot \frac{V}{V_t} + \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1 \cdot R_1}{[a(t)-1]}\right) \cdot [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}] + \left[I_{sc} \cdot \frac{V}{V_t} + I_{sc} - I_{se}\right] \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]}$$

$$\begin{aligned} \Xi(V, a(t)) = {} & \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1 \cdot R_1}{[a(t) - 1]}\right) \\ & \cdot \left\{[1 - (k_1 + 1) \cdot \alpha_{f1}] \cdot \frac{V}{V_t} + [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]\right\} \\ & + I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{V}{V_t} + (I_{sc} - I_{se}) \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \end{aligned}$$

$$X_2 = I_{C1} = \frac{\Xi(V, a(t))}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t}\right]}$$

For simplicity we define the following functions and global parameters:

$$\psi_1(a(t)) = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)}; B_1 = [1-(k_1+1)\cdot\alpha_{f1}]\cdot\frac{1}{V_t};$$

$$B_2 = [2 + k_1\cdot(1-\alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]$$

$$B_3 = I_{sc}\cdot(\alpha_{r1}\cdot\alpha_{f1}-1)\cdot\frac{1}{V_t}; B_4 = (I_{sc}-I_{se})\cdot(\alpha_{r1}\cdot\alpha_{f1}-1); \psi_1 = \psi_1(a(t))$$

$$\Xi(V,a(t)) = \psi_1(t)\cdot\left(V + \frac{\xi_1\cdot R_1}{[a(t)-1]}\right)\cdot(B_1\cdot V + B_2) + B_3\cdot V + B_4; \psi_1(t) = \psi_1(a(t))$$

$$\xi_1(V,a(t)) = \frac{1}{R_{CQ1}\cdot\left[1 + \frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{R_1}{\left(R_1+\frac{V_t}{I_0}\right)}\right]}\cdot\left\{V_{cc} - V\cdot\left(1 + \frac{V_t}{I_0}\cdot\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\right)\right\}$$

$$B_5 = \frac{1}{R_{CQ1}\cdot\left[1 + \frac{1}{R_{CQ1}}\cdot\frac{V_t}{I_0}\cdot\frac{R_1}{(R_1+\frac{V_t}{I_0})}\right]}; \xi_1(V,a(t)) = B_5\cdot\left\{V_{cc} - V\cdot\left(1+\frac{V_t}{I_0}\cdot\psi_1(a(t))\right)\right\}$$

$$\xi_1 = \xi_1(V,a(t)); \psi_1(t) = \psi_1(a(t)); \xi_1 = B_5\cdot\left\{V_{cc} - V\cdot\left[1+\frac{V_t}{I_0}\cdot\psi_1(t)\right]\right\}$$

$$\Xi(V,a(t)) = \psi_1(t)\cdot\left(V + \frac{B_5\cdot\left\{V_{cc} - V\cdot\left[1+\frac{V_t}{I_0}\cdot\psi_1(t)\right]\right\}\cdot R_1}{[a(t)-1]}\right)\cdot(B_1\cdot V+B_2) + B_3\cdot V + B_4$$

$$\begin{aligned}\Xi(V,a(t)) &= \psi_1(t)\\ &\quad\cdot\left(V + \frac{\left\{V_{cc}\cdot B_5\cdot R_1 - V\cdot B_5\cdot R_1 - V\cdot\frac{V_t}{I_0}\cdot\psi_1(t)\cdot B_5\cdot R_1\right\}}{[a(t)-1]}\right)\\ &\quad\cdot(B_1\cdot V+B_2) + B_3\cdot V + B_4\end{aligned}$$

$$\begin{aligned}\Xi(V,a(t)) &= \psi_1(t)\\ &\quad\cdot\left\{B_1\cdot V^2 + V\cdot B_2 + \frac{\left\{V_{cc}\cdot B_5\cdot R_1 - V\cdot B_5\cdot R_1 - V\cdot\frac{V_t}{I_0}\cdot\psi_1(t)\cdot B_5\cdot R_1\right\}\cdot(B_1\cdot V+B_2)}{[a(t)-1]}\right\}\\ &\quad+ B_3\cdot V + B_4\end{aligned}$$

$$\Xi(V,a(t)) = \psi_1(t) \cdot \left\{ B_1 \cdot V^2 + V \cdot B_2 + \frac{\left[V_{cc} \cdot B_5 \cdot R_1 \cdot B_1 \cdot V - V^2 \cdot B_5 \cdot R_1 \cdot B_1 - V^2 \cdot \frac{V_t}{I_0} \cdot \psi_1(t) \cdot B_5 \cdot R_1 \cdot B_1 + V_{cc} \cdot B_5 \cdot R_1 \cdot B_2 - V \cdot B_5 \cdot R_1 \cdot B_2 - V \cdot \frac{V_t}{I_0} \cdot \psi_1(t) \cdot B_5 \cdot R_1 \cdot B_2 \right]}{[a(t) - 1]} \right\} + B_3 \cdot V + B_4$$

$$\Xi(V,a(t)) = \psi_1(t) \cdot \left\{ B_1 \cdot V^2 + V \cdot B_2 + \frac{\left[B_5 \cdot R_1 \cdot (V_{cc} \cdot B_1 - B_2) \cdot V - V^2 \cdot B_5 \cdot R_1 \cdot B_1 - V^2 \cdot \psi_1(t) \cdot \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_1 + V_{cc} \cdot B_5 \cdot R_1 \cdot B_2 - V \cdot \psi_1(t) \cdot \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_2 \right]}{[a(t) - 1]} \right\} + B_3 \cdot V + B_4$$

For simplicity we define the following parameters:

$$A_1 = B_5 \cdot R_1 \cdot (V_{cc} \cdot B_1 - B_2); A_2 = B_5 \cdot R_1 \cdot B_1; A_3 = \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_1;$$

$$A_3 = \frac{V_t}{I_0} \cdot A_2; A_4 = V_{cc} \cdot B_5 \cdot R_1 \cdot B_2$$

$$A_5 = \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_2; A_3 = \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_1 \Rightarrow \frac{V_t}{I_0} \cdot B_5 \cdot R_1 = \frac{A_3}{B_1} \Rightarrow A_5 = \frac{A_3}{B_1} \cdot B_2$$

$$\Xi(V,a(t)) = \psi_1(t) \cdot \left\{ B_1 \cdot V^2 + V \cdot B_2 + \frac{[A_1 \cdot V - V^2 \cdot A_2 - V^2 \cdot \psi_1(t) \cdot A_3 + A_4 - V \cdot \psi_1(t) \cdot A_5]}{[a(t) - 1]} \right\} + B_3 \cdot V + B_4$$

$$X_2 = \frac{\Xi(V,a(t))}{\left[2 - \alpha_{r1} - \alpha_{f1} + (1 - \alpha_{f1}) \cdot \frac{V}{V_t} \right]}; A_6 = 2 - \alpha_{r1} - \alpha_{f1}; A_7 = (1 - \alpha_{f1}) \cdot \frac{1}{V_t}$$

$$X_2 = \frac{\psi_1(t) \cdot \left\{ B_1 \cdot V^2 + V \cdot B_2 + \frac{[A_1 \cdot V - V^2 \cdot A_2 - V^2 \cdot \psi_1(t) \cdot A_3 + A_4 - V \cdot \psi_1(t) \cdot A_5]}{[a(t) - 1]} \right\} + B_3 \cdot V + B_4}{[A_6 + A_7 \cdot V]}$$

$$X_2 \cdot [A_6 + A_7 \cdot V] = \psi_1(t) \cdot \left\{B_1 \cdot V^2 + V \cdot B_2 + \frac{[A_1 \cdot V - V^2 \cdot A_2 - V^2 \cdot \psi_1(t) \cdot A_3 + A_4 - V \cdot \psi_1(t) \cdot A_5]}{[a(t) - 1]}\right\} + B_3 \cdot V + B_4$$

$$X_2 \cdot [A_6 + A_7 \cdot V] \cdot [a(t) - 1] = \psi_1(t) \cdot \{B_1 \cdot [a(t) - 1] \cdot V^2 + V \cdot B_2 \cdot [a(t) - 1] + [A_1 \cdot V - V^2 \cdot A_2 - V^2 \cdot \psi_1(t) \cdot A_3 + A_4 - V \cdot \psi_1(t) \cdot A_5]\} + B_3 \cdot [a(t) - 1] \cdot V + B_4 \cdot [a(t) - 1]$$

$$X_2 \cdot A_6 \cdot [a(t) - 1] + X_2 \cdot A_7 \cdot [a(t) - 1] \cdot V = \psi_1(t) \cdot B_1 \cdot [a(t) - 1] \cdot V^2 + V \cdot \psi_1(t) \cdot B_2 \cdot [a(t) - 1] + \psi_1(t) \cdot A_1 \cdot V - V^2 \cdot \psi_1(t) \cdot A_2 - V^2 \cdot \psi_1(t) \cdot \psi_1(t) \cdot A_3 + \psi_1(t) \cdot A_4 - V \cdot \psi_1(t) \cdot \psi_1(t) \cdot A_5 + B_3 \cdot [a(t) - 1] \cdot V + B_4 \cdot [a(t) - 1]$$

$$\psi_1(t) \cdot B_1 \cdot [a(t) - 1] \cdot V^2 + V \cdot \psi_1(t) \cdot B_2 \cdot [a(t) - 1] + \psi_1(t) \cdot A_1 \cdot V - V^2 \cdot \psi_1(t) \cdot A_2 - V^2 \cdot \psi_1^2(t) \cdot A_3 + \psi_1(t) \cdot A_4 - V \cdot \psi_1^2(t) \cdot A_5 + B_3 \cdot [a(t) - 1] \cdot V + B_4 \cdot [a(t) - 1] - X_2 \cdot A_6 \cdot [a(t) - 1] - X_2 \cdot A_7 \cdot [a(t) - 1] \cdot V = 0$$

$$\psi_1(t) \cdot V^2 \cdot \{B_1 \cdot [a(t) - 1] - A_2 - \psi_1(t) \cdot A_3\} + V \cdot \{\psi_1(t) \cdot B_2 \cdot [a(t) - 1] + \psi_1(t) \cdot A_1 - \psi_1^2(t) \cdot A_5 + B_3 \cdot [a(t) - 1] - X_2 \cdot A_7 \cdot [a(t) - 1]\} + \psi_1(t) \cdot A_4 + B_4 \cdot [a(t) - 1] - X_2 \cdot A_6 \cdot [a(t) - 1] = 0$$

We define the following functions: $\chi_1(t) \cdot V^2 + \chi_2(t) \cdot V + \chi_3(t) = 0$

$$\chi_1(t) = \psi_1(t) \cdot \{B_1 \cdot [a(t) - 1] - A_2 - \psi_1(t) \cdot A_3\}$$
$$\chi_2(t) = \psi_1(t) \cdot B_2 \cdot [a(t) - 1] + \psi_1(t) \cdot A_1 - \psi_1^2(t) \cdot A_5 + B_3 \cdot [a(t) - 1] - X_2 \cdot A_7 \cdot [a(t) - 1]$$
$$\chi_3(t) = \psi_1(t) \cdot A_4 + B_4 \cdot [a(t) - 1] - X_2 \cdot A_6 \cdot [a(t) - 1]$$

$$\chi_1(t) \cdot V^2 + \chi_2(t) \cdot V + \chi_3(t) = 0 \Rightarrow V_{\#,\#\#} = \frac{-\chi_2(t) \pm \sqrt{\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)}}{2 \cdot \chi_1(t)}$$

We can summery our additional functions and parameters in the following Table 5.14.

Table 5.14 Summary of our additional functions and parameters

Function/parameter	Expression
$\psi_1(a(t)); \frac{\partial \psi_1(a(t))}{\partial t}$	$\psi_1(t) = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)}; \frac{\partial \psi_1(t)}{\partial t} = \frac{\partial a(t)/\partial t}{\left(R_1 + \frac{V_t}{I_0}\right)}$
$B_1; \frac{\partial B_1}{\partial t} = 0$	$B_1 = [1 - (k_1 + 1) \cdot \alpha_{f1}] \cdot \frac{1}{V_t}$
$B_2; \frac{\partial B_2}{\partial t} = 0$	$B_2 = [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]$
$B_3; \frac{\partial B_3}{\partial t}$	$B_3 = I_{sc} \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1) \cdot \frac{1}{V_t}$
$B_4; \frac{\partial B_4}{\partial t} = 0$	$B_4 = (I_{sc} - I_{se}) \cdot (\alpha_{r1} \cdot \alpha_{f1} - 1)$
$B_5; \frac{\partial B_5}{\partial t} = 0$	$B_5 = \frac{1}{R_{CQ1} \cdot \left[1 + \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{R_1}{\left(R_1 + \frac{V_t}{I_0}\right)}\right]}$
$\xi_1(t)$	$\xi_1 = B_5 \cdot \left\{V_{cc} - V \cdot \left[1 + \frac{V_t}{I_0} \cdot \psi_1(t)\right]\right\}$ $\xi_1(V, a(t)) = \frac{1}{R_{CQ1} \cdot \left[1 + \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{R_1}{\left(R_1 + \frac{V_t}{I_0}\right)}\right]} \cdot \left\{V_{cc} - V \cdot \left(1 + \frac{V_t}{I_0} \cdot \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)}\right)\right\}$
$\frac{\partial \xi_1(t)}{\partial t}$	$\frac{d\xi_1}{dt} = -\frac{1}{R_{CQ1} \cdot \left[1 + \frac{1}{R_{CQ1}} \cdot \frac{V_t}{I_0} \cdot \frac{R_1}{\left(R_1 + \frac{V_t}{I_0}\right)}\right]} \cdot \left\{\frac{dV}{dt} \cdot \left(1 + \frac{V_t}{I_0} \cdot \frac{[a(t)-1]}{(R_1 + \frac{V_t}{I_0})}\right) + V \cdot \frac{V_t}{I_0} \cdot \frac{da(t)/dt}{\left(R_1 + \frac{V_t}{I_0}\right)}\right\}$
$\eta_1(V, a(t))$	$\eta_1(V, a(t)) = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t)-1]}\right) \cdot (\alpha_{r1} - k_1 - 1)$
$\frac{\partial \eta_1(V, a(t))}{\partial t}$	$\frac{\partial \eta_1}{\partial t} = \left\{\frac{da(t)/dt}{(R_1 + \frac{V_t}{I_0})} \cdot \left(V + \frac{\xi_1 \cdot R_1}{[a(t) - 1]}\right) + \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(\frac{dV}{dt} + \frac{\frac{d\xi_1}{dt} \cdot R_1 \cdot [a(t) - 1] - \xi_1 \cdot R_1 \cdot \frac{da(t)}{dt}}{[a(t) - 1]^2}\right)\right\} \cdot (\alpha_{r1} - k_1 - 1)$
$\eta_2(V, a(t))$	$\eta_2(V, a(t)) = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t)-1]}\right) \cdot [1 - (k_1 + 1) \cdot \alpha_{f1}]$
$\frac{d\eta_2}{dt}$	$\frac{d\eta_2}{dt} = \left\{\frac{da(t)/dt}{(R_1 + \frac{V_t}{I_0})} \cdot \left(V + \frac{\xi_1 \cdot R_1}{[a(t) - 1]}\right) + \frac{[a(t) - 1]}{(R_1 + \frac{V_t}{I_0})} \cdot \left(\frac{dV}{dt} + \frac{\frac{d\xi_1}{dt} \cdot R_1 \cdot [a(t) - 1] - \xi_1 \cdot R_1 \cdot \frac{da(t)}{dt}}{[a(t) - 1]^2}\right)\right\} \cdot [1 - (k_1 + 1) \cdot \alpha_{f1}]$
$\Delta(t) = \eta_2(t) - \eta_1(t)$	$\Delta(t) = \frac{[a(t)-1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t)-1]}\right) \cdot [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]$
$\frac{d\Delta}{dt} = \left[\frac{da(t)/dt}{(R_1 + \frac{V_t}{I_0})} \cdot \left(V + \frac{\xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]}\right) + \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left(\frac{dV}{dt} + \frac{\frac{d\xi_1(V, a(t))}{dt} \cdot R_1 \cdot [a(t) - 1] - \frac{da(t)}{dt} \cdot \xi_1(V, a(t)) \cdot R_1}{[a(t) - 1]^2}\right)\right] \cdot [2 + k_1 \cdot (1 - \alpha_{f1}) - \alpha_{f1} - \alpha_{r1}]$	
A_1	$A_1 = B_5 \cdot R_1 \cdot (V_{cc} \cdot B_1 - B_2)$
A_2	$A_2 = B_5 \cdot R_1 \cdot B_1$
A_3	$A_3 = \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_1$
A_4	$A_4 = V_{cc} \cdot B_5 \cdot R_1 \cdot B_2$
A_5	$A_5 = \frac{V_t}{I_0} \cdot B_5 \cdot R_1 \cdot B_2$
A_6	$A_6 = 2 - \alpha_{r1} - \alpha_{f1}$
A_7	$A_7 = (1 - \alpha_{f1}) \cdot \frac{1}{V_t}$

$$\chi_1(t) = \psi_1(t)\cdot\{B_1\cdot[a(t)-1] - A_2 - \psi_1(t)\cdot A_3\}$$

$$\dot{\chi}_1(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot\{B_1\cdot[a(t)-1] - A_2 - \psi_1(t)\cdot A_3\} + \psi_1(t)\cdot\left\{B_1\cdot\frac{\partial a(t)}{\partial t} - \frac{\partial\psi_1(t)}{\partial t}\cdot A_3\right\}$$

$$\chi_2(t) = \psi_1(t)\cdot B_2\cdot[a(t)-1] + \psi_1(t)\cdot A_1 - \psi_1^2(t)\cdot A_5 + B_3\cdot[a(t)-1] - X_2\cdot A_7\cdot[a(t)-1]$$

$$\dot{\chi}_2(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot B_2\cdot[a(t)-1] + \psi_1(t)\cdot B_2\cdot\frac{\partial a(t)}{\partial t} + \frac{\partial\psi_1(t)}{\partial t}\cdot A_1 - 2\cdot\psi_1(t)\cdot\frac{\partial\psi_1(t)}{\partial t}\cdot A_5 + B_3\cdot\frac{\partial a(t)}{\partial t} - \frac{\partial X_2}{\partial t}\cdot A_7\cdot[a(t)-1] - X_2\cdot A_7\cdot\frac{\partial a(t)}{\partial t}$$

$$\chi_3(t) = \psi_1(t)\cdot A_4 + B_4\cdot[a(t)-1] - X_2\cdot A_6\cdot[a(t)-1]$$

$$\dot{\chi}_3(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot A_4 + B_4\cdot\frac{\partial a(t)}{\partial t} - \frac{\partial X_2}{\partial t}\cdot A_6\cdot[a(t)-1] - X_2\cdot A_6\cdot\frac{\partial a(t)}{\partial t}$$

We can summarize our $\chi_i, \dot{\chi}_i (i = 1, 2, 3)$ expressions in Table 5.15:

Table 5.15 Summary of our expressions $\chi_i, \dot{\chi}_i (i = 1, 2, 3)$

$\chi_i(i = 1, 2, 3)$	$\dot{\chi}_i(i = 1, 2, 3)$
$\chi_1(t) = \psi_1(t)\cdot\{B_1\cdot[a(t)-1] - A_2 - \psi_1(t)\cdot A_3\}$	$\dot{\chi}_1(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot\{B_1\cdot[a(t)-1] - A_2 - \psi_1(t)\cdot A_3\} + \psi_1(t)\cdot\left\{B_1\cdot\frac{\partial a(t)}{\partial t} - \frac{\partial\psi_1(t)}{\partial t}\cdot A_3\right\}$
$\chi_2(t) = \psi_1(t)\cdot B_2\cdot[a(t)-1] + \psi_1(t)\cdot A_1 - \psi_1^2(t)\cdot A_5 + B_3\cdot[a(t)-1] - X_2\cdot A_7\cdot[a(t)-1]$	$\dot{\chi}_2(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot B_2\cdot[a(t)-1] + \psi_1(t)\cdot B_2\cdot\frac{\partial a(t)}{\partial t} + \frac{\partial\psi_1(t)}{\partial t}\cdot A_1 - 2\cdot\psi_1(t)\cdot\frac{\partial\psi_1(t)}{\partial t}\cdot A_5 + B_3\cdot\frac{\partial a(t)}{\partial t} - \frac{\partial X_2}{\partial t}\cdot A_7\cdot[a(t)-1] - X_2\cdot A_7\cdot\frac{\partial a(t)}{\partial t}$ $\frac{\partial X_2}{\partial t} \Rightarrow \frac{dX_2}{dt}; \frac{dX_2}{dt} = X_1$ $\dot{\chi}_2(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot B_2\cdot[a(t)-1] + \psi_1(t)\cdot B_2\cdot\frac{\partial a(t)}{\partial t} + \frac{\partial\psi_1(t)}{\partial t}\cdot A_1 - 2\cdot\psi_1(t)\cdot\frac{\partial\psi_1(t)}{\partial t}\cdot A_5 + B_3\cdot\frac{\partial a(t)}{\partial t} - X_1\cdot A_7\cdot[a(t)-1] - X_2\cdot A_7\cdot\frac{\partial a(t)}{\partial t}$
$\chi_3(t) = \psi_1(t)\cdot A_4 + B_4\cdot[a(t)-1] - X_2\cdot A_6\cdot[a(t)-1]$	$\dot{\chi}_3(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot A_4 + B_4\cdot\frac{\partial a(t)}{\partial t} - \frac{\partial X_2}{\partial t}\cdot A_6\cdot[a(t)-1] - X_2\cdot A_6\cdot\frac{\partial a(t)}{\partial t}$ $\frac{\partial X_2}{\partial t} \Rightarrow \frac{dX_2}{dt}; \frac{dX_2}{dt} = X_1$ $\dot{\chi}_3(t) = \frac{\partial\psi_1(t)}{\partial t}\cdot A_4 + B_4\cdot\frac{\partial a(t)}{\partial t} - X_1\cdot A_6\cdot[a(t)-1] - X_2\cdot A_6\cdot\frac{\partial a(t)}{\partial t}$

$$V_{\#,\#\#} = \frac{-\chi_2(t) \pm [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{\frac{1}{2}}}{2 \cdot \chi_1(t)}$$

$$\Rightarrow \frac{\mathrm{d}V_{\#,\#\#}}{\mathrm{d}t} = \frac{d}{\mathrm{d}t}\left[\frac{-\chi_2(t) \pm [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{\frac{1}{2}}}{2 \cdot \chi_1(t)}\right]$$

$$\frac{\mathrm{d}V_{\#,\#\#}}{\mathrm{d}t} = \frac{\begin{matrix}\left\{-\dot{\chi}_2(t) \pm \frac{1}{2} \cdot [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{-\frac{1}{2}} \cdot [2 \cdot \chi_2(t) \cdot \dot{\chi}_2(t) - 4 \cdot \dot{\chi}_1(t) \cdot \chi_3(t)\right. \\ \left. -4 \cdot \chi_1(t) \cdot \dot{\chi}_3(t)]\right\} \cdot 2 \cdot \chi_1(t) - \left\{-\chi_2(t) \pm [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{\frac{1}{2}}\right\} \cdot 2 \cdot \dot{\chi}_1(t)\end{matrix}}{4 \cdot \chi_1^2(t)}$$

We define the following function:

$$\frac{\mathrm{d}V_{\#,\#\#}}{\mathrm{d}t} = g(X_1, X_2, \dot{X}_1, \dot{X}_2, a(t), \frac{\partial a(t)}{\partial t}, \psi_1(t), \frac{\partial \psi_1(t)}{\partial t})$$

$$g\left(X_1, X_2, \dot{X}_1, \dot{X}_2, a(t), \frac{\partial a(t)}{\partial t}, \psi_1(t), \frac{\partial \psi_1(t)}{\partial t}\right)$$

$$= \frac{\begin{matrix}\left\{-\dot{\chi}_2(t) \pm \frac{1}{2} \cdot [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{-\frac{1}{2}} \cdot [2 \cdot \chi_2(t) \cdot \dot{\chi}_2(t) - 4 \cdot \dot{\chi}_1(t) \cdot \chi_3(t)\right. \\ \left. -4 \cdot \chi_1(t) \cdot \dot{\chi}_3(t)]\right\} \cdot 2 \cdot \chi_1(t) - \left\{-\chi_2(t) \pm [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{\frac{1}{2}}\right\} \cdot 2 \cdot \dot{\chi}_1(t)\end{matrix}}{4 \cdot \chi_1^2(t)}$$

We already got the following expression: $\frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1} \cdot \left\{\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{1}{C_1} \cdot X_2\right\}$

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1} \cdot \left\{\frac{\mathrm{d}V_{\#,\#\#}}{\mathrm{d}t} - \frac{1}{C_1} \cdot X_2\right\}; \frac{\mathrm{d}X_2}{\mathrm{d}t} = X_1 \Rightarrow \frac{\mathrm{d}X_2}{\mathrm{d}t} = X_1;$$
$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = \frac{1}{L_1} \cdot \left\{g(X_1, X_2, a(t), \ldots) - \frac{1}{C_1} \cdot X_2\right\}$$

We can define out OptoNDR differential equations:

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = f_1(X_1, X_2, a(t), \ldots); \frac{\mathrm{d}X_2}{\mathrm{d}t} = f_2(X_1, X_2, a(t), \ldots)$$

$$f_1(X_1, X_2, a(t), \ldots) = \frac{1}{L_1} \cdot \{g(X_1, X_2, a(t), \ldots) - \frac{1}{C_1} \cdot X_2\}; f_2(X_1, X_2, a(t), \ldots) = X_1$$

Next we need to find our system Jacobian matrix at limit cycle A(t).

$$A(t) = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix}_{(X_1(t),X_2(t))\text{limitcycle}} \Bigg|_{\frac{\partial f_2}{\partial X_2}=0;\frac{\partial f_2}{\partial X_1}=1} = -\frac{\partial f_1}{\partial X_2}\Bigg|_{(X_1(t),X_2(t))\text{limitcycle}}$$
$$= -\frac{1}{L_1} \cdot \left\{ \frac{\partial g(X_1, X_2, a(t), \ldots)}{\partial X_2} - \frac{1}{C_1} \right\} \Bigg|_{(X_1(t),X_2(t))\text{limitcycle}}$$

$$\frac{\partial f_1}{\partial X_1} = \frac{1}{L_1} \cdot \frac{\partial g(X_1, X_2, a(t), \ldots)}{\partial X_1}; \frac{\partial f_1}{\partial X_2} = \frac{1}{L_1} \cdot \left\{ \frac{\partial g(X_1, X_2, a(t), \ldots)}{\partial X_2} - \frac{1}{C_1} \right\}; \frac{\partial f_2}{\partial X_1} = 1; \frac{\partial f_2}{\partial X_2} = 0$$

We need to find the expressions for $\frac{\partial g(X_1,X_2,a(t),\ldots)}{\partial X_1}; \frac{\partial g(X_1,X_2,a(t),\ldots)}{\partial X_2}$

$$\frac{\partial g\left(X_1, X_2, a(t), \frac{\partial a(t)}{\partial t}\right)}{\partial X_1} = \frac{\partial}{\partial X_1} \left[\frac{\begin{array}{l} \left\{-\dot{\chi}_2(t) \pm \frac{1}{2} \cdot [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{-\frac{1}{2}} \cdot [2 \cdot \chi_2(t) \cdot \dot{\chi}_2(t) - 4 \cdot \dot{\chi}_1(t) \cdot \chi_3(t) \right. \\ \left. -4 \cdot \chi_1(t) \cdot \dot{\chi}_3(t)]\right\} \cdot 2 \cdot \chi_1(t) - \left\{-\chi_2(t) \pm [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{\frac{1}{2}}\right\} \cdot 2 \cdot \dot{\chi}_1(t) \end{array}}{4 \cdot \chi_1^2(t)} \right]$$

$$\frac{\partial g\left(X_1, X_2, a(t), \frac{\partial a(t)}{\partial t}\right)}{\partial X_2} = \frac{\partial}{\partial X_2} \left[\frac{\begin{array}{l} \left\{-\dot{\chi}_2(t) \pm \frac{1}{2} \cdot [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{-\frac{1}{2}} \cdot [2 \cdot \chi_2(t) \cdot \dot{\chi}_2(t) - 4 \cdot \dot{\chi}_1(t) \cdot \chi_3(t) \right. \\ \left. -4 \cdot \chi_1(t) \cdot \dot{\chi}_3(t)]\right\} \cdot 2 \cdot \chi_1(t) - \left\{-\chi_2(t) \pm [\chi_2^2(t) - 4 \cdot \chi_1(t) \cdot \chi_3(t)]^{\frac{1}{2}}\right\} \cdot 2 \cdot \dot{\chi}_1(t) \end{array}}{4 \cdot \chi_1^2(t)} \right]$$

$$\psi_1(t) = \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)}; \frac{\partial \psi_1(t)}{\partial t} = \frac{\partial a(t)/\partial t}{\left(R_1 + \frac{V_t}{I_0}\right)}$$

$$\chi_1(t) = \psi_1(t) \cdot \{B_1 \cdot [a(t) - 1] - A_2 - \psi_1(t) \cdot A_3\}$$
$$= \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot \left\{ B_1 \cdot [a(t) - 1] - A_2 - \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot A_3 \right\}$$
$$\chi_2(t) = \psi_1(t) \cdot B_2 \cdot [a(t) - 1] + \psi_1(t) \cdot A_1 - \psi_1^2(t) \cdot A_5 + B_3 \cdot [a(t) - 1] - X_2 \cdot A_7 \cdot [a(t) - 1]$$

$$\chi_2(t) = \frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \cdot (B_2 \cdot [a(t) - 1] + A_1) - \left[\frac{[a(t) - 1]}{\left(R_1 + \frac{V_t}{I_0}\right)} \right]^2 \cdot A_5 + B_3 \cdot [a(t) - 1] - X_2 \cdot A_7 \cdot [a(t) - 1]$$

$$\chi_3(t) = \psi_1(t)\cdot A_4 + B_4\cdot\,[a(t)-1] - X_2\cdot A_6\cdot\,[a(t)-1] = [a(t)-1]\cdot\left(\frac{A_4}{\left(R_1+\frac{V_t}{I_0}\right)} + B_4 - X_2\cdot A_6\right)$$

$$\begin{aligned}\dot{\chi}_1(t) = \frac{\partial a(t)}{\partial t}\cdot\Bigg[&\frac{1}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left\{B_1\cdot[a(t)-1] - A_2 - \frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot A_3\right\}\\ &+\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\left\{B_1 - \frac{A_3}{(R_1+\frac{V_t}{I_0})}\right\}\Bigg]\end{aligned}$$

$$\begin{aligned}\dot{\chi}_2(t) = \frac{\partial a(t)}{\partial t}\cdot\Bigg\{&\frac{B_2\cdot[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)} + \frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot B_2\\ &+\frac{1}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot A_1 - 2\cdot\frac{[a(t)-1]}{\left(R_1+\frac{V_t}{I_0}\right)}\cdot\frac{A_5}{\left(R_1+\frac{V_t}{I_0}\right)} + B_3 - X_2\cdot A_7\Bigg\}\\ &- X_1\cdot A_7\cdot[a(t)-1]\end{aligned}$$

$$\dot{\chi}_3(t) = \frac{\partial a(t)}{\partial t}\cdot\left[\frac{A_4}{\left(R_1+\frac{V_t}{I_0}\right)} + B_4 - X_2\cdot A_6\right] - X_1\cdot A_6\cdot\,[a(t)-1]$$

We need to find the Jacobian Matrix elements at limit cycle.

$$\begin{aligned}\left.\frac{\partial f_1}{\partial X_1}\right|_{(X_1(t),X_2(t))\text{limitcycle}} &= \frac{1}{L_1}\cdot\left.\frac{\partial g(X_1,X_2,a(t),\ldots)}{\partial X_1}\right|_{(X_1(t),X_2(t))\text{limitcycle}}\\ \left.\frac{\partial f_1}{\partial X_2}\right|_{(X_1(t),X_2(t))\text{limitcycle}} &= \frac{1}{L_1}\cdot\left\{\left.\frac{\partial g(X_1,X_2,a(t),\ldots)}{\partial X_2}\right|_{(X_1(t),X_2(t))\text{limitcycle}} - \frac{1}{C_1}\right\}\\ \left.\frac{\partial f_2}{\partial X_1}\right|_{(X_1(t),X_2(t))\text{limitcycle}} &= 1;\left.\frac{\partial f_2}{\partial X_2}\right|_{(X_1(t),X_2(t))\text{limitcycle}} = 0\end{aligned}$$

We study the stability of the limit cycle solution, first we have to locate it. One point on the limit cycles, as well as its period T, such that $X_1(t+T) = X_1(t)$; $X_2(t+T) = X_2(t)$. First we need to prove that the system has periodic orbits and it is done by changing system cartesian coordinates ($X_1(t)$, $X_2(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. We approve that there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers [46, 47]. One of them is unity. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation

z = 0), and the cylindrical axis is the Cartesian z axis. In our system we refer to Cartesian X_1−X_2 plane (with equation X_3 = 0). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X_1, X_2) are the same as for polar coordinates, namely $X_1(t) = r(t) \cdot \cos[\theta(t)]$; $X_2(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X_1^2 + X_2^2}$. $\theta(t) = 0$ if $X_1 = 0$ and $X_2 = 0$. $\theta(t) = \arcsin(X_2/r)$ if X_1 >=0.

$\theta(t) = -\arcsin(X_2/r) + \pi$ if $X_1 < 0$. We represent our system equation:

$\frac{dX_1}{dt} = \frac{1}{L_1} \cdot \left\{ g(X_1, X_2, a(t), \ldots) - \frac{1}{C_1} \cdot X_2 \right\}; \frac{dX_2}{dt} = X_1$ by using cylindrical coordinates ($r(t)$, $\theta(t)$) [78–80].

$$X_1(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{dX_1(t)}{dt} = \frac{dr(t)}{dt} \cdot \cos[\theta(t)] - r(t) \cdot \frac{d\theta(t)}{dt} \cdot \sin[\theta(t)]$$

$$X_2(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{dX_2(t)}{dt} = \frac{dr(t)}{dt} \cdot \sin[\theta(t)] + r(t) \cdot \frac{d\theta(t)}{dt} \cdot \cos[\theta(t)]$$

$$\frac{dX_1(t)}{dt} = \frac{dX_1}{dt}; \frac{dr(t)}{dt} = r'; \frac{d\theta(t)}{dt} = \theta'; \theta(t) = \theta; r(t) = r$$

We get the equations:

$$\frac{dX_1}{dt} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta; \frac{dX_2}{dt} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$

$$\frac{dX_2}{dt} = X_1 \Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = r \cdot \cos\theta$$
$$\Rightarrow r' \cdot \sin\theta = -r \cdot \theta' \cdot \cos\theta + r \cdot \cos\theta$$

$$r' \cdot \sin\theta = -r \cdot \theta' \cdot \cos\theta + r \cdot \cos\theta$$
$$\Rightarrow r' \cdot \sin\theta = r \cdot \cos\theta \cdot (1 - \theta') \Rightarrow tg\theta = \frac{r \cdot (1 - \theta')}{r'}$$

$$tg\theta = \frac{r \cdot (1 - \theta')}{r'} \Rightarrow \theta = arctg\left\{\frac{r \cdot (1 - \theta')}{r'}\right\}; X_1 = r \cdot \cos\theta;$$
$$X_2 = r \cdot \sin\theta$$

To find our solution constant radius for system with constant source: $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$ we set $\frac{dr}{dt} = 0$ which yield the following outcome:

$$\frac{dX_2}{dt} = X_1 \Rightarrow tg\theta = \frac{r \cdot (1 - \theta')}{r'} \Rightarrow r' = \frac{r \cdot (1 - \theta')}{tg\theta}; \frac{dr}{dt} = 0 \Leftrightarrow \frac{r \cdot (1 - \theta')}{tg\theta} = 0$$

Since $r \neq 0$, we get two possible

1. $tg\theta \to \infty \Rightarrow \theta = \frac{\pi}{2} + \pi \cdot n = \pi \cdot \left(\frac{1}{2} + n\right) \forall n = 0, 1, 2, \ldots$
2. $1 - \theta' = 0 \Rightarrow \frac{d\theta}{dt} = 1 \Rightarrow \theta = t + \text{const}; \frac{d\theta}{dt} = \frac{2 \cdot \pi}{T} = 1 \Rightarrow T = 2 \cdot \pi$.

We need to find $g(X_1, X_2, a(t), \ldots)$ in cylindrical coordinates ($r(t)$, $\theta(t)$) for system with constant source: $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$. The transformation: $g(X_1, X_2, a(t) = 1, \ldots) \to g(r(t), \theta(t), a(t) = 1, \ldots)$.

$$\chi_1(t)|_{a(t)=1;\frac{\partial a(t)}{\partial t}=0} = 0;\ \chi_2(t)|_{a(t)=1;\frac{\partial a(t)}{\partial t}=0} = 0;\ \chi_3(t)|_{a(t)=1;\frac{\partial a(t)}{\partial t}=0} = 0$$

$$\dot{\chi}_1(t)|_{a(t)=1;\frac{\partial a(t)}{\partial t}=0} = 0;\ \dot{\chi}_2(t)|_{a(t)=1;\frac{\partial a(t)}{\partial t}=0} = 0;\ \dot{\chi}_3(t)|_{a(t)=1;\frac{\partial a(t)}{\partial t}=0} = 0$$

We get undefined expression: $\frac{\partial g\left(X_1, X_2, a(t), \frac{\partial a(t)}{\partial t}\right)}{\partial X_2}$ for $a(t) = 1; \frac{\partial a(t)}{\partial t} = 0$.

Our solution constant radius for system can not be with constant source. We choose periodic source and find Jacobian matrix at limit cycle $A(t)$.

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = f_1(X_1, X_2, a(t), \ldots) = f_1; \frac{\mathrm{d}X_2}{\mathrm{d}t} = f_2(X_1, X_2, a(t), \ldots) = f_2$$

We get our system equations:

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = f_1(X_1, X_2, a(t), \ldots); \frac{\mathrm{d}X_2}{\mathrm{d}t} = f_2(X_1, X_2, a(t), \ldots)$$

We need to find $\left(\frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2}\right)\Big|_{(X_1(t), X_2(t))\text{limitcycle}}$ at a solution with constant radius ($\mathrm{d}r/\mathrm{d}t = 0$).

$$\rho_1 \cdot \rho_2 = \mathrm{e}^{\int_0^T tr(A(s))\cdot ds}\Bigg|_{\rho_1=1} \Rightarrow \rho_2 = \mathrm{e}^{\int_0^T tr(A(s))\cdot ds} = \mathrm{e}^{\int_0^T \left(\frac{\partial f_1}{\partial X_1} + \frac{\partial f_2}{\partial X_2}\right)\Big|_{r=r_{\text{const}}}\cdot ds}$$

As a result, the limit cycle with radius r_{const}. Is stable if $\int_0^T tr(A(s)) \cdot ds < 0$ or $\rho_2 < 1$ and unstable if $\int_0^T tr(A(s)) \cdot ds > 0$ or $\rho_2 > 1$.

Remark In the last analysis our solution constant radius for system can not be with constant source; $\partial a(t)/\partial t = 0$. It is reader exercise to do the limit cycle stability analysis for all other cases $\partial a(t)/\partial t \neq 0$, periodic source by using Floquet theory [46, 47].

The stability of a limit cycle is analyzed by changing OptoNDR system's parameters and verify when $\max(\mathrm{Re}(\rho_i))$ value pass through "−1" value. It is signifying a period doubling bifurcation of a limit cycle.

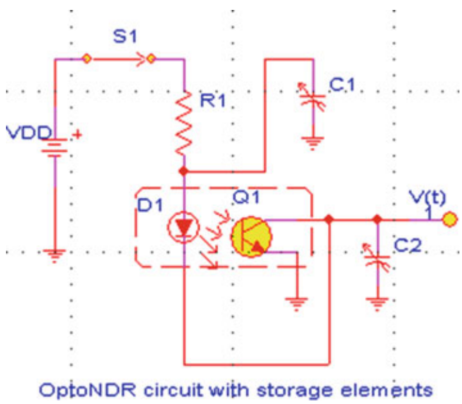
5.7 Exercises

1. We consider an OptoNDR circuit with storage elements, variable capacitors (see Fig. 5.16). At $t = 0$ switch $S1$ moves his position from OFF state to ON state. Circuit dynamic starts and $V(t)$ is the main system variable. We consider that initially capacitor $C1$ is charged to $V_{C1}(t = 0)$ and capacitor $C2$ is charged to $V_{C2}(t = 0)$ $V_{DD} \gg V_{Q1(\text{sustain})}$ and photo transistor $Q1$ reaches his sustaining voltage (breakover voltage) after we move switch $S1$ to ON state. After we move $S1$ to ON state capacitors $C1$ and $C2$ start to charge. Circuit output voltage $V_{CEQ1} = V(t)$ is growing up.

   1.1 Find optoisolation circuit differential equations.
   1.2 Check in which conditions the system has periodic orbits. Hint: It is done by changing system Cartesian coordinates to cylindrical coordinates $r(t)$, $\theta(t)$. Find system periodic T.
   1.3 Get system constant radius as a function of circuit parameters (C_1, C_2, α_f, α_r, I_{se}, I_{sc}, etc.).
   1.4 Find the expression for Floquet multipliers and discuss stability.
   1.5 We set $C = C_1 = 2 \cdot C_2$. How system periodic orbit dependents on C capacitor values? How system constant radius changes when we change C capacitor values? $r_{\text{const-radius}} = f(C)$.

2. We consider an OptoNDR circuit with storage elements, variable capacitors (see Fig. 5.17). At $t = 0$ switch $S1$ moves his position from OFF state to ON state. Circuit dynamic starts and $V(t)$ is the main system variable. We consider that initially capacitor $C1$ is charged to $V_{C1}(t = 0)$ and capacitor $C2$ is charged to $V_{C2}(t = 0)$ $V_{DD} \gg V_{Q1(\text{sustain})}$ and photo transistor $Q1$ reaches his sustaining voltage (breakover voltage) after we move switch $S1$ to ON state. After we

Fig. 5.16 OptoNDR circuit with storage elements, variable capacitors



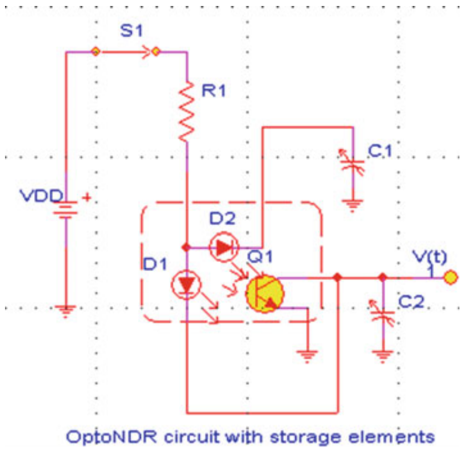


Fig. 5.17 OptoNDR circuit with storage elements, variable capacitors

move $S1$ to ON state capacitors $C1$ and $C2$ start to charge. Circuit output voltage $V_{CEQ1} = V(t)$ is growing up. Photo transistor $Q1$ has two coupling LEDs D1 and D2.

$$\mathrm{I_{BQ1}} = \mathrm{k1}\cdot \mathrm{I}_{D1} + \mathrm{k2}\cdot \mathrm{I_{D2}} \text{ or } \mathrm{I_{BQ1}} = \mathrm{k1}\cdot \xi_1(\mathrm{I_{D1}}) + \mathrm{k2}\cdot \xi_2(\mathrm{I_{D2}}).$$

2.1 Find optoisolation circuit differential equations for $I_{BQ1} = k1 \cdot I_{D1} + k2 \cdot I_{D2}$.
2.2 Check in which conditions the system has periodic orbits for $I_{BQ1} = k1 \cdot I_{D1} + k2 \cdot I_{D2}$. Hint: It is done by changing system Cartesian coordinates to cylindrical coordinates $r(t)$, $\theta(\mathrm{t})$.
Find system periodic T.
2.3 Get system constant radius as a function of circuit parameters (C_1, C_2, α_f, α_r, I_{se}, I_{sc}, etc.) for $I_{BQ1} = k1 \cdot I_{D1} + k2 \cdot I_{D2}$.
2.4 Find the expression for Floquet multipliers and discuss stability for $I_{BQ1} = k1 \cdot I_{D1} + k2 \cdot I_{D2}$.
2.5 We set $C = C_1 = 2 \cdot C_2$ and $\xi_1(I_{D1}) = I_{D1}^2; \xi_2(I_{D2}) = I_{D2}\cdot\Gamma - 1$. How system periodic orbit dependents on C capacitor values? How system constant radius changes when we change C capacitor values and Γ parameter values? $r_{\text{const-radius}} = f(C,\Gamma)$. $I_{BQ1} = k1 \cdot \xi 1(I_{D1}) + k2 \cdot \xi 1 (I_{D2})$.

3. We have autonomous Chua's circuit with OptoNDR element which contains phototransistor $Q1$ coupled with two LEDs ($D1$, $D2$). The circuit contains three energy storage elements (Inductance $L1$, Capacitors $C1$, and $C2$). Chua's circuit exhibits different and unique variety of bifurcations and chaos among the chaos-producing mechanism with OptoNDR element. OptoNDR element is a current-controlled nonlinear element characterized by $V_{\text{OptoNDR}} = g(i_{NDR}, i_{C1})$. $i_{C1} = i_{D2}$; $i_{D1} = i_{CQ1}$. We consider i_{BQ1} as a combination of two circuit's

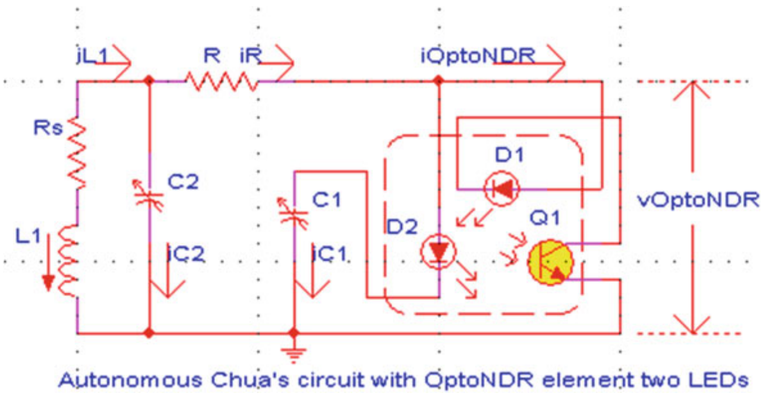


Fig. 5.18 Autonomous Chua's circuit with OptoNDR element which contains phototransistor *Q*1 coupled with two LEDs (*D*1, *D*2)

currents with *k*1 and *k*2 coupling coefficients, respectively. i_{BQ1} equation is $I_{BQ1} = k_1 \cdot I_{D1} + k_2 \cdot I_{C1}; k_1 = n \cdot k_2; k_2 = k; k_1, k_2 < 1$ (Fig. 5.18).

3.1 Find circuit's differential equations and represent the main variables as *X*(*t*), *Y*(*t*), *Z*(*t*).
3.2 Find circuit's fixed points. How circuit's fixed points are dependent on *k*, *n* circuit's parameters?
3.3 Find Chua's circuit OptoNDR Jacobian matrix and elements.
3.4 Discuss stability for possible circuit's fixed points.
3.5 How system stability changes for various *k*, *n* parameters values.
3.6 We turn over the polarity of *D*2. How circuit's fixed points and stability analysis change.

4. We have autonomous Chua's circuit with OptoNDR element which contains phototransistor *Q*1 coupled with three LEDs (*D*1, *D*2, *D*3). The circuit contains three energy storage elements (Inductance *L*1, Capacitors *C*1, and *C*2). Chua's circuit exhibits different and unique variety of bifurcations and chaos among the chaos-producing mechanism with OptoNDR element. OptoNDR element is a current-controlled nonlinear element characterized by $V_{\text{OptoNDR}} = g(i_{NDR}, i_{C1}, i_{EQ1})$. $i_{C1} = i_{D2}$; $i_{D1} = i_{CQ1}$. We consider I_{BQ1} as a combination of three circuit's currents with *k*1, *k*2, and *k*3 coupling coefficients, respectively. I_{BQ1} equation is $I_{BQ1} = k_1 \times I_{D1} + k_2 \times I_{C1} + k_3 \times I_{EQ1}; k_1 = n \cdot k_2; k_2 = \sqrt{n} \cdot k; \quad k_3 = k; k_1 < 1; k_2 < 1; k_3 < 1$ (Fig. 5.19).

4.1 Find circuit's differential equations and represent the main variables as *X*(*t*), *Y*(*t*), *Z*(*t*).
4.2 Find circuit's fixed points. How circuit's fixed points are dependent on *k*, *n* circuit's parameters?

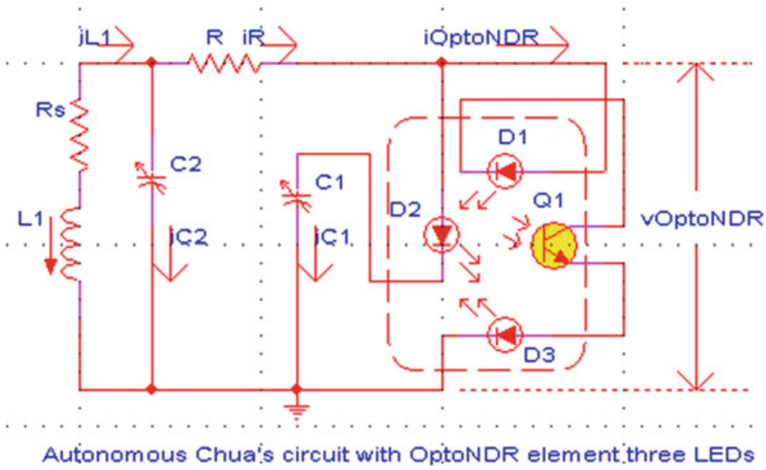


Fig. 5.19 Autonomous Chua's circuit with OptoNDR element which contains phototransistor *Q*1 coupled with three LEDs (*D*1, *D*2, *D*3)

4.3 Find Chua's circuit OptoNDR Jacobian matrix and elements.
4.4 Discuss stability for possible circuit's fixed points.
4.5 How system stability changes for various *k*, *n* parameters values.
4.6 We turn over the polarity of *D*3. How circuit's fixed points and stability analysis change.

5. We have autonomous Chua's circuit with two OptoNDR elements, OptoNDR elements (No. 1 and No. 2) contain phototransistor *Q*1 coupled with LED *D*1 and phototransistor *Q*2 coupled with LED *D*2, respectively. The circuit contains three energy storage elements (Inductance *L*1, Capacitors *C*1, and *C*2). Chua's circuit exhibits different and unique variety of bifurcations and chaos among the chaos-producing mechanism with OptoNDR elements. OptoNDR elements are a current-controlled nonlinear elements characterized by $V_{\text{OptoNDR1}} = g_1(i_{NDR1})$ and $V_{\text{OptoNDR2}} = g_1(i_{NDR2})$ (Fig. 5.20).
We consider that the two optocouplers are not the same $\alpha_{r1} \neq \alpha_{r2}$; $\alpha_{f1} \neq \alpha_{f2;}$ $k_1 \neq k_2$. $i_R = i_{\text{OptoNDR1.}}$ $I_{BQi=}k_i \cdot I_{Di}$ for $i = 1, 2$.

5.1 Find circuit's differential equations and represent the main variables as *X*(*t*), *Y*(*t*), *Z*(*t*).
5.2 Find circuit's fixed points. How circuit's fixed points are dependent on k_1, k_2 circuit's parameters?
5.3 Find Chua's circuit OptoNDR's Jacobian matrix and elements. Discuss stability for possible circuit's fixed points.

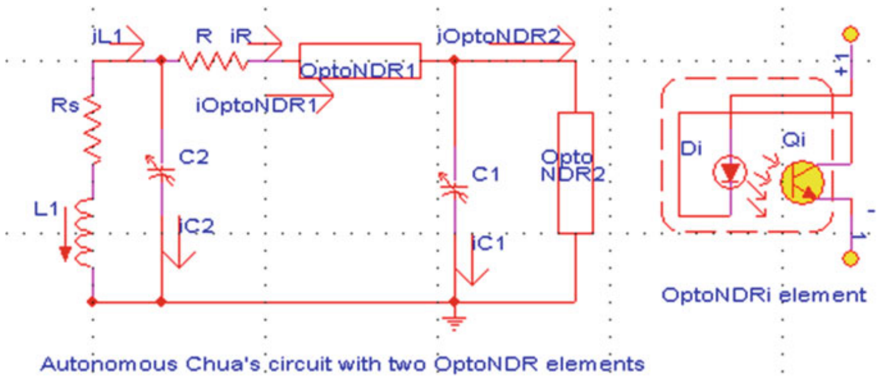
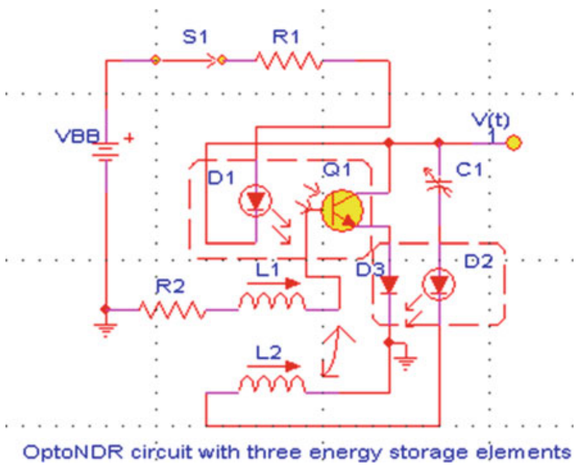


Fig. 5.20 Autonomous Chua's circuit with two OptoNDR elements, OptoNDR elements (No. 1 and No. 2) contain phototransistor $Q1$ coupled with LED $D1$ and phototransistor $Q2$ coupled with LED $D2$, respectively

5.4 How system stability changes for various k_1, k_2 parameters values. We disconnect OptoNDR2 element. How circuit's fixed points and stability analysis change.

6. We have an OptoNDR circuit with three storage elements (variable capacitor and variable Inductances see Fig. 5.21). At $t = 0$ switch $S1$ moves his position from OFF state to ON state. Circuit dynamic starts and $V(t)$ is the main system variable. We consider that initially capacitor $C1$ is charged to $VC1(t = 0)$ and inductances $L1$ and $L2$ are charged to $IL1(t = 0)$ and $IL1(t = 0)$, respectively. $VBB \gg VQ1$(sustain) and photo transistor $Q1$ can reach his sustaining voltage (breakover voltage) after we move switch $S1$ to ON state. k_1 is the coupling coefficient between LED $D1$ and photo transistor $Q1$. $D2$ (LED) is coupled with $D3$ (photo diode), k_2 is the coupling coefficient ($I_{D3} = k_2 \cdot I_{D2}$). $2{\cdot}L_m$ element

Fig. 5.21 OptoNDR circuit with three storage elements (variable capacitor and variable Inductances)



represents the mutual inductance between $L1$ and $L2$. $L_m = \gamma \cdot \sqrt{L_1 \cdot L_2}$, γ is the coupling coefficient of two inductors $0 \le \gamma \le 1$. After we move $S1$ to ON state capacitor $C1$ starts to charge and once $VCEQ1$ reaches sustain voltage the current through inductor $L1$ rise up.

6.1 Write system equations and differential equations
6.2 How system stability is dependent on coupling coefficient γ?
6.3 Discuss stability of system periodic orbit by using Floquet theory.
6.4 How our system Floquet multipliers are dependent on parameters k_1, k_2, and γ?
6.5 Discuss system parameters variation which cause to periodic orbit transition from unstable to stable.

7. We have OptoNDR system with three global variables: X, Y, Z.

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \psi_1(X, Y, Z); \frac{\mathrm{d}Y}{\mathrm{d}t} = \psi_2(X, Y, Z); \frac{\mathrm{d}Z}{\mathrm{d}t} = \psi_3(X, Y, Z) + \sqrt{\Xi}$$

7.1 Draw possible system circuits implementation which fulfils OptoNDR system with three global parameters.
7.2 Find $\psi_i (i = 1,2,3)$ functions and find how system stability is dependent on these functions.
7.3 Express Ξ global parameter as a function of circuit parameters.
7.4 Discuss stability of system periodic orbit by using Floquet theory.
7.5 Discuss system parameters variation which cause to periodic orbit transition from stable to unstable.
7.6 Find circuit implementation which represents by only two global variables $\frac{\mathrm{d}X}{\mathrm{d}t} = \psi_1(X, Z); \frac{\mathrm{d}Z}{\mathrm{d}t} = \psi_3(X, Z) + \frac{\sqrt{\Xi}}{\sin\left(\frac{2 \cdot \pi}{\Xi^2} \cdot t\right)}$. Discuss stability switching for various values of Ξ global parameter.

8. We have optoisolation system with two global variables $X_1(t)$ and $X_2(t)$. The global variables grow or shrink exponentially at rate $r_i(t)$ with $r_1(t) = \cos(2 \cdot \pi \cdot t); r_2(t) = -\cos(2 \cdot \pi \cdot t)$, so that they change from sources ($r > 0$) to sinks ($r < 0$) perfectly out of phase with period $T = 1$. The global variables $X_1(t)$ and $X_2(t)$ are coupled by random disposal with rate ξ. The model for this situation is $\frac{\mathrm{d}X_1}{\mathrm{d}t} = r_1(t) \cdot X_1 + \xi \cdot (X_2 - X_1)$ $\frac{\mathrm{d}X_2}{\mathrm{d}t} = r_2(t) \cdot X_2 + \xi \cdot (X_1 - X_2)$.

8.1 Find possible circuits implementation for our system by using optoisolation elements and discrete components.
8.2 If the two global variables are completely uncouples ($\xi = 0$), then the growth rate for each global variable varies from -1 to $+1$. The average growth rate is zero. Discuss circuit's parameters, stability, Floquet exponents, and Floquet multipliers. Find $\max(\mathrm{Re}(\mu_i))$ as a function of optoisolation circuit parameters.

8.3 What happened when we exchange between periodic sources $r_1(t)$ and $r_2(t)$, $r_1(t) \rightleftharpoons r_2(t)$? How system dynamic and stability behavior change?

8.4 If the two global variables are completely well-mixed ($\xi \to \infty$), what changes happened? Draw circuit implementation and discuss all related Eqs.

8.5 We have new functions for periodic sources: $r_1(t) = \sin(2 \cdot \pi \cdot t) + \cos(2 \cdot \pi \cdot t)$ $r_2(t) = \sin(2 \cdot \pi \cdot t) - \cos(2 \cdot \pi \cdot t)$. Analyze system dynamic and limit cycle stability by using Floquet theory.

9. Figure 5.22 OptoNDR circuit with periodic source $a(t)$ includes two resistors (R_1, R_{cQ1}), two inductors ($L1$, $L2$), and one capacitor $C1$. The circuit includes two voltage sources $V_{cc}1$, $V_{cc}2$ which there voltages are much bigger than photo transistors break voltages $V_{CC1} \gg V_{Q1break} V_{CC2} \gg V_{Q2break}$. The multiplication element ([]*[]) is implemented by using op-amps, resistors, capacitors, diodes, etc. Multiplication element's input current is zero since his input impedance is infinite $I_{in\{[]*[]\}} \to \varepsilon$ $R_{in\{[]*[]\}} \to \infty$. At $t = 0$ switch $S1$ move from OFF state to ON state and after τ s switch $S2$ move from OFF state to ON state $S1 = U(t)$, $S2 = U(t - \tau)$. You can neglect the coupling between the two inductors.

9.1 Find circuit differential equations and define global parameters.

9.2 Find system Jacobian $A(t)$ matrix at limit cycle.

9.3 Find Floquet multipliers and discuss the system stability of a limit cycle.

9.4 Return your calculations (9.1, 9.2, 9.3) for $\tau = 0$ and $\tau \to \infty$.

9.5 How our system stability of a limit cycle is dependent on τ parameter?

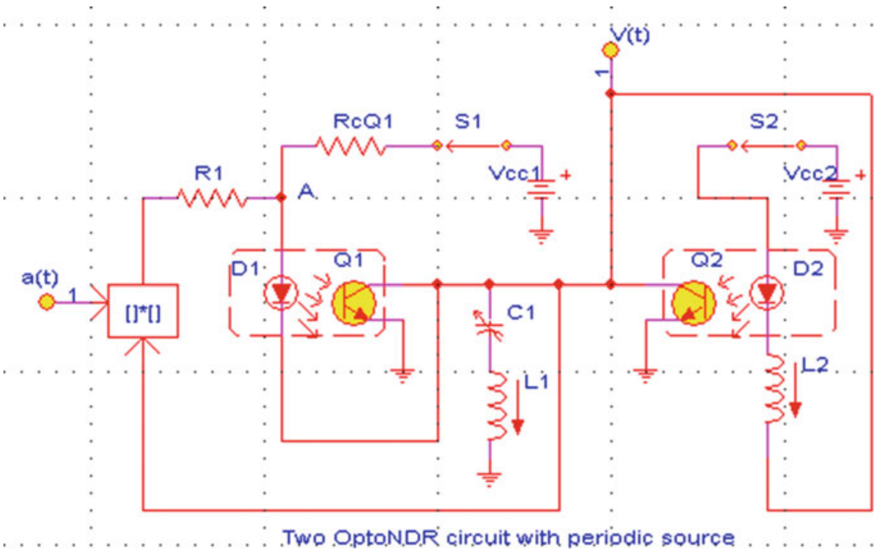


Fig. 5.22 OptoNDR circuit with periodic source $a(t)$ includes two resistors (R_1, R_{cQ1}), two inductors ($L1$, $L2$), and one capacitor $C1$

9.6 How our circuit analysis results and system stability of a limit cycle change if, first we switch *S*2 OFF → ON and after τ s switch *S*1 OFF → ON.

Remark Two photo transistors and LEDs have different parameters.

10. We have optoisolation system with three global variables: X_1, X_2,X_3, and global parameters: Γ_i, Ξ_i, Ω_i for i = 1, 2. Our system differential equations:

$$\frac{dX_1}{dt} = X_1 \cdot (1 - X_1) - \frac{\Gamma_1 \cdot X_1}{1 + \Omega_1 \cdot X_1} \cdot X_2; \frac{dX_3}{dt} = \frac{\Gamma_2 \cdot X_2}{1 + \Omega_2 \cdot X_2} \cdot X_3 - \Xi_2 \cdot X_3$$
$$\frac{dX_2}{dt} = \frac{\Gamma_1 \cdot X_1}{1 + \Omega_1 \cdot X_1} \cdot X_2 - \frac{\Gamma_2 \cdot X_2}{1 + \Omega_2 \cdot X_2} \cdot X_3 + sign(\Xi_1) \cdot \Xi_1 \cdot X_2$$

$$sign(\Xi_1) = +1 \text{ for } \Xi_1 > 0; sign(\Xi_1) = -1 \text{ for } \Xi_1 < 0$$

10.1 Draw possible system circuits implementation which fulfils OptoNDR system with three global parameters.
10.2 Find Γ_i, Ξ_i, Ω_i for i = 1, 2 global parameters and find how system stability is dependent on these parameters.
10.3 Discuss stability of system periodic orbit by using Floquet theory.
10.4 Express Γ_i, Ξ_i, Ω_i for i = 1, 2 global parameters as a function of circuit parameters.
10.5 Discuss system parameters variation which cause to periodic orbit transition from stable to unstable.
10.6 Find circuit implementation which represents by only two global variables:

$$\frac{dX_1}{dt} = X_1 \cdot (1 - X_1) - \frac{\Gamma_1 \cdot X_1}{1 + \Omega_1 \cdot X_1} \cdot X_2$$
$$\frac{dX_2}{dt} = \frac{\Gamma_1 \cdot X_1}{1 + \Omega_1 \cdot X_1} \cdot X_2 - \frac{\Gamma_2 \cdot X_2}{1 + \Omega_2 \cdot X_2} \cdot X_1 - sign(\sqrt{\Xi_1}) \cdot \Xi_1 \cdot X_2$$

Discuss stability switching for various values of global parameters.

Chapter 6
Optoisolation Circuits with Periodic Limit Cycle Solutions Orbital Stability

Optoisolation systems periodic orbits are frequently encountered as trajectories. We use solutions of initial value problems, as a mean of finding stable periodic orbits of vector fields. We can represent our system as a vector fields in Euclidean space, systems of differential equations: $\frac{dX}{dt} = f(X, \Omega);\ X \in \mathbb{R}^n;\ \Omega \in \mathbb{R}^k$ with Ω a vector of optoisolation circuit parameters. Periodic orbits are nonequilibrium trajectories $X(t)$ that satisfy $X(T) = X(0)$ for some $T > 0$. The smallest such T is the period of the orbit. The local dynamics near a periodic orbit are typically determined by return map. The return map has a fixed point at its intersection with a periodic orbit. If the Jacobian of the return map at this fixed point has eigenvalues inside the unit circle, the orbit is asymptotically stable. Initial conditions in the Basin of Attraction (BOA) of the periodic orbit have trajectories whose limit set is the periodic orbit. It is the right circuit design issue, by choosing the optoisolation elements and discrete components, topological circuit structure which fulfill system periodic orbits [5–8].

6.1 Planar Cubic Vector Field and Van der Pol Equation

We can represent our planar cubic vector field system as two differential equations: $\frac{dX_1}{dt} = X_2;\ \frac{dX_2}{dt} = -(X_1^3 + r \cdot X_1^2 + n \cdot X_1 + m) + (\Gamma - X_1^2) \cdot X_2$. The system contains the unfolding of a co-dimension two bifurcation of an equilibrium point with a double eigenvalue zero in the presence of a rotational symmetry of the plane. To find system fixed points we set $dX_1/dt = 0$; $dX_2/dt = 0$. $\frac{dX_1}{dt} = 0\ \&\ \frac{dX_2}{dt} = 0 \Rightarrow X_2 = 0 \Rightarrow X_1^3 + r \cdot X_1^2 + n \cdot X_1 + m = 0$. We need to find the roots of $P(X_1) = 0;\ P(X_1) = X_1^3 + r \cdot X_1^2 + n \cdot X_1 + m$. Every cubic equation with real coefficients has at least one solution X_1 among the real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant, $\Delta = -4 \cdot r^3 \cdot m + r^2 \cdot n^2 - 4 \cdot n^3 + 18 \cdot r \cdot n \cdot m - 27 \cdot m^2$. If $\Delta > 0$,

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DOI 10.1007/978-3-319-55316-0_6

then the equation has three distinct real roots. If $\Delta < 0$, then the equation has one real root and a pair of complex conjugate roots. If $\Delta = 0$, then (at least) two roots coincide. We consider our system roots are only real [9]. We define the root of $P(X_1) = 0 \Rightarrow X_1^*$ and we define our system fixed points :

$$E^{(j)} = (X_1^*, 0).\ \frac{dX_1}{dt} = f_1(X_1, X_2);\ f_1(X_1, X_2) = X_2;\ \frac{dX_2}{dt} = f_2(X_1, X_2)$$

$f_2(X_1, X_2) = -(X_1^3 + r \cdot X_1^2 + n \cdot X_1 + m) + (\Gamma - X_1^2) \cdot X_2$. Our Jacobian Matrix at the fixed point $E^{(j)} = (X_1^*, 0)$ is $A = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix}_{E^{(j)}=(X_1^*,0)}$. $\frac{\partial f_1}{\partial X_1} = 0;\ \frac{\partial f_1}{\partial X_2} = 1$

$$\frac{\partial f_2}{\partial X_1} = -(3 \cdot X_1^2 + r \cdot 2 \cdot X_1 + n) - 2 \cdot X_1 \cdot X_2;\ \frac{\partial f_2}{\partial X_2} = \Gamma - X_1^2$$

$$\frac{\partial f_2}{\partial X_1}|_{E^{(j)}=(X_1^*,0)} = -(3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n);\ \frac{\partial f_2}{\partial X_2}|_{E^{(j)}=(X_1^*,0)} = \Gamma - [X_1^*]^2$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix}_{E^{(j)}=(X_1^*,0)} = \begin{pmatrix} 0 & 1 \\ -(3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n) & \Gamma - [X_1^*]^2 \end{pmatrix};\ \det|A - \lambda \cdot I| = 0$$

$$A - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 \\ -(3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n) & \Gamma - [X_1^*]^2 - \lambda \end{pmatrix}$$

$$\det|A - \lambda \cdot I| = 0 \Rightarrow -\lambda \cdot (\Gamma - [X_1^*]^2 - \lambda) + (3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n) = 0$$
$$\lambda^2 - \lambda \cdot (\Gamma - [X_1^*]^2) + (3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n) = 0$$

$\lambda_{1,2} = \frac{(\Gamma - [X_1^*]^2) \pm \sqrt{(\Gamma - [X_1^*]^2)^2 - 4 \cdot (3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n)}}{2}$. We have the following possibilities for our system eigenvalues: $\lambda_1 > 0, \lambda_2 > 0$ our fixed point is unstable spiral. $\lambda_1 < 0, \lambda_2 < 0$ our fixed point is stable node. $\lambda_1 \cdot \lambda_2 < 0$ then our fixed point is saddle node. $\lambda_{1,2}$ complex then $\lambda_{1,2} = \alpha \pm j \cdot \omega$ and $\alpha = \mathrm{Re}(\lambda_{1,2}) < 0$ then decaying oscillations spiral (stable point spiral). $\lambda_{1,2}$ complex then $\lambda_{1,2} = \alpha \pm j \cdot \omega$ and $\alpha = \mathrm{Re}(\lambda_{1,2}) > 0$ then growing oscillations spiral (unstable point spiral). $\lambda_{1,2}$ complex then $\lambda_{1,2} = \alpha \pm j \cdot \omega$ and $\alpha = \mathrm{Re}(\lambda_{1,2}) = 0$ then eigenvalues are pure imaginary and the solutions are periodic with period $T = 2\pi/\omega$. Our interest is in periodic solution and we need to find the system parameters conditions. For getting complex eigenvalues we get the condition: $(\Gamma - [X_1^*]^2)^2 - 4 \cdot (3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n) < 0$; $\alpha = \frac{(\Gamma - [X_1^*]^2)}{2}$ and periodic solution $\alpha = 0 \Rightarrow \frac{(\Gamma - [X_1^*]^2)}{2} = 0 \Rightarrow \Gamma = [X_1^*]^2$; $\omega = \frac{\sqrt{(\Gamma - [X_1^*]^2)^2 - 4 \cdot (3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n)}}{2}$ $T = \frac{2 \cdot \pi}{\omega} = \frac{4 \cdot \pi}{\sqrt{(\Gamma - [X_1^*]^2)^2 - 4 \cdot (3 \cdot [X_1^*]^2 + r \cdot 2 \cdot X_1^* + n)}}$. The oscillations have fixed amplitude and the fixed point is a center. We need to prove that the system has periodic orbits and it is done by changing system cartesian coordinates ($X_1(t)$, $X_2(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z axis. In our system we refer to Cartesian $X_1 - X_2$ plane (with equation $X_3 = 0$). Then the z- coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X_1, X_2) are the same as for polar coordinates, namely $X_1(t) = r(t) \cdot \cos[\theta(t)]$; $X_2(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X_1^2 + X_2^2}$. $\theta(t) = 0$ if $X_1 = 0$ and $X_2 = 0$. $\theta(t) = \arcsin(X_2/r)$ if $X_1 \geq 0$. $x \to X_1$, $y \to X_2$. $X_1(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{dX_1(t)}{dt} = \frac{dr(t)}{dt} \cdot \cos[\theta(t)] - r(t) \cdot \frac{d\theta(t)}{dt} \cdot \sin[\theta(t)]$

$$X_2(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{dX_2(t)}{dt} = \frac{dr(t)}{dt} \cdot \sin[\theta(t)] + r(t) \cdot \frac{d\theta(t)}{dt} \cdot \cos[\theta(t)]$$
$$\frac{dX_1(t)}{dt} = \frac{dX_1}{dt}; \ \frac{dr(t)}{dt} = r'; \ \frac{d\theta(t)}{dt} = \theta'; \ \theta(t) = \theta; \ r(t) = r$$

We get the following equations:

$$\frac{dX_1}{dt} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta; \ \frac{dX_2}{dt} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$
$$\frac{dX_1}{dt} = X_2 \Rightarrow r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = r \cdot \sin\theta \Rightarrow r' \cdot \cos\theta = r \cdot \theta' \cdot \sin\theta + r \cdot \sin\theta$$
$$r' \cdot \cos\theta = r \cdot \theta' \cdot \sin\theta + r \cdot \sin\theta \Rightarrow r' \cdot \cos\theta = r \cdot \sin\theta \cdot (\theta' + 1) \Rightarrow tg\theta = \frac{r'}{r \cdot (\theta' + 1)}$$
$$tg\theta = \frac{r'}{r \cdot (\theta' + 1)} \Rightarrow \theta = \mathrm{arctg}\left\{\frac{r'}{r \cdot (\theta' + 1)}\right\}; \ X_1 = r \cdot \cos\theta; \ X_2 = r \cdot \sin\theta$$
$$\frac{dX_2}{dt} = -(X_1^3 + r_{\#} \cdot X_1^2 + n \cdot X_1 + m) + (\Gamma - X_1^2) \cdot X_2; \ r \to r_{\#}$$

$$r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = -(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) + (\Gamma - r^2 \cdot \cos^2\theta) \cdot r \cdot \sin\theta$$

We can summarize our system as two differential equations in r, θ.

(*) $tg\theta = \frac{r'}{r \cdot (\theta' + 1)} \Rightarrow r' = r \cdot (\theta' + 1) \cdot tg\theta$

(**) $r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = -(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) + (\Gamma - r^2 \cdot \cos^2\theta) \cdot r \cdot \sin\theta$

(*) → (**)

$$r \cdot (\theta' + 1) \cdot tg\theta \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = -(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) + (\Gamma - r^2 \cdot \cos^2\theta) \cdot r \cdot \sin\theta$$

$$r \cdot \theta' \cdot [tg\theta \cdot \sin\theta + \cos\theta] = -(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) + (\Gamma - r^2 \cdot \cos^2\theta) \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta$$

$$tg\theta \cdot \sin\theta + \cos\theta = \frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$$

$$\theta' = -\frac{1}{r} \cdot (r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - \sin^2\theta$$

$$r' = r \cdot (\theta' + 1) \cdot tg\theta \Rightarrow r' = \{-(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + r \cdot (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - r \cdot \sin^2\theta + r\} \cdot tg\theta$$

Finally, we get two differential equations in r and θ:

$$\theta' = -\frac{1}{r} \cdot (r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - \sin^2\theta$$

$$r' = \{-(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + r \cdot (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - r \cdot \sin^2\theta + r\} \cdot tg\theta$$

If there is a stable limit cycle as specific value of radius r, dr/dt = 0 then

$$r' = 0 \Rightarrow \{-(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + r \cdot (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - r \cdot \sin^2\theta + r\} \cdot tg\theta = 0.$$

Option I $tg\,\theta = 0 \Rightarrow \theta = k \cdot \pi \quad \forall k = \ldots -2, -1, 0, 1, 2, \ldots$ k is integer number. We have two cases: k is even (k = 0, 2, 4, …), $\sin\theta = 0\,\&\,\cos\theta = 1 \Rightarrow \theta' = -\frac{1}{r} \cdot (r^3 + r_{\#} \cdot r^2 + n \cdot r + m)$. k is odd (k = 1, 3, 5, …), $\sin\theta = 0\,\&\,\cos\theta = -1 \Rightarrow \theta' = \frac{1}{r} \cdot (-r^3 + r_{\#} \cdot r^2 - n \cdot r + m)$.

Option II

$$-(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + r \cdot (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - r \cdot \sin^2\theta + r = 0$$

$$(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta = r \cdot (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - r \cdot \sin^2\theta + r$$

$$\theta' = -\frac{1}{r}\cdot\{r\cdot(\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - r\cdot\sin^2\theta + r\}$$
$$+(\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - \sin^2\theta$$

$$\theta' = -\{(\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - \sin^2\theta + 1\}$$
$$+(\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - \sin^2\theta$$

$$\theta' = -(\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - \cos^2\theta + (\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - \sin^2\theta$$
$$\Rightarrow \theta' = -1$$

$$\theta' = -1 \Rightarrow \frac{d\theta}{dt} = -1 \Rightarrow \theta = -t + \text{const};\ \frac{d\theta}{dt} = \frac{2\cdot\pi}{T} = -1 \Rightarrow T = -2\cdot\pi$$

$$r^3\cdot\cos^4\theta + r_{\#}\cdot r^2\cdot\cos^3\theta + n\cdot r\cdot\cos^2\theta + m\cdot\cos\theta = r\cdot\Gamma\cdot\sin\theta\cdot\cos\theta$$
$$- r^3\cdot\cos^2\theta\cdot\sin\theta\cdot\cos\theta - r\cdot\sin^2\theta + r$$

$$r^3\cdot\cos^3\theta\cdot[\cos\theta + \sin\theta] + r^2\cdot r_{\#}\cdot\cos^3\theta + r\cdot[n\cdot\cos^2\theta - \Gamma\cdot\sin\theta\cdot\cos\theta + \sin^2\theta - 1]$$
$$+ m\cdot\cos\theta = 0$$

Other case is when system limit cycle is resided in annulus $0 < r_{\min} < r < r_{\max}$ and there is no constant r in limit cycle $dr/dt \neq 0$. We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$. $dr/dt < 0$ on the outer circle and $dr/dt > 0$ on the inner circle. To find $r_{\min}$, we require $dr/dt > 0$ for all values of θ:

$$\{-(r^3\cdot\cos^3\theta + r_{\#}\cdot r^2\cdot\cos^2\theta + n\cdot r\cdot\cos\theta + m)\cdot\cos\theta + r\cdot(\Gamma - r^2\cdot\cos^2\theta)$$
$$\cdot\sin\theta\cdot\cos\theta - r\cdot\sin^2\theta + r\}\cdot tg\theta > 0.$$

We have two subcases to fulfill $dr/dt > 0$ for all values of θ (find $r_{\min}$):

(1) $\{-(r^3\cdot\cos^3\theta + r_{\#}\cdot r^2\cdot\cos^2\theta + n\cdot r\cdot\cos\theta + m)\cdot\cos\theta + r\cdot(\Gamma - r^2\cdot\cos^2\theta)\cdot\sin\theta\cdot\cos\theta - r\cdot\sin^2\theta + r\} > 0$ and $tg\theta > 0$. $tg\theta > 0 \Rightarrow \pi\cdot k < \theta < \pi\cdot\left(\frac{1}{2} + k\right)\ \ \forall\, k = \ldots, -1, 0, +1, \ldots$ then $0 < \cos\theta < 1\ \&\ 0 < \sin\theta < 1$ or $-1 < \cos\theta < 0\ \&\ -1 < \sin\theta < 0$. First, we choose the option: $0 < \cos\theta < 1\ \&\ 0 < \sin\theta < 1$, the first limit $\sin\theta = 1$ and $\cos\theta = 0$ then $\{-r + r\} > 0$ which is not true, second limit $\sin\theta = 0$ and $\cos\theta = 1$ $\{-(r^3 + r_{\#}\cdot r^2 + n\cdot r + m) + r\} > 0 \Rightarrow r^3 + r_{\#}\cdot r^2 + r\cdot(n-1) + m < 0$, we need to investigate the cubic function. If $r_{\#} = 0;\ n = 1;\ m = 0 \Rightarrow r^3 < 0 \Rightarrow r \to r_{\min} \Rightarrow r_{\min} < 0$ and since $r_{\min}$ is always positive this is not true in our case. The cubic equation with real coefficients has at least one solution $r_{\min}$ among real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant (Δ) [64]. $\Delta = 18\cdot r_{\#}\cdot(n-1)\cdot m - 4\cdot r_{\#}^3\cdot m + r_{\#}^2\cdot(n-1)^2 - 4\cdot(n-1)^3 - 27\cdot m^2$. If $\Delta > 0$ we have three distinct real roots. If $\Delta = 0$ then we have multiple root

and all its roots are real. If $\Delta < 0$ then the equation has one real root and two non-real complex conjugate roots. In case of three distinct real roots, $m = 0$ then $r^3 + r_{\#} \cdot r^2 + r \cdot (n-1) = 0 \Rightarrow r \cdot [r^2 + r_{\#} \cdot r + (n-1)] = 0 \Rightarrow r_1 = 0;$ $r_{2,3} = \frac{-r_{\#} \pm \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$. We have two graph peaks: $r_{\text{peaks}} = \frac{-r_{\#} \pm \sqrt{r_{\#}^2 - 3 \cdot (n-1)}}{3}$; upon location of the peaks we have two possibilities: $0 < r_{\min} < \frac{-r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$ or $\frac{-r_{\#} - \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2} < r_{\min} < 0$ not exist since $r_{\min}$ is always positive. The only possible solution is $0 < r_{\min} < \frac{-r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$. In case of $m \neq 0$ then we need to solve it numerically and find $r_{\min}$ real values which cubic function get negative values. Second, we choose the option $-1 < \cos\theta < 0 \;\&\; -1 < \sin\theta < 0$ the first limit $\sin\theta = 0$ and $\cos\theta = -1$ then $-r^3 + r_{\#} \cdot r^2 - n \cdot r + m + r > 0 \Rightarrow r^3 - r_{\#} \cdot r^2 + r \cdot (n-1) - m < 0 \quad r \to r_{\min}$. if $r_{\#} = 0;\; n = 1;\; m = 0 \Rightarrow r^3 < 0 \Rightarrow r \to r_{\min} \Rightarrow r_{\min} < 0$ and since $r_{\min}$ is always positive this is not true in our case. The cubic equation with real coefficients has at least one solution $r_{\min}$ among real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant (Δ). $\Delta = 18 \cdot r_{\#} \cdot (n-1) \cdot m - 4 \cdot r_{\#}^3 \cdot m + r_{\#}^2 \cdot (n-1)^2 - 4 \cdot (n-1)^3 - 27 \cdot m^2$. If $\Delta > 0$ we have three distinct real roots. If $\Delta = 0$ then we have multiple root and all its roots are real. If $\Delta < 0$ then the equation has one real root and two non-real complex conjugate roots. In case of three distinct real roots, $m = 0$ then $r^3 - r_{\#} \cdot r^2 + r \cdot (n-1) = 0 \Rightarrow r \cdot [r^2 - r \cdot r_{\#} + (n-1)] = 0 \Rightarrow r_1 = 0;\; r_{2,3} = \frac{r_{\#} \pm \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$. We have two graph peaks: $r_{\text{peaks}} = \frac{r_{\#} \pm \sqrt{r_{\#}^2 - 3 \cdot (n-1)}}{3}$; upon location of the peaks we have two possibilities: $0 < r_{\min} < \frac{r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$ or $\frac{r_{\#} - \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2} < r_{\min} < 0$ not exist since $r_{\min}$ is always positive. The only possible solution is $0 < r_{\min} < \frac{r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$. In case of $m \neq 0$ then we need to solve it numerically and find $r_{\min}$ real values which cubic function get negative values. Second limit $\sin\theta = -1$ and $\cos\theta = 0$ then $\{-r + r\} > 0$ which is impossible.

Remark The upper limit of $r_{\min}$ must be positive value and $r_{\#}^2 > 4 \cdot (n-1)$, we get two possible $r_{\min}$ upper limits: $\frac{-r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$ and $\frac{r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$, choose the positive one of them. In case $r_{\#}^2 = 4 \cdot (n-1)$. Then possible $r_{\min}$ upper limits are $\frac{-r_{\#}}{2}, \frac{r_{\#}}{2}$ and we chose the positive of them.

(2) $\{-(r^3 \cdot \cos^3\theta + r_{\#} \cdot r^2 \cdot \cos^2\theta + n \cdot r \cdot \cos\theta + m) \cdot \cos\theta + r \cdot (\Gamma - r^2 \cdot \cos^2\theta) \cdot \sin\theta \cdot \cos\theta - r \cdot \sin^2\theta + r\} < 0$

And $tg\theta < 0$. $tg\theta < 0 \Rightarrow \pi \cdot (k - \frac{1}{2}) < \theta < \pi \cdot k \quad \forall\, k = \ldots, -1, 0, +1, \ldots$, then $0 < \sin\theta < 1\ \&\ -1 < \cos\theta < 0$ or $1 > \cos\theta > 0\ \&\ -1 < \sin\theta < 0$. First, we choose the option: $0 < \sin\theta < 1\ \&\ -1 < \cos\theta < 0$ the first limit $\sin\theta = 1$ and $\cos\theta = 0$ then $\{-r + r\} < 0$ which is not true. Second limit $\sin\theta = 0$ and $\cos\theta = -1$ $-r^3 + r_{\#} \cdot r^2 - n \cdot r + m + r < 0 \Rightarrow r^3 - r_{\#} \cdot r^2 + r \cdot (n-1) - m > 0$. We need to investigate the cubic function. If $r_{\#} = 0$; $n = 1$; $m = 0 \Rightarrow r^3 > 0 \Rightarrow r \to r_{\min} \Rightarrow r_{\min} > 0$ anyway this is the base condition. The cubic equation with real coefficients has at least one solution $r_{\min}$ among real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant (Δ). $\Delta = 18 \cdot r_{\#} \cdot (n-1) \cdot m - 4 \cdot r_{\#}^3 \cdot m + r_{\#}^2 \cdot (n-1)^2 - 4 \cdot (n-1)^3 - 27 \cdot m^2$. If $\Delta > 0$ we have three distinct real roots. If $\Delta = 0$ then we have multiple root and all its roots are real. If $\Delta < 0$ then the equation has one real root and two non-real complex conjugate roots. In case of three distinct real roots, $m = 0$ then $r^3 + r_{\#} \cdot r^2 + r \cdot (n-1) = 0 \Rightarrow r \cdot [r^2 - r_{\#} \cdot r + (n-1)] = 0 \Rightarrow r_1 = 0$; $r_{2,3} = \frac{r_{\#} \pm \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$.

We have two graph peaks: $r_{\text{peaks}} = \frac{r_{\#} \pm \sqrt{r_{\#}^2 - 3 \cdot (n-1)}}{3}$; upon location of the peaks we have two possibilities: $0 < r_{\min} < \frac{r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$ or $\frac{r_{\#} - \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2} < r_{\min} < 0$ not exist since $r_{\min}$ is always positive [6, 7]. In case of $m \neq 0$ then we need to solve it numerically and find $r_{\min}$ real values which cubic function get positive values. Second, we choose the option $1 > \cos\theta > 0\ \&\ -1 < \sin\theta < 0$ the first limit $\sin\theta = 1$ and $\cos\theta = 1$ then $-r^3 - r_{\#} \cdot r^2 - n \cdot r - m + r < 0 \Rightarrow r^3 + r_{\#} \cdot r^2 + r \cdot (n-1) + m > 0$. We need to investigate the cubic function. If $r_{\#} = 0$; $n = 1$; $m = 0 \Rightarrow r^3 > 0 \Rightarrow r \to r_{\min} \Rightarrow r_{\min} > 0$ anyway this is the base condition. The cubic equation with real coefficients has at least one solution $r_{\min}$ among real numbers; this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant (Δ). $\Delta = 18 \cdot r_{\#} \cdot (n-1) \cdot m - 4 \cdot r_{\#}^3 \cdot m + r_{\#}^2 \cdot (n-1)^2 - 4 \cdot (n-1)^3 - 27 \cdot m^2$. If $\Delta > 0$ we have three distinct real roots. If $\Delta = 0$ then we have multiple root and all its roots are real. If $\Delta < 0$ then the equation has one real root and two non-real complex conjugate roots. In case of three distinct real roots, $m = 0$ then

$$r^3 + r_{\#} \cdot r^2 + r \cdot (n-1) - m = 0|_{m=0} \Rightarrow r \cdot [r^2 + r_{\#} \cdot r + (n-1)] = 0$$

$$r^3 + r_{\#} \cdot r^2 + r \cdot (n-1) = 0 \Rightarrow r \cdot [r^2 + r_{\#} \cdot r + (n-1)] = 0 \Rightarrow r_1 = 0;$$

$$r_{2,3} = \frac{-r_{\#} \pm \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$$

We have two graph peaks: $r_{\text{peaks}} = \frac{-r_{\#} \pm \sqrt{r_{\#}^2 - 3 \cdot (n-1)}}{3}$; upon location of the peaks we have two possibilities: $0 < r_{\min} < \frac{-r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$ or $\frac{-r_{\#} - \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2} < r_{\min} < 0$ not exist since $r_{\min}$ is always positive. In case of $m \neq 0$ then we need to solve it numerically and find $r_{\min}$ real values which cubic function get positive values. Second limit $\sin \theta = -1$ and $\cos \theta = 0$, then $\{-r + r\} < 0$ which is not true.

Remark The upper limit of $r_{\min}$ must be positive value and $r_{\#}^2 > 4 \cdot (n-1)$, we get two possible $r_{\min}$ upper limits: $\frac{r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$ and $\frac{-r_{\#} + \sqrt{r_{\#}^2 - 4 \cdot (n-1)}}{2}$, choose the positive one of them. In case $r_{\#}^2 = 4 \cdot (n-1)$ then possible $r_{\min}$ upper limits: $\frac{-r_{\#}}{2}, \frac{r_{\#}}{2}$ and we chose the positive of them.

The same analysis we do for finding $r_{\max}$ in the outer circle, $dr/dt < 0$. To find $r_{\max}$, we require $dr/dt < 0$ for all values of θ:

$$r' = \{-(r^3 \cdot \cos^3 \theta + r_{\#} \cdot r^2 \cdot \cos^2 \theta + n \cdot r \cdot \cos \theta + m) \cdot \cos \theta + r \cdot (\Gamma - r^2 \cdot \cos^2 \theta) \cdot \sin \theta \cdot \cos \theta - r \cdot \sin^2 \theta + r\} \cdot tg\theta$$

$$r' < 0 \Rightarrow r' = \{-(r^3 \cdot \cos^3 \theta + r_{\#} \cdot r^2 \cdot \cos^2 \theta + n \cdot r \cdot \cos \theta + m) \cdot \cos \theta + r \cdot (\Gamma - r^2 \cdot \cos^2 \theta) \cdot \sin \theta \cdot \cos \theta - r \cdot \sin^2 \theta + r\} \cdot tg\theta < 0$$

We have two subcases to fulfill $dr/dt < 0$ for all values of θ (find $r_{\max}$):

(1) $\{-(r^3 \cdot \cos^3 \theta + r_{\#} \cdot r^2 \cdot \cos^2 \theta + n \cdot r \cdot \cos \theta + m) \cdot \cos \theta + r \cdot (\Gamma - r^2 \cdot \cos^2 \theta) \cdot \sin \theta \cdot \cos \theta - r \cdot \sin^2 \theta + r\} > 0$ and $tg\theta < 0$. The next step is to finding $r_{\max}$ expression.

(2) $\{-(r^3 \cdot \cos^3 \theta + r_{\#} \cdot r^2 \cdot \cos^2 \theta + n \cdot r \cdot \cos \theta + m) \cdot \cos \theta + r \cdot (\Gamma - r^2 \cdot \cos^2 \theta) \cdot \sin \theta \cdot \cos \theta - r \cdot \sin^2 \theta + r\} < 0$ and $tg\theta > 0$. The next step is to finding $r_{\max}$ expression.

The Van der Pol oscillator can be given by the following equations: $\frac{dX_1}{dt} = X_2$ and $\frac{dX_2}{dt} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2$. We can express it by second-degree differential equation: $\frac{d^2 X_1}{dt^2} + \mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{dt} + X_1 = 0$. To find system fixed points we set $\frac{dX_1}{dt} = 0$; $\frac{dX_2}{dt} = 0$ then $X_1^* = 0$; $X_2^* = 0$. We consider the system $\frac{dX_1}{dt} = f_1(X_1, X_2)$; $\frac{dX_2}{dt} = f_2(X_1, X_2)$; $f_1(X_1, X_2) = X_2$; $f_2(X_1, X_2) = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2$.

And we suppose that (X_1^*, X_2^*) is a fixed point $f_1(X_1^*, X_2^*) = 0 \,\&\, f_2(X_1^*, X_2^*) = 0$. Let $u = X_1 - X_1^*$; $v = X_2 - X_2^*$ and denote the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v. $\frac{du}{dt} = \frac{dX_1}{dt}$ since X_1^* is constant and by substitution $\frac{du}{dt} = \frac{dX_1}{dt} = f_1(X_1^* + u, X_2^* + v)$. Using Taylor series expansion $\frac{du}{dt} =$

$\frac{dX_1}{dt} = f_1(X_1^* + u, X_2^* + v) = f_1(X_1^*, X_2^*) + u \cdot \frac{\partial f_1}{\partial X_1} + v \cdot \frac{\partial f_1}{\partial X_2} + O(u^2, v^2, u \cdot v)$.

Since $f_1(X_1^*, X_2^*) = 0$ we get $\frac{du}{dt} = \frac{dX_1}{dt} = u \cdot \frac{\partial f_1}{\partial X_1} + v \cdot \frac{\partial f_1}{\partial X_2} + O(u^2, v^2, u \cdot v)$. The partial derivatives are to be evaluated at the fixed point (X_1^*, X_2^*); thus they are numbers, not functions. $O(u^2, v^2, u \cdot v)$ denotes quadratic terms in u and v. Since u and v are small, these quadratic terms are extremely small. Also $\frac{dv}{dt} = \frac{dX_2}{dt} = u \cdot \frac{\partial f_2}{\partial X_1} + v \cdot \frac{\partial f_2}{\partial X_2} + O(u^2, v^2, u \cdot v)$; hence, the disturbance (u, v) evolves according to $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix} + \text{quadratic term}$. The matrix $A = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix}_{(X_1^*, X_2^*)}$.

Is called the Jacobian at the fixed point (X_1^*, X_2^*). We obtain our Van der Pol system linearized system $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix}$. Next is to get the values for matrix A elements. $\frac{\partial f_1}{\partial X_1}|_{X_1^*=0, X_2^*=0} = 0$; $\frac{\partial f_1}{\partial X_2}|_{X_1^*=0, X_2^*=0} = 1$; $\frac{\partial f_2}{\partial X_1}|_{X_1^*=0, X_2^*=0} = -1$; $\frac{\partial f_2}{\partial X_2}|_{X_1^*=0, X_2^*=0} = \mu$

$$A - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{pmatrix} \Rightarrow \det|A - \lambda \cdot I| = \lambda \cdot (\lambda - \mu) + 1. \ \det|A - \lambda \cdot I| = 0$$
$$\Rightarrow \lambda^2 - \lambda \cdot \mu + 1 = 0.$$

$\lambda^2 - \lambda \cdot \mu + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$, we get eigenvalues $\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$. We see that if $0 < \mu < 2$, the equilibrium is an unstable spiral. For $\mu > 2$, the equilibrium is an unstable node. Because there are no other equilibria , the only possible attractors for this system are the point at infinity, or periodic solutions. If $\mu = 0$ then $\lambda_{1,2} = \pm j$ (eigenvalues are pure imaginary), solutions are periodic with period $T = 2\pi$. If $-2 < \mu < 0$, then $\lambda_{1,2} = A \pm j \cdot B$; $A < 0$ & $B > 0$ the equilibrium is stable spiral. If $\mu = -2$ then $\lambda_{1,2} = -1$, then equilibrium is stable node. If $\mu < -2$ then we have two negative eigenvalues and the equilibrium is stable node. If we look on equation $\frac{d^2X_1}{dt^2} + \mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{dt} + X_1 = 0$, its looks like a simple harmonic oscillator, but with nonlinear damping term $\mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{dt}$. This term acts like ordinary positive damping for $|X_1| > 1$, but like negative damping for $|X_1| < 1$. The system settles into a self-sustained oscillation where the energy dissipated over one cycle balances the energy pumped in. The Van der Pol equation has a unique, stable limit cycle for each $\mu > 0$. Van der Pol equation can be with forcing function $f(t)$. The oscillator of Van der Pol is structurally stable under small autonomous perturbations. There are changes in the structure of the solutions when oscillator is subjected to non-autonomous periodic perturbations. We consider the non-autonomous system: $\frac{dX_1}{dt} = X_2$; $\frac{dX_2}{dt} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2 + \Gamma \cdot f(t)$. $f(t)$ is a T periodic function of the independent variable t, and Γ is a real parameter. The term $\Gamma \cdot f(t)$ is called the forcing function. When $\Gamma = 0$, there is no forcing and the system is the oscillator of Van der Pol equivalently, we may consider the orbits of the three-dimensional autonomous system: $\frac{dX_1}{dt} = X_2$; $\frac{dX_2}{dt} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2 + \Gamma \cdot f(X_3)$; $\frac{dX_3}{dt} = 1$. The case of $\Gamma = 0$: the limit cycle, the isolated periodic orbit, of unforced oscillator of Van der Pol becomes a cylinder, homeomorphism to $S^1 \times R$. This cylinder is an invariant manifold in the sense that any solution starting on the cylinder remains on it for all positive time. For $\Gamma = 0$, the invariant cylinder is filled with a family of periodic solutions. Let us consider the case of periodic forcing with small amplitude, that is, $|\Gamma|$ small. There is still a cylinder in $R^2 \times R$ close to the invariant cylinder of the unforced oscillator. This cylinder is an invariant manifold of solutions of the forced equation and attracts all nearby solutions. More sophisticated system is coupler Van der Pol oscillators. We can express our coupled Van der Pol oscillators by two differential equations:

$$\frac{d^2X_1}{dt^2} + X_1 - \mu_1 \cdot (1 - X_1^2) \cdot \frac{dX_1}{dt} = \alpha_1 \cdot (X_2 - X_1); \; \frac{d^2X_2}{dt^2} + (1 + \Delta) \cdot X_2 - \mu_2 \cdot (1 - X_2^2) \cdot \frac{dX_2}{dt} = \alpha_2 \cdot (X_1 - X_2),$$

where Δ, μ_1, μ_2, α_1, and α_2 are parameters. Δ is related to the small difference in linearized frequencies, and α_1 and α_2 represent the strength of the coupling for the first and second Van der Pol systems respectively. μ_1 and μ_2 are the parameters which establish the behavior of each Van der Pol system respectively.

Remark System is one Van der Pol oscillator. We can separate each second-degree Van der Pol differential equation to two first-degree differential equations. Totally, we have four differential equations for our two coupled Van der Pol systems. Our new variables are as follows: $X_1 = X_1(t)$; $X_1' = X_1'(t)$; $X_2 = X_2(t)$; $X_2' = X_2'(t)$

$$X_1' = \frac{dX_1}{dt}; \; \frac{dX_1'}{dt} + X_1 - \mu_1 \cdot (1 - X_1^2) \cdot X_1' = \alpha_1 \cdot (X_2 - X_1)$$
$$X_2' = \frac{dX_2}{dt}; \; \frac{dX_2'}{dt} + (1 + \Delta) \cdot X_2 - \mu_2 \cdot (1 - X_2^2) \cdot X_2' = \alpha_2 \cdot (X_1 - X_2)$$

To find system fixed points we set $\frac{dX_1}{dt} = 0$; $\frac{dX_1'}{dt} = 0$; $\frac{dX_2}{dt} = 0$; $\frac{dX_2'}{dt} = 0$

$X_1'^* = 0$; $X_2'^* = 0$; $(1 + \Delta) \cdot X_2^* = \alpha_2 \cdot (X_1^* - X_2^*)$; $X_1^* = \alpha_1 \cdot (X_2^* - X_1^*)$. And at fixed points: $X_1^* = \frac{\alpha_1}{(1 + \alpha_1)} \cdot X_2^*; 1 + \Delta + \frac{\alpha_2}{(1 + \alpha_1)} = 0$; $X_1^* = -\frac{\alpha_1}{\alpha_2} \cdot (1 + \Delta) \cdot X_2^*$.

Stability Analysis Our coupled Van der Pol oscillator's Jacobian matrix (A): $A = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X'_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_4}{\partial X_1} & \cdots & \frac{\partial f_4}{\partial X'_2} \end{pmatrix}$

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = X'_1$$
$$\frac{\mathrm{d}X'_1}{\mathrm{d}t} = \alpha_1 \cdot (X_2 - X_1) - X_1 + \mu_1 \cdot (1 - X_1^2) \cdot X'_1$$
$$\frac{\mathrm{d}X_2}{\mathrm{d}t} = X'_2$$
$$\frac{\mathrm{d}X'_2}{\mathrm{d}t} = \alpha_2 \cdot (X_1 - X_2) - (1 + \Delta) \cdot X_2 + \mu_2 \cdot (1 - X_2^2) \cdot X'_2$$

$$f_1 = X'_1;\ f_2 = \alpha_1 \cdot (X_2 - X_1) - X_1 + \mu_1 \cdot (1 - X_1^2) \cdot X'_1;\ f_3 = X'_2$$
$$f_4 = \alpha_2 \cdot (X_1 - X_2) - (1 + \Delta) \cdot X_2 + \mu_2 \cdot (1 - X_2^2) \cdot X'_2$$

$$f_1 = X'_1 \Rightarrow \frac{\partial f_1}{\partial X_1} = 0;\ \frac{\partial f_1}{\partial X'_1} = 1;\ \frac{\partial f_1}{\partial X_2} = 0;\ \frac{\partial f_1}{\partial X_2} = 0;\ \frac{\partial f_2}{\partial X_1} = -\alpha_1 - 1 - 2 \cdot \mu_1 \cdot X'_1 \cdot X_1$$
$$\frac{\partial f_2}{\partial X'_1} = \mu_1 \cdot (1 - X_1^2);\ \frac{\partial f_2}{\partial X_2} = \alpha_1;\ \frac{\partial f_2}{\partial X'_2} = 0;\ \frac{\partial f_3}{\partial X_1} = 0;\ \frac{\partial f_3}{\partial X'_1} = 0;\ \frac{\partial f_3}{\partial X_2} = 0;\ \frac{\partial f_3}{\partial X'_2} = 1$$
$$\frac{\partial f_4}{\partial X_1} = \alpha_2;\ \frac{\partial f_4}{\partial X'_1} = 0;\ \frac{\partial f_4}{\partial X_2} = -\alpha_2 - (1 + \Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X'_2;\ \frac{\partial f_4}{\partial X'_2} = \mu_2 \cdot (1 - X_2^2)$$

$$\det|A - \lambda \cdot I| = -\lambda \cdot \det\begin{pmatrix} \mu_1 \cdot (1 - X_1^2) - \lambda & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\alpha_2 - (1 + \Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X'_2 & \mu_2 \cdot (1 - X_2^2) - \lambda \end{pmatrix}$$
$$- \det\begin{pmatrix} -\alpha_1 - 1 - 2 \cdot \mu_1 \cdot X'_1 \cdot X_1 & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ \alpha_2 & -\alpha_2 - (1 + \Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X'_2 & \mu_2 \cdot (1 - X_2^2) - \lambda \end{pmatrix}$$

$$\det\begin{pmatrix} \mu_1 \cdot (1 - X_1^2) - \lambda & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ 0 & [-\alpha_2 - (1 + \Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X'_2] & [\mu_2 \cdot (1 - X_2^2) - \lambda] \end{pmatrix} =$$
$$[\mu_1 \cdot (1 - X_1^2) - \lambda] \cdot \det\begin{pmatrix} -\lambda & 1 \\ [-\alpha_2 - (1 + \Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X'_2] & [\mu_2 \cdot (1 - X_2^2) - \lambda] \end{pmatrix}$$
$$- \alpha_1 \cdot \det\begin{pmatrix} 0 & 1 \\ 0 & [\mu_2 \cdot (1 - X_2^2) - \lambda] \end{pmatrix}; \det\begin{pmatrix} 0 & 1 \\ 0 & [\mu_2 \cdot (1 - X_2^2) - \lambda] \end{pmatrix} = 0$$

$$\det\begin{pmatrix} \mu_1\cdot(1-X_1^2)-\lambda & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ 0 & [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$
$$= [\mu_1\cdot(1-X_1^2)-\lambda]\cdot\det\begin{pmatrix} -\lambda & 1 \\ [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$

For simplicity we define the following functions:

$$\Upsilon_1=\mu_1\cdot(1-X_1^2);\ \ \Upsilon_2=[-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'];\ \ \Upsilon_3=\mu_2\cdot(1-X_2^2)$$

$$\det\begin{pmatrix} \mu_1\cdot(1-X_1^2)-\lambda & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ 0 & [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$
$$= [\Upsilon_1-\lambda]\cdot\det\begin{pmatrix} -\lambda & 1 \\ \Upsilon_2 & [\Upsilon_3-\lambda] \end{pmatrix} = [\Upsilon_1-\lambda]\cdot\{-\lambda\cdot[\Upsilon_3-\lambda]-\Upsilon_2\}$$
$$= -\lambda^3+\lambda^2\cdot(\Upsilon_1+\Upsilon_3)+\lambda\cdot(\Upsilon_2-\Upsilon_3\cdot\Upsilon_1)-\Upsilon_2\cdot\Upsilon_1$$

We define the following functions parameters: Ω_i; i = 0, …, 3

$$\Omega_0=-\Upsilon_2\cdot\Upsilon_1;\ \Omega_1=(\Upsilon_2-\Upsilon_3\cdot\Upsilon_1);\ \Omega_2=(\Upsilon_1+\Upsilon_3);\ \Omega_3=-1$$

$$\det\begin{pmatrix} \mu_1\cdot(1-X_1^2)-\lambda & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ 0 & [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$
$$= \lambda^3\cdot\Omega_3+\lambda^2\cdot\Omega_2+\lambda\cdot\Omega_1+\Omega_0=\sum_{i=0}^{3}\lambda^i\cdot\Omega_i$$

$$\det\begin{pmatrix} -\alpha_1-1-2\cdot\mu_1\cdot X_1'\cdot X_1 & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ \alpha_2 & [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$
$$= [-\alpha_1-1-2\cdot\mu_1\cdot X_1'\cdot X_1]\cdot\det\begin{pmatrix} -\lambda & 1 \\ [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$
$$-\alpha_1\cdot\begin{pmatrix} 0 & 1 \\ \alpha_2 & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$

$$\det\begin{pmatrix} -\alpha_1-1-2\cdot\mu_1\cdot X_1'\cdot X_1 & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ \alpha_2 & [-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2'] & [\mu_2\cdot(1-X_2^2)-\lambda] \end{pmatrix}$$
$$= [-\alpha_1-1-2\cdot\mu_1\cdot X_1'\cdot X_1]\cdot\{-\lambda\cdot[\mu_2\cdot(1-X_2^2)-\lambda]-[-\alpha_2-(1+\Delta)-2\cdot\mu_2\cdot X_2\cdot X_2']\}+\alpha_1\cdot\alpha_2$$
$$= [-\alpha_1-1-2\cdot\mu_1\cdot X_1'\cdot X_1]\cdot\{-\lambda\cdot\mu_2\cdot(1-X_2^2)+\lambda^2+(\alpha_2+(1+\Delta)+2\cdot\mu_2\cdot X_2\cdot X_2')\}+\alpha_1\cdot\alpha_2$$

For simplicity we define the following functions: $\Upsilon_4 = [-\alpha_1 - 1 - 2 \cdot \mu_1 \cdot X_1' \cdot X_1]$

$$\Upsilon_3 = \mu_2 \cdot (1 - X_2^2);\;\; \Upsilon_5 = \alpha_2 + (1+\Delta) + 2 \cdot \mu_2 \cdot X_2 \cdot X_2';\;\; \Upsilon_6 = \alpha_1 \cdot \alpha_2$$

$$\det\begin{pmatrix} -\alpha_1 - 1 - 2 \cdot \mu_1 \cdot X_1' \cdot X_1 & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ \alpha_2 & [-\alpha_2 - (1+\Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X_2'] & [\mu_2 \cdot (1 - X_2^2) - \lambda] \end{pmatrix}$$
$$= \Upsilon_4 \cdot (-\lambda \cdot \Upsilon_3 + \lambda^2 + \Upsilon_5) + \Upsilon_6 = \lambda^2 \cdot \Upsilon_4 - \lambda \cdot \Upsilon_3 \cdot \Upsilon_4 + (\Upsilon_5 \cdot \Upsilon_4 + \Upsilon_6)$$

We define the following functions parameters: Ξ_i; $i = 0, \ldots, 3$

$$\Xi_0 = \Upsilon_5 \cdot \Upsilon_4 + \Upsilon_6;\;\; \Xi_1 = -\Upsilon_3 \cdot \Upsilon_4;\;\; \Xi_2 = \Upsilon_4$$

$$\det\begin{pmatrix} -\alpha_1 - 1 - 2 \cdot \mu_1 \cdot X_1' \cdot X_1 & \alpha_1 & 0 \\ 0 & -\lambda & 1 \\ \alpha_2 & [-\alpha_2 - (1+\Delta) - 2 \cdot \mu_2 \cdot X_2 \cdot X_2'] & [\mu_2 \cdot (1 - X_2^2) - \lambda] \end{pmatrix}$$
$$= \lambda^2 \cdot \Upsilon_4 - \lambda \cdot \Upsilon_3 \cdot \Upsilon_4 + (\Upsilon_5 \cdot \Upsilon_4 + \Upsilon_6) = \sum_{i=0}^{2} \lambda^i \cdot \Xi_i$$

Summary of our last calculations:

$$\det|A - \lambda \cdot I| = -\lambda \cdot \sum_{i=0}^{3} \lambda^i \cdot \Omega_i - \sum_{i=0}^{2} \lambda^i \cdot \Xi_i = -\left[\sum_{i=0}^{3} \lambda^{i+1} \cdot \Omega_i + \sum_{i=0}^{2} \lambda^i \cdot \Xi_i\right]$$

To find our characteristic equation: $\det|A - \lambda \cdot I| = 0 \Rightarrow \sum_{i=0}^{3} \lambda^{i+1} \cdot \Omega_i + \sum_{i=0}^{2} \lambda^i \cdot \Xi_i = 0$

$$\sum_{i=0}^{3} \lambda^{i+1} \cdot \Omega_i = \lambda^4 \cdot \Omega_3 + \lambda^3 \cdot \Omega_2 + \lambda^2 \cdot \Omega_1 + \lambda \cdot \Omega_0;$$
$$\sum_{i=0}^{2} \lambda^i \cdot \Xi_i = \lambda^2 \cdot \Xi_2 + \lambda \cdot \Xi_1 + \Xi_0$$

$$\sum_{i=0}^{3} \lambda^{i+1} \cdot \Omega_i + \sum_{i=0}^{2} \lambda^i \cdot \Xi_i = \lambda^4 \cdot \Omega_3 + \lambda^3 \cdot \Omega_2 + \lambda^2 \cdot [\Omega_1 + \Xi_2] + \lambda \cdot [\Omega_0 + \Xi_1] + \Xi_0.$$

We can define new parameters: $\sum_{i=0}^{3} \lambda^{i+1} \cdot \Omega_i + \sum_{i=0}^{2} \lambda^i \cdot \Xi_i = \sum_{k=0}^{4} \lambda^k \cdot \Phi_k$

$$\Phi_4 = \Omega_3;\ \Phi_3 = \Omega_2;\ \Phi_2 = \Omega_1 + \Xi_2;\ \Phi_1 = \Omega_0 + \Xi_1;\ \Phi_0 = \Xi_0;\ \det|A - \lambda \cdot I| = 0$$
$$\Rightarrow \sum_{k=0}^{4} \lambda^k \cdot \Phi_k = 0$$

Fixed point are $E^{(i)}$, i = 0, 1, … $E^{(i)}(X_1^*, X_1'^*, X_2^*, X_2'^*) = (X_1^*, 0, X_2^*, 0)$

$$\Upsilon_1|_{(X_1^*,0,X_2^*,0)} = \mu_1 \cdot (1 - [X_1^*]^2);\ \Upsilon_2|_{(X_1^*,0,X_2^*,0)} = [-\alpha_2 - (1 + \Delta)];$$
$$\Upsilon_3|_{(X_1^*,0,X_2^*,0)} = \mu_2 \cdot (1 - [X_2^*]^2)$$
$$\Upsilon_4|_{(X_1^*,0,X_2^*,0)} = -(-\alpha_1 + 1);\ \Upsilon_5|_{(X_1^*,0,X_2^*,0)} = \alpha_2 + (1 + \Delta);\ \Upsilon_6|_{(X_1^*,0,X_2^*,0)} = \alpha_1 \cdot \alpha_2$$

$$\Omega_0|_{(X_1^*,0,X_2^*,0)} = -[-\alpha_2 - (1 + \Delta)] \cdot \mu_1 \cdot (1 - [X_1^*]^2);$$
$$\Omega_1|_{(X_1^*,0,X_2^*,0)} = [-\alpha_2 - (1 + \Delta)] - \mu_2 \cdot (1 - [X_2^*]^2) \cdot \mu_1 \cdot (1 - [X_1^*]^2)$$
$$\Omega_2|_{(X_1^*,0,X_2^*,0)} = \mu_1 \cdot (1 - [X_1^*]^2) + \mu_2 \cdot (1 - [X_2^*]^2);\ \Omega_3|_{(X_1^*,0,X_2^*,0)} = -1$$

$$\Xi_0|_{(X_1^*,0,X_2^*,0)} = [\alpha_2 + (1 + \Delta)] \cdot (\alpha_1 - 1) + \alpha_1 \cdot \alpha_2;$$
$$\Xi_1|_{(X_1^*,0,X_2^*,0)} = \mu_2 \cdot (1 - [X_2^*]^2) \cdot (-\alpha_1 + 1);\ \Xi_2|_{(X_1^*,0,X_2^*,0)} = \alpha_1 - 1.$$

For getting our system eigenvalues, we need to find eigenvalues for specific system's parameters (Δ, μ_1, μ_2, α_1, α_2) values and analyze stability behavior $\det|A - \lambda \cdot I| = 0 \Rightarrow \sum_{k=0}^{4} \lambda^k \cdot \Phi_k = 0$.

Limit Cycle Discussion, One Van der Pol Oscillator System

The unforced Van der Pol oscillator can be given by the following equations: $\frac{dX_1}{dt} = X_2$ and $\frac{dX_2}{dt} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2$. We need to prove that Van der Pol system has periodic orbits and it is done by changing system cartesian coordinates ($X_1(t)$, $X_2(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation z = 0), and the cylindrical axis is the Cartesian z axis. In our system we refer to Cartesian $X_1 - X_2$ plane (with equation X_3 = 0). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X_1, X_2) are the same as for polar coordinates, namely $X_1(t) = r(t) \cdot \cos[\theta(t)]$; $X_2(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X_1^2 + X_2^2}$. $\theta(t) = 0$ if $X_1 = 0$ and $X_2 = 0$. $\theta(t) = \arcsin(X_2/r)$ if $X_1 \geq 0$. $x \to X_1$, $y \to X_2$.

$$X_1(t) = r(t)\cdot\cos[\theta(t)] \Rightarrow \frac{dX_1(t)}{dt} = \frac{dr(t)}{dt}\cdot\cos[\theta(t)] - r(t)\cdot\frac{d\theta(t)}{dt}\cdot\sin[\theta(t)]$$
$$X_2(t) = r(t)\cdot\sin[\theta(t)] \Rightarrow \frac{dX_2(t)}{dt} = \frac{dr(t)}{dt}\cdot\sin[\theta(t)] + r(t)\cdot\frac{d\theta(t)}{dt}\cdot\cos[\theta(t)]$$
$$\frac{dX_1(t)}{dt} = \frac{dX_1}{dt};\ \frac{dr(t)}{dt} = r';\ \frac{d\theta(t)}{dt} = \theta';\ \theta(t) = \theta;\ r(t) = r$$

We get the equations:

$$\frac{dX_1}{dt} = r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta;\ \frac{dX_2}{dt} = r'\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta$$
$$\frac{dX_1}{dt} = X_2 \Rightarrow r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta = r\cdot\sin\theta \Rightarrow r'\cdot\cos\theta = r\cdot\theta'\cdot\sin\theta + r\cdot\sin\theta$$
$$r'\cdot\cos\theta = r\cdot\theta'\cdot\sin\theta + r\cdot\sin\theta \Rightarrow r'\cdot\cos\theta = r\cdot\sin\theta\cdot(\theta'+1) \Rightarrow tg\theta = \frac{r'}{r\cdot(\theta'+1)}$$
$$tg\theta = \frac{r'}{r\cdot(\theta'+1)} \Rightarrow \theta = \operatorname{arctg}\left\{\frac{r'}{r\cdot(\theta'+1)}\right\};\ X_1 = r\cdot\cos\theta;\ X_2 = r\cdot\sin\theta$$
$$\frac{dX_2}{dt} = -X_1 - \mu\cdot(X_1^2-1)\cdot X_2 \Rightarrow r'\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta = -r\cdot\cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot r\cdot\sin\theta$$
$$tg\theta = \frac{r'}{r\cdot(\theta'+1)} \Rightarrow r' = r\cdot(\theta'+1)\cdot tg\theta$$

then for the second equation, we get

$$r\cdot(\theta'+1)\cdot tg\theta\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta = -r\cdot\cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot r\cdot\sin\theta$$
$$r\cdot\theta'\cdot tg\theta\cdot\sin\theta + r\cdot tg\theta\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta = -r\cdot\cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot r\cdot\sin\theta$$
$$r\cdot\theta'\cdot[tg\theta\cdot\sin\theta + \cos\theta] + r\cdot tg\theta\cdot\sin\theta = -r\cdot\cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot r\cdot\sin\theta$$
$$tg\theta\cdot\sin\theta + \cos\theta = \frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$$
$$r\cdot\theta'\cdot\frac{1}{\cos\theta} + r\cdot tg\theta\cdot\sin\theta = -r\cdot\cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot r\cdot\sin\theta$$
$$\theta'\cdot\frac{1}{\cos\theta} + tg\theta\cdot\sin\theta = -\cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot\sin\theta \Rightarrow \theta'\cdot$$
$$\frac{1}{\cos\theta} = -tg\theta\cdot\sin\theta - \cos\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot\sin\theta$$
$$\theta' = -\sin^2\theta - \cos^2\theta - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot\sin\theta\cdot\cos\theta;\ -\sin^2\theta - \cos^2\theta = -1$$
$$\theta' = -1 - \mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot\sin\theta\cdot\cos\theta;\ \sin[2\cdot\theta] = 2\cdot\sin\theta\cdot\cos\theta$$
$$\Rightarrow \sin\theta\cdot\cos\theta = \frac{1}{2}\cdot\sin[2\cdot\theta]$$
$$\theta' = -1 - \frac{1}{2}\cdot\mu\cdot(r^2\cdot\cos^2\theta - 1)\cdot\sin[2\cdot\theta]$$

We need to get the expression for r'.

$$tg\theta = \frac{r'}{r \cdot (\theta' + 1)} \Rightarrow \theta' = \frac{r'}{r \cdot tg\theta} - 1.$$

Back to our second equation:

$$\frac{dX_2}{dt} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2 \Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$
$$= -r \cdot \cos\theta - \mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta$$

$$r' \cdot \sin\theta + r \cdot \left(\frac{r'}{r \cdot tg\theta} - 1\right) \cdot \cos\theta = -r \cdot \cos\theta - \mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta$$
$$r' \cdot \sin\theta + r \cdot \frac{r'}{r \cdot tg\theta} \cdot \cos\theta - r \cdot \cos\theta = -r \cdot \cos\theta - \mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta$$
$$r' \cdot \left[\sin\theta + r \cdot \frac{1}{r \cdot tg\theta} \cdot \cos\theta\right] - r \cdot \cos\theta = -r \cdot \cos\theta - \mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta$$
$$r' \cdot \left[\sin\theta + r \cdot \frac{1}{r \cdot tg\theta} \cdot \cos\theta\right] = -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta$$
$$r' \cdot \left[\sin\theta + \frac{1}{\sin\theta} \cdot \cos^2\theta\right] = -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta;\ \sin\theta + \frac{1}{\sin\theta} \cdot \cos^2\theta = \frac{1}{\sin\theta}$$
$$r' \cdot \frac{1}{\sin\theta} = -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin\theta \Rightarrow r' = -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin^2\theta$$

We can summarize our last results:

(*) $\theta' = -1 - \frac{1}{2} \cdot \mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot \sin[2 \cdot \theta]$; (**) $r' = -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin^2\theta$

We define $\xi_1(r,\theta) = -1 - \frac{1}{2} \cdot \mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot \sin[2 \cdot \theta]$; $\xi_2(r,\theta) = -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin^2\theta$

$\theta' = \xi_1(r,\theta)$; $r' = \xi_2(r,\theta)$. If there is a stable limit cycle as specific value of radius r, $dr/dt = 0$ then $r' = \xi_2(r,\theta) = 0 \Rightarrow -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin^2\theta = 0$; $r \neq 0$; $\mu \neq 0$. Then $r^2 \cdot \cos^2\theta - 1 = 0$ (Case A) or $\sin^2\theta = 0$ (Case B). $\Theta(t) = 0.2\pi$ then only Case A exist.

$$r^2 \cdot \cos^2\theta - 1 = 0 \Rightarrow (r \cdot \cos\theta + 1) \cdot (r \cdot \cos\theta - 1) \Rightarrow r_{1,2} = \frac{\pm 1}{\cos\theta}\ ;\ r > 0$$
$$\Rightarrow r(t) = \frac{1}{|\cos\theta(t)|}.$$

This is contradiction with the assumption that $dr/dt = 0$ since $r(\theta)$. Other case is when system limit cycle is resided in annulus $0 < r_{min} < r < r_{max}$ and there is no constant r in limit cycle $dr/dt \neq 0$. We seek two concentric circles with radii r_{min} and r_{max}. $dr/dt<0$ on the outer circle and $dr/dt > 0$ on the inner circle. To find r_{min}, we require $dr/dt > 0$ for all values of θ: $-\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot$

$\sin^2\theta > 0$; $\sin^2\theta \geq 0$. $r > 0$ then we have two subcases: $-\mu \cdot (r^2 \cdot \cos^2\theta - 1) > 0 \Rightarrow \mu \cdot (r^2 \cdot \cos^2\theta - 1) < 0 \quad r \to r_{\min}$. $\mu \cdot (r^2 \cdot \cos^2\theta - 1) < 0$. Can be divided into two subcases: (1) $\mu > 0$ and $(r^2 \cdot \cos^2\theta - 1) < 0$, (2) $\mu < 0$ and $(r^2 \cdot \cos^2\theta - 1) > 0$ [5, 6].

(1) $\mu > 0;\ (r^2 \cdot \cos^2\theta - 1) < 0 \Rightarrow (r \cdot \cos\theta - 1) \cdot (r \cdot \cos\theta + 1) < 0$

(1.1) $(r \cdot \cos\theta - 1) < 0$; $(r \cdot \cos\theta + 1) > 0$; $\cos\theta \in [-1, +1]$, and we check it for $\cos\theta = -1, 0, +1$.

(1.1.1) $\cos\theta = -1 \Rightarrow r = r_{\min};\ (-r_{\min} - 1) < 0\ ;\ (-r_{\min} + 1) > 0$
$\Rightarrow -1 < r_{\min} < 1$
And since $r_{\min} > 0 \Rightarrow 1 > r_{\min} > 0$

(1.1.2) $\cos\theta = 0 \Rightarrow r = r_{\min} \Rightarrow -1 < 0\ ;\ +1 > 0$ can't exist.

(1.1.3) $\cos\theta = +1 \Rightarrow r = r_{\min};\ (r_{\min} - 1) < 0\ ;\ (r_{\min} + 1) > 0$
$\Rightarrow -1 < r_{\min} < 1$
And since $r_{\min} > 0 \Rightarrow 1 > r_{\min} > 0$.
OR

(1.2) $(r \cdot \cos\theta - 1) > 0$; $(r \cdot \cos\theta + 1) < 0$; $\cos\theta \in [-1, +1]$, and we check it
for $\cos\theta = -1, 0, +1$.

(1.2.1) $\cos\theta = -1 \Rightarrow r = r_{\min};\ (-r_{\min} - 1) > 0\ ;\ (-r_{\min} + 1) < 0$
$\Rightarrow r_{\min} < -1; r_{\min} > 1$
Not exist.

(1.2.2) $\cos\theta = 0 \Rightarrow r = r_{\min};\ -1 > 0\ ;\ +1 < 0$ can't exist.

(1.2.3) $\cos\theta = 1 \Rightarrow r = r_{\min};\ (r_{\min} - 1) > 0\ ;\ (r_{\min} + 1) \quad < 0 \Rightarrow r_{\min} < -1; r_{\min} > 1$ can't exist.

(2) $\mu < 0;\ (r^2 \cdot \cos^2\theta - 1) > 0 \Rightarrow (r \cdot \cos\theta - 1) \cdot (r \cdot \cos\theta + 1) > 0$

(2.1) $(r \cdot \cos\theta - 1) > 0$; $(r \cdot \cos\theta + 1) > 0$; $\cos\theta \in [-1, +1]$, and we check it for $\cos\theta = -1, 0, +1$.

(2.1.1) $\cos\theta = -1 \Rightarrow r = r_{\min} > 0; (-r_{\min} - 1) > 0\ ;\ (-r_{\min} + 1) > 0$
$\Rightarrow r_{\min} < -1;\ r_{\min} < 1$
$r_{\min} < -1$ can't exist.

(2.1.2) $\cos\theta = 0 \Rightarrow r = r_{\min} \Rightarrow -1 > 0\ ;\ +1 > 0$ can't exist.

(2.1.3) $\cos\theta = 1 \Rightarrow r = r_{\min} > 0; (r_{\min} - 1) > 0\ ;\ (r_{\min} + 1) > 0$
$\Rightarrow r_{\min} > -1;\ r_{\min} > 1$
Which yield $r_{\min} > 1$.
OR

(2.2) $(r \cdot \cos\theta - 1) < 0$; $(r \cdot \cos\theta + 1) < 0$; $\cos\theta \in [-1, +1]$, and we check it for $\cos\theta = -1, 0, +1$.

(2.2.1) $\cos\theta = -1 \Rightarrow r = r_{\min} > 0; (-r_{\min} - 1) < 0\ ;\ (-r_{\min} + 1) < 0$
$\Rightarrow r_{\min} > -1;\ r_{\min} > 1$
Which yield to $r_{\min} > 1$.
(2.2.2) $\cos\theta = 0 \Rightarrow r = r_{\min} \Rightarrow -1 < 0\ ;\ +1 < 0$ can't exist.
(2.2.3) $\cos\theta = 1 \Rightarrow r = r_{\min} > 0; (r_{\min} - 1) < 0\ ;\ (r_{\min} + 1) < 0$
$\Rightarrow r_{\min} < -1;\ r_{\min} < 1$
Which yield to $r_{\min} < -1$, not exist.

Result (A) $\{\mu > 0;\ 0 < r_{\min} < 1\}$ OR $\{\mu < 0;\ r_{\min} > 1\}$ and $\{\mu > 0\} \cup \{\mu < 0\} = \mu \in \mathbb{R},\ \mu \neq 0$

To find $r_{\max}$, we require dr/dt < 0 for all values of θ: $-\mu \cdot (r^2 \cdot \cos^2\theta - 1) \cdot r \cdot \sin^2\theta < 0 \quad \sin^2\theta \geq 0\ ;\ r > 0 \Rightarrow -\mu \cdot (r^2 \cdot \cos^2\theta - 1) \quad < 0 \Rightarrow \mu \cdot (r^2 \cdot \cos^2\theta - 1) > 0$. Can be divided to two subcases: (3) $\mu > 0$ and $(r^2 \cdot \cos^2\theta - 1) > 0$, (4) $\mu < 0$ and $(r^2 \cdot \cos^2\theta - 1) < 0$.

(3). $\mu > 0;\ (r^2 \cdot \cos^2\theta - 1) > 0 \Rightarrow (r \cdot \cos\theta - 1) \cdot (r \cdot \cos\theta + 1) > 0$

(1.1) $(r \cdot \cos\theta - 1) > 0\ ;\ (r \cdot \cos\theta + 1) > 0;\ \cos\theta \in [-1, +1]$, and we check it for $\cos\theta = -1, 0, +1$.

(1.1.4) $\cos\theta = -1 \Rightarrow r = r_{\max};\ (-r_{\max} - 1) > 0\ ;\ (-r_{\max} + 1) > 0$
$\Rightarrow r_{\max} < -1$
Can't exist since $r_{\max} > 0$.
(1.1.5) $\cos\theta = 0 \Rightarrow r = r_{\max} \Rightarrow -1 > 0\ ;\ +1 > 0$ can't exist.
(1.1.6) $\cos\theta = +1 \Rightarrow r = r_{\max};\ (r_{\max} - 1) > 0\ ;\ (r_{\max} + 1) > 0$
$\Rightarrow r_{\max} > 1$

OR

(1.2) $(r \cdot \cos\theta - 1) < 0\ ;\ (r \cdot \cos\theta + 1) < 0;\ \cos\theta \in [-1, +1]$, and we check it
for $\cos\theta = -1, 0, +1$.

(1.2.1) $\cos\theta = -1 \Rightarrow r = r_{\max}$
$;\ (-r_{\max} - 1) < 0\ ;\ (-r_{\max} + 1) < 0 \Rightarrow r_{\max} > 1$
(1.2.2) $\cos\theta = 0 \Rightarrow r = r_{\max};\ -1 < 0\ ;\ +1 < 0$ can't exist.
(1.2.3) $\cos\theta = 1 \Rightarrow r = r_{\max}\ ;\ (r_{\max} - 1) < 0\ ;\ (r_{\max} + 1) < 0 \Rightarrow r_{\max} < -1$ can't exist.

(4). $\mu < 0;\ (r^2 \cdot \cos^2\theta - 1) < 0 \Rightarrow (r \cdot \cos\theta - 1) \cdot (r \cdot \cos\theta + 1) < 0$

(2.1) $(r \cdot \cos\theta - 1) > 0\ ;\ (r \cdot \cos\theta + 1) < 0\ ;\ \cos\theta \in [-1, +1]$, and we check it for $\cos\theta = -1, 0, +1$.

(2.1.1) $\cos\theta = -1 \Rightarrow r = r_{\max} > 0; (-r_{\max} - 1) > 0;$
$(-r_{\max} + 1) < 0 \Rightarrow r_{\max} < -1;\ r_{\max} > 1$
can't exist.

Table 6.1 Combining results ($\mu < 0$ and $\mu > 0$)

$\mu < 0$	$\mu > 0$	
$r_{min} > 1; \frac{dr}{dt} > 0$	$0 < r_{min} < 1; \frac{dr}{dt} > 0$	Inner circle
$0 < r_{max} < 1 \frac{dr}{dt} < 0$	$r_{max} > 1; \frac{dr}{dt} < 0$	Outer circle

(2.1.2) $\cos\theta = 0 \Rightarrow r = r_{max} \Rightarrow -1 > 0\ ;\ +1 < 0$ can't exist.
(2.1.3) $\cos\theta = 1 \Rightarrow r = r_{max} > 0; (r_{max} - 1) > 0\ ;\ (r_{max} + 1) < 0 \Rightarrow r_{max} > 1;\ r_{max} < -1$
can't exist.

OR

(2.2) $(r \cdot \cos\theta - 1) < 0\ ;\ (r \cdot \cos\theta + 1) > 0;\ \cos\theta \in [-1, +1]$, and we check it for $\cos\theta = -1, 0, +1$.

(2.2.1) $\cos\theta = -1 \Rightarrow r = r_{max} > 0; (-r_{max} - 1) < 0;$
$(-r_{max} + 1) > 0 \Rightarrow r_{max} > -1;\ r_{max} < 1$

$$r_{max} \neq 0\ ;\ r_{max} > 0 \Rightarrow 0 < r_{max} < 1$$

(2.2.2) $\cos\theta = 0 \Rightarrow r = r_{max} \Rightarrow -1 < 0\ ;\ +1 > 0$ can't exist.
(2.2.3) $\cos\theta = 1 \Rightarrow r = r_{max} > 0; (r_{max} - 1) < 0\ ;\ (r_{max} + 1) > 0 \Rightarrow r_{max} < 1;\ r_{max} > -1$
$r_{max} \neq 0\ ;\ r_{max} > 0 \Rightarrow 0 < r_{max} < 1$.

Result (B) $\{\mu < 0;\ 0 < r_{max} < 1\}$ OR $\{\mu > 0;\ r_{max} > 1\}$ and $\{\mu > 0\} \cup \{\mu < 0\} = \mu \in \mathbb{R},\ \mu \neq 0$ (Table 6.1).

We can see that one Van der Pol system has a unique, stable limit cycle for each $\mu > 0$.

Limit Cycle Discussion, One Van der Pol Oscillator System, μ = 0

The Van der Pol oscillator can be given by the following equations: $\frac{dX_1}{dt} = X_2$ and $\frac{dX_2}{dt} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2 \Rightarrow |_{\mu=0} \frac{dX_2}{dt} = -X_1$. To find system fixed point we set $\frac{dX_1}{dt} = 0;\ \frac{dX_2}{dt} = 0 \Rightarrow E^*(X_1^*, X_2^*) = (0, 0)$. $\begin{pmatrix} \frac{dX_1}{dt} \\ \frac{dX_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$;
$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $A - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \Rightarrow \det|A - \lambda \cdot I| = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow$ $\lambda_{1,2} = \pm j$ (Eigenvalues are pure imaginary), solutions are periodic with period T = 2π. We need to prove that Van der Pol system ($\mu = 0$) has periodic orbits and it is done by changing system cartesian coordinates ($X_1(t)$, $X_2(t)$) to cylindrical

coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z axis. In our system we refer to Cartesian $X_1 - X_2$ plane (with equation $X_3 = 0$). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X_1, X_2) are the same as for polar coordinates, namely $X_1(t) = r(t) \cdot \cos[\theta(t)]$; $X_2(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X_1^2 + X_2^2}$. $\theta(t) = 0$ if $X_1 = 0$ and $X_2 = 0$. $\theta(t) = \arcsin(X_2/r)$ if $X_1 \geq 0$. $x \to X_1$, $y \to X_2$.

$$X_1(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{dX_1(t)}{dt} = \frac{dr(t)}{dt} \cdot \cos[\theta(t)] - r(t) \cdot \frac{d\theta(t)}{dt} \cdot \sin[\theta(t)]$$

$$X_2(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{dX_2(t)}{dt} = \frac{dr(t)}{dt} \cdot \sin[\theta(t)] + r(t) \cdot \frac{d\theta(t)}{dt} \cdot \cos[\theta(t)]$$

$$\frac{dX_1(t)}{dt} = \frac{dX_1}{dt};\ \frac{dr(t)}{dt} = r';\ \frac{d\theta(t)}{dt} = \theta';\ \theta(t) = \theta;\ r(t) = r$$

We get the following equations:

$$\frac{dX_1}{dt} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta;\ \frac{dX_2}{dt} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$

$$\frac{dX_1}{dt} = X_2 \Rightarrow r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = r \cdot \sin\theta \Rightarrow r' \cdot \cos\theta = r \cdot \theta' \cdot \sin\theta + r \cdot \sin\theta$$

$$r' \cdot \cos\theta = r \cdot \theta' \cdot \sin\theta + r \cdot \sin\theta \Rightarrow r' \cdot \cos\theta = r \cdot \sin\theta \cdot (\theta' + 1) \Rightarrow tg\theta = \frac{r'}{r \cdot (\theta' + 1)}$$

$$tg\theta = \frac{r'}{r \cdot (\theta' + 1)} \Rightarrow \theta = \text{arctg}\left\{\frac{r'}{r \cdot (\theta' + 1)}\right\};\ X_1 = r \cdot \cos\theta;\ X_2 = r \cdot \sin\theta$$

$$tg\theta = \frac{r'}{r \cdot (\theta' + 1)} \Rightarrow r' = r \cdot (\theta' + 1) \cdot tg\theta \Rightarrow \theta' = \frac{r'}{r \cdot tg\theta} - 1$$

$$\frac{dX_2}{dt} = -X_1$$

$$\Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = -r \cdot \cos\theta \Rightarrow r' \cdot \sin\theta + r \cdot \left[\frac{r'}{r \cdot tg\theta} - 1\right] \cdot \cos\theta = -r \cdot \cos\theta$$

$$r' \cdot \sin\theta + \frac{r'}{tg\theta} \cdot \cos\theta - r \cdot \cos\theta = -r \cdot \cos\theta \Rightarrow r' \cdot \left[\sin\theta + \frac{\cos\theta}{tg\theta}\right] = 0;$$

$$\sin\theta + \frac{\cos\theta}{tg\theta} = \frac{1}{\sin\theta}$$

$$r' \cdot \frac{1}{\sin\theta} = 0 \Rightarrow r' = 0 \Rightarrow r = \text{const.}$$

6.2 OptoNDR Circuit Van der Pol Limit Cycle Solution

The Van der Pol oscillator can be given by the following equations: $\frac{dX_1}{d_t} = X_2$ and $\frac{dX_2}{d_t} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2$. We can express it by second-degree differential equation: $\frac{d^2X_1}{d_t^2} + \mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{d_t} + X_1 = 0$. The parameter μ is non-negative real number $\mu \geq 0;\ \mu \in \mathbb{R}$. The equation is related to nonlinear electrical circuits and related to simple harmonic oscillator which includes a nonlinear damping term $\mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{d_t}$. Practically, this term is like ordinary positive damping for $|X_1| > 1$, but like negative damping for $|X_1| < 1$. The behavior, $\mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{d_t}$ term causes large amplitude oscillations to decay, but it pumps them back up if they become too small. Our system settles into a self-sustained oscillation and the special phenomena happened. The energy is dissipated over one cycle which balances the energy pumped in. The Van der Pol equation has a unique, stable limit cycle for $\mu > 0;\ \mu \in \mathbb{R}$. We have a Van der Pol oscillator circuit. The active element of the circuit is semiconductor device (OptoNDR circuit/device). It acts like an ordinary resistor when current $I(t)$ is high ($I(t) > I_{sat}$), but like negative resistor (energy source) or negative differential resistance (NDR) when $I(t)$ is low $I(t) > I_{break}$ and $I(t) < I_{sat}$. Our circuit current–voltage characteristic $V = f(I) \quad \forall \frac{dI}{d_t} = 0$ resembles a cubic function. We consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below I_{break} ($I(t) < I_{break}$) [1, 2, 4] (Fig. 6.1).

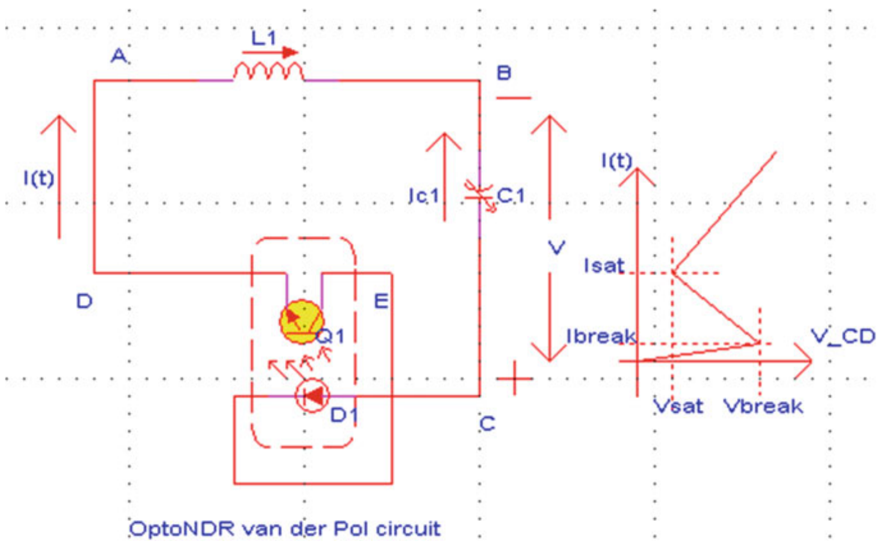

Fig. 6.1 OptoNDR van der Pol circuit current–voltage characteristic $V = f(I) \quad \forall \frac{dI}{d_t} = 0$ resembles a cubic function

$V_C > V_B;\ V = V_{CB} = V_C - V_B = -V_{BC}$, denote the voltage drop from point C to point B in the circuit. $V = V_{C_1};\ I_{C_1} = C_1 \cdot \frac{dV}{dt};\ I = -I_{C_1};\ \frac{dV}{dt} = \frac{I_{C_1}}{C_1} = -\frac{I}{C_1};$ $V_{L_1} = V_{AB} = L_1 \cdot \frac{dI}{dt}$ KVL (Kircoff's Voltage Law): $V_{CD} + V_{AB} - V = 0;$ $V_{CD} = V_{D_1} + V_{CEQ_1} = f(I);\ V_{AB} = V_{L_1}$

$$f(I) + V_{L_1} - V = 0;\ V_{CD} = V_C - V_D = f(I);\ f(I) + L_1 \cdot \frac{dI}{dt} - V = 0;\ V_A = V_D$$

We want to show that the system circuit is equivalent to Van der Pol basic equations, $\frac{dV}{dt} = -\frac{I}{C_1} \Rightarrow \frac{dV}{dt} \cdot \sqrt{C_1} \cdot \sqrt{C_1} = -I;\ \frac{dV}{dt} \cdot \sqrt{C_1 \cdot L_1} \cdot \sqrt{C_1} = -I \cdot \sqrt{L_1}.$ We define new time variable $\tau = \frac{1}{\sqrt{C_1 \cdot L_1}} \cdot t \Rightarrow t = \tau \cdot \sqrt{C_1 \cdot L_1}$; $dt = d\tau \cdot \sqrt{C_1 \cdot L_1}.$ $\frac{dV}{d\tau \cdot \sqrt{C_1 \cdot L_1}} \cdot \sqrt{C_1 \cdot L_1} \cdot \sqrt{C_1} = -I \cdot \sqrt{L_1} \Rightarrow \frac{dV}{d\tau} \cdot \sqrt{C_1} = -I \cdot \sqrt{L_1}.$ We define new variables: $x = I \cdot \sqrt{L_1};\ w = V \cdot \sqrt{C_1};\ I = \frac{x}{\sqrt{L_1}};\ V = \frac{w}{\sqrt{C_1}}$ then the first differential equation$\frac{dV}{d\tau} \cdot \sqrt{C_1} = -I \cdot \sqrt{L_1} \Rightarrow \frac{dw}{d\tau} \frac{1}{\sqrt{C_1}} \cdot \sqrt{C_1} = -\frac{x}{\sqrt{L_1}} \cdot \sqrt{L_1};\ \frac{dw}{d\tau} = -x.$

$$V = f(I) + L_1 \cdot \frac{dI}{dt};\ f(I) = \mu \cdot \frac{1}{\sqrt{C_1}} \cdot F(I \cdot \sqrt{L_1});\ x = I \cdot \sqrt{L_1} \Rightarrow I = \frac{x}{\sqrt{L_1}}$$
$$f\left(\frac{x}{\sqrt{L_1}}\right) = \mu \cdot \frac{1}{\sqrt{C_1}} \cdot F\left(\frac{x}{\sqrt{L_1}} \cdot \sqrt{L_1}\right) \Rightarrow f\left(\frac{x}{\sqrt{L_1}}\right) = \mu \cdot \frac{1}{\sqrt{C_1}} \cdot F(x);\ F(x) = \frac{\sqrt{C_1}}{\mu} \cdot f\left(\frac{x}{\sqrt{L_1}}\right)$$

$$V = f(I) + L_1 \cdot \frac{dI}{dt} \Rightarrow V = \mu \cdot \frac{1}{\sqrt{C_1}} \cdot F(I \cdot \sqrt{L_1}) + L_1 \cdot \frac{dI}{dt};\ V = \mu \cdot \frac{1}{\sqrt{C_1}} \cdot F(x) + L_1 \cdot \frac{dI}{dt}$$
$$V \cdot \sqrt{C_1} = \mu \cdot F(x) + L_1 \cdot \sqrt{C_1} \cdot \frac{dI}{dt};\ w = V \cdot \sqrt{C_1};\ w = \mu \cdot F(x) + L_1 \cdot \sqrt{C_1} \cdot \frac{dI}{dt}$$
$$w = \mu \cdot F(x) + L_1 \cdot \sqrt{C_1} \cdot \frac{dI}{dt} \Rightarrow w = \mu \cdot F(x) + \sqrt{L_1} \cdot \sqrt{C_1 \cdot L_1} \cdot \frac{dI}{dt}\ ;\ dt = d\tau \cdot \sqrt{C_1 \cdot L_1}$$
$$w = \mu \cdot F(x) + \sqrt{L_1} \cdot \sqrt{C_1 \cdot L_1} \cdot \frac{dI}{d\tau \cdot \sqrt{C_1 \cdot L_1}} \Rightarrow w = \mu \cdot F(x) + \sqrt{L_1} \cdot \frac{dI}{d\tau};\ I = \frac{x}{\sqrt{L_1}}$$
$$w = \mu \cdot F(x) + \sqrt{L_1} \cdot \frac{dx}{d\tau} \cdot \frac{1}{\sqrt{L_1}} \Rightarrow w = \mu \cdot F(x) + \frac{dx}{d\tau};\ \frac{dx}{d\tau} = w - \mu \cdot F(x)$$

We can summarize our new Van der Pol equations (w and x variables):

$$\frac{dw}{d\tau} = -x;\ \frac{dx}{d\tau} = w - \mu \cdot F(x);\ \frac{dx}{d\tau} = w - \mu \cdot F(x) \Rightarrow w = \frac{dx}{d\tau} + \mu \cdot F(x)$$

$$\frac{dw}{d\tau} = \frac{d^2x}{d\tau^2} + \mu \cdot \frac{dF(x)}{d\tau};\ \frac{dw}{d\tau} = -x \Rightarrow \frac{d^2x}{d\tau^2} + \mu \cdot \frac{dF(x)}{d\tau} = -x;\ \frac{d^2x}{d\tau^2} + \mu \cdot \frac{dF(x)}{d\tau} + x = 0$$

The classical two-degree differential equation of Van der Pol equation: $\frac{d^2 x}{d\tau^2} + \mu \cdot \frac{dF(x)}{d\tau} + x = 0$; $x \to X_1$; $\frac{dF(x)}{d\tau} \to (X_1^2 - 1) \cdot \frac{dX_1}{d\tau}$ and we get our classical Van der Pol equation $\frac{d^2 X_1}{d\tau^2} + \mu \cdot (X_1^2 - 1) \cdot \frac{dX_1}{d\tau} + X_1 = 0$.

OptoNDR Element Mathematical Analysis

Our OptoNDR element/circuit is constructed from LED and phototransistor in series. The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant ($I_{BQ_1} = I_{\text{LED}} \cdot k = I_{CQ_1} \cdot k$) and is the phototransistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations. The basic Ebers–Moll schematic for NPN bipolar transistor is shown in the next figure. We need to implement the regular Ebers–Moll model to the optocoupler circuit (transistor Q_1 and LED D_1) and get a complete final expression for the negative differential resistance (NDR) characteristics of that circuit [18] (Fig. 6.2).

$$i_{DE} + i_{DC} = i_{bQ_1} + \alpha_f \cdot i_{DEQ_1} + \alpha_r \cdot i_{DCQ_1};\; i_{DCQ_1} + I_{CQ_1} = \alpha_f \cdot i_{DEQ_1};\; i_{DEQ_1} = \alpha_r \cdot i_{DCQ_1} + i_{EQ_1}$$

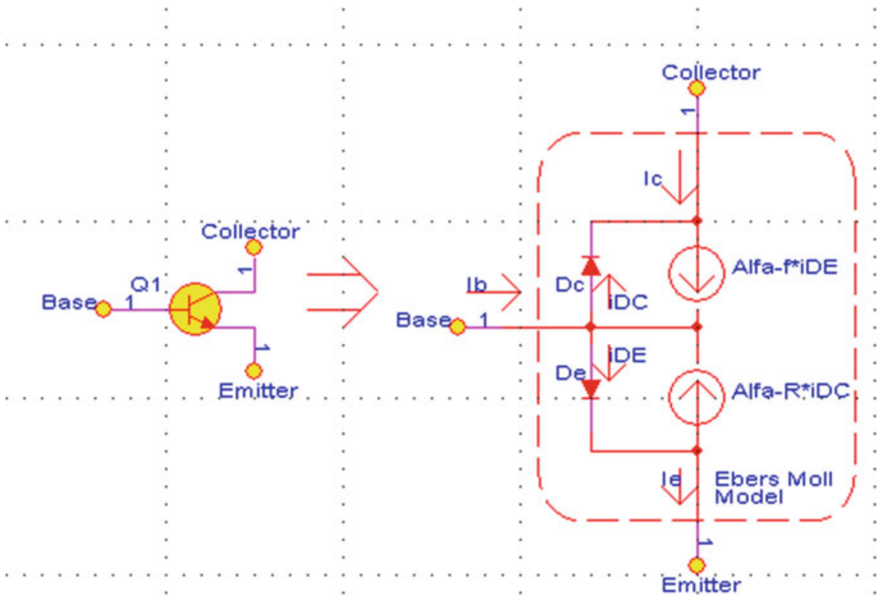

Fig. 6.2 NPN BJT Ebers-Moll model circuit

$$i_{DCQ_1} + I_{CQ_1} = \alpha_f \cdot i_{DEQ_1} \Rightarrow i_{DCQ_1} = \alpha_f \cdot i_{DEQ_1} - I_{CQ_1};\; i_{DEQ_1} = \alpha_r \cdot (\alpha_f \cdot i_{DEQ_1} - I_{CQ_1}) + i_{EQ_1}$$

$$i_{DEQ_1} = \alpha_r \cdot \alpha_f \cdot i_{DEQ_1} - \alpha_r \cdot I_{CQ_1} + i_{EQ_1};\; i_{DEQ_1} - \alpha_r \cdot \alpha_f \cdot i_{DEQ_1} = i_{EQ_1} - \alpha_r \cdot I_{CQ_1}$$

$$i_{DEQ_1} = \frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f};\; i_{DCQ_1} = \alpha_f \cdot i_{DEQ_1} - I_{CQ_1} = \alpha_f \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) - I_{CQ_1}$$

$$i_{DCQ_1} = \alpha_f \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) - I_{CQ_1} = \frac{\alpha_f \cdot (i_{EQ_1} - \alpha_r \cdot I_{CQ_1}) - I_{CQ_1} \cdot (1 - \alpha_r \cdot \alpha_f)}{1 - \alpha_r \cdot \alpha_f}$$

$$i_{DCQ_1} = \frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f};$$

$$V_{\text{Base-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_1} + 1\right];\; V_{\text{Base-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Base-Collector}} = V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot i_{DCQ_1} + 1\right];\; V_{\text{Base-Collector}} = V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot \left(\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Collector-Emitter}} = V_{\text{Collector-Base}} + V_{\text{Base-Emitter}};\; V_{\text{Collector-Base}} = -V_{\text{Base-Collector}}$$

$$V_{\text{Collector-Emitter}} = V_{\text{Collector-Base}} + V_{\text{Base-Emitter}} = V_{\text{Base-Emitter}} - V_{\text{Base-Collector}}$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right] - V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot \left(\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \ln\left[\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\left\{\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\} \cdot \left\{\frac{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\left\{\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\} \cdot \left\{\frac{I_{sc}}{I_{se}}\right\}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right];$$

$$I_{CQ_1} = I_{D_1}\;;\; I_{EQ_1} = I(t) = I$$

$$V_{D_1} = V_{CE} = V_t \cdot \ln\left[\frac{I_{D_1}}{I_0} + 1\right]; \; V_{D_1} = V_{CE} = V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right];$$
$$i_{EQ_1} \to I_{EQ_1}; \; i_{CQ_1} \to I_{CQ_1}$$

The optical coupling between the LED (D_1) and the phototransistor (Q_1) is represented as transistor-dependent base current on LED (D_1) current:

$$I_{BQ_1} = I_{D_1} \cdot k; \; I_{D_1} = I_{CQ_1}; \; I_{BQ_1} = I_{CQ_1} \cdot k; \; I_{EQ_1} = I; \; I_{EQ_1} = I_{CQ_1} + I_{BQ_1} = I_{CQ_1} \cdot (1+k)$$
$$I_{EQ_1} = I \Rightarrow I = I_{CQ_1} \cdot (1+k) \Rightarrow I_{CQ_1} = \frac{I}{(1+k)}; \; I_{BQ_1} = I_{CQ_1} \cdot k = \frac{I \cdot k}{(1+k)}$$

As long as the phototransistor (Q_1) is in cut-off region, the currents I_{CQ_1}, I_{EQ_1}, and I_{BQ_1} are very low. When the phototransistor (Q_1) reaches breakover voltage it enters saturation region ($V_{\text{Collector-Emitter}}$ decreases and I_{CQ_1} increases). The region which $V_{\text{Collector-Emitter}}$ decreases and I_{CQ_1} increases is the Negative Differential Resistance area of V_{CD}–I_{CQ1} characteristics. The positive feedback in which the phototransistor collector current I_{CQ_1} increases and then I_{BQ_1} increases ($I_{BQ_1} = I_{CQ_1} \cdot k$) is repeated in increasing cycles [1]. The positive feedback ends when the phototransistor reaches saturation state. Finally, we arrive at an expression which is the voltage$V_{\text{Collector-Emitter}}$ as a function of the current (I_{CQ_1}) for NDR circuit ($V_{CD} = V_{\text{Collector-Emitter}} + V_{D_1}$).

$$\begin{aligned} V_{CD} &= V_{\text{Collector-Emitter}} + V_{D_1} \\ &= V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] \\ &\quad + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right] \end{aligned}$$

Assume, $I_{sc} \approx I_{se}; \; V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] \approx 0$

$$V_{CD} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]$$

$$I_{CQ_1} = \frac{I}{(1+k)}; \; I_{BQ_1} = I_{CQ_1} \cdot k = \frac{I \cdot k}{(1+k)}$$

$$\begin{aligned} V_{CD} &= f(I); \; V_{CD} \\ &= V_t \cdot \ln\left[\frac{I - \alpha_r \cdot \frac{I}{(1+k)} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot I - \frac{I}{(1+k)} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] \end{aligned}$$

$$f(I) = V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

The typical values for our BJT transistor and LED diode parameters are

$$V_t = 0.026 = 26\,\text{mV};\; k = 0.02;\; I_{se} = 1\;\mu\text{A} = 10^{-6};\; I_{sc} = 2\;\mu\text{A} = 2 \times 10^{-6}$$
$$\alpha_r = 0.5;\; \alpha_f = 0.98;\; I_0 = 1\;\mu\text{A} = 10^{-6};\; 1 - \alpha_r \cdot \frac{1}{(1+k)} = 0.51;\; V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] = 0.018$$
$$\alpha_f - \frac{1}{(1+k)} = -0.000392;\; I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) = 0.51 \times 10^{-6};\; I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) = 1.02 \times 10^{-6}$$

$\frac{1}{(1+k) \cdot I_0} = 980,392.15$. We need to plot the graph *f*(*I*) versus *I* (Fig. 6.3).

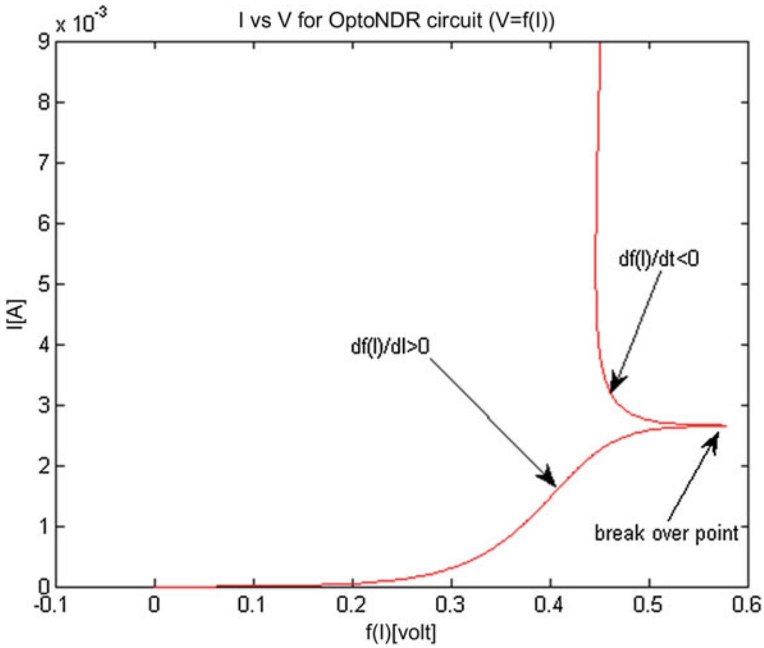

Fig. 6.3 *I* vs *V* for OptoNDR circuit ($V = f(I)$)

MATLAB Script

```
I=0:0.00001:0.009;
a=0.018;b=0.026*log(I*980392.15+1);
d=I*0.51+0.51*0.000001;e=-I*0.000392+1.02*0.00000102;
c=0.026*log(d./e)+a+b;
plot(c,I,'-r');
```

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t \cdot \frac{d}{\mathrm{d}I}\left\{\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]\right\} + V_t \cdot \frac{d}{\mathrm{d}I}\left\{\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right]\right\}$$

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t \cdot \frac{1}{I+(1+k)\cdot I_0} + V_t \cdot \left\{\frac{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}\right\} \cdot \left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot\left\{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right\} - \left[\alpha_f-\frac{1}{(1+k)}\right]\cdot\left\{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)\right\}}{\left\{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right\}^2}\right\}$$

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t \cdot \frac{1}{I+(1+k)\cdot I_0} + V_t \cdot \left\{\frac{(1-\alpha_r\cdot\alpha_f)\cdot\left\{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-\left[\alpha_f-\frac{1}{(1+k)}\right]\cdot I_{se}\right\}}{\left\{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f\right\}}\right\} \cdot \left\{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right\}$$

We get the conditions in NDR region: $I \neq -(1+k)\cdot I_0$

$$I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)\neq 0 \Rightarrow I \neq -\frac{I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]}$$

$$I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) \neq 0 \Rightarrow I \neq -\frac{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{\left[\alpha_f - \frac{1}{(1+k)}\right]}$$

We can demonstrate the $\frac{df(I)}{dI}$ equation as a parametric function with some constant. Let us define the constants first.

$$\Gamma_1 = (1+k) \cdot I_0;\ \Gamma_2 = (1 - \alpha_r \cdot \alpha_f) \cdot \left\{\left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] \cdot I_{sc} - \left[\alpha_f - \frac{1}{(1+k)}\right] \cdot I_{se}\right\}$$

$$\Gamma_3 = \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right];\ \Gamma_4 = I_{se} \cdot (1 - \alpha_r \cdot \alpha_f);\ \Gamma_5 = \left[\alpha_f - \frac{1}{(1+k)}\right];\ \Gamma_6 = I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)$$

$$\frac{df(I)}{dI} = V_t \cdot \frac{1}{I + \Gamma_1} + V_t \cdot \left\{\frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}}\right\};\ \Gamma_1, \Gamma_2, \ldots, \Gamma_6 \in \mathbb{R}.$$

We need to analyze the above equation for regions which are near the saturation region and cut-off region. For the region which is after the breakover voltage but near enough to the cut-off region:

$$Q_{1(\text{cutoff})}(\Delta f(I) \to \varepsilon \ll 1,\ \Delta I \to 0) \Rightarrow \text{NDR} = \frac{\Delta f(I)}{\Delta I} \to \infty.$$

For the region which is near and in the phototransistor saturation state:

$$Q_{1(\text{saturation})}(\Delta f(I) \to \varepsilon \ll 1,\ \Delta I \to \infty) \Rightarrow \text{NDR} = \frac{\Delta f(I)}{\Delta I} \to 0.$$

For the cut-off region before the breakover $k = 0$ then we get the expression $\frac{df(I)}{dI}$.

$$\frac{df(I)}{dI}\Big|_{k=0} = V_t \cdot \frac{1}{I + I_0} + V_t \cdot \left\{\frac{(1 - \alpha_r \cdot \alpha_f) \cdot \{[1 - \alpha_r] \cdot I_{sc} - [\alpha_f - 1] \cdot I_{se}\}}{\{I \cdot [1 - \alpha_r] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)\} \cdot \{I \cdot [\alpha_f - 1] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\}}\right\}$$

Back to our circuit Van der Pol differential equations:

$$\frac{dV}{dt} = -\frac{I}{C_1};\ f(I) + L_1 \cdot \frac{dI}{dt} - V = 0$$

$$\frac{dV}{dt} = -\frac{I}{C_1};$$

$$V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right]$$

$$+ V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + L_1 \cdot \frac{dI}{dt} - V = 0$$

Limit cycle discussion, Van der Pol circuit oscillator: $V(t) = V;\ I(t) = I$.

We need to prove that our Van der Pol circuit has periodic orbits and it is done by changing system Cartesian coordinates ($V(t)$, $I(t)$) and show that the cylinder is invariant. For the conversion between cylindrical and Cartesian circuits system and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z axis. In our circuit system we refer to Cartesian V–I plane (with equation, addition coordinate equal to zero) [7, 8]. Then the z coordinate is the same in both circuits system, and the correspondence between cylindrical (r, θ) and Cartesian (V, I) are the same as polar coordinates, namely $V(t) = r(t) \cdot \cos[\theta(t)];\ I(t) = r(t) \cdot \sin[\theta(t)];\ r = \sqrt{V^2 + I^2}$.

$\theta(t) = 0$ if $V = 0$ and $I = 0$. $\theta(t) = \arcsin(I/r)$ if $V > 0$, $x \to V;\ y \to I$.

$$V(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{dV(t)}{dt} = \frac{dr(t)}{dt} \cdot \cos[\theta(t)] - r(t) \cdot \frac{d\theta(t)}{dt} \cdot \sin[\theta(t)]$$

$$I(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{dI(t)}{dt} = \frac{dr(t)}{dt} \cdot \sin[\theta(t)] + r(t) \cdot \frac{d\theta(t)}{dt} \cdot \cos[\theta(t)]$$

$$\frac{dV(t)}{dt} = \frac{dV}{dt};\ \frac{dr(t)}{dt} = r';\ \frac{d\theta(t)}{dt} = \theta';\ \theta(t) = \theta;\ r(t) = r$$

We get the equations:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta; \ \frac{\mathrm{d}I}{\mathrm{d}t} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{C_1} \cdot I \Rightarrow r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = -\frac{1}{C_1} \cdot r \cdot \sin\theta;$$

$$r' \cdot \cos\theta = r \cdot \theta' \cdot \sin\theta - \frac{1}{C_1} \cdot r \cdot \sin\theta$$

$$r' \cdot \cos\theta = r \cdot \sin\theta \cdot \left[\theta' - \frac{1}{C_1}\right] \Rightarrow r' = r \cdot tg\theta \cdot \left[\theta' - \frac{1}{C_1}\right]; \ tg\theta = \frac{r'}{r \cdot \left[\theta' - \frac{1}{C_1}\right]}$$

$$tg\theta = \frac{r'}{r \cdot \left[\theta' - \frac{1}{C_1}\right]} \Rightarrow \theta = \mathrm{arctg}\left\{\frac{r'}{r \cdot \left[\theta' - \frac{1}{C_1}\right]}\right\};$$

$$V = r \cdot \cos\theta; \ I = r \cdot \sin\theta$$

$$V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t$$

$$\cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - V = 0$$

$$V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$

$$+ V_t \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$+ L_1 \cdot (r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta) - r \cdot \cos\theta = 0$$

$$r' = r \cdot tg\theta \cdot \left[\theta' - \frac{1}{C_1}\right]$$

$$V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right]$$

$$+ V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + L_1 \cdot \left(r \cdot tg\theta \cdot \left[\theta' - \frac{1}{C_1}\right] \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta\right) - r \cdot \cos\theta = 0$$

$$V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right]$$

$$+ V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + L_1 \cdot r \cdot tg\theta \cdot \sin\theta \cdot \left[\theta' - \frac{1}{C_1}\right] + L_1 \cdot r \cdot \theta' \cdot \cos\theta - r \cdot \cos\theta = 0$$

$$V_t \cdot \ln\left[\frac{r\cdot \sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]+V_t\cdot\ln\left[\frac{r\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]+V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]$$
$$+L_1\cdot r\cdot\theta'\cdot[tg\theta\cdot\sin\theta+\cos\theta]-L_1\cdot r\cdot tg\theta\cdot\sin\theta\cdot\frac{1}{C_1}-r\cdot\cos\theta=0$$

$$tg\theta\cdot\sin\theta+\cos\theta=\frac{\sin^2\theta}{\cos\theta}+\cos\theta=\frac{\sin^2\theta+\cos^2\theta}{\cos\theta}=\frac{1}{\cos\theta}$$

$$V_t \cdot \ln\left[\frac{r\cdot \sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]+V_t\cdot\ln\left[\frac{r\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]$$
$$+V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]+L_1\cdot r\cdot\theta'\cdot\frac{1}{\cos\theta}-L_1\cdot r\cdot tg\theta\cdot\sin\theta\cdot\frac{1}{C_1}-r\cdot\cos\theta=0$$

$$L_1\cdot r\cdot\theta'\cdot\frac{1}{\cos\theta}=L_1\cdot r\cdot tg\theta\cdot\sin\theta\cdot\frac{1}{C_1}+r\cdot\cos\theta-V_t$$
$$\cdot\ln\left[\frac{r\cdot \sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]$$
$$-V_t\cdot\ln\left[\frac{r\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]-V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$\theta'=\sin^2\theta\cdot\frac{1}{C_1}+\frac{1}{L_1}\cdot\cos^2\theta-\frac{1}{L_1\cdot r}\cdot\cos\theta\cdot V_t$$
$$\cdot\ln\left[\frac{r\cdot \sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]$$
$$-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r}\cdot\ln\left[\frac{r\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r}\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]$$

We need to get the expression for r': $r'=r\cdot tg\theta\cdot\left[\theta'-\frac{1}{C_1}\right]$

$$r'=r\cdot tg\theta\cdot\left\{\sin^2\theta\cdot\frac{1}{C_1}+\frac{1}{L_1}\cdot\cos^2\theta-\frac{1}{L_1\cdot r}\cdot\cos\theta\cdot V_t\right.$$
$$\cdot\ln\left[\frac{r\cdot \sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]$$
$$\left.-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r}\cdot\ln\left[\frac{r\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r}\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]-\frac{1}{C_1}\right\}$$

We can summarize our last results:

$$\theta' = \sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$r' = r \cdot tg\theta \cdot \left\{\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1}\right\}$$

We define $\theta' = \xi_1(r, \theta);\ r' = \xi_2(r, \theta)$

$$\xi_1(r, \theta) = \sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$\xi_2(r, \theta) = r \cdot tg\theta \cdot \left\{\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1}\right\}$$

If there is a stable limit cycle as specific value of radius r, $\mathrm{d}r/\mathrm{d}t = 0$ then

$$r' = \xi_2(r,\theta);\ r' = 0;\ r \neq 0$$

$$r \cdot tg\theta \cdot \left\{ \sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \right.$$
$$\cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$
$$\left. - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1} \right\} = 0$$

Case A $tg\theta = 0 \Rightarrow \theta = k \cdot \pi \quad \forall\ k = \ldots, -2, -1, 0, 1, 2, \ldots$ the $tg\theta$ function is undefined for $\theta = \frac{\pi}{2} + \pi \cdot k = \pi \cdot \left(\frac{1}{2} + k\right)$.

Case B

$$\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t$$
$$\cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$
$$- V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1} = 0$$

$$\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{C_1} - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$
$$= V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right]$$
$$+ \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$

The above function is difficult for analytic solution and numerical solution can be very good. In our solutions, we ignore all negative values and complex values of r ($r > 0;\ r \in \mathbb{R}_+$). Additionally, θ must be real number and we ignore complex values $\theta \in \mathbb{R}$ [54, 55]:

$$r' = r \cdot tg\theta \cdot \left[\theta' - \frac{1}{C_1}\right];\ r' = 0 \Rightarrow r \cdot tg\theta \cdot \left[\theta' - \frac{1}{C_1}\right] = 0 \Rightarrow \theta' = \frac{1}{C_1};$$
$$\theta = \frac{1}{C_1} \cdot t + \text{Const.}$$

Other case is when system limit cycle is resided in annulus $0 < r_{\min} < r < r_{\max}$ and there is no constant r in the limit cycle $\frac{dr}{dt} \neq 0$. We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$. Exist $\frac{dr}{dt} < 0$. On the outer circle and $\frac{dr}{dt} > 0$ on the inner circle. To find $r_{\min}$, we require $\frac{dr}{dt} > 0$ for all values of θ: $r' > 0 \Rightarrow \xi_2(r, \theta) > 0$.

$$r \cdot tg\theta \cdot \left\{ \sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1} \right\} > 0$$

$r > 0$ always then $tg\theta \cdot \psi(r, \theta) > 0$ is the condition we need to check. We have two subcases, Case 1.1: $tg\theta > 0$; $\psi(r, \theta) > 0$ and Case 1.2: $tg\theta < 0$; $\psi(r, \theta) < 0$.

$$\psi(r, \theta) = \sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1}$$

Case 1.1

$$r \to r_{\min}\,;\ tg\theta > 0\,\&\,\psi(r_{\min}, \theta) > 0;\ tg\theta > 0 \Rightarrow \pi \cdot (k-1) < \theta < \pi \cdot \left(k - \frac{1}{2}\right)$$

$$\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r_{\min}} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r_{\min} \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r_{\min} \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r_{\min}} \cdot \ln\left[\frac{r_{\min} \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r_{\min}} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1} > 0$$

The above function with condition is difficult for analytic solution and numerical solution can be very good to find the limits for $r_{\min}$. In our solutions, we ignore all negative $r_{\min}$ values and complex $r_{\min}$ values of $r_{\min}$ ($r_{\min} > 0$; $r_{\min} \in \mathbb{R}_+$). Additionally, θ must be real number and we ignore complex values $\theta \in \mathbb{R}$.

Case 1.2 $r \to r_{\min};\ tg\theta < 0\ \&\ \psi(r_{\min}, \theta) < 0;\ tg\theta < 0 \Rightarrow \pi \cdot (k - \frac{1}{2}) < \theta < \pi \cdot k$

$$\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r_{\min}} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r_{\min} \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r_{\min} \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r_{\min}} \cdot \ln\left[\frac{r_{\min} \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r_{\min}} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1} < 0$$

The above function with condition is difficult for analytic solution and numerical solution can be very good to find the limits for $r_{\min}$. In our solutions, we ignore all negative $r_{\min}$ values and complex $r_{\min}$ values of $r_{\min}$ ($r_{\min} > 0$; $r_{\min} \in \mathbb{R}_+$). Additionally, θ must be real number and we ignore complex values $\theta \in \mathbb{R}$.

To find $r_{\max}$, we require $\frac{dr}{dt} < 0$ for all values of θ: $r' < 0 \Rightarrow \xi_2(r, \theta) < 0$

$$r \cdot tg\theta \cdot \left\{\sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1}\right\} < 0$$

$r > 0$ always then $tg\theta \cdot \psi(r, \theta) < 0$ is the condition we need to check. We have two subcases, Case 2.1: $tg\theta > 0;\ \psi(r, \theta) < 0$ and Case 2.2: $tg\theta < 0;\ \psi(r, \theta) > 0$.

$$\psi(r, \theta) = \sin^2\theta \cdot \frac{1}{C_1} + \frac{1}{L_1} \cdot \cos^2\theta - \frac{1}{L_1 \cdot r} \cdot \cos\theta \cdot V_t \cdot \ln\left[\frac{r \cdot \sin\theta \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{r \cdot \sin\theta \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{r \cdot \sin\theta}{(1+k) \cdot I_0} + 1\right] - V_t \cdot \cos\theta \cdot \frac{1}{L_1 \cdot r} \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] - \frac{1}{C_1}$$

Case 2.1

$$r \to r_{\max};\ tg\theta > 0\ \&\ \psi(r_{\max}, \theta) < 0;\ tg\theta > 0 \Rightarrow \pi \cdot (k - 1) < \theta < \pi \cdot \left(k - \frac{1}{2}\right)$$

$$\sin^2\theta\cdot\frac{1}{C_1}+\frac{1}{L_1}\cdot\cos^2\theta-\frac{1}{L_1\cdot r_{\max}}\cdot\cos\theta\cdot V_t$$
$$\cdot\ln\left[\frac{r_{\max}\cdot\sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r_{\max}\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]$$
$$-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r_{\max}}\cdot\ln\left[\frac{r_{\max}\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r_{\max}}\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]-\frac{1}{C_1}<0$$

The above function with condition is difficult for analytic solution and numerical solution can be very good to find the limits for $r_{\max}$. In our solutions, we ignore all negative $r_{\max}$ values and complex $r_{\max}$ values of $r_{\max}$ ($r_{\max}>0$; $r_{\max}\in\mathbb{R}_+$). Additionally, θ must be real number and we ignore complex values $\theta\in\mathbb{R}$.

Case 2.2 $r\to r_{\max}$; $tg\theta<0$ & $\psi(r_{\max},\theta)>0$; $tg\theta<0\Rightarrow\pi\cdot(k-\frac{1}{2})<\theta<\pi\cdot k$

$$\sin^2\theta\cdot\frac{1}{C_1}+\frac{1}{L_1}\cdot\cos^2\theta-\frac{1}{L_1\cdot r_{\max}}\cdot\cos\theta\cdot V_t$$
$$\cdot\ln\left[\frac{r_{\max}\cdot\sin\theta\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{r_{\max}\cdot\sin\theta\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]$$
$$-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r_{\max}}\cdot\ln\left[\frac{r_{\max}\cdot\sin\theta}{(1+k)\cdot I_0}+1\right]-V_t\cdot\cos\theta\cdot\frac{1}{L_1\cdot r_{\max}}\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]-\frac{1}{C_1}>0$$

The above function with condition is difficult for analytic solution and numerical solution can be very good to find the limits for $r_{\max}$. In our solutions, we ignore all negative $r_{\max}$ values and complex $r_{\max}$ values of $r_{\max}$ ($r_{\max}>0$; $r_{\max}\in\mathbb{R}_+$). Additionally, θ must be real number and we ignore complex values $\theta\in\mathbb{R}$.

OptoNDR Van der Pol Circuit System Stability Analysis

Our circuit Van der Pol differential equations: $\frac{dV}{dt}=-\frac{I}{C_1}$; $f(I)+L_1\cdot\frac{dI}{dt}-V=0$

$$\frac{dV}{dt}=-\frac{I}{C_1};\;V_t\cdot\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]$$
$$+V_t\cdot\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right]+V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]+L_1\cdot\frac{dI}{dt}-V=0$$

$$\frac{dI}{dt}=\frac{1}{L_1}\cdot V-\frac{1}{L_1}\cdot V_t\cdot\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]-\frac{1}{L_1}\cdot V_t$$
$$\cdot\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right]-\frac{1}{L_1}\cdot V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]$$

To find Van der Pol circuit system fixed points we set $\frac{dV}{dt} = 0; \frac{dI}{dt} = 0$

$$\frac{dV}{dt} = 0 \Rightarrow -\frac{I^*}{C_1} = 0;\ I^* = 0;\ \frac{dI}{dt} = 0;\ \ln\left[\frac{I^*}{(1+k)\cdot I_0} + 1\right]|_{I^*=0} = 0;\ E^*(V^*, I^*) = (0, 0)$$

$$\frac{1}{L_1}\cdot V^* - \frac{1}{L_1}\cdot V_t\cdot \ln\left[\frac{I_{se}}{I_{sc}}\right] - \frac{1}{L_1}\cdot V_t\cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] = 0 \Rightarrow V^* = V_t\cdot\left(\ln\left[\frac{I_{se}}{I_{sc}}\right] + \ln\left[\frac{I_{sc}}{I_{se}}\right]\right)$$
$$= V_t\cdot \ln\left[\frac{I_{se}}{I_{sc}}\cdot\frac{I_{sc}}{I_{se}}\right];\ V^* = 0.$$

We define our system differential equations: $\frac{dV}{dt} = \vartheta_1(V, I); \frac{dI}{dt} = \vartheta_2(V, I)$

$$\vartheta_1(V, I) = -\frac{I}{C_1}$$

$$\vartheta_2(V, I) = \frac{1}{L_1}\cdot V - \frac{1}{L_1}\cdot V_t\cdot \ln\left[\frac{I\cdot\left[1 - \alpha_r\cdot\frac{1}{(1+k)}\right] + I_{se}\cdot(1 - \alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc}\cdot(1 - \alpha_r\cdot\alpha_f)}\right]$$
$$- \frac{1}{L_1}\cdot V_t\cdot \ln\left[\frac{I}{(1+k)\cdot I_0} + 1\right] - \frac{1}{L_1}\cdot V_t\cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

We need to establish the stability of our OptoNDR Van der Pol system fixed point. We can ask for other quantitative measure of stability like rate of decay to the stable fixed point. There is a need for linearization about the fixed point. We implement the linearization technique to our dimensional system [4, 85, 86]. Suppose that the system differential equations are $\frac{dV}{dt} = \vartheta_1(V, I); \frac{dI}{dt} = \vartheta_2(V, I)$ and suppose that (V^*, I^*) is a fixed point, then $\vartheta_1(V^*, I^*) = 0; \vartheta_2(V^*, I^*) = 0$. Let $u = V - V^*; v = I - I^*$ denote the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v. The u equation is $\frac{du}{dt} = \frac{dV}{dt}$ since V^* is constant and by substitution $\frac{du}{dt} = \frac{dV}{dt} = \vartheta_1(V^* + u, I^* + v)$. Using Taylor series expansion $\frac{du}{dt} = \frac{dV}{dt} = \vartheta_1(V^* + u, I^* + v) = \vartheta_1(V^*, I^*) + u\cdot\frac{\partial\vartheta_1}{\partial V} + v\cdot\frac{\partial\vartheta_1}{\partial I} + O(u^2, v^2, u\cdot v)$. Since $\vartheta_1(V^*, I^*) = 0$, we can write $\frac{du}{dt} = \frac{dV}{dt} = u\cdot\frac{\partial\vartheta_1}{\partial V} + v\cdot\frac{\partial\vartheta_1}{\partial I} + O(u^2, v^2, u\cdot v)$. The partial derivatives $\frac{\partial\vartheta_1}{\partial V}, \frac{\partial\vartheta_1}{\partial I}$ are to be evaluating at the fixed point (V^*, I^*); thus they are numbers and not functions. The notation $O(u^2, v^2, u\cdot v)$ denotes quadratic term in u and v. Since u and v are small, the quadratic terms are very small and similarly we can write $\frac{dV}{dt} = \frac{dI}{dt} = v\cdot\frac{\partial\vartheta_2}{\partial V} + v\cdot\frac{\partial\vartheta_2}{\partial I} + O(u^2, v^2, u\cdot v)$. Hence, the disturbance $(u,\ v)$ evolves according to $\begin{pmatrix}\frac{du}{dt}\\ \frac{dV}{dt}\end{pmatrix} = \begin{pmatrix}\frac{\partial\vartheta_1}{\partial V} & \frac{\partial\vartheta_1}{\partial I}\\ \frac{\partial\vartheta_2}{\partial V} & \frac{\partial\vartheta_2}{\partial I}\end{pmatrix}\cdot\begin{pmatrix}u\\ v\end{pmatrix} +$ $O(u^2, v^2, u\cdot v)$.

The matrix $A = \begin{pmatrix} \frac{\partial \vartheta_1}{\partial V} & \frac{\partial \vartheta_1}{\partial I} \\ \frac{\partial \vartheta_2}{\partial V} & \frac{\partial \vartheta_2}{\partial I} \end{pmatrix}_{(V^*,I^*)}$ is called the Jacobian matrix at the fixed point (V^*, I^*). Since the quadratic terms $O(u^2, v^2, u \cdot v)$ are very small, we can neglect them and we get the linearized system: $\begin{pmatrix} \frac{du}{dt} \\ \frac{dV}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial \vartheta_1}{\partial V} & \frac{\partial \vartheta_1}{\partial I} \\ \frac{\partial \vartheta_2}{\partial V} & \frac{\partial \vartheta_2}{\partial I} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$.

$$\vartheta_1(V,I) = -\frac{I}{C_1} \Rightarrow \frac{\partial \vartheta_1}{\partial V} = 0;\ \frac{\partial \vartheta_1}{\partial I} = -\frac{1}{C_1};\ \frac{\partial \vartheta_2}{\partial V} = \frac{1}{L_1}$$

$$\vartheta_2(V,I) = \frac{1}{L_1} \cdot V - \frac{1}{L_1} \cdot V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$
$$- \frac{1}{L_1} \cdot V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] - \frac{1}{L_1} \cdot V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$\frac{\partial \vartheta_2}{\partial I} = -\frac{\partial}{\partial I}\left\{\frac{1}{L_1} \cdot V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]\right.$$
$$\left. + \frac{1}{L_1} \cdot V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right]\right\}$$

$$\frac{\partial}{\partial I}\left\{\ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]\right\}$$
$$= \left\{\frac{\left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] \cdot \left\{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right\} - \left\{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)\right\} \cdot \left[\alpha_f - \frac{1}{(1+k)}\right]}{\left\{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right\}^2}\right\}$$
$$\cdot \left\{\frac{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\}$$

$$\frac{\partial}{\partial I}\left\{\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]\right\}$$
$$=\left\{\frac{(1-\alpha_r\cdot\alpha_f)\cdot\left\{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]\right\}}{\left\{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right\}\cdot\left\{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)\right\}}\right\}$$

$$\frac{\partial}{\partial I}\left\{\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]\right\}_{@I^*=0}$$
$$=\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right\}$$

$$\frac{\partial}{\partial I}\left\{\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right]\right\}=\frac{\frac{1}{(1+k)\cdot I_0}}{\frac{I}{(1+k)\cdot I_0}+1}=\left\{\frac{\frac{1}{(1+k)\cdot I_0}}{\frac{I}{(1+k)\cdot I_0}+1}\right\}\cdot\frac{(1+k)\cdot I_0}{(1+k)\cdot I_0}$$
$$=\frac{1}{I+(1+k)\cdot I_0}$$

$$\frac{\partial}{\partial I}\left\{\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right]\right\}|_{(V^*,I^*)}=\frac{1}{(1+k)\cdot I_0}$$

$$\frac{\partial\vartheta_2}{\partial I}=-\frac{V_t}{L_1}\cdot\left\{\frac{\partial}{\partial I}\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right]+\frac{\partial}{\partial I}\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right]\right\}$$

$$\frac{\partial\vartheta_2}{\partial I}|_{(V^*,I^*)}=-\frac{V_t}{L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}+\frac{1}{(1+k)\cdot I_0}\right\}$$

$$\frac{\partial\vartheta_1}{\partial V}=\frac{\partial\vartheta_1}{\partial V}|_{(V^*,I^*)}=0;\ \frac{\partial\vartheta_1}{\partial I}=\frac{\partial\vartheta_1}{\partial I}|_{(V^*,I^*)}=-\frac{1}{C_1};\ \frac{\partial\vartheta_2}{\partial V}=\frac{\partial\vartheta_2}{\partial V}|_{(V^*,I^*)}=\frac{1}{L_1}$$

$$A=\begin{pmatrix}\frac{\partial\vartheta_1}{\partial V} & \frac{\partial\vartheta_1}{\partial I}\\ \frac{\partial\vartheta_2}{\partial V} & \frac{\partial\vartheta_2}{\partial I}\end{pmatrix}_{(V^*,I^*)}=\begin{pmatrix}0 & -\frac{1}{C_1}\\ \frac{1}{L_1} & -\frac{V_t}{L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}+\frac{1}{(1+k)\cdot I_0}\right\}\end{pmatrix}$$

The characteristic equation becomes $\det(A-\lambda\cdot I)=0$

$$\det(A-\lambda\cdot I)=\det\left[\begin{pmatrix}\frac{\partial\vartheta_1}{\partial V} & \frac{\partial\vartheta_1}{\partial I}\\ \frac{\partial\vartheta_2}{\partial V} & \frac{\partial\vartheta_2}{\partial I}\end{pmatrix}_{(V^*,I^*)}-\lambda\cdot I\right]=\det\begin{pmatrix}\frac{\partial\vartheta_1}{\partial V}-\lambda & \frac{\partial\vartheta_1}{\partial I}\\ \frac{\partial\vartheta_2}{\partial V} & \frac{\partial\vartheta_2}{\partial I}-\lambda\end{pmatrix}_{(V^*,I^*)}=0$$

$$\det\begin{pmatrix}-\lambda & -\frac{1}{C_1}\\ \frac{1}{L_1} & -\frac{V_t}{L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}+\frac{1}{(1+k)\cdot I_0}\right\}-\lambda\end{pmatrix}=0$$

$$(-\lambda)\cdot\left(-\frac{V_t}{L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}+\frac{1}{(1+k)\cdot I_0}\right\}-\lambda\right)+\frac{1}{L_1\cdot C_1}=0$$

$$\lambda^2+\lambda\cdot\frac{V_t}{L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}+\frac{1}{(1+k)\cdot I_0}\right\}+\frac{1}{L_1\cdot C_1}=0$$

$$\operatorname{trace}(A)=-\frac{V_t}{L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc}-I_{se}\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}+\frac{1}{(1+k)\cdot I_0}\right\};$$

$$\Delta=\det(A)=\frac{1}{L_1\cdot C_1}$$

$$\lambda_1=\frac{\operatorname{trace}(A)+\sqrt{[\operatorname{trace}(A)]^2-4\cdot\frac{1}{L_1\cdot C_1}}}{2};\ \lambda_2=\frac{\operatorname{trace}(A)-\sqrt{[\operatorname{trace}(A)]^2-4\cdot\frac{1}{L_1\cdot C_1}}}{2}.$$

Eigenvalues λ_1 and λ_2 are the solutions of the quadratic equation. The eigenvalues depend on the trace and determinant of the matrix A. The typical situation is for system eigenvalues to be distinct $\lambda_1 \neq \lambda_2$, and eigenvectors linearly independent and span entire plane. If the eigenvalues are complex, the fixed point is either a center or a spiral. A center fixed point is a simple harmonic oscillator; the origin is surrounded by a family of closed orbits. The centers are neutrally stable, since trajectories are neither attracted to nor repelled from the fixed point. A spiral would occur if the harmonic oscillator were lightly damped. Then the trajectory would fail to close, the oscillator loses a bit of energy on each cycle [7]. The complex eigenvalues occur when $[\operatorname{trace}(A)]^2-4\cdot\frac{1}{L_1\cdot C_1}<0$, and we can simplify the notation by writing the eigenvalues as $\lambda_1=\Omega_1+i\cdot\Omega_2;\ \lambda_2=\Omega_1-i\cdot\Omega_2$. Where $\Omega_1=\frac{\operatorname{trace}(A)}{2}$ $\Omega_2=\frac{1}{2}\cdot\sqrt{4\cdot\frac{1}{L_1\cdot C_1}-[\operatorname{trace}(A)]^2}$ and by assuming $\Omega_2\neq 0$ then the eigenvalues are distinct. Exponentially decaying oscillations if $\Omega_1=\operatorname{Re}(\lambda_1)<0;\ \Omega_1=\operatorname{Re}(\lambda_1)=\operatorname{Re}(\lambda_2)$ and growing oscillations if $\Omega_1=\operatorname{Re}(\lambda_1)>0;$

$\Omega_1 = \text{Re}(\lambda_1) = \text{Re}(\lambda_2)$. The corresponding fixed points are stable and unstable spiral, respectively. If the eigenvalues are pure imaginary ($\Omega_1 = 0$), then all solutions are periodic with period $T = \frac{2\cdot\pi}{\Omega_2}$. The oscillations have fixed amplitude and the fixed point is a center.

$$\Omega_1 = \frac{\text{trace}(A)}{2}$$
$$= -\frac{V_t}{2\cdot L_1}\cdot\left\{\frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc} - I_{se}\cdot\left[\alpha_f - \frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)} + \frac{1}{(1+k)\cdot I_0}\right\}$$

$$\Omega_1 = 0 \Rightarrow \frac{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc} - I_{se}\cdot\left[\alpha_f - \frac{1}{(1+k)}\right]}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)} + \frac{1}{(1+k)\cdot I_0} = 0$$

$$\Omega_1 = 0 \Rightarrow \frac{\left\{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc} - I_{se}\cdot\left[\alpha_f - \frac{1}{(1+k)}\right]\right\}\cdot(1+k)\cdot I_0 + I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}{I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\cdot(1+k)\cdot I_0} = 0$$

$$I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\cdot(1+k)\cdot I_0 \neq 0$$

$$\left\{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc} - I_{se}\cdot\left[\alpha_f - \frac{1}{(1+k)}\right]\right\}\cdot(1+k)\cdot I_0$$
$$+ I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f) = 0$$

$$k = \frac{I_0\cdot(\alpha_r\cdot I_{sc} - I_{se}) - I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}{I_0\cdot(I_{sc} - I_{se}\cdot\alpha_f)} - 1;\ T = \frac{2\cdot\pi}{\Omega_2}$$
$$= \frac{2\cdot\pi}{\frac{1}{2}\cdot\sqrt{4\cdot\frac{1}{L_1\cdot C_1} - [\text{trace}(A)]^2}}.$$

Result Discussion By choosing the right proportional k constant value, we guarantee our OptoNDR Van der Pol system oscillations.

$$k > 0 \Rightarrow \frac{I_0\cdot(\alpha_r\cdot I_{sc} - I_{se}) - I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}{I_0\cdot(I_{sc} - I_{se}\cdot\alpha_f)} > 1;\ I_{sc} > I_{se}\cdot\alpha_f$$

$$I_0\cdot(\alpha_r\cdot I_{sc} - I_{se}) - I_{se}\cdot I_{sc}\cdot(1-\alpha_r\cdot\alpha_f) > I_0\cdot(I_{sc} - I_{se}\cdot\alpha_f)$$

Assumption $k > 1$, for keeping the saturation process after breakover occurs and accordingly the parameters condition.

6.3 Glycolytic Oscillator Periodic Limit Cycle and Stability

Glycolytic oscillation is the repetitive fluctuation of in the concentrations of metabolites. The problem of modeling glycolytic oscillation has been studied in control theory and dynamical systems. The behavior depends on the rate of substrate injection. Early models used two variables (X, Y), but the most complex behavior they could demonstrate was period oscillations due to the Poincaré–Bendixson theorem. The Poincaré–Bendixson theorem is a statement about the long-term behavior of orbits of continuous dynamical systems on the plane, cylinder, or two-sphere [7, 8]. The condition that the dynamical system is on the plane is necessary to the theorem. Chaotic behavior can only arise in continuous dynamical systems whose phase space has three or more dimensions. However, the theorem does not apply to discrete dynamical systems, where chaotic behavior can arise in two- or even one-dimensional systems. A two-dimensional continuous dynamical system cannot give rise to a strange attractor. Periodic oscillations in biochemical systems are receiving much attention. The existence of sustained oscillations in yeast cell extracts and in the whole yeast cell as well as in heart muscle cell extracts has been proved. The model explains sustained oscillations in the yeast glycolytic system. The model has no limit cycle for those values of its parameters with which self-oscillations are observed. The model represents an enzyme reaction with substrate inhibition and product activation. Practically, the dynamical process called Glycolysis can proceed in an oscillatory fashion. The differential equations model of glycolytic oscillator is presented as follows:

$$\frac{dX}{dt} = -X + \mu_1 \cdot Y + X^2 \cdot Y; \ \frac{dY}{dt} = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y; \ f_1(X,Y) = -X + \mu_1 \cdot Y + X^2 \cdot Y$$

$$f_2(X,Y) = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y; \ \frac{dX}{dt} = f_1(X,Y); \frac{dY}{dt} = f_2(X,Y)$$

To find fixed points , we set $\frac{dX}{dt} = 0$; $\frac{dY}{dt} = 0$; $f_1(X^*, Y^*) = 0$; $f_2(X^*, Y^*) = 0$.

$$f_1(X^*, Y^*) = 0 \Rightarrow -X^* + \mu_1 \cdot Y^* + [X^*]^2 \cdot Y^* = 0;$$

$$f_2(X^*, Y^*) = 0 \Rightarrow \mu_2 - \mu_1 \cdot Y^* - [X^*]^2 \cdot Y^* = 0$$

$$-X^* + \mu_1 \cdot Y^* + [X^*]^2 \cdot Y^* = 0 \Rightarrow Y^* = \frac{X^*}{\mu_1 + [X^*]^2}; \ X^* = \mu_2; \ Y^* = \frac{\mu_2}{\mu_1 + [\mu_2]^2}$$

$$\frac{\partial f_1(X,Y)}{\partial X} = -1 + 2 \cdot X \cdot Y; \ \frac{\partial f_1(X,Y)}{\partial Y} = \mu_1 + X^2;$$

$$\frac{\partial f_2(X,Y)}{\partial X} = -2 \cdot X \cdot Y; \ \frac{\partial f_2(X,Y)}{\partial Y} = -\mu_1 - X^2$$

$$\frac{\partial f_1(X,Y)}{\partial X}|_{X^*,Y^*} = -1 + 2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2};\ \frac{\partial f_1(X,Y)}{\partial Y}|_{X^*,Y^*} = \mu_1+[\mu_2]^2$$

$$\frac{\partial f_2(X,Y)}{\partial X}|_{X^*,Y^*} = -2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2};\ \frac{\partial f_2(X,Y)}{\partial Y}|_{X^*,Y^*} = -\mu_1-[\mu_2]^2.$$

Classify our system fixed points as stable (spiral, node), unstable (spiral, node) is done by inspecting of glycolytic oscillator system characteristic equation:

$$A-\lambda\cdot I = \begin{pmatrix} \frac{\partial f_1(X,Y)}{\partial X} & \frac{\partial f_1(X,Y)}{\partial Y} \\ \frac{\partial f_2(X,Y)}{\partial X} & \frac{\partial f_2(X,Y)}{\partial Y} \end{pmatrix}_{(X^*,Y^*)} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\lambda$$

$$= \begin{pmatrix} \frac{\partial f_1(X,Y)}{\partial X}-\lambda & \frac{\partial f_1(X,Y)}{\partial Y} \\ \frac{\partial f_2(X,Y)}{\partial X} & \frac{\partial f_2(X,Y)}{\partial Y}-\lambda \end{pmatrix}_{(X^*,Y^*)}$$

$$A = \begin{pmatrix} 2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2}-1 & \mu_1+[\mu_2]^2 \\ -2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2} & -(\mu_1+[\mu_2]^2) \end{pmatrix}$$

$$\det(A-\lambda\cdot I)=0\ \text{Then}\ \det\left|\begin{pmatrix} \frac{\partial f_1(X,Y)}{\partial X}-\lambda & \frac{\partial f_1(X,Y)}{\partial Y} \\ \frac{\partial f_2(X,Y)}{\partial X} & \frac{\partial f_2(X,Y)}{\partial Y}-\lambda \end{pmatrix}_{(X^*,Y^*)}\right| = 0$$

$$\det\left|\begin{pmatrix} 2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2}-1-\lambda & \mu_1+[\mu_2]^2 \\ -2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2} & -\mu_1-[\mu_2]^2-\lambda \end{pmatrix}_{\left(X^*=\mu_2,\,Y^*=\frac{\mu_2}{\mu_1+[\mu_2]^2}\right)}\right| = 0$$

$$\left(2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2}-[1+\lambda]\right)\cdot(-(\mu_1+[\mu_2]^2)-\lambda)+2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2}\cdot(\mu_1+[\mu_2]^2)=0$$

$$-2\cdot\frac{[\mu_2]^2\cdot\lambda}{\mu_1+[\mu_2]^2}+[1+\lambda]\cdot(\mu_1+[\mu_2]^2)+[1+\lambda]\cdot\lambda=0$$

$$-2\cdot\frac{[\mu_2]^2\cdot\lambda}{\mu_1+[\mu_2]^2}+(\mu_1+[\mu_2]^2)+\lambda\cdot(\mu_1+[\mu_2]^2)+\lambda+\lambda^2=0$$

$$\lambda^2+\lambda\cdot\left\{(\mu_1+[\mu_2]^2)-2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2}+1\right\}+\mu_1+[\mu_2]^2=0$$

$$(\mu_1 + [\mu_2]^2) - 2 \cdot \frac{[\mu_2]^2}{\mu_1 + [\mu_2]^2} = \frac{2 \cdot [\mu_2]^2 \cdot (\mu_1 - 1) + \mu_1^2 + [\mu_2]^4}{\mu_1 + [\mu_2]^2}$$

$$\frac{2 \cdot [\mu_2]^2 \cdot (\mu_1 - 1) + \mu_1^2 + [\mu_2]^4}{\mu_1 + [\mu_2]^2} + 1 = \frac{(2 \cdot \mu_1 - 1) \cdot [\mu_2]^2 + \mu_1^2 + [\mu_2]^4 + \mu_1}{\mu_1 + [\mu_2]^2}.$$

Glycolytic oscillator system characteristic equation:

$$\lambda^2 + \lambda \cdot \left\{ (\mu_1 + [\mu_2]^2) - 2 \cdot \frac{[\mu_2]^2}{\mu_1 + [\mu_2]^2} + 1 \right\} + \mu_1 + [\mu_2]^2 = 0$$

$$\lambda^2 - \lambda \cdot \{2 \cdot \frac{[\mu_2]^2}{\mu_1 + [\mu_2]^2} - (\mu_1 + [\mu_2]^2) - 1\} + (\mu_1 + [\mu_2]^2) = 0$$

$$\tau = \mathrm{trace}(A) = 2 \cdot \frac{[\mu_2]^2}{\mu_1 + [\mu_2]^2} - (\mu_1 + [\mu_2]^2) - 1; \; \Delta = \mu_1 + [\mu_2]^2.$$

The solution of the quadratic equation (characteristic equation):

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4 \cdot \Delta}}{2}; \; \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4 \cdot \Delta}}{2}.$$

The eigenvalues depend only on the trace and the determinant of the matrix A. In case, eigenvalues to be distinct $\lambda_1 \neq \lambda_2$ corresponding eigenvectors are linearly independent, and hence span the entire plane. If the eigenvalues are complex, the fixed point is either a center and the origin is surrounded by a family of closed orbits. The centers are stable and trajectories are neither attracted to nor repelled from the fixed point. A spiral occurs if oscillations are lightly damped. Complex eigenvalues occur when $\Gamma_1(\tau, \Delta) = \tau^2 - 4 \cdot \Delta < 0$.

$$\begin{aligned} \Gamma_1(\tau(\mu_1, \mu_2), \Delta(\mu_1, \mu_2)) &= \Gamma_1(\mu_1, \mu_2) \\ &= \left\{ 2 \cdot \frac{[\mu_2]^2}{\mu_1 + [\mu_2]^2} - (\mu_1 + [\mu_2]^2) - 1 \right\}^2 - 4 \\ &\quad \cdot (\mu_1 + [\mu_2]^2) < 0. \end{aligned}$$

We plot 3D $\Gamma_1(\mu_1, \mu_2)$ function: $\Gamma_1(\mu_1, \mu_2) \rightarrow Gama - 1$; $\mu_1 \rightarrow k_1, Mui - 1$; $\mu_2 \rightarrow k_2, Mui - 2$ (Fig. 6.4).

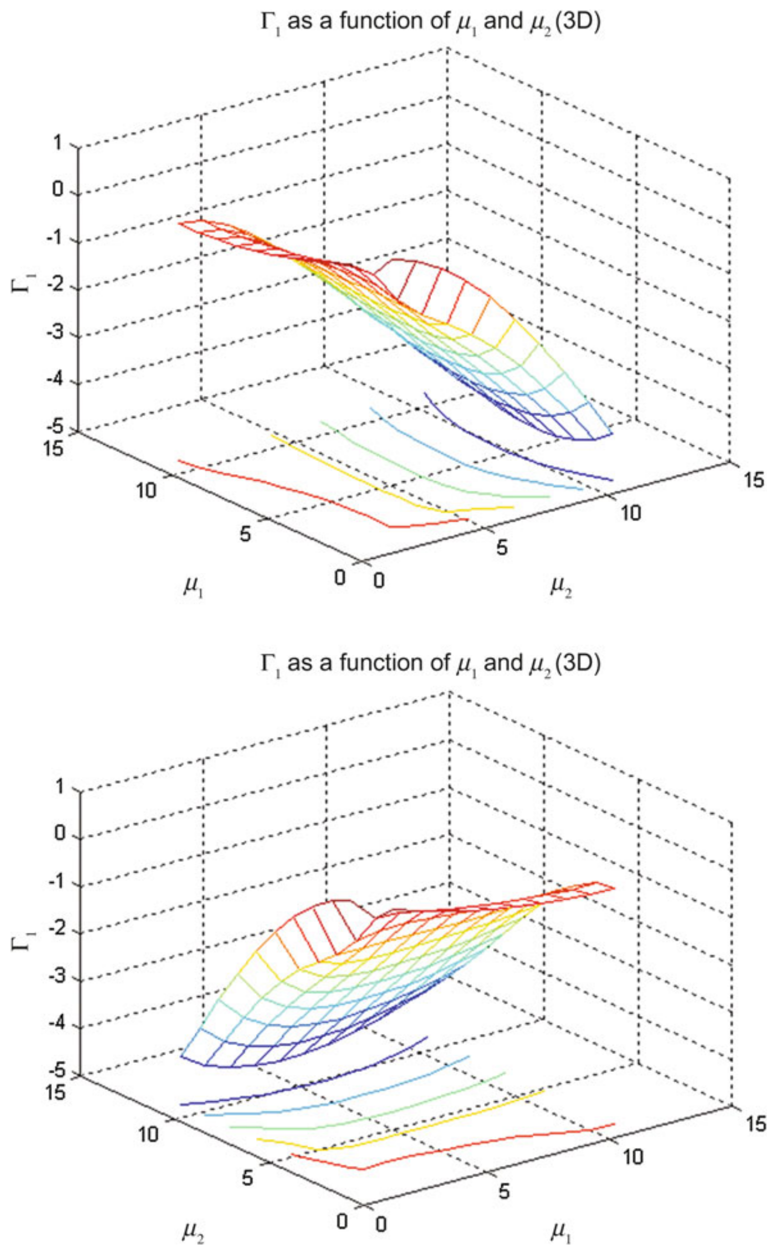

Fig. 6.4 Γ_1 as a function of μ_1 and μ_2 (3D)

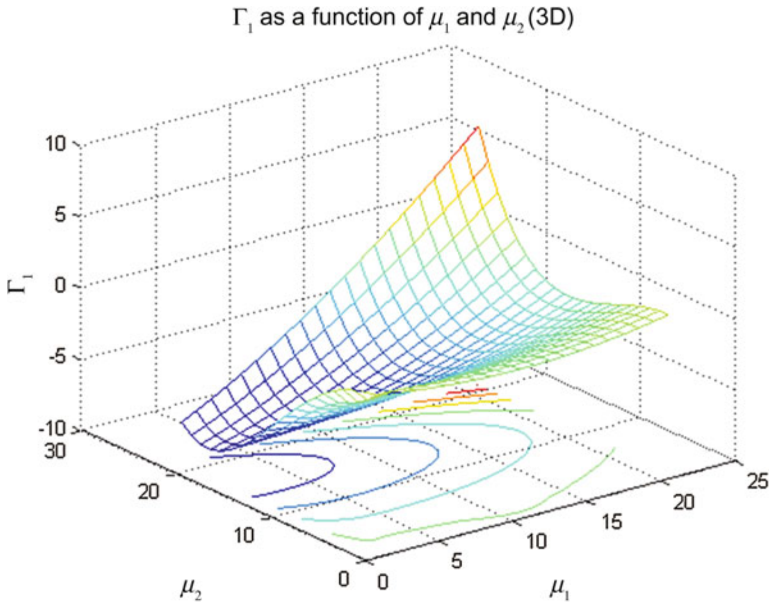

Fig. 6.4 (continued)

MATLAB Script

```
[k1,k2]=meshgrid(0:0.1:2,0:0.1:2);
e1=2*(k2.*k2)./(k1+k2.*k2)-(k1+k2.*k2)-1;
e2=e1.*e1;e3=4*(k1+k2.*k2);
e4=e2-e3;
meshc(e4);
```

Classification of fixed points : $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4 \cdot \Delta}}{2}$; $\Delta = \prod_{k=1}^{2} \lambda_k$; $\tau = \sum_{k=1}^{2} \lambda_k$. If $\Delta < 0$, the eigenvalues are real and have opposite signs and the fixed point is a saddle point $(\Delta < 0 \Rightarrow \mu_1 + [\mu_2]^2 < 0 \Rightarrow [\mu_2]^2 < -\mu_1;\ \mu_1 < 0;\ [\mu_2]^2 < |\mu_1|)$. If $\Delta > 0$, the eigenvalues are either real with the same sign (nodes), or complex conjugate (spiral and center). $\Delta > 0 \Rightarrow \mu_1 + [\mu_2]^2 > 0 \Rightarrow [\mu_2]^2 > -\mu_1$. The parabola $\Gamma_1(\tau, \Delta) = \tau^2 - 4 \cdot \Delta = 0$ is the borderline between nodes and spirals; star nodes and degenerate nodes lie on this parabola. The stability of nodes and spirals is determined by τ. When $\tau < 0$ both eigenvalues have negative real parts and the fixed point is stable. If $\tau > 0$ then spiral nodes are unstable. Stable centers lie on the borderline $\tau = \sum_{k=1}^{2} \lambda_k = 0$, where the eigenvalues are purely imaginary.

$\tau = \text{trace}(A) = 2 \cdot \frac{[\mu_2]^2}{\mu_1 + [\mu_2]^2} - (\mu_1 + [\mu_2]^2) - 1 = 0$. We need to plot $\tau(\mu_1, \mu_2)$ 3D function (Fig. 6.5).

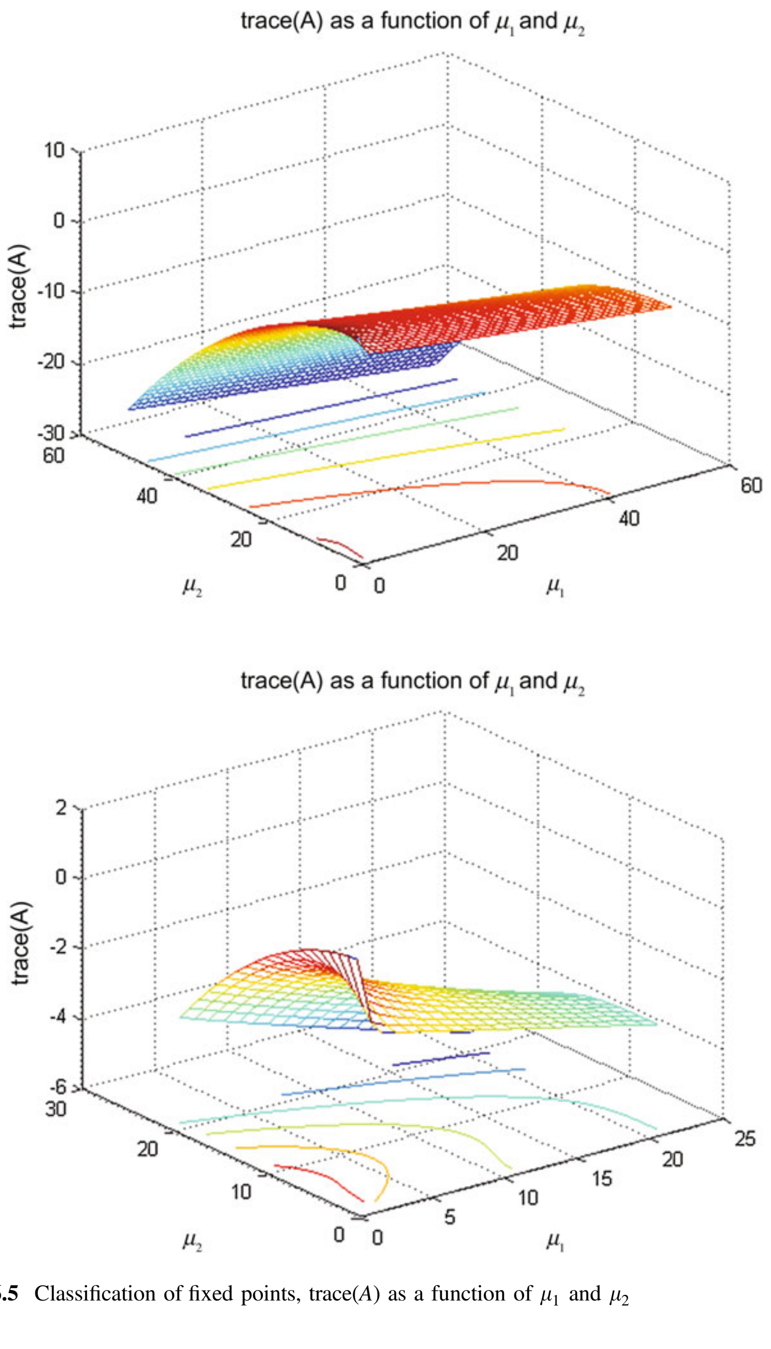

Fig. 6.5 Classification of fixed points, trace(A) as a function of μ_1 and μ_2

MATLAB Script

$\mu_1 \rightarrow k_1, Mui-1;\ \mu_2 \rightarrow k_2, Mui-2$

```
[k1,k2]=meshgrid(0:0.1:2,0:0.1:2);
e1=2*(k2.*k2)./(k1+k2.*k2)-(k1+k2.*k2)-1;
meshc(e1);
```

If $\Delta = \prod_{k=1}^{2} \lambda_k = 0$ ($\Delta = 0 \Rightarrow \mu_1 + [\mu_2]^2 = 0$), at least one of the eigenvalues is zero then the origin is not an isolated fixed point . There is either a whole line of fixed points or a plane of fixed points. Stable centers lie on the borderline $\tau = \sum_{k=1}^{2} \lambda_k = 0$, where the eigenvalues are purely imaginary and all the solutions are periodic with period $T = \frac{2\cdot\pi}{\frac{1}{2}\cdot\sqrt{4\cdot\Delta-\tau^2}} = \frac{4\cdot\pi}{\sqrt{4\cdot\Delta-\tau^2}}$.

$$\frac{\sqrt{4\cdot\Delta-\tau^2}}{2} = \frac{1}{2}\cdot\sqrt{4\cdot(\mu_1+[\mu_2]^2) - \left\{2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2} - (\mu_1+[\mu_2]^2) - 1\right\}^2}$$

$$T = \frac{4\cdot\pi}{\sqrt{4\cdot(\mu_1+[\mu_2]^2) - \left\{2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2} - (\mu_1+[\mu_2]^2) - 1\right\}^2}}$$

Remark $4\cdot(\mu_1+[\mu_2]^2) - \left\{2\cdot\frac{[\mu_2]^2}{\mu_1+[\mu_2]^2} - (\mu_1+[\mu_2]^2) - 1\right\}^2 > 0$.

The oscillations have fixed amplitude and the fixed point is a center. We need to prove that the system has periodic orbits and it is done by changing system cartesian coordinates ($X(t)$, $Y(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z axis. In our system we refer to Cartesian X–Y plane (with equation third coordinate equal to zero). Then the z coordinate is the same in both systems, and the correspondences between cylindrical (r, θ) and Cartesian (X, Y) are the same as for polar coordinates, namely $X(t) = r(t)\cdot\cos[\theta(t)]$; $Y(t) = r(t)\cdot\sin[\theta(t)]$; $r = \sqrt{X^2+Y^2}$. $\theta(t) = 0$ if $X = 0$ and $Y = 0$.

$\theta(t) = \arcsin(Y/r)$ if $X \geq 0$. $x \rightarrow X$, $y \rightarrow Y$. $X(t) = r(t)\cdot\cos[\theta(t)]$

$$X(t) = r(t)\cdot\cos[\theta(t)] \Rightarrow \frac{dX(t)}{dt} = \frac{dr(t)}{dt}\cdot\cos[\theta(t)] - r(t)\cdot\frac{d\theta(t)}{dt}\cdot\sin[\theta(t)]$$
$$Y(t) = r(t)\cdot\sin[\theta(t)] \Rightarrow \frac{dY(t)}{dt} = \frac{dr(t)}{dt}\cdot\sin[\theta(t)] + r(t)\cdot\frac{d\theta(t)}{dt}\cdot\cos[\theta(t)]$$
$$\frac{dX(t)}{dt} = \frac{dX}{dt};\ \frac{dr(t)}{dt} = r';\ \frac{d\theta(t)}{dt} = \theta';\ \theta(t) = \theta;\ r(t) = r$$

We get the equations: $\frac{dX}{dt} = r'\cdot\cos\theta - r\cdot\theta'\cdot sin\theta$; $\frac{dY}{dt} = r'\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta$

$X = r \cdot \cos\theta$; $Y = r \cdot \sin\theta$. The differential equations model of glycolytic oscillator is presented $\frac{dX}{dt} = -X + \mu_1 \cdot Y + X^2 \cdot Y$; $\frac{dY}{dt} = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y$, and we move to polar coordinates.

First differential equation: $\frac{dX}{dt} = \ldots$

$$r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = -r \cdot \cos\theta + \mu_1 \cdot r \cdot \sin\theta + r^2 \cdot \cos^2\theta \cdot r \cdot \sin\theta$$

$$r' = r \cdot \theta' \cdot \frac{\sin\theta}{\cos\theta} - r + \mu_1 \cdot r \cdot \frac{\sin\theta}{\cos\theta} + r^2 \cdot \cos\theta \cdot r \cdot \sin\theta$$

Second differential equation: $\frac{dY}{dt} = \ldots$

$$r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta = \mu_2 - \mu_1 \cdot r \cdot \sin\theta - r^2 \cdot \cos^2\theta \cdot r \cdot \sin\theta$$

$$r' = \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - r \cdot \theta' \cdot \frac{\cos\theta}{\sin\theta}$$

&&&

$$\begin{aligned} r \cdot \theta' \cdot \frac{\sin\theta}{\cos\theta} - r + \mu_1 \cdot r \cdot \frac{\sin\theta}{\cos\theta} + r^2 \cdot \cos\theta \cdot r \cdot \sin\theta \\ = \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - r \cdot \theta' \cdot \frac{\cos\theta}{\sin\theta} \end{aligned}$$

$$\begin{aligned} r \cdot \theta' \cdot (tg\theta + ctg\theta) = \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r + r - \mu_1 \cdot r \cdot tg\theta - r^2 \\ \cdot \cos\theta \cdot r \cdot \sin\theta. \end{aligned}$$

Finally, we get two differential equations in r and θ:

$$\begin{aligned} \theta' = \left\{ \mu_2 \cdot \frac{1}{r \cdot \sin\theta} - \mu_1 - r^2 \cdot \cos^2\theta + 1 - \mu_1 \cdot tg\theta - r^2 \cdot \cos\theta \cdot \sin\theta \right\} \\ \cdot \frac{1}{(tg\theta + ctg\theta)} \end{aligned}$$

$$\begin{aligned} r' = \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - r \cdot \{\mu_2 \cdot \frac{1}{r \cdot \sin\theta} - \mu_1 - r^2 \cdot \cos^2\theta \\ + 1 - \mu_1 \cdot tg\theta - r^2 \cdot \cos\theta \cdot \sin\theta\} \cdot \frac{1}{(tg\theta + ctg\theta)} \cdot \frac{\cos\theta}{\sin\theta} \end{aligned}$$

If there is a stable limit cycle as specific value of radius r, $dr/dt = 0$ then

$$r' = 0 \Rightarrow \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - r \cdot \left\{ \mu_2 \cdot \frac{1}{r \cdot \sin\theta} - \mu_1 - r^2 \cdot \cos^2\theta \right.$$
$$\left. + 1 - \mu_1 \cdot tg\theta - r^2 \cdot \cos\theta \cdot \sin\theta \right\} \cdot \frac{1}{(tg\theta + ctg\theta)} \cdot \frac{\cos\theta}{\sin\theta} = 0$$

$$\frac{1}{(tg\theta + ctg\theta)} \cdot \frac{\cos\theta}{\sin\theta} = \sin\theta \cdot \cos\theta \cdot \frac{\cos\theta}{\sin\theta} = \cos^2\theta$$

$$\mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - \mu_2 \cdot \frac{\cos^2\theta}{\sin\theta} + r \cdot \cos^2\theta \cdot \mu_1 + r^3 \cdot \cos^4\theta$$
$$- r \cdot \cos^2\theta + r \cdot \cos^2\theta \cdot \mu_1 \cdot tg\theta + r^3 \cdot \cos^3\theta \cdot \sin\theta = 0$$

$$r \cdot [\mu_1 \cdot (\cos^2\theta - 1) - \cos^2\theta + \cos^2\theta \cdot \mu_1 \cdot tg\theta] + r^3 \cdot [(\cos^2\theta - 1) \cdot \cos^2\theta + \cos^3\theta \cdot \sin\theta]$$
$$= \mu_2 \cdot \frac{1}{\sin\theta} \cdot [\cos^2\theta - 1]$$

$$r \cdot [\mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta] + r^3 \cdot \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta]$$
$$+ \mu_2 \cdot \sin\theta = 0$$

$$r^3 \cdot \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta] + r \cdot [\mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta]$$
$$+ \mu_2 \cdot \sin\theta = 0$$

$$\Omega_1(\mu_1, \mu_2, \theta) = \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta];$$
$$\Omega_2(\mu_1, \mu_2, \theta) = \mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta$$
$$\Omega_3(\mu_1, \mu_2, \theta) = \mu_2 \cdot \sin\theta; \; r^3 \cdot \Omega_1(\mu_1, \mu_2, \theta) + r \cdot \Omega_2(\mu_1, \mu_2, \theta) + \Omega_3(\mu_1, \mu_2, \theta) = 0$$

We get cubic function of radius (r) variable:

$$r^3 \cdot \Omega_1(\mu_1, \mu_2, \theta) + r \cdot \Omega_2(\mu_1, \mu_2, \theta) + \Omega_3(\mu_1, \mu_2, \theta) = 0; \; \mu_1, \mu_2, \theta \in \mathbb{R}$$

$$r_{(1)\_\text{limitcycle}} = -\frac{1}{3 \cdot \Omega_1} \cdot \sqrt[3]{\frac{27 \cdot \Omega_1^2 \cdot \Omega_3 + \sqrt{(27 \cdot \Omega_1^2 \cdot \Omega_3)^2 + 108 \cdot \Omega_1^3 \cdot \Omega_2^3}}{2}}$$
$$- \frac{1}{3 \cdot \Omega_1} \cdot \sqrt[3]{\frac{27 \cdot \Omega_1^2 \cdot \Omega_3 - \sqrt{(27 \cdot \Omega_1^2 \cdot \Omega_3)^2 + 108 \cdot \Omega_1^3 \cdot \Omega_2^3}}{2}}$$

$$r_{(2)\_\text{limitcycle}} = \frac{1 + i \cdot \sqrt{3}}{6 \cdot \Omega_1} \cdot \sqrt[3]{\frac{27 \cdot \Omega_1^2 \cdot \Omega_3 + \sqrt{(27 \cdot \Omega_1^2 \cdot \Omega_3)^2 + 108 \cdot \Omega_1^3 \cdot \Omega_2^3}}{2}}$$
$$+ \frac{1 - i \cdot \sqrt{3}}{6 \cdot \Omega_1} \cdot \sqrt[3]{\frac{27 \cdot \Omega_1^2 \cdot \Omega_3 - \sqrt{(27 \cdot \Omega_1^2 \cdot \Omega_3)^2 + 108 \cdot \Omega_1^3 \cdot \Omega_2^3}}{2}}$$

$$r_{(3)\_\text{limitcycle}} = \frac{1 - i \cdot \sqrt{3}}{6 \cdot \Omega_1} \cdot \sqrt[3]{\frac{27 \cdot \Omega_1^2 \cdot \Omega_3 + \sqrt{(27 \cdot \Omega_1^2 \cdot \Omega_3)^2 + 108 \cdot \Omega_1^3 \cdot \Omega_2^3}}{2}} + \frac{1 + i \cdot \sqrt{3}}{6 \cdot \Omega_1} \cdot \sqrt[3]{\frac{27 \cdot \Omega_1^2 \cdot \Omega_3 - \sqrt{(27 \cdot \Omega_1^2 \cdot \Omega_3)^2 + 108 \cdot \Omega_1^3 \cdot \Omega_2^3}}{2}}$$

We know $r_{(i)\_\text{limitcycle}} > 0; r_{(i)\_\text{limitcycle}} \in \mathbb{R}_+ \quad \forall\, i = 1,2,3$ so we ignore all improper results. System limit cycle radius $r_{(i)\_\text{limitcycle}}$ must be real and positive number.

$$\Omega_1(\mu_1, \mu_2, \theta) \neq 0 \Rightarrow \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta] \neq 0;\ \sin\theta \neq 0 \Rightarrow \theta\ / = k \cdot \pi;\ \ (k = \ldots, -2, -1, 0, 1, 2, \ldots)$$

$$\cos^2\theta \neq 0 \Rightarrow \cos\theta \neq 0 \Rightarrow \theta \neq \pi \cdot \left(k + \frac{1}{2}\right);\ (k = \ldots, -2, -1, 0, 1, 2, \ldots)$$

$$\cos\theta - \sin\theta \neq 0 \Rightarrow \sin\theta \neq \cos\theta \Rightarrow tg\theta \neq 1 \Rightarrow \theta \neq \pi \cdot \left(k + \frac{1}{4}\right);$$
$$(k = \ldots, -2, -1, 0, 1, 2, \ldots)$$

The coefficients $\Omega_1(\mu_1, \mu_2, \theta), \Omega_2(\mu_1, \mu_2, \theta), \Omega_3(\mu_1, \mu_2, \theta)$ are real numbers. Every cubic equation with real coefficients has at least one solution r among the real numbers ($r > 0$); this is a consequence of the intermediate value theorem. We can distinguish several possible cases using determinant (Det).

$$\text{Det} = -4 \cdot \Omega_1 \cdot \Omega_2^3 - 27 \cdot \Omega_1^2 \cdot \Omega_3^2;\ \Omega_1 = \Omega_1(\mu_1, \mu_2, \theta), \Omega_2 = \Omega_2(\mu_1, \mu_2, \theta),$$
$$\Omega_3 = \Omega_3(\mu_1, \mu_2, \theta)$$

Case a: If Det > 0, then the equation has three distinct real roots.
Case b: If Det < 0, then the equation has one real root and pair of complex conjugate roots (not the case since $r > 0;\ r \in \mathbb{R}_+$).
Case c: If Det $= 0$ then (at least) two roots coincide. It may be that the equation has a double real root and another distinct single real root; alternatively, all three roots coincide yielding a triple real root. A possible way to decide between these subcases is to compute the resultant of the cubic and its second derivative; a triple root exists if and only if this resultant vanishes [7].

Other case is when system limit cycle is resided in annulus $0 < r_{\min} < r < r_{\max}$ and there is no constant r in limit cycle $\mathrm{d}r/\mathrm{d}t \neq 0$. We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$. $\mathrm{d}r/\mathrm{d}t < 0$ on the outer circle and $\mathrm{d}r/\mathrm{d}t > 0$ on the inner circle. To find $r_{\min}$, we require $\mathrm{d}r/\mathrm{d}t > 0$ for all values of θ:

$$r' > 0 \Rightarrow \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - r \cdot \left\{\mu_2 \cdot \frac{1}{r \cdot \sin\theta} - \mu_1 - r^2 \cdot \cos^2\theta + 1 - \mu_1 \cdot tg\theta - r^2 \cdot \cos\theta \cdot \sin\theta\right\} \cdot \frac{1}{(tg\theta + ctg\theta)} \cdot \frac{\cos\theta}{\sin\theta} > 0$$

$$r^3 \cdot \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta] + r \cdot [\mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta] + \mu_2 \cdot \sin\theta > 0$$

$$r \to r_{\min};\ r_{\min}^3 \cdot \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta] + r_{\min} \cdot [\mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta] + \mu_2 \cdot \sin\theta > 0$$

The same analysis we do for finding $r_{\max}$ in the outer circle, dr/dt < 0. To find $r_{\max}$, we require dr/dt < 0 for all values of θ:

$$r' < 0 \Rightarrow \mu_2 \cdot \frac{1}{\sin\theta} - \mu_1 \cdot r - r^2 \cdot \cos^2\theta \cdot r - r \cdot \left\{\mu_2 \cdot \frac{1}{r \cdot \sin\theta} - \mu_1 - r^2 \cdot \cos^2\theta + 1 - \mu_1 \cdot tg\theta - r^2 \cdot \cos\theta \cdot \sin\theta\right\} \cdot \frac{1}{(tg\theta + ctg\theta)} \cdot \frac{\cos\theta}{\sin\theta} < 0$$

$$r^3 \cdot \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta] + r \cdot \left[\mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta\right] + \mu_2 \cdot \sin\theta < 0$$

$$r \to r_{\max};\ r_{\max}^3 \cdot \sin\theta \cdot \cos^2\theta \cdot [\cos\theta - \sin\theta] + r_{\max} \cdot \left[\mu_1 \cdot \sin\theta \cdot (\cos\theta - \sin\theta) - \cos^2\theta\right] + \mu_2 \cdot \sin\theta < 0.$$

6.4 Optoisolation Glycolytic Circuits Limit Cycle Solution

The differential equations model of glycolytic oscillator is presented:

$$\frac{dX}{dt} = -X + \mu_1 \cdot Y + X^2 \cdot Y;$$
$$\frac{dY}{dt} = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y;\ f_1(X,Y) = -X + \mu_1 \cdot Y + X^2 \cdot Y$$

$$f_2(X,Y) = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y;\ \frac{dX}{dt} = f_1(X,Y); \frac{dY}{dt} = f_2(X,Y).$$

We need to implement it using operational amplifiers, capacitors, resistors, inductors, diodes, optic couplers, etc. Opt couplers consist of gallium arsenide infrared LED and a silicon NPN phototransistor [85, 86]:

$$\frac{dX}{dt} = -X + \mu_1 \cdot Y + X^2 \cdot Y \Rightarrow \frac{dX}{dt} + X = (\mu_1 + X^2) \cdot Y; \; Y = \frac{\frac{dX}{dt} + X}{\mu_1 + X^2}$$

$$\frac{dY}{dt} = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y \Rightarrow \frac{dY}{dt} = \mu_2 - \left(\mu_1 + X^2\right) \cdot Y; \; \frac{dY}{dt} = \mu_2 - \left(\mu_1 + X^2\right) \cdot \frac{\left(\frac{dX}{dt} + X\right)}{(\mu_1 + X^2)}$$

$$\frac{dY}{dt} = \mu_2 - \left(\frac{dX}{dt} + X\right) \Rightarrow \frac{dY}{dt} = \mu_2 - \frac{dX}{dt} - X; \; X = \mu_2 - \frac{dX}{dt} - \frac{dY}{dt}$$

We can summarize our glycolytic oscillator X and Y equations (Fig. 6.6):

$$Y = \frac{\frac{dX}{dt} + X}{\mu_1 + X^2}; \; X = \mu_2 - \frac{dX}{dt} - \frac{dY}{dt}$$

In our dynamical optoisolation glycolytic circuit we use operational amplifiers in various topologies (inverter amplifier, non-inverting amplifier, difference amplifier,

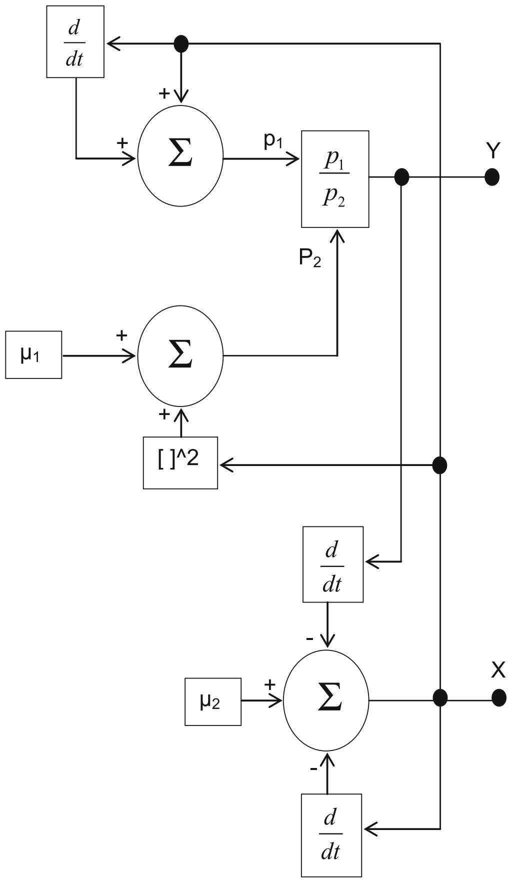

Fig. 6.6 Block diagram of dynamical optoisolation glycolytic circuit

inverting summing amplifier, differentiator, integrator, etc.) [132]. The operational amplifiers are used in the amplification loop and feedback loop elements in our optoisolation circuit system. The operational amplifier is a direct-coupled high-gain amplifier to which feedback is added to control its overall response characteristic. It is used to perform a wide variety of linear functions and is referred to as the basic linear integrated circuit. The integrated operational amplifier offers all the advantages of monolithic integrated circuits, small size, high reliability, reduced cost, temperature tracking, and low offset voltage and current. A large number of operational amplifiers have a different input, with voltages V_1 and V_2 applied to the inverting and non-inverting terminals, respectively. A single-ended amplifier may be considered as a special case where one of the input terminals is grounded. Nearly, all operational amplifiers have only one output terminal. In our optoisolation circuit system we consider op-amp as ideal operational amplifier. The ideal op-amp has the following characteristics; Input resistance $R_i \to \infty$, Output resistance $R_o = 0$, Voltage gain $A_v = -\infty$, Bandwidth = ∞, Perfect balance: $V_O = 0$ when $V_1 = V_2$, characteristics do not drift with temperature [85, 86]:

$$X \to V_x;\ Y \to V_y;\ \mu_1 \to V_{\mu_1};\ \mu_2 \to V_{\mu_2}$$

$$V_{A_1} = V_x;\ V_{A_{16}} = V_y;\ V_{A_3} = -R_4 \cdot C_1 \cdot \frac{dV_x}{dt};\ V_{A_{18}} = -\frac{R_{18}}{R_{17}} \cdot V_x;\ \frac{R_{18}}{R_{17}} = 1 \Rightarrow V_{A_{18}} = -V_x$$

$$V_{A_5} = -R_1 \cdot \left(\frac{V_{A_3}}{R_2} + \frac{V_{A_{18}}}{R_3}\right) = -R_1 \cdot \left(-\frac{R_4}{R_2} \cdot C_1 \cdot \frac{dV_x}{dt} - \frac{V_x}{R_3}\right);$$

$$R_2 = R_3 = R_1;\ V_{A_5} = R_4 \cdot C_1 \cdot \frac{dV_x}{dt} + V_x$$

$$R_4 \cdot C_1 = 1 \Rightarrow V_{p_1} = V_{A_5} = \frac{dV_x}{dt} + V_x;\ V_{p_2} = V_{A_7} = -R_5 \cdot \left(\frac{-V_{\mu_1}}{R_6} - \frac{V_x^2}{R_7}\right);$$

$$R_5 = R_6 = R_7;\ V_{p_2} = V_{A_7} = V_{\mu_1} + V_x^2$$

$$V_{A_{12}} = -R_{16} \cdot C_2 \cdot \frac{dV_y}{dt};\ V_{A_{13}} = -R_{15} \cdot C_3 \cdot \frac{dV_x}{dt};\ V_{A_9} = -R_{12} \cdot \left(\frac{V_{A_{12}}}{R_{13}} + \frac{V_{A_{13}}}{R_{14}}\right) = -(V_{A_{12}} + V_{A_{13}})$$

$$R_{12} = R_{13} = R_{14};\ V_{A_9} = -R_{12} \cdot \left(\frac{V_{A_{12}}}{R_{13}} + \frac{V_{A_{13}}}{R_{14}}\right) = -(V_{A_{12}} + V_{A_{13}})$$

$$= -\left(-R_{16} \cdot C_2 \cdot \frac{dV_y}{dt} - R_{15} \cdot C_3 \cdot \frac{dV_x}{dt}\right)$$

$$V_{A_9} = R_{16} \cdot C_2 \cdot \frac{dV_y}{dt} + R_{15} \cdot C_3 \cdot \frac{dV_x}{dt};\ R_{16} \cdot C_2 = 1;\ R_{15} \cdot C_3 = 1;\ V_{A_9} = \frac{dV_y}{dt} + \frac{dV_x}{dt}$$

$$V_{A_x} = \left(\frac{R_9 + R_8}{R_{10} + R_{11}}\right) \cdot \frac{R_{11}}{R_9} \cdot V_{\mu_2} - \frac{R_8}{R_9} \cdot V_{A_9};\ R_9 = R_{10};\ R_8 = R_{11};$$

$$R_8 = R_9;\ V_{A_x} = \frac{R_8}{R_9} \cdot \left(V_{\mu_2} - V_{A_9}\right)$$

$$R_8 || R_9 = R_{10} || R_{11};\ V_{A_x} = V_{\mu_2} - V_{A_9} = V_{\mu_2} - \left(\frac{dV_y}{dt} + \frac{dV_x}{dt}\right)$$

We can summarize our results: $V_{p_2} = V_{A_7} = V_{\mu_1} + V_x^2;\ V_{p_1} = V_{A_5} = \frac{dV_x}{dt} + V_x$ (Fig. 6.7)

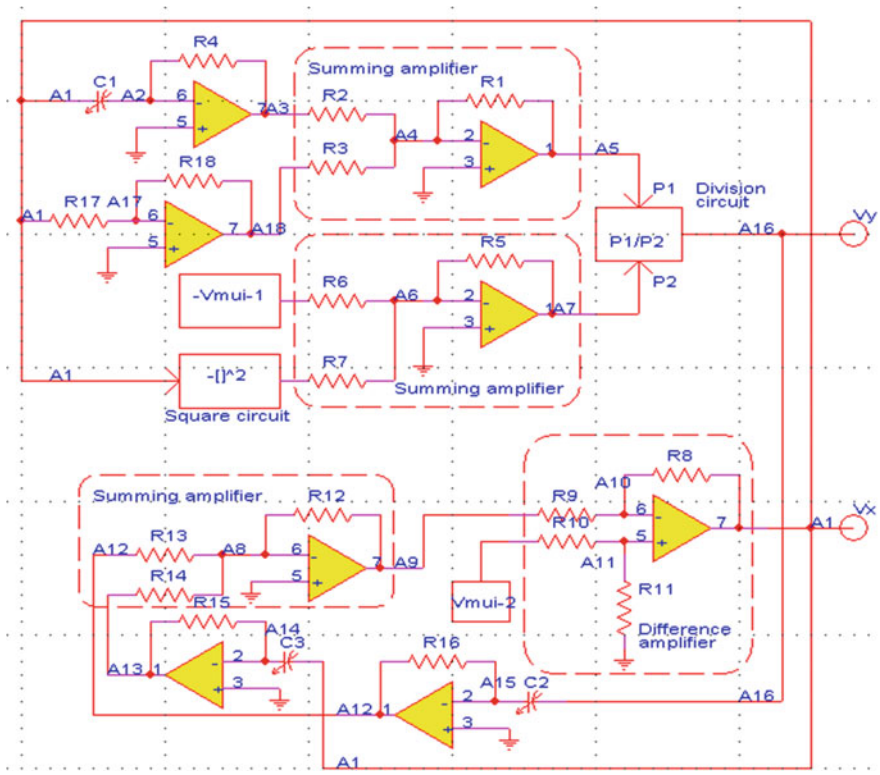

Fig. 6.7 Dynamical glycolytic circuit

$$V_{A_x} = V_{\mu_2} - V_{A_9} = V_{\mu_2} - \left(\frac{\mathrm{d}V_y}{\mathrm{d}t} + \frac{\mathrm{d}V_x}{\mathrm{d}t}\right).$$

Division Circuit In our division circuit implementation, we use Log and Antilog amplifiers, discrete components (resistors, capacitors), LEDs, and phototransistors. Log and Antilog amplifiers are nonlinear circuits in which the output voltage is proportional to the logarithmic (or exponent) of the input. It is well known that some process such as multiplication and division can be performed by addition and subtraction of logs. They have numerous applications in electronics, such as multiplication and division, powers and roots, compression and decompression, true RMS detection, and process control. There are two basic circuits for logarithmic amplifiers: Tran's diode and diode-connected transistor. Most logarithmic amplifiers are based on the inherent logarithmic relationship between the collector current I_c, and the base–emitter voltage, V_{be} in silicon bipolar transistor. The input voltage is converted by input resistor into a current, which flows through the transistor's collector modulation, the base–emitter voltage according to the input voltage. The op-amp forces the collector voltage to that at non-inverting input, zero voltage. We have in the circuit three current sources ($I_{cont1}, I_{cont2}, I_{cont3}$), which inject current to LEDs (D_1, D_2, and D_3). LEDs lights strike the phototransistors (Q_1, Q_2, and Q_3) base window and can be represented as a dependent current source respectively. BJT transistors base current is constructed from feedback loop partial current and the current which caused by the light strike on the phototransistors base window. Operational amplifiers virtual ground $V_{B_1} = 0$; $V_{B_6} = 0$; $V_{B_3} = 0$; $V_{B_8} = 0$. A resistor is often inserted between the non-inverting input and ground, reducing the input offset voltage due to different voltage drops due to bias current, and may reduce distortion in op-amp. The potential at the operational amplifier inputs remains virtually constant (near ground) in the inverting configuration. An operational amplifier is a DC-coupled high-gain electronic voltage amplifier with a different input and a single-ended output. In this configuration, an op-amp produces an output potential (relative to circuit ground) that is typically hundreds of thousands of times larger than the potential difference between its input terminals (Fig. 6.8).

Logarithmic Amplifier Function We have logarithmic amplifiers in our division circuit . Logarithmic amplifier is constructed from operational amplifier, resistor, and phototransistor on the feedback loop . We short phototransistor collector–base junctions and practically our phototransistor is operated as a base–emitter junction diode. The first logarithmic amplifier is market by index (i = 1) and the second logarithmic amplifier is marked by index (i = 2).

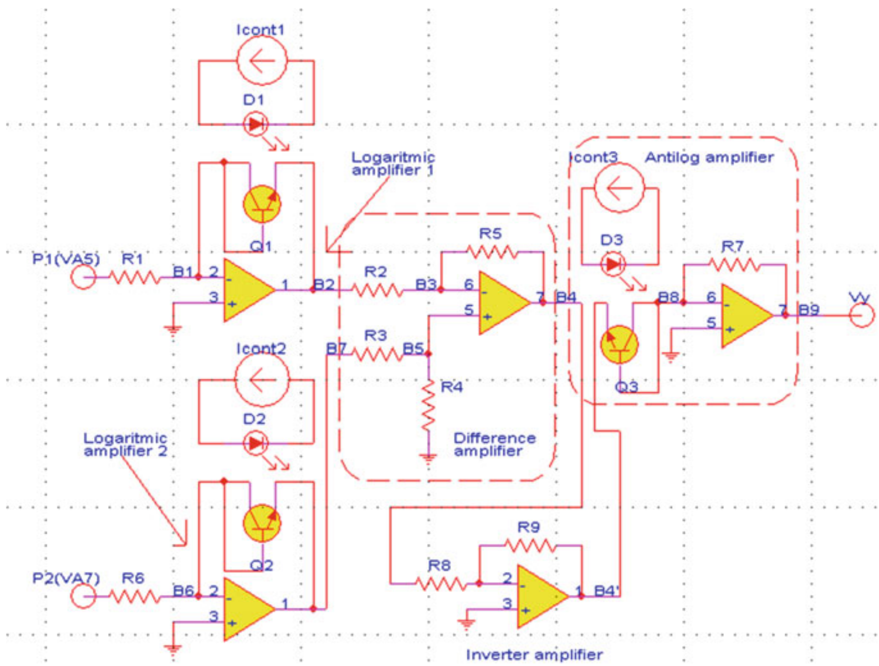

Fig. 6.8 Glycolytic circuit logarithmic simplifiers and antilog amplifier circuit schematic

$$V_{CEQ_i} = V_{CBQ_i} + V_{BEQ_i};\ V_{CBQ_i} = 0 \Rightarrow V_{CEQ_i} = V_{BEQ_i};\ \alpha_{f1} \neq \alpha_{f2};\ \alpha_{r1} \neq \alpha_{r2};\ k_1 \neq k_2;\ i = 1,2$$

Remark Optocouplers (Q_1–D_1 and Q_2–D_2) are not identical.

The op-amp forces the collector voltage to that at the non-inverting input, zero volts. We use the transistor Ebers–Moll model to get the logarithmic amplifier function. I_i is the current that flows from operational amplifier inverting port to the phototransistor (i). The output voltage of logarithmic amplifier is a negative voltage compared to the input voltage.

$$V_{CBQ_i} = 0 \Rightarrow I_{CQ_i} = 0;\ I_{EQ_i} = I_{CQ_i} + I_{BQ_i};\ I_{CQ_i} = 0(\rightarrow \varepsilon) \Rightarrow I_{EQ_i} \simeq I_{BQ_i}$$

$$I_{BQ_i} = I_i + k_i \cdot I_{cont-i};\ I_{EQ_i} = I_i + k_i \cdot I_{cont-i};\ i_{DCQ_i} = 0.$$

Ebers–Moll model:

$$i_{DCQ_i} = \frac{\alpha_{f_i} \cdot I_{EQ_i} - I_{CQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}};\ V_{\text{Base-EmitterQ}_i} = V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_i} + 1\right]$$

$$i_{DEQ_i} = \frac{I_{EQ_i} - \alpha_{r_i} \cdot I_{CQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}};\ V_{\text{Base-EmitterQ}_i} = V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i} - \alpha_{r_i} \cdot I_{CQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1\right]$$

$$I_{CQ_i} = 0 \Rightarrow V_{\text{Base-EmitterQ}_i} = V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1\right]$$

$$V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1\right] \Rightarrow \exp\left[\frac{V_{BEQ_i}}{V_t}\right] = \frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1$$

$$\exp\left[\frac{V_{BEQ_i}}{V_t}\right] - 1 = \frac{1}{I_{se}} \cdot \left(\frac{I_i + k_i \cdot I_{cont-i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right);$$

$$\left(\exp\left[\frac{V_{BEQ_i}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_i} \cdot \alpha_{f_i}) = I_i + k_i \cdot I_{cont-i}$$

$$I_i = \left(\exp\left[\frac{V_{BEQ_i}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_i} \cdot \alpha_{f_i}) - k_i \cdot I_{cont-i}.$$

For the first logarithmic amplifier ($i = 1$), R_1 is connected to the op-amp inverting port:

$$\frac{V_{p_1}}{R_1} = \frac{V_{A_5}}{R_1} = I_1;\ \frac{V_{p_1}}{R_1} = \left(\exp\left[\frac{V_{BEQ_1}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1}) - k_1 \cdot I_{cont-1}$$

$$\frac{V_{p_1}}{R_1} = \left(\exp\left[\frac{V_{BEQ_1}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1}) - k_1 \cdot I_{cont-1};$$
$$\exp\left[\frac{V_{BEQ_1}}{V_t}\right] \gg 1; \exp\left[\frac{V_{BEQ_1}}{V_t}\right] - 1 \approx \exp\left[\frac{V_{BEQ_1}}{V_t}\right]$$

$$\frac{V_{p_1}}{R_1} = \exp\left[\frac{V_{BEQ_1}}{V_t}\right] \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1}) - k_1 \cdot I_{cont-1};\ \exp\left[\frac{V_{BEQ_1}}{V_t}\right] \simeq \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}$$

$$\exp\left[\frac{V_{BEQ_1}}{V_t}\right] \simeq \frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}.$$

V_{BEQ_1} is the output voltage from the logarithmic amplifier ($-V_{BEQ_1} = V_{B_2}$).

$$\frac{V_{BEQ_1}}{V_t} \simeq \ln\left[\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}\right]$$

$$V_{B_2} = -V_{BEQ_1} \simeq -V_t \cdot \ln\left[\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}\right]$$

In the same manner the calculation is done for the second logarithmic amplifier:

$$V_{B_7} = -V_{BEQ_2} \simeq -V_t \cdot \ln\left[\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}\right].$$

Current sources I_{cont-1}, I_{cont-2} function as a setting current for logarithmic amplifiers.

$$I_{cont-1} = 0 \Rightarrow V_{B_2} \simeq -V_t \cdot \ln\left\{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}\right\};\ I_{cont-2} = 0 \Rightarrow V_{B_7} \simeq -V_t \cdot \ln\left\{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}\right\}.$$

If $R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1}) = 1$ and $R_6 \cdot I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2}) = 1$ then $V_{B_2} \simeq -V_t \cdot \ln(V_{p_1})$ and $V_{B_7} \simeq -V_t \cdot \ln(V_{p_2})$.

Antilogarithmic Amplifier Function We have antilogarithmic amplifier in our division circuit . Antilogarithmic amplifier is constructed from operational amplifier, resistor, and optocoupler. The resistor is in our operational amplifier feedback loop . The optocoupler is connected to the operational amplifier inverting terminal. We short phototransistor collector–base junctions and practically our phototransistor is operated as a base–emitter junction diode.

$$V_{CEQ_3} = V_{CBQ_3} + V_{BEQ_3};\ V_{CBQ_3} = 0 \Rightarrow V_{CEQ_3} = V_{BEQ_3}$$

The op-amp forces the collector voltage to that at the non-inverting input, zero volts. We use the transistor Ebers–Moll model to get the antilogarithmic amplifier function. I_3 is the current that flows from operational amplifier inverting port to phototransistor Q_3. The output voltage of antilogarithmic amplifier is a negative voltage compared to the input voltage [85, 86].

$$V_{CBQ_3} = 0 \Rightarrow I_{CQ_3} = 0;\ I_{EQ_3} = I_{CQ_3} + I_{BQ_3};\ I_{CQ_3} = 0(\rightarrow \varepsilon) \Rightarrow I_{EQ_3} \simeq I_{BQ_3}$$
$$I_{BQ_3} = I_3 + k_3 \cdot I_{cont-3};\ I_{EQ_3} = I_3 + k_3 \cdot I_{cont-3};\ i_{DCQ_3} = 0$$

Ebers–Moll model:

$$i_{DCQ_3} = \frac{\alpha_{f_3} \cdot I_{EQ_3} - I_{CQ_3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}};\ V_{\text{Base-EmitterQ}_3} = V_{BEQ_3} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_3} + 1\right]$$

$$i_{DEQ_3} = \frac{I_{EQ_3} - \alpha_{r_3} \cdot I_{CQ_3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}};\ V_{\text{Base-EmitterQ}_3} = V_{BEQ_3} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_3} - \alpha_{r_3} \cdot I_{CQ_3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}}\right) + 1\right]$$
$$I_{CQ_3} = 0 \Rightarrow V_{\text{Base-EmitterQ}_3} = V_{BEQ_3} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}}\right) + 1\right]$$
$$V_{BEQ_3} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}}\right) + 1\right] \Rightarrow \exp\left[\frac{V_{BEQ_3}}{V_t}\right] = \frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}}\right) + 1$$
$$\exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1 = \frac{1}{I_{se}} \cdot \left(\frac{I_3 + k_3 \cdot I_{cont-3}}{1 - \alpha_{r_3} \cdot \alpha_{f_3}}\right);$$
$$\left(\exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1\right) \cdot I_{se} \cdot \left(1 - \alpha_{r_3} \cdot \alpha_{f_3}\right) = I_3 + k_3 \cdot I_{cont-3}$$
$$I_3 = \left(\exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1\right) \cdot I_{se} \cdot \left(1 - \alpha_{r_3} \cdot \alpha_{f_3}\right) - k_3 \cdot I_{cont-3}$$

For the antilogarithmic amplifier, R_7 is connected to the op-amp inverting port:

$$\frac{V_{B_9}}{R_7} = I_3;\ \frac{V_{B_9}}{R_7} = \left(\exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) - k_3 \cdot I_{cont-3}$$

$$\frac{V_{B_9}}{R_7} = \left(\exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) - k_3 \cdot I_{cont-3};$$
$$\exp\left[\frac{V_{BEQ_3}}{V_t}\right] \gg 1;\ \exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1 \approx \exp\left[\frac{V_{BEQ_3}}{V_t}\right]$$
$$\frac{V_{B_9}}{R_7} = \left(\exp\left[\frac{V_{BEQ_3}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) - k_3 \cdot I_{cont-3};$$
$$\frac{V_{B_9}}{R_7} = \exp\left[\frac{V_{BEQ_3}}{V_t}\right] \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) - k_3 \cdot I_{cont-3}$$
$$\exp\left[\frac{V_{BEQ_3}}{V_t}\right] \simeq \frac{V_{B_9}}{R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3})} + \frac{k_3 \cdot I_{cont-3}}{I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3})}$$

$$\frac{V_{B_9}}{R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3})} \simeq \exp\left[\frac{V_{BEQ_3}}{V_t}\right] - \frac{k_3 \cdot I_{cont-3}}{I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3})}$$

$$V_{B_9} \simeq R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \exp\left[\frac{V_{BEQ_3}}{V_t}\right] - k_3 \cdot I_{cont-3} \cdot R_7.$$

Current source I_{cont-3} functions as a setting current for antilogarithmic amplifier.

$$I_{cont-3} = 0 \Rightarrow V_{B_9} \simeq R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \exp\left[\frac{V_{BEQ_3}}{V_t}\right]$$

If $I_{se} \cdot R_7 \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) = 1$ then $V_{B_9} \simeq \exp\left[\frac{V_{BEQ_3}}{V_t}\right]$

$$\begin{aligned} V_{B_4} &= \frac{(R_2 + R_5)}{(R_3 + R_4)} \cdot \frac{R_4}{R_2} \cdot V_{B_7} - \frac{R_5}{R_2} \cdot V_{B_2};\ R_2 = R_3;\ R_5 = R_4 \Rightarrow V_{B_4} \\ &= \frac{R_5}{R_2} \cdot (V_{B_7} - V_{B_2}) \end{aligned}$$

$$R_2||R_5 = R_3||R_4;\ \frac{R_5}{R_2} = 1 \Rightarrow V_{B_4} = V_{B_7} - V_{B_2};\ V_{B'_4} = -V_{B_4} = -(V_{B_7} - V_{B_2})$$

$$R_8 = R_9 \Rightarrow V_{B'_4} = -V_{B_4}$$

$$\begin{aligned} V_{B_4} = V_{B_7} - V_{B_2} &= -V_t \cdot \ln\left[\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}\right] \\ &- \left\{-V_t \cdot \ln\left[\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}\right]\right\} \end{aligned}$$

$$\begin{aligned} V_{B_4} = -V_t \cdot \Bigg\{ &\ln\left[\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}\right] \\ &- \ln\left[\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}\right]\Bigg\} \end{aligned}$$

$$V_{B_4} = -V_t \cdot \ln\left\{\frac{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}}{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}}\right\}$$

$$I_{cont-1} = 0;\ I_{cont-2} = 0;\ V_{B_4} = -V_t \cdot \ln\left\{\frac{V_{p_2}}{V_{p_1}} \cdot \frac{R_1 \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}{R_6 \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}\right\}$$

$$\begin{aligned} V_{B_4} &= -V_t \cdot \ln\left(\frac{V_{p_2}}{V_{p_1}}\right) - V_t \cdot \ln\left[\frac{R_1 \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}{R_6 \cdot (1 - \alpha_{r_2} \cdot \alpha_{f_2})}\right];\ \frac{R_1 \cdot (1 - \alpha_{r_1} \cdot \alpha_{f_1})}{R_6 \cdot \left(1 - \alpha_{r_2} \cdot \alpha_{f_2}\right)} \approx 1 \\ &\Rightarrow V_{B_4} \simeq -V_t \cdot \ln\left(\frac{V_{p_2}}{V_{p_1}}\right) \end{aligned}$$

$$-V_t \cdot \ln\left(\frac{V_{p_2}}{V_{p_1}}\right) = V_t \cdot \ln\left(\frac{V_{p_1}}{V_{p_2}}\right);\; V_{B_4} = V_t \cdot \ln\left(\frac{V_{p_1}}{V_{p_2}}\right) - V_t \cdot \ln\left[\frac{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}\right]$$

$$V_{B_9} \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1-\alpha_{r_3} \cdot \alpha_{f_3}) \cdot \exp\left[\frac{V_{BEQ_3}}{V_t}\right];$$

$$V_{B_4} = -V_{B'_4};\; V_{BEQ_3} = -V_{B'_4} = V_{B_4}$$

$$-V_{B'_4} = V_{BEQ_3} = V_{B_4} = V_t \cdot \ln\left\{\left[\frac{V_{p_1}}{V_{p_2}}\right] \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right]\right\}$$

$$V_{B_9} \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1-\alpha_{r_3} \cdot \alpha_{f_3}) \cdot \exp\left[\ln\left\{\left[\frac{V_{p_1}}{V_{p_2}}\right] \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right]\right\}\right]$$

$$\exp\left[\ln\left\{\left[\frac{V_{p_1}}{V_{p_2}}\right] \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right]\right\}\right] = \left[\frac{V_{p_1}}{V_{p_2}}\right] \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right]$$

$$V_{B_9} \simeq R_7 \cdot I_{se} \cdot (1-\alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left[\frac{V_{p_1}}{V_{p_2}}\right] \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right] - k_3 \cdot I_{cont-3} \cdot R_7$$

$$V_y = V_{B_9} \simeq R_7 \cdot I_{se} \cdot (1-\alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left[\frac{V_{p_1}}{V_{p_2}}\right] \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right] - k_3 \cdot I_{cont-3} \cdot R_7$$

If $R_7 \cdot I_{se} \cdot (1-\alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left[\frac{R_6 \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}{R_1 \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}\right] = 1$ and $I_{cont-3} = 0$ then $V_y = \frac{V_{p_1}}{V_{p_2}}$.

In case all LED's control currents are bigger than zero ($I_{cont-k} > 0 \quad \forall\, k = 1,2,3$)

$$V_{B_4} = -V_t \cdot \ln\left\{\frac{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}\right\}$$

$$V_{B_4} = V_t \cdot \ln\left\{\frac{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}\right\}$$

$$V_{B_9} \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1-\alpha_{r_3} \cdot \alpha_{f_3}) \cdot \exp\left[\frac{V_{BEQ_3}}{V_t}\right];\; V_{B_4} = -V_{B'_4};$$

$$V_{BEQ_3} = -V_{B'_4} = V_{B_4}$$

$$V_{B_9} \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \exp\left[\ln\left\{\frac{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}\right\}\right]$$

$$\exp\left[\ln\left\{\frac{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}\right\}\right] = \frac{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}$$

$$V_{B_9} \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{\frac{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}\right\}$$

$$V_y = V_{B_9} \Rightarrow V_y = V_{B_9} \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{\frac{\frac{V_{p_1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p_2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}}\right\}.$$

Square Circuit In our square circuit implementation, we use Log and Antilog amplifiers, discrete components (resistors, capacitors), LEDs, and phototransistors. Log and Antilog amplifiers are nonlinear circuits in which the output voltage is proportional to the logarithmic (or exponent) of the input. It is well known that some process such as multiplication and division can be performed by addition and subtraction of logs. There are two basic circuits for logarithmic amplifiers: Tran's diode and diode connected transistor. Most logarithmic amplifiers are based on the inherent logarithmic relationship between the collector current I_c, and the base–emitter voltage, V_{be} in silicon bipolar transistor. The input voltage is converted by input resistor into a current, which flows through the transistor's collector modulation the base–emitter voltage according to the input voltage. The op-amp forces the collector voltage to that at non-inverting input, zero voltage. We have in the circuit three current sources ($I_{cont4}, I_{cont5}, I_{cont6}$), which injects current to LEDs (D_4, D_5, and D_6). LEDs light strikes the phototransistors (Q_4, Q_5, and Q_6) base window and can be represented as a dependent current source, respectively. BJT transistors base current is constructed from feedback loop partial current and the current which caused by the light strike on the phototransistors base window. Operational amplifiers virtual ground are $V_{E_1}, V_{E_3}, V_{E_5}, V_{E_8}, V_{E_7}$ (Fig. 6.9).

Logarithmic Amplifier Function We have logarithmic amplifiers in our square circuit. Logarithmic amplifier is constructed from operational amplifier, resistor, and phototransistor on the feedback loop. We short phototransistor collector–base junctions and practically our phototransistor is operated as a base–emitter junction diode. The first logarithmic amplifier is market by index (i = 4) and the second logarithmic amplifier is marked by index (i = 5).

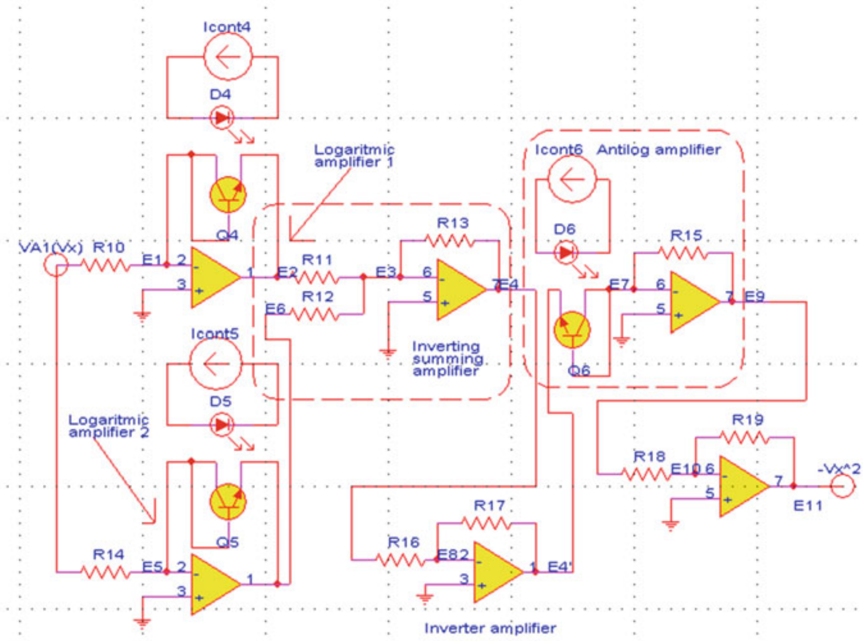

Fig. 6.9 Square circuit

$$V_{CEQ_i} = V_{CBQ_i} + V_{BEQ_i};\ V_{CBQ_i} = 0 \Rightarrow V_{CEQ_i} = V_{BEQ_i};\ \alpha_{f4} \neq \alpha_{f5};\ \alpha_{r4} \neq \alpha_{r5};$$
$$k_4 \neq k_5;\ i = 4, 5$$

Remark Optocouplers (Q_4–D_4 and Q_5–D_5) are not identical.

The op-amp forces the collector voltage to that at the non-inverting input, zero volts. We use the transistor Ebers–Moll model to get the logarithmic amplifier function. I_i is the current that flows from operational amplifier inverting port to the phototransistor (i). The output voltage of logarithmic amplifier is a negative voltage compared to the input voltage:

$$V_{CBQ_i} = 0 \Rightarrow I_{CQ_i} = 0;\ I_{EQ_i} = I_{CQ_i} + I_{BQ_i};\ I_{CQ_i} = 0(\rightarrow \varepsilon) \Rightarrow I_{EQ_i} \simeq I_{BQ_i}$$
$$I_{BQ_i} = I_i + k_i \cdot I_{cont-i};\ I_{EQ_i} = I_i + k_i \cdot I_{cont-i};\ i_{DCQ_i} = 0$$

Ebers–Moll model:

$$i_{DCQ_i} = \frac{\alpha_{f_i} \cdot I_{EQ_i} - I_{CQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}};\ V_{\text{Base-Emitter}Q_i} = V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_i} + 1\right]$$

$$i_{DEQ_i} = \frac{I_{EQ_i} - \alpha_{r_i} \cdot I_{CQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}; \; V_{\text{Base}-\text{Emitter}Q_i} = V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i} - \alpha_{r_i} \cdot I_{CQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1\right]$$

$$I_{CQ_i} = 0 \Rightarrow V_{\text{Base}-\text{Emitter}Q_i} = V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1\right]$$

$$V_{BEQ_i} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1\right] \Rightarrow \exp\left[\frac{V_{BEQ_i}}{V_t}\right] = \frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right) + 1$$

$$\exp\left[\frac{V_{BEQ_i}}{V_t}\right] - 1 = \frac{1}{I_{se}} \cdot \left(\frac{I_i + k_i \cdot I_{cont-i}}{1 - \alpha_{r_i} \cdot \alpha_{f_i}}\right);$$

$$\left(\exp\left[\frac{V_{BEQ_i}}{V_t}\right] - 1\right) \cdot I_{se} \cdot \left(1 - \alpha_{r_i} \cdot \alpha_{f_i}\right) = I_i + k_i \cdot I_{cont-i}$$

$$I_i = \left(\exp\left[\frac{V_{BEQ_i}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_i} \cdot \alpha_{f_i}) - k_i \cdot I_{cont-i}.$$

For the first logarithmic amplifier (i = 4), R_{10} is connected to the op-amp inverting port:

$$\frac{V_x}{R_{10}} = \frac{V_{A_1}}{R_{10}} = I_4; \; \frac{V_x}{R_{10}} = \left(\exp\left[\frac{V_{BEQ_4}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4}) - k_4 \cdot I_{cont-4}$$

$$\frac{V_x}{R_{10}} = \left(\exp\left[\frac{V_{BEQ_4}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4}) - k_4 \cdot I_{cont-4}; \; \exp\left[\frac{V_{BEQ_4}}{V_t}\right] \gg 1; \; \exp\left[\frac{V_{BEQ_4}}{V_t}\right] - 1 \approx \exp\left[\frac{V_{BEQ_4}}{V_t}\right]$$

$$\frac{V_x}{R_{10}} = \exp\left[\frac{V_{BEQ_4}}{V_t}\right] \cdot I_{se} \cdot \left(1 - \alpha_{r_4} \cdot \alpha_{f_4}\right) - k_4 \cdot I_{cont-4}; \; \exp\left[\frac{V_{BEQ_4}}{V_t}\right] \simeq \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot \left(1 - \alpha_{r_4} \cdot \alpha_{f_4}\right)} + \frac{V_x}{R_{10} \cdot I_{se} \cdot \left(1 - \alpha_{r_4} \cdot \alpha_{f_4}\right)}$$

$$\exp\left[\frac{V_{BEQ_4}}{V_t}\right] \simeq \frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-1}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}.$$

V_{BEQ_4} is the output voltage from the logarithmic amplifier ($-V_{BEQ_4} = V_{E_2}$).

$$\frac{V_{BEQ_4}}{V_t} \simeq \ln\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right]$$

$$V_{E_2} = -V_{BEQ_4} \simeq -V_t \cdot \ln\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right].$$

In the same manner the calculation is done for the second (i = 5) logarithmic amplifier:

$$V_{E_6} = -V_{BEQ_5} \simeq -V_t \cdot \ln\left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right]$$

Current sources I_{cont-4}, I_{cont-5} function as a setting current for logarithmic amplifiers.

$$I_{cont-4} = 0 \Rightarrow V_{E_2} \simeq -V_t \cdot \ln\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right];\ I_{cont-5} = 0 \Rightarrow V_{B_7} \simeq -V_t \cdot \ln\left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right].$$

If $R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4}) = 1$ and $R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5}) = 1$ then $V_{E_2} \simeq -V_t \cdot \ln[V_x]$ and $V_{B_7} \simeq -V_t \cdot \ln[V_x]$.

Inverting summing amplifier: $V_{E_4} = -R_{13} \cdot \left(\frac{V_{E_2}}{R_{11}} + \frac{V_{E_6}}{R_{12}}\right)$

$$R_{16} = R_{17} \Rightarrow V_{E'_4} = -V_{E_4};\ V_{E'_4} = R_{13} \cdot \left(\frac{V_{E_2}}{R_{11}} + \frac{V_{E_6}}{R_{12}}\right)$$

$$V_{E'_4} = -R_{13} \cdot V_t \cdot \left(\frac{1}{R_{11}} \cdot \ln\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right] + \frac{1}{R_{12}} \cdot \ln\left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right]\right)$$

$$\frac{1}{R_{11}} = \frac{1}{R_{12}} \Rightarrow V_{E'_4} = -\frac{R_{13}}{R_{11}} \cdot V_t \cdot \left(\ln\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right] + \ln\left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right]\right)$$

$$V_{E'_4} = -\frac{R_{13}}{R_{11}} \cdot V_t \cdot \ln\left\{\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot \left(1 - \alpha_{r_4} \cdot \alpha_{f_4}\right)} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right]\right\}$$

If $I_{cont-4} = 0$ and $I_{cont-5} = 0$ then

$$V_{E'_4} = -\frac{R_{13}}{R_{11}} \cdot V_t \cdot \ln\left\{\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right]\right\}$$

If $I_{cont-4} = 0$; $I_{cont-5} = 0$ and $R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4}) = 1$; $R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5}) = 1$ then

$$V_{E'_4} = -\frac{R_{13}}{R_{11}} \cdot V_t \cdot \ln[V_x^2]$$

Antilogarithmic Amplifier Function We have antilogarithmic amplifier in our square circuit. Antilogarithmic amplifier is constructed from operational amplifier, resistor, and optocoupler. The resistor is in our operational amplifier feedback loop . The optocoupler is connected to the operational amplifier inverting terminal. We short phototransistor collector–base junctions and practically our phototransistor is operated as a base–emitter junction diode.

$$V_{CEQ_6} = V_{CBQ_6} + V_{BEQ_6};\ V_{CBQ_6} = 0 \Rightarrow V_{CEQ_6} = V_{BEQ_6}.$$

The op-amp forces the collector voltage to that at the non-inverting input, zero volts. We use the transistor Ebers–Moll model to get the antilogarithmic amplifier function. I_6 is the current that flows from operational amplifier inverting port to phototransistor Q_6. The output voltage of antilogarithmic amplifier is a negative voltage compared to the input voltage:

$$V_{CBQ_6} = 0 \Rightarrow I_{CQ_6} = 0;\ I_{EQ_6} = I_{CQ_6} + I_{BQ_6};\ I_{CQ_6} = 0(\to \varepsilon) \Rightarrow I_{EQ_6} \simeq I_{BQ_6}$$

$$I_{BQ_6} = I_6 + k_6 \cdot I_{cont-6};\ I_{EQ_6} = I_6 + k_6 \cdot I_{cont-6};\ i_{DCQ_6} = 0.$$

Ebers–Moll model:

$$i_{DCQ_6} = \frac{\alpha_{f_6} \cdot I_{EQ_6} - I_{CQ_6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}};\ V_{\text{Base-EmitterQ}_6} = V_{BEQ_6} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_6} + 1\right]$$

$$i_{DEQ_6} = \frac{I_{EQ_6} - \alpha_{r_6} \cdot I_{CQ_6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}};\ V_{\text{Base-EmitterQ}_6} = V_{BEQ_6} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_6} - \alpha_{r_6} \cdot I_{CQ_6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}}\right) + 1\right]$$

$$I_{CQ_6} = 0 \Rightarrow V_{\text{Base-EmitterQ}_6} = V_{BEQ_6} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}}\right) + 1\right]$$

$$V_{BEQ_6} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}}\right) + 1\right] \Rightarrow \exp\left[\frac{V_{BEQ_6}}{V_t}\right] = \frac{1}{I_{se}} \cdot \left(\frac{I_{EQ_6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}}\right) + 1$$

$$\exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1 = \frac{1}{I_{se}} \cdot \left(\frac{I_6 + k_6 \cdot I_{cont-6}}{1 - \alpha_{r_6} \cdot \alpha_{f_6}}\right);$$

$$\left(\exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) = I_6 + k_6 \cdot I_{cont-6}$$

$$I_6 = \left(\exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) - k_6 \cdot I_{cont-6}$$

For the antilogarithmic amplifier, R_{15} is connected to the op-amp inverting port:

$$\frac{V_{E_9}}{R_{15}} = I_6;\ \frac{V_{E_9}}{R_{15}} = \left(\exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) - k_6 \cdot I_{cont-6}$$

$$\frac{V_{E_9}}{R_{15}} = \left(\exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1\right) \cdot I_{se} \cdot \left(1 - \alpha_{r_6} \cdot \alpha_{f_6}\right) - k_6 \cdot I_{cont-6};$$

$$\exp\left[\frac{V_{BEQ_6}}{V_t}\right] \gg 1;\ \exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1 \approx \exp\left[\frac{V_{BEQ_6}}{V_t}\right]$$

$$\frac{V_{E_9}}{R_{15}} = \left(\exp\left[\frac{V_{BEQ_6}}{V_t}\right] - 1\right) \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) - k_6 \cdot I_{cont-6};$$

$$\frac{V_{E_9}}{R_{15}} = \exp\left[\frac{V_{BEQ_6}}{V_t}\right] \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) - k_6 \cdot I_{cont-6}$$

$$\exp\left[\frac{V_{BEQ_6}}{V_t}\right] = \frac{V_{E_9}}{R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6})} + \frac{k_6 \cdot I_{cont-6}}{I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6})}$$

$$\frac{V_{E_9}}{R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6})} = \exp\left[\frac{V_{BEQ_6}}{V_t}\right] - \frac{k_6 \cdot I_{cont-6}}{I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6})}$$

$$V_{E_9} \simeq R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \exp\left[\frac{V_{BEQ_6}}{V_t}\right] - k_6 \cdot I_{cont-6} \cdot R_{15}.$$

Current source I_{cont-6} functions as a setting current for antilogarithmic amplifier:

$$I_{cont-6} = 0 \Rightarrow V_{E_9} \simeq R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \exp\left[\frac{V_{BEQ_6}}{V_t}\right]$$

If $R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) = 1$ then $V_{E_9} \simeq \exp\left[\frac{V_{BEQ_6}}{V_t}\right]$

$$V_{BEQ_6} = -V_{E'_4};\ V_{E'_4} = -V_{E_4};\ V_{BEQ_6} = V_{E_4}$$

$$V_{E_9} \simeq R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \exp\left[\frac{V_{E_4}}{V_t}\right] - k_6 \cdot I_{cont-6} \cdot R_{15}$$

$$V_{E_9} \simeq R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \exp\left[\frac{V_{E_4}}{V_t}\right] - k_6 \cdot I_{cont-6} \cdot R_{15}$$

$$V_{E'_4} = -V_{E_4} = -\frac{R_{13}}{R_{11}} \cdot V_t \cdot \ln\left\{\left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right]\right.$$
$$\left. \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right]\right\}$$

$$V_{E_4} = \frac{R_{13}}{R_{11}} \cdot V_t \cdot \ln\left\{ \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right] \right\}$$

$$V_{E_9} \simeq R_{15} \cdot I_{se} \cdot \left(1 - \alpha_{r_6} \cdot \alpha_{f_6}\right) \cdot \exp\left[\frac{R_{13}}{R_{11}} \cdot \ln\left\{ \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot \left(1 - \alpha_{r_4} \cdot \alpha_{f_4}\right)} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot \left(1 - \alpha_{r_4} \cdot \alpha_{f_4}\right)} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right] \right\} \right] - k_6 \cdot I_{cont-6} \cdot R_{15}$$

$$\frac{R_{13}}{R_{11}} = 1 \Rightarrow V_{E_9} \simeq R_{15} \cdot I_{se} \cdot \left(1 - \alpha_{r_6} \cdot \alpha_{f_6}\right) \cdot \exp\left[\ln\left\{ \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right] \right\} \right] - k_6 \cdot I_{cont-6} \cdot R_{15}$$

$$\exp\left[\ln\left\{ \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right] \right\} \right] = \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right]$$

$$V_{E_9} \simeq R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right] - k_6 \cdot I_{cont-6} \cdot R_{15}$$

$$R_{18} = R_{19} \Rightarrow V_{E_{11}} = -V_{E_9};\; V_{E_{11}} \sim -V_x^2$$

$$V_{E_{11}} = -R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} \right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} \right] + k_6 \cdot I_{cont-6} \cdot R_{15}$$

Special case: $I_{cont-4} = 0;\; I_{cont-5} = 0;\; I_{cont-6} = 0;\; R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) = 1$

$$R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4}) = 1;\; R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5}) = 1.$$

Then $V_{E_{11}} = -V_x^2$.

We can summarize our optoisolation glycolytic circuit equations:

$$V_{p_2} = V_{A_7} = V_{\mu_1} + V_x^2;\; V_{p_1} = V_{A_5} = \frac{dV_x}{dt} + V_x;\; V_{A_x} = V_{\mu_2} - V_{A_9} = V_{\mu_2} - \left(\frac{dV_y}{dt} + \frac{dV_x}{dt}\right)$$

$$V_x = V_{\mu_2} - V_{A_9} = V_{\mu_2} - \left(\frac{dV_y}{dt} + \frac{dV_x}{dt}\right);\; V_x = V_{\mu_2} - \left(\frac{dV_y}{dt} + \frac{dV_x}{dt}\right);\; V_y \simeq V_{A_{16}};\; V_x \simeq V_{A_1}$$

$$V_y \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{ \frac{\frac{V_{p1}}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{p2}}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}} \right\}$$

$$V_y \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{ \frac{\frac{\frac{dV_x}{dt} + V_x}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{\mu_1} + V_x^2}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}} \right\}$$

$$R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right] - k_6 \cdot I_{cont-6} \cdot R_{15} \Leftrightarrow V_x^2$$

We define $\upsilon(V_x, I_{cont-4}, I_{cont-5}, I_{cont-6}) = \upsilon(V_x)$ function for simplicity:

$$\upsilon(V_x, I_{cont-4}, I_{cont-5}, I_{cont-6}) = R_{15} \cdot I_{se} \cdot (1 - \alpha_{r_6} \cdot \alpha_{f_6}) \cdot \left[\frac{V_x}{R_{10} \cdot I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})} + \frac{k_4 \cdot I_{cont-4}}{I_{se} \cdot (1 - \alpha_{r_4} \cdot \alpha_{f_4})}\right] \cdot \left[\frac{V_x}{R_{14} \cdot I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})} + \frac{k_5 \cdot I_{cont-5}}{I_{se} \cdot (1 - \alpha_{r_5} \cdot \alpha_{f_5})}\right] - k_6 \cdot I_{cont-6} \cdot R_{15}$$

$$V_y \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{ \frac{\frac{\frac{dV_x}{dt} + V_x}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{\mu_1} + \upsilon(V_x)}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}} \right\}.$$

We get two sets of glycolytic oscillator differential equations: first, mathematical differential equations and second, optoisolation circuit implementation differential equations [131, 132]. The second set includes setting current sources $I_{cont-i};\ i = 1, 2, \ldots, 6$.

We can move from the mathematical differential equations variables to optoisolation circuit differential equations variables ($X \to V_x;\ Y \to V_y$). Additionally, $V_{\mu_1} \Leftrightarrow \mu_1;\ V_{\mu_2} \Leftrightarrow \mu_2$.

Optoisolation glycolytic circuit equations cylindrical (r, θ) and Cartesian (V_x, V_y) are the same as for polar coordinates, namely $V_x(t) = r(t) \cdot \cos[\theta(t)]$;

$V_y(t) = r(t)\cdot\sin[\theta(t)]$; $r = \sqrt{V_x^2 + V_y^2}$. $\theta(t) = 0$ if $V_x = 0$ and $V_y = 0$.
$\theta(t) = \arcsin(V_y/r)$ if $V_x \geq 0$. $x \to V_x$, $y \to V_y$. $V_x(t) = r(t)\cdot\cos[\theta(t)]$

$$V_x(t) = r(t)\cdot\cos[\theta(t)] \Rightarrow \frac{\mathrm{d}V_x(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t}\cdot\cos[\theta(t)] - r(t)\cdot\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\cdot\sin[\theta(t)]$$
$$V_y(t) = r(t)\cdot\sin[\theta(t)] \Rightarrow \frac{\mathrm{d}V_y(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t}\cdot\sin[\theta(t)] + r(t)\cdot\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\cdot\cos[\theta(t)]$$
$$\frac{\mathrm{d}V_x(t)}{\mathrm{d}t} = \frac{\mathrm{d}V_x}{\mathrm{d}t};\ \frac{\mathrm{d}V_y(t)}{\mathrm{d}t} = \frac{\mathrm{d}V_y}{\mathrm{d}t};\ \frac{\mathrm{d}r(t)}{\mathrm{d}t} = r';\ \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \theta';\ \theta(t) = \theta;\ r(t) = r$$

We get the following equations:

$$\frac{\mathrm{d}V_x}{\mathrm{d}t} = r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta;\ \frac{\mathrm{d}V_y}{\mathrm{d}t} = r'\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta$$

$$V_x = r\cdot\cos\theta;\ V_y = r\cdot\sin\theta.$$

$$\begin{aligned} V_x = V_{\mu_2} - \left(\frac{\mathrm{d}V_y}{\mathrm{d}t} + \frac{\mathrm{d}V_x}{\mathrm{d}t}\right) &\Rightarrow r\cdot\cos\theta \\ &= V_{\mu_2} - (r'\cdot\sin\theta + r\cdot\theta'\cdot\cos\theta + r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta). \end{aligned}$$

If there is a stable limit cycle as specific value of radius r, $\mathrm{d}r/\mathrm{d}t = 0$ ($r' = 0$).

$$r' = 0 \Rightarrow r\cdot\cos\theta = V_{\mu_2} - r\cdot\theta'\cdot\cos\theta + r\cdot\theta'\cdot\sin\theta$$
$$r\cdot[\cos\theta + \theta'\cdot(\cos\theta - \sin\theta)] = V_{\mu_2};\ r = \frac{V_{\mu_2}}{\cos\theta + \theta'\cdot(\cos\theta - \sin\theta)}$$

$$r\cdot[\cos\theta + \theta'\cdot(\cos\theta - \sin\theta)] = V_{\mu_2} \Rightarrow \theta' = \frac{V_{\mu_2}}{r\cdot(\cos\theta - \sin\theta)} - \frac{\cos\theta}{(\cos\theta - \sin\theta)}$$

$$\begin{aligned} &\upsilon(r, \theta, I_{cont-4}, I_{cont-5}, I_{cont-6}) \\ &= R_{15}\cdot I_{se}\cdot(1 - \alpha_{r_6}\cdot\alpha_{f_6})\cdot\left[\frac{r\cdot\cos\theta}{R_{10}\cdot I_{se}\cdot(1 - \alpha_{r_4}\cdot\alpha_{f_4})} + \frac{k_4\cdot I_{cont-4}}{I_{se}\cdot(1 - \alpha_{r_4}\cdot\alpha_{f_4})}\right] \\ &\quad\cdot\left[\frac{r\cdot\cos\theta}{R_{14}\cdot I_{se}\cdot(1 - \alpha_{r_5}\cdot\alpha_{f_5})} + \frac{k_5\cdot I_{cont-5}}{I_{se}\cdot(1 - \alpha_{r_5}\cdot\alpha_{f_5})}\right] - k_6\cdot I_{cont-6}\cdot R_{15} \end{aligned}$$

$$\begin{aligned} V_y \simeq -k_3\cdot I_{cont-3}\cdot R_7 + R_7\cdot I_{se}\cdot(1 - \alpha_{r_3}\cdot\alpha_{f_3}) \\ \cdot\left\{\frac{\frac{r'\cdot\cos\theta - r\cdot\theta'\cdot\sin\theta + r\cdot\cos\theta}{R_1\cdot I_{se}\cdot(1-\alpha_{r_1}\cdot\alpha_{f_1})} + \frac{k_1\cdot I_{cont-1}}{I_{se}\cdot(1-\alpha_{r_1}\cdot\alpha_{f_1})}}{\frac{V_{\mu_1} + \upsilon(r,\theta)}{R_6\cdot I_{se}\cdot(1-\alpha_{r_2}\cdot\alpha_{f_2})} + \frac{k_2\cdot I_{cont-2}}{I_{se}\cdot(1-\alpha_{r_2}\cdot\alpha_{f_2})}}\right\} \end{aligned}$$

$$r' = 0 \Rightarrow V_y(r,\theta,\theta') \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{ \frac{\frac{-r \cdot \theta' \cdot \sin\theta + r \cdot \cos\theta}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{\mu_1} + \upsilon(r,\theta)}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}} \right\}$$

$$r' = 0 \Rightarrow V_y(r,\theta) \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot (1 - \alpha_{r_3} \cdot \alpha_{f_3}) \cdot \left\{ \frac{\left(\frac{r \cdot \cos\theta - r \cdot \left[\frac{V_{\mu_2}}{r \cdot (\cos\theta - \sin\theta)} - \frac{\cos\theta}{(\cos\theta - \sin\theta)} \right] \cdot \sin\theta}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} \right) + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{\mu_1} + \upsilon(r,\theta)}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}} \right\}$$

$$V_y(r,\theta) = r \cdot \sin\theta$$

$$r' = 0 \Rightarrow r \cdot \sin\theta \simeq -k_3 \cdot I_{cont-3} \cdot R_7 + R_7 \cdot I_{se} \cdot \left(1 - \alpha_{r_3} \cdot \alpha_{f_3}\right) \cdot \left\{ \frac{\left(\frac{r \cdot \cos\theta - r \cdot \left[\frac{V_{\mu_2}}{r \cdot (\cos\theta - \sin\theta)} - \frac{\cos\theta}{(\cos\theta - \sin\theta)} \right] \cdot \sin\theta}{R_1 \cdot I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})} \right) + \frac{k_1 \cdot I_{cont-1}}{I_{se} \cdot (1-\alpha_{r_1} \cdot \alpha_{f_1})}}{\frac{V_{\mu_1} + \upsilon(r,\theta)}{R_6 \cdot I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})} + \frac{k_2 \cdot I_{cont-2}}{I_{se} \cdot (1-\alpha_{r_2} \cdot \alpha_{f_2})}} \right\}.$$

To find the glycolytic oscillator limit cycle radius (r) analytically is very difficult and it is recommended to calculate it numerically.

6.5 Exercises

1. We have coupled planar cubic vector systems as four differential equations (Ω_1, Ω_2 are coupling parameters):

$$\frac{dX_1}{dt} = X_2\,;\ \frac{dX_2}{dt} = -(X_1^3 + r\cdot X_1^2 + n\cdot X_1 + m) + (\Gamma - X_1^2)\cdot X_2 + \Omega_1\cdot(X_4 - X_2)$$

$$\frac{dX_3}{dt} = X_4\,;\ \frac{dX_4}{dt} = -(X_3^3 + (1-r)\cdot X_3^2 + (1-\sqrt{n})\cdot X_3 + m^2) + (\sqrt{\Gamma} - X_3^2)\cdot X_4 + \Omega_2\cdot(X_2 - X_4)$$

1.1 Check if the systems contain the unfolding of a co-dimension two bifurcation of an equilibrium point with a double eigenvalue zero in the presence of a rotational symmetry of the plane.
1.2 Find systems fixed points and Jacobian.
1.3 Discuss systems stability and Basin of Attraction (BOA) for the selected fixed points.
1.4 Prove that the systems have a periodic orbit. Find $d\theta_i/dt = \xi_{1i}(r_i, \theta_i)$, $dr_i/dt = \xi_{2i}(r_i, \theta_i)$ expressions (i = 1, 2—system index).
1.5 Consider that systems limit cycle is resided in annulus $0 < r_{\min,i} < r_i < r_{\max,i}$ and there is no constant r_i in limit cycle $dr_i/dt \neq 0$. Find the conditions for $r_{\min,i}$ and $r_{\max,i}$ as a function of coupled system parameters (i = 1, 2—system index).
1.6 How Ω_1, Ω_2 coupling parameters influence our systems stability and limit cycle?

2. We have three coupler Van der Pol systems which can be characterize by the following differential equations:

$$\frac{d^2X_1}{dt^2} + X_1 - \mu_1\cdot(1 - X_1^2)\cdot\frac{dX_1}{dt} = \alpha_1\cdot(X_2 - X_1/X_2)$$
$$\frac{d^2X_2}{dt^2} + (1+\Delta)\cdot X_2 - \mu_2\cdot(1 - X_2^2)\cdot\frac{dX_2}{dt} = \alpha_2\cdot(X_3 - X_2/X_3)$$
$$\frac{d^2X_3}{dt^2} + (1+\sqrt{\Delta})\cdot X_3 - |\mu_1 - \mu_2|\cdot(1 - X_3^2)\cdot\frac{dX_3}{dt} = |\alpha_1 - \alpha_2|\cdot(X_3/X_2 - X_3)$$

α_1 and α_2 are coupling coefficients. Δ, μ_1, μ_2, α_1, and α_2 are parameters. Δ is related to the small difference in linearized frequencies, and α_1 and α_2 represent the strength of the coupling for Van der Pol systems. μ_1 and μ_2 are parameters which establish the behavior of each Van der Pol system.

2.1 Check if the systems contain the unfolding of a co-dimension two bifurcation of an equilibrium point with a double eigenvalue zero in the presence of a rotational symmetry of the plane.
2.2 Find systems fixed points and Jacobian.

2.3 Discuss systems stability and Basin of Attraction (BOA) for the selected fixed points.
2.4 Prove that the systems have a periodic orbits. Find $d\theta_i/dt = \xi_{1i}(r_i, \theta_i)$, $dr_i/dt = \xi_{2i}(r_i, \theta_i)$ expressions (i = 1, 2, 3—system index).
2.5 Consider that system limit cycle is resided in annulus $0 < r_{\min,i} < r_i < r_{\max,i}$ and there is no constant r_i in limit cycle $dr_i/dt \neq 0$. Find the conditions for $r_{\min,i}$ and $r_{\max,i}$ as a function of coupled system parameters (i = 1, 2, 3—system index).
2.6 How α_1 and α_2 coupling parameters influence our systems stability and limit cycle?

3. Consider the system $\frac{d^2X}{dt^2} + \mu \cdot \frac{dX}{dt} \cdot \left[X^2 + \left(\frac{dX}{dt}\right)^2 - 1\right] + X = 0;\ \mu > 0;\ \mu \in \mathbb{R}$ μ value established the dynamical of our system.

3.1 Find system fixed points and discuss stability and stability switching for different values of μ parameter.
3.2 Show that system has a circular limit cycle, and find amplitude and period.
3.3 Determine stability and limit cycle for different values of μ parameter and show that limit cycle is unique.
3.4 Implement the system using op-amp, resistors, capacitors, optocouplers, etc.
3.5 Discuss system circuit stability switching for different values of optocouplers parameters $(\alpha_{f_i}, \alpha_{r_i}, k_i\,;\ i = 1, 2, 3, \ldots)$.

4. Consider the system $\frac{dX}{dt} = Y + \mu_1 \cdot X \cdot (1 - 2 \cdot \mu_2 - X^2 - Y^2)$

$$\frac{dY}{dt} = -X + \mu_1 \cdot Y \cdot (1 - X^2 - Y^2)\,;\ \mu_1, \mu_2 \in \mathbb{R}\,;\ 0 < \mu_1 \leq 1\,;\ 0 < \mu_2 \leq \frac{1}{2}$$

4.1 Write polar coordinate of the system. Find limit cycle radius (r) and discuss his variation for different values of μ_1, μ_2 parameters.
4.2 Prove that there is at list one limit cycle or several but the same period $T(\mu_1, \mu_2)$. If $\mu_2 = 0$ there is only one limit cycle, prove it.
4.3 When system limit cycle is resided in annulus $0 < r_{\min} < r < r_{\max}$ and there is no constant r in limit cycle $dr/dt \neq 0$. We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$. $dr/dt < 0$ on the outer circle and $dr/dt > 0$ on the inner circle. Find the expressions for $r_{\min}$ and $r_{\max}$ and they change for different values of μ_1, μ_2 parameters.
4.4 Discuss stability and stability switching for different values of μ_1, μ_2 parameters.
4.5 Implement the system using op-amp, resistors, capacitors, optocouplers, etc.
4.6 Discuss system circuit stability switching for different values of optocouplers parameters $(\alpha_{f_i}, \alpha_{r_i}, k_i\,;\ i = 1, 2, 3, \ldots)$.

5. We have circuit implementation of Van der Pol system. The active elements of the circuit are semiconductor devices (two OptoNDR circuits/devices in series). Capacitor C_2 is placed between the first OptoNDR circuit (Q_1–D_1) and the second OptoNDR circuit (Q_2–D_2). Active elements act like an ordinary resistors when current $I(t)$ is high ($I(t) > I_{sat}$), but like negative resistors (energy source) or negative differential resistances (NDRs) when $I(t)$ is low ($I_{sat} > I(t) > I_{break}$). Parameters k_1 and k_2 are the coupling coefficients between LEDs (D_1, D_2) current and phototransistors (Q_1, Q_2) base window current, respectively. Our circuit current voltage characteristic is $V = f(I) \quad \forall \ \frac{\mathrm{d}I}{\mathrm{d}t} = 0$. We consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below $I_{break}(I(t) < I_{break})$ (Fig. 6.10).

5.1 Write circuit differential equations, and find the analogy to Van der Pol's mathematical differential equations. Use BJT transistor Ebers–Moll equations.
5.2 Find fixed points and discuss stability and stability switching for different values of C_1, C_2, and L_1.
5.3 Discuss Van der Pol circuit limit cycle and find limit cycle radius (r).
5.4 When our system circuit limit cycle is resided in annulus line $0 < r_{min} < r < r_{max}$ and there is no constant (r) in the limit cycle $\frac{\mathrm{d}r}{\mathrm{d}t} \neq 0$. Find the expressions for r_{min} and r_{max}. How the expressions are influenced when we change optocouplers parameters ($\alpha_{f_i}, \alpha_{r_i}, k_i$; $i = 1, 2$).
5.5 We short capacitor C_2. How it influences our system circuit dynamics? Find fixed points and discuss limit cycle.

Remark The optical coupling between the LEDs (D_1, D_2) to the phototransistors (Q_1 and Q_2) is represented as transistor-dependent base current on LED (D_1) current and LED (D_2) current respectively, "k_i" is the multiplication coefficient (first optocoupler i = 1 and second optocoupler i = 2).

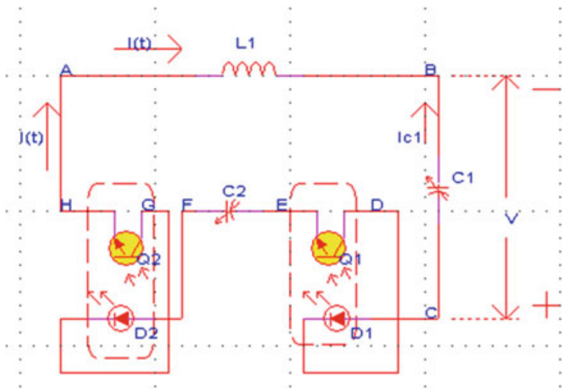

Fig. 6.10 Circuit implementation of van der Pol system

6. Consider the Lienard system $\frac{d^2X}{dt^2} + \mu \cdot (X^2 - 1) \cdot \frac{dX}{dt} + tgh(X) = 0\,;\ \mu > 0$. Lienard equation is a second-order differential equation and it can be used to model oscillating circuits. Lienard's theorem guarantees the uniqueness and existence of a limit cycle for such a system. The basic Lienard equation is constructed from two continuously differentiable functions $g_1(X)$ and $g_2(X)$ on $\mathbb{R}$, with $g_2(X)$ an odd function and $g_1(X)$ an even function. Then the second-order ordinary differential equation of form $\frac{d^2X}{dt^2} + g_1(X) \cdot \frac{dX}{dt} + g_2(X) = 0$ is called Lienard equation and $g_1(X) = \mu \cdot (X^2 - 1)\,;\ g_2(X) = tgh(X)$. Hyperbolic tangent ($tgh$, tanh):

$$tgh(X) = \frac{\sinh(X)}{\cosh(X)} = \frac{e^X - e^{-X}}{e^X + e^{-X}};\ tgh(X) = \frac{e^{2\cdot X} - 1}{e^{2\cdot X} + 1} = \frac{1 - e^{-2\cdot X}}{1 + e^{-2\cdot X}};\ tgh(0) = 0$$

$$tgh(X) = 1 \quad \forall\ X \to \infty;\ tgh(X) = -1 \quad \forall\ X \to -\infty;\ tgh(-X) = -tgh(X).$$

6.1 Find system fixed points and discuss stability and stability switching for different values of μ parameter.
6.2 Find limit cycle and radius (r). Show that the system has exactly one period solution.
6.3 What happened for the case $\mu = 0$? Find fixed points and discuss stability and limit cycles.
6.4 Implement Lienard system circuit using op-amps, capacitors, resistors, optocouplers, etc.
6.5 Discuss Lienard system circuit stability switching for different values of optocouplers parameters ($\alpha_{f_i}, \alpha_{r_i}, k_i;\ i = 1, 2, 3, \ldots$).

Remark A Lienard system has a unique and stable limit cycle surrounding the origin if it satisfied the following additional properties: $g_2(X) > 0$ for all $X > 0$, $\lim_{X\to\infty} G_1(X) = \lim_{X\to\infty} \int_0^X g_1(\xi) \cdot d\xi = \infty$. Function $G_1(X)$ has exactly one positive root at some value ρ, where $G_1(X) < 0$ for $0 < X < \rho$. Function $G_1(X) > 0$ monotonic for $X > \rho$.

7. Consider the system with two variables: X, Y and is represented by two differential equations: $\frac{dX}{dt} = \mu_1 \cdot X - Y - \mu_2 \cdot X \cdot (X^2 + Y^2) - X \cdot (X^2 + Y^2)^2$

$$\frac{dY}{dt} = X + \mu_1 \cdot Y - \mu_2 \cdot Y \cdot (X^2 + Y^2) - Y \cdot (X^2 + Y^2)^2;\ \mu_1, \mu_2 \in \mathbb{R}.$$

7.1 Investigate the stability of the fixed point (X^*, Y^*) = (0, 0) and find any bifurcation families of periodic orbits that may exist.

7.2 Draw a bifurcation diagram (a graph over (μ_1, μ_2) space) indicating the fixed points and periodic orbits.
7.3 Discuss stability and stability switching for different values of μ_1, μ_2 parameters. If $\mu_1 = 0$, how the dynamic of the system is changed. Find fixed points and discuss stability.
7.4 Implement the system circuit using op-amp, capacitors, resistors, optocouplers, etc.
7.5 Discuss system circuit stability and stability switching for different values of optocouplers parameters $(\alpha_{f_i}, \alpha_{r_i}, k_i;\ i = 1, 2, 3, \ldots)$.

8. Rossler system is a system of three differential equations with a simple strange attractor than Lorenz's. $\frac{dX}{dt} = -Y - Z;\ \frac{dY}{dt} = X + \mu_1 \cdot Y$ and $\frac{dZ}{dt} = \mu_2 + Z \cdot (X - \mu_3);\ \mu_1, \mu_2, \mu_3 \in \mathbb{R}$. In one direction there is compression toward the attractor, and in the other direction there is divergence along the attractor. There are expanding directions along which stretching takes place, where μ_1, μ_2 and μ_3 are parameters. This system contains only one nonlinear term $Z \cdot X$, and is simpler than the Lorenz system. The system shows periodic relaxation oscillations in dimensional two and higher types of relaxation behavior. It has chaotic behavior if the motion is spiraling out on one branch of the slow manifold. System oscillations can be amplified if $\mu_1 > 0$, which results into a spiraling out motion. The system possesses two steady states: one at the origin $X^* = 0;\ Y^* = 0;\ Z^* = 0$, around which the motion spirals out, and another one at some distance of the origin due to the quadratic nonlinearity. This system presents stationary, periodic, quasiperiodic, and chaotic attractors depending on the value of the parameters μ_1, μ_2, μ_3.

8.1 Find system fixed points and discuss stability for different values of parameters μ_1, μ_2, μ_3.
8.2 Discuss system limit cycle and find radius (r). When system's limit cycle is resided in annulus $0 < r_{\min} < r < r_{\max}$ and there is no constant r in limit cycle $dr/dt \neq 0$. We seek two concentric circles with radii $r_{\min}$ and $r_{\max}$. $dr/dt < 0$ on the outer circle and $dr/dt > 0$ on the inner circle. Find the expressions for $r_{\min}$ and $r_{\max}$ and they change for different values of μ_1, μ_2 and μ_3 parameters.
8.3 If $\mu_3 = 0$, how it influences system dynamic? Find fixed point and discuss stability. Discuss limit cycle and find radius (r).
8.4 Implement the system circuit using op-amp, capacitors, resistors, optocouplers, etc.
8.5 Discuss system circuit stability and stability switching for different values of optocouplers parameters $(\alpha_{f_i}, \alpha_{r_i}, k_i;\ i = 1, 2, 3, \ldots)$.

9. We have circuit implementation of Van der Pol system with additional discrete components. The active elements of the circuit are semiconductor devices (two OptoNDR circuits/devices). Capacitor C_2 is placed in parallel to Q_1 collector emitter. Inductor L_2 is placed between the two OptoNDR circuits. Active

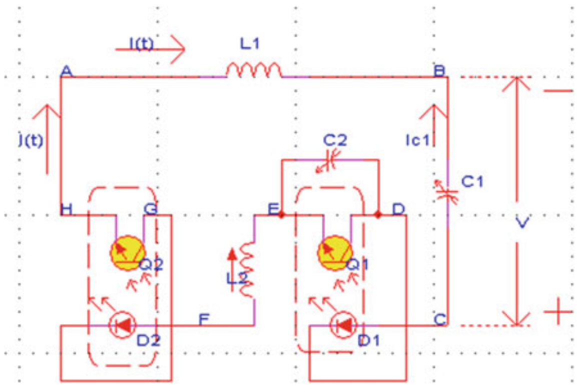

Fig. 6.11 Circuit implementation of van der Pol system with additional discrete components

elements act like an ordinary resistors when current $I(t)$ is high ($I(t) > I_{sat}$), but like negative resistors (energy source) or negative differential resistances (NDRs) when $I(t)$ is low ($I_{sat} > I(t) > I_{break}$). Parameters k_1 and k_2 are the coupling coefficients between LEDs (D_1, D_2) current and phototransistors (Q_1, Q_2) base window current, respectively. Our circuit current voltage characteristic is $V = f(I) + \psi(I) \quad \forall \frac{dI}{dt} = 0$. We consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below $I_{break}(I(t) < I_{break})$ (Fig. 6.11).

9.1 Write circuit differential equations, and find the analogy to Van der Pol's mathematical differential equations. Use BJT transistor Ebers–Moll equations. Find the functions $f(I), \psi(I)$.

9.2 Find fixed points and discuss stability and stability switching for different values of L_1, L_2 inductors.

9.3 Discuss Van der Pol circuit limit cycle and find limit cycle radius (r).

9.4 When our system circuit limit cycle is resided in annulus line $0 < r_{min} < r < r_{max}$ and there is no constant (r) in the limit cycle $\frac{dr}{dt} \neq 0$. Find the expressions for r_{min} and r_{max}. How the expressions are influenced when we change optocouplers parameters ($\alpha_{f_i}, \alpha_{r_i}, k_i;\ i = 1, 2$).

9.5 We short inductor L_2. How it influences our system circuit dynamics? Find fixed points and discuss limit cycle.

Remark The optical coupling between the LEDs (D_1, D_2) to the phototransistors (Q_1 and Q_2) is represented as transistor-dependent base current on LED (D_1) current and LED (D_2) current, respectively, "k_i" is the multiplication coefficient (first optocoupler i = 1 and second optocoupler i = 2).

10. Consider the system differential equations, variables X and Y.

$$\frac{\mathrm{d}X}{\mathrm{d}t} = X \cdot (1 - X^2 - Y^2) \cdot (X^2 + Y^2 - \mu_1) + Y \cdot (X^2 + Y^2 + \mu_2 \cdot X - \mu_3)$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = Y \cdot (1 - X^2 - Y^2) \cdot (X^2 + Y^2 + \mu_1) - X \cdot (X^2 + Y^2 + \mu_2 \cdot X - \mu_3)$$

$$\mu_1, \mu_2, \mu_3 \in \mathbb{R}.$$

10.1 Find critical points of the system and prove that there is an asymptotically stable focus at (0, 0).

10.2 Move to system polar coordinates and discuss limit cycle. Find radius (r) and discuss stability.

10.3 Find and discuss from the equations of $\frac{\mathrm{d}r}{\mathrm{d}t} = \ldots;\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \ldots$ the cases: $\frac{\mathrm{d}r}{\mathrm{d}t} < 0;\ \frac{\mathrm{d}r}{\mathrm{d}t} > 0$ and find the trajectories.

10.4 Find in which conditions all trajectories inside the unit circle converge to the critical point (0, 0) which is an asymptotically stable focus.

10.5 Implement the system circuit using op-amp, capacitors, resistors, optocouplers, etc.

10.6 Discuss system circuit stability and stability switching for different values of optocouplers parameters $(\alpha_{f_i}, \alpha_{r_i}, k_i;\ i = 1, 2, 3, \ldots)$.

Chapter 7
Optoisolation Circuits Poincare Maps and Periodic Orbit

Poincare maps is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower dimensional subspace, called the Poincaré section, transversal to the flow of the system. Poincare map is a discrete dynamical system with a state space that is one dimension smaller than the original continuous dynamical system. We use it for analyzing the original system. We use Poincare maps to study the flow near a periodic orbit, the flow in some chaotic system. We define n-dimensional system $\frac{dx}{dt} = f(X)$. Let "S" be an $(n-1)$ dimensional surface of section. "S" is required to be transverse to the flow and all trajectories starting on "S" flow through it, not parallel to it. The Poincare map ψ is a mapping from "S" to itself and trajectories from one intersection with "S" to the next. If $X_k \in S$ denotes the kth intersection, then Poincare map is defined by $X_{k+1} = \psi(X_k)$. If X^* is a fixed point then $\psi(X^*) = X^*$. Then a trajectory starting at X^* returns to X^* after time T and a closed orbit for the original system $\frac{dx}{dt} = f(X)$. Many systems are presented by set of differential equations and analysis is done by using Poincare maps. We implement these dynamical systems by using opto couplers, Op-amps, and discrete components (resistors, capacitors, etc.,) and investigation is done by using Poincare maps [5–8].

7.1 Poincare Maps and Periodic Orbit Flow

In every dynamical system we need to use the global existence theorems. The dynamical system describes the physical behavior in time. Many dynamical systems described by its position and velocity as functions of time and the initial conditions. A dynamical system is a function $\phi(t, X)$, defined for all $t \in \mathbb{R}$ and $X \in E \in \mathbb{R}^n$. It describes how points $X \in E$ move with respect to time. The families of maps $\phi_t(X) = \phi(t, X)$ have the properties of a flow. Integral part of our analysis is

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periodic orbits or cycles, limit cycles and separatrix cycles of a dynamical system $\phi(t, X)$ defined by $\frac{dX}{dt} = f(X)$. Periodic orbits have stable and unstable manifold just as equilibrium points do. A finite union of compatibly oriented separatrix cycles is called a compound separatrix cycle or graphic. We need to discuss the stability and bifurcations of periodic orbits by the Poincare map or first return map. A separatrix is the boundary separating two modes of behavior in a differential equation and an equation to determine the borders of a system. A phase curve meets a hyperbolic fixed point or connects the unstable and stable manifolds of a pair of hyperbolic or parabolic fixed points. A separatrix marks a boundary between phase curves with different properties. Poincare map is very useful tool to study the stability and bifurcations of periodic orbit. If Γ is periodic orbit of the system $\frac{dX}{dt} = f(X)$ through the point X_0 and Σ is a hyperplane perpendicular to Γ at X_0, then any point $X \in \Sigma$ sufficiently near X_0, the solution $\frac{dX}{dt} = f(X)$ through X at $t = 0$, $\phi_t(X)$ will cross the Σ again at a point $\psi(X)$ near X_0. The mapping $X \to \psi(X)$ is called the Poincare map. When the surface Σ intersects the curve Γ transversally at X_0, Σ is a smooth surface, through a point $X_0 \in \Gamma$, which is not tangent to Γ at X_0. Additionally, we can discuss the existence and continuity of the Poincare map $\psi(X)$ and of its first derivative $D\psi(X)$. Consider the system $\frac{d^2X}{dt^2} + \mu_1 \cdot \frac{dX}{dt} + \sin(X) = \mu_2;\ \mu_1, \mu_2 \in \mathbb{R}$. Let $Y = \frac{dX}{dt}$ then the system becomes $\frac{dX}{dt} = Y; \frac{dY}{dt} + \mu_1 \cdot Y + \sin(X) = \mu_2$. The fixed points satisfy $\frac{dX}{dt} = 0;\ \frac{dY}{dt} = 0$ then $Y^* = 0;\ \sin(X^*) = \mu_2 \Rightarrow X^* = \arcsin(\mu_2)$. Hence there are two fixed points on the cylinder if $\mu_2 < 1$, and none if $\mu_2 > 1$ ($\mu_2 > 1 \Rightarrow \sin(X^*) > 1$), cannot exist. When the fixed point exists, one is a saddle and the other is a sink. We define the following functions: $g_1(X, Y) = Y;\ g_2(X, Y) = -\mu_1 \cdot Y - \sin(X) + \mu_2$ then $\frac{dX}{dt} = g_1(X, Y)\ \frac{dY}{dt} = g_2(X, Y)$. We need to use linearization technique about the fixed point.

$g_1(X^*, Y^*) = 0;\ g_2(X^*, Y^*) = 0$ Let $u = X - X^*;\ v = Y - Y^*$ denote the components of a small disturbance from the fixed point. We need to analyze whether the disturbance grows or decays. We derive differential equations for u and v

$$\frac{du}{dt} = \frac{dX}{dt} = g_1(X^* + u, Y^* + v);$$
$$\frac{du}{dt} = \frac{dX}{dt} = g_1(X^*, Y^*) + u \cdot \frac{\partial g_1}{\partial X} + v \cdot \frac{\partial g_1}{\partial Y} + O(u^2, v^2, u \cdot v)$$

$g_1(X^*, Y^*) = 0 \Rightarrow \frac{du}{dt} = \frac{dX}{dt} = u \cdot \frac{\partial g_1}{\partial X} + v \cdot \frac{\partial g_1}{\partial Y} + O(u^2, v^2, u \cdot v)$. The partial derivatives are to be evaluated at the fixed point (X^*, Y^*). Expression $O(u^2, v^2, u \cdot v)$ denotes quadratic term in u and v. Since u and v are small, these quadratic terms are small. Similarly, we find $\frac{dv}{dt} = \frac{dY}{dt} = u \cdot \frac{\partial g_2}{\partial X} + v \cdot \frac{\partial g_2}{\partial Y} + O(u^2, v^2, u \cdot v)$. Hence the disturbance (u, v) evolves according to $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} +$ quadratic terms.

The matrix $A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} \end{pmatrix}_{(X^*, Y^*)}$ is called the Jacobian matrix at the fixed point (X^*, Y^*). Since the quadratic terms $O(u^2, v^2, u \cdot v)$ are tiny, we neglect them altogether. We obtain the linearized system:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix};$$

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial X} & \frac{\partial g_1}{\partial Y} \\ \frac{\partial g_2}{\partial X} & \frac{\partial g_2}{\partial Y} \end{pmatrix}_{(X^*, Y^*)}; \quad \frac{\partial g_1(X, Y)}{\partial X} = 0; \quad \frac{\partial g_1(X, Y)}{\partial Y} = 1$$

$$\frac{\partial g_2(X, Y)}{\partial X} = -\cos(X); \quad \frac{\partial g_2(X, Y)}{\partial Y} = -\mu_1; \quad A = \begin{pmatrix} 0 & 1 \\ -\cos(X) & -\mu_1 \end{pmatrix}$$

$$A_{(X^*, Y^*)} = \begin{pmatrix} 0 & 1 \\ -\cos(X) & -\mu_1 \end{pmatrix}_{(X^* = \arcsin(\mu_2), Y^* = 0)} = \begin{pmatrix} 0 & 1 \\ -\cos(\arcsin(\mu_2)) & -\mu_1 \end{pmatrix}$$

$$\cos(\arcsin(\mu_2)) = \sin(\arccos(\mu_2)) = \sqrt{1 - \mu_2^2}; \quad \mathrm{X} \in [-1, 1]$$

$$A_{(X^*, Y^*)} = \begin{pmatrix} 0 & 1 \\ -\cos(X) & -\mu_1 \end{pmatrix}_{(X^* = \arcsin(\mu_2),\, Y^* = 0)} = \begin{pmatrix} 0 & 1 \\ -\sqrt{1 - \mu_2^2} & -\mu_1 \end{pmatrix}$$

The eigenvalues of matrix A are given by the characteristic equation $\det(A - \lambda \cdot I) = 0$, where "$I$" is the identity matrix [5, 7].

$\det(A - \lambda \cdot I) = \det\begin{pmatrix} -\lambda & 1 \\ -\sqrt{1 - \mu_2^2} & -\mu_1 - \lambda \end{pmatrix} = 0$. Expanding the determinant yields:

$$-\lambda \cdot (-\mu_1 - \lambda) + \sqrt{1 - \mu_2^2} = 0;$$

$$\lambda^2 + \lambda \cdot \mu_1 + \sqrt{1 - \mu_2^2} = 0;$$

$$\tau = \mathrm{trace}(A) = -\mu_1$$

$\Delta = \det(A) = \sqrt{1-\mu_2^2}$; $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\cdot\Delta}}{2}$. The eigenvalues depend only on the trace and determinant of the matrix A.

$$\lambda_1 = \frac{-\mu_1 + \sqrt{\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2}}}{2}; \quad \lambda_2 = \frac{-\mu_1 - \sqrt{\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2}}}{2}$$

The above expressions are the solutions of characteristic equation. The typical situation of the characteristic equation is for the eigenvalues to be distinct: $\lambda_1 \neq \lambda_2$. If the eigenvalues are complex, the fixed point is either center or a spiral ($\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2} < 0$). The origin is surrounded by a family of closed orbits and the centers are neutrally stable. Nearby trajectories are neither attracted to nor repelled from the fixe point. Thus complex eigenvalues occur when $\tau^2 - 4\cdot\Delta < 0 \Rightarrow [\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2}] < 0$; $\mu_1^2 < 4\cdot\sqrt{1-\mu_2^2}$.

$$\lambda_1 = \frac{-\mu_1 + i\cdot\sqrt{4\cdot\sqrt{1-\mu_2^2} - \mu_1^2}}{2}; \quad \lambda_2 = \frac{-\mu_1 - i\cdot\sqrt{4\cdot\sqrt{1-\mu_2^2} - \mu_1^2}}{2}$$

$$\lambda_1 = -\frac{\mu_1}{2} + i\cdot\frac{1}{2}\cdot\sqrt{4\cdot\sqrt{1-\mu_2^2} - \mu_1^2}; \quad \lambda_2 = -\frac{\mu_1}{2} - i\cdot\frac{1}{2}\cdot\sqrt{4\cdot\sqrt{1-\mu_2^2} - \mu_1^2}.$$

By assumption $\frac{1}{2}\cdot\sqrt{4\cdot\sqrt{1-\mu_2^2} - \mu_1^2} \neq 0 \Rightarrow 4\cdot\sqrt{1-\mu_2^2} - \mu_1^2 \neq 0 \Rightarrow \mu_1^2 \neq 4\cdot\sqrt{1-\mu_2^2}$.

Then the eigenvalues are distinct $\Delta = \prod_{k=1}^{2} \lambda_k$; $\tau = \sum_{k=1}^{2} \lambda_k$.

If $\Delta < 0 \Rightarrow \sqrt{1-\mu_2^2} < 0$, the eigenvalues are real and have opposite sign; hence the fixed point is a saddle point (cannot exist since the graph of function $\sqrt{1-\mu_2^2}$ made up of half a parabola with vertical directrix). If $\Delta > 0 \Rightarrow \sqrt{1-\mu_2^2} > 0$ then $1-\mu_2^2 > 0 \Rightarrow \mu_2^2 - 1 < 0; -1 < \mu_2 < 1$ and the eigenvalues are either real with the same sign (nodes), or complex conjugate (spirals and centers). Nodes satisfy

$\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2} > 0$ And spirals satisfy $\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2} < 0$. The parabola $\mu_1^2 - 4\cdot\sqrt{1-\mu_2^2} = 0$ is the borderline between nodes and spirals. Star nodes and degenerate nodes live on this parabola. The stability of the nodes and spirals is determined by $\tau = \mathrm{trace}(A) = -\mu_1$. When $-\mu_1 < 0 \Rightarrow \mu_1 > 0$, both eigenvalues have negative real parts and the fixed point is stable. If $-\mu_1 > 0 \Rightarrow \mu_1 < 0$ then we get unstable spirals and nodes. Stable centers live on the borderline $\mu_1 = 0$ and the eigenvalues are purely imaginary. If $\Delta = 0 \Rightarrow \sqrt{1-\mu_2^2} = 0 \Rightarrow 1-\mu_2^2 = 0 \Rightarrow \mu_2 = \pm 1$ then at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point and there is either a whole line of fixed points or a plane of fixed points. If the eigenvalues are pure imaginary $-\frac{\mu_1}{2} = 0 \Rightarrow \mu_1 = 0$, then all solutions are periodic with period $T = \frac{4 \cdot \pi}{\sqrt{4 \cdot \sqrt{1-\mu_2^2} - \mu_1^2}}$; $\mu_1^2 < 4 \cdot \sqrt{1 - \mu_2^2}$. We need to investigate the existence of a system closed orbit. There are two fixed points on the cylinder if $\mu_2 < 1$, and none if $\mu_2 > 1$ ($\mu_2 > 1 \Rightarrow \sin(X^*) > 1$), cannot exist. If $\mu_2 > 1$ ($\mu_2 > 1 \Rightarrow \sin(X^*) > 1$) there are no more fixed points available. Isocline is a curve through points at which the parent function's slope will always be the same, regardless of initial conditions. Null clines in mathematics analysis called zero growth isoclines. Nullcline of our system $\frac{dY}{dt} + \mu_1 \cdot Y + \sin(X) = \mu_2$; $\frac{dY}{dt} = 0$. $\mu_1 \cdot Y + \sin(X) = \mu_2 \Rightarrow Y = \frac{\mu_2}{\mu_1} - \frac{1}{\mu_1} \cdot \sin(X)$.

$\sin(X) \in [-1, 1] \Rightarrow Y_{\max} = \frac{\mu_2 + 1}{\mu_1}$; $Y_{\min} = \frac{\mu_2 - 1}{\mu_1}$. All trajectories enter the strip $Y_1 \le Y \le Y_2$; $Y > 0 \Rightarrow \frac{dX}{dt} > 0$; $0 < Y_1 < Y_{\min} \Rightarrow 0 < Y_1 < \frac{\mu_2 - 1}{\mu_1}$; $Y_2 > Y_{\max} \Rightarrow Y_2 > \frac{\mu_2 + 1}{\mu_1}$. Inside the strip, the flow is always to the right. Fact, $X = 0$; $X = 2 \cdot \pi$ are equivalent on the cylinder $\sin(0) = \sin(2 \cdot \pi) = 0$. We define rectangular box $0 \le X \le 2 \cdot \pi$; $Y_1 \le Y \le Y_2$. The trajectory starts at the height Y on the left side of the box and flows until it intersects the right side of the box at new height $\psi(Y)$. The mapping from the Y to $\psi(Y)$ is called Poincare map. We get information on height of the trajectory changes after one lap around the cylinder. The Poincare map is also called first return map. If there is a point Y^*; $\psi(Y^*) = Y^*$, then the corresponding trajectory will be closed orbit. Function $\psi(Y)$ is a monotonic function. We define $\Omega(\mu_1, \mu_2)$ function (3D). The function $\Omega(\mu_1, \mu_2) = \mu_1^2 - 4 \cdot \sqrt{1 - \mu_2^2}$ and we plot it in three dimensional ($\mu_1 \to m_1$, $\mu_2 \to m_2$); $0 < \mu_1 < 10$; $-1 < \mu_2 < 1$ [54].

MATLAB [m2,m1] = meshgrid(-1:0.01:1,0:0.1:10);
e = m1.*m1-4*sqrt(1-m2.*m2);
meshc(e); (Fig. 7.1)

System's limit cycle discussion

System differential equations: $\frac{dX}{dt} = Y$; $\frac{dY}{dt} + \mu_1 \cdot Y + \sin(X) = \mu_2$. We need to prove that the system has periodic orbits and it is done by changing system Cartesian coordinates ($X(t)$, $Y(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z-axis. In our system we refer to Cartesian X–Y plane (with equation $Z = 0$). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X, Y) are the same as for polar coordinates, namely $X(t) = r(t) \cdot \cos[\theta(t)]$; $Y(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X^2 + Y^2}$. $\theta(t) = 0$ if $X = 0$ and $Y = 0$. $\theta(t) = \arcsin(Y/r)$ if $X >= 0$. $x \to X$, $y \to Y$.

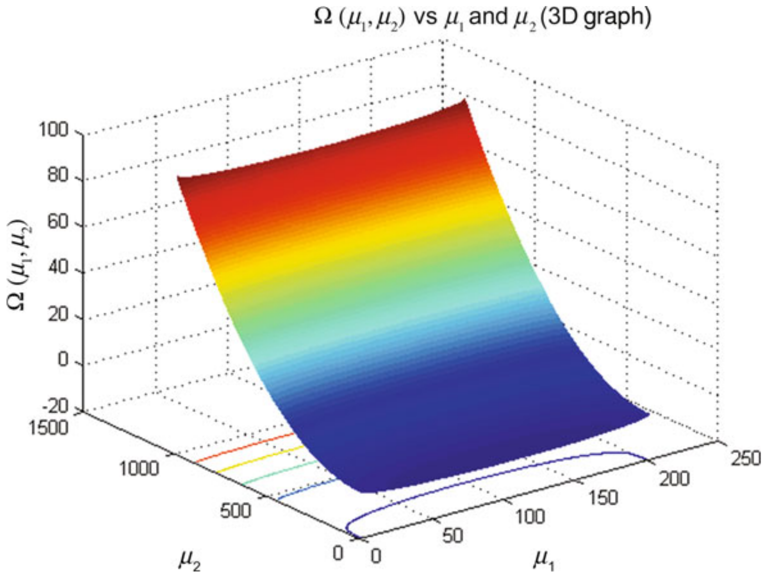


Fig. 7.1 $\Omega(\mu_1, \mu_2)$ vs μ_1 and μ_2 (3D graph)

$$X(t) = r(t) \cdot \cos[\theta(t)] \Rightarrow \frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t} \cdot \cos[\theta(t)] - r(t) \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \cdot \sin[\theta(t)]$$

$$Y(t) = r(t) \cdot \sin[\theta(t)] \Rightarrow \frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \frac{\mathrm{d}r(t)}{\mathrm{d}t} \cdot \sin[\theta(t)] + r(t) \cdot \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \cdot \cos[\theta(t)]$$

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t}; \quad \frac{\mathrm{d}r(t)}{\mathrm{d}t} = r'; \quad \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \theta'; \quad \theta(t) = \theta; \quad r(t) = r.$$

We get the equations: $\frac{\mathrm{d}X}{\mathrm{d}t} = r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta$; $\frac{\mathrm{d}Y}{\mathrm{d}t} = r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = Y \Rightarrow r' \cdot \cos\theta - r \cdot \theta' \cdot \sin\theta = r \cdot \sin\theta; \quad r' \cdot \cos\theta = r \cdot \sin\theta \cdot [\theta' + 1]$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} + \mu_1 \cdot Y + \sin(X) = \mu_2$$
$$\Rightarrow r' \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta + \mu_1 \cdot r \cdot \sin\theta + \sin(r \cdot \cos\theta) = \mu_2$$

$$r' \cdot \cos\theta = r \cdot \sin\theta \cdot [\theta' + 1] \Rightarrow r' = r \cdot tg\theta \cdot [\theta' + 1]$$

$$r \cdot tg\theta \cdot [\theta' + 1] \cdot \sin\theta + r \cdot \theta' \cdot \cos\theta + \mu_1 \cdot r \cdot \sin\theta + \sin(r \cdot \cos\theta) = \mu_2$$

$$\theta' = \frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]}$$

$$r' = r \cdot tg\theta \cdot [\theta' + 1]$$
$$= r \cdot tg\theta \cdot \left\{ \frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]} + 1 \right\}.$$

Finally, we get two differential equations in r and θ:

$$\theta' = \frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]}$$

$$r' = r \cdot tg\theta \cdot \left\{ \frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]} + 1 \right\}.$$

If there is a stable limit cycle as specific value of radius r, $\frac{\mathrm{d}r}{\mathrm{d}t} = 0;\ r \neq 0;\ r > 0.$

$$r' = 0 \Rightarrow r \cdot tg\theta \cdot \left\{ \frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]} + 1 \right\} = 0$$

Case A $r \neq 0;\ tg\theta = 0 \Rightarrow \theta = k \cdot \pi\, \forall\, k = 0, 1, 2, \ldots$

Case B $\frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]} + 1 = 0$

$$\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta = -r \cdot tg\theta \cdot \sin\theta - r \cdot \cos\theta$$

$$\mu_2 - \sin(r \cdot \cos\theta) = r \cdot [\mu_1 \cdot \sin\theta - \cos\theta]$$

Case $r \neq 0;\ tg\theta = 0 \Rightarrow \theta = k \cdot \pi\, \forall\, k = 0, 1, 2, \ldots$

$$\sin\theta = \pm \frac{tg\theta}{\sqrt{1 + tg^2\theta}}; \quad \cos\theta = \pm \frac{1}{\sqrt{1 + tg^2\theta}};$$
$$tg\theta = 0 \Rightarrow \sin\theta = 0; \quad \cos\theta = \pm 1$$

Equation: $\theta' = \frac{\mu_2 - \sin(r \cdot \cos\theta) - \mu_1 \cdot r \cdot \sin\theta - r \cdot tg\theta \cdot \sin\theta}{r \cdot [tg\theta \cdot \sin\theta + \cos\theta]}$

$$\theta' = \frac{\mu_2 - \sin(r \cdot [\pm 1])}{r \cdot (\pm 1)};\ \theta' = 0 \Rightarrow \mu_2 - \sin(r^* \cdot [\pm 1]) = 0;$$
$$\sin(r^* \cdot [\pm 1]) = \pm \sin(r^*)$$

At fixed point : $r' = 0;\ \theta' = 0 \Rightarrow \mu_2 - [\pm \sin(r^*)] = 0 \Rightarrow \quad \mu_2 \mp \sin(r^*) = 0$ $\Rightarrow \sin(r^*) = \pm\mu_2$

$$r^* \neq 0; \quad r^* > 0; \quad \mu_2 > 0 \Rightarrow r^* = \arcsin(\mu_2); \quad \mu_2 < 0 \Rightarrow r^* = \arcsin(\mu_2)$$

We find one possible fixed point: $r^* = \arcsin(\mu_2);\ tg\theta^* = 0 \Rightarrow \theta^* = k \cdot \pi \,\forall\, k = 0, 1, 2, \ldots$

System Cartesian coordinate fixed point: $Y^* = 0;\ \sin(X^*) = \mu_2 \Rightarrow X^* = \arcsin(\mu_2)$

$$r^* = \sqrt{(X^*)^2 + (Y^*)^2}; \quad X^* = \arcsin(\mu_2); \quad Y^* = 0 \Rightarrow r^* = X^* = \arcsin(\mu_2)$$

&&&

Consider the system vector field given in polar coordinates: $\frac{d\theta}{dt} = 1$ $\frac{dr}{dt} = r \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r)$ and n-dimensional system $\frac{dX}{dt} = \xi(X)$. Let "S" be an $(n - 1)$ dimensional surface of section and all trajectories starting on "S" flow through it. In our case "S" be the positive x-axis, and compute Poincare map. We define the initial condition on "S" as r_0. Since $\frac{d\theta}{dt} = 1;\ \theta = \omega \cdot t = 2 \cdot \pi \cdot f \cdot t;\ \frac{d\theta}{dt} = 1;\ \theta = \frac{2 \cdot \pi}{T} \cdot t;$ $\frac{d\theta}{dt} = \frac{2 \cdot \pi}{T} = 1 \Rightarrow T = 2 \cdot \pi$ then the first return to "S" occurs after a time of flight $T = 2 \cdot \pi$. Then we define the Poincare map ψ as the mapping from "S" (positive x-axis) to itself. It is obtained by trajectories from one intersection with "S" (positive x-axis) to the next. If $X_k \in S$ ($X_k \in$ positive x-axis) denotes the k-th intersection, then the Poincare map is defined by $X_{k+1} = \psi(X_k)$ supposes that r^* is a fixed point of ψ then $r^* = \psi(r^*)$ then a trajectory starting at r^* return to r^* after some time $T = 2 \cdot \pi$. It is a closed orbit for the original system $\frac{dX}{dt} = \xi(X)$. By inspecting the behavior of ψ near this fixed point r^*, we can determine the stability of the closed orbit. Our Poincare map converts, closed orbits analysis into fixed points of a mapping analysis.

$$\frac{dr}{dt} = r \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r); \quad \frac{dr}{r \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r)} = dt;$$
$$r_1 = \psi(r_0); \quad r_2 = \psi(r_1); \quad r_3 = \psi(r_2)$$

$r_{k+1} = \psi(r_k)$. For $r_1 = \psi(r_0) \Rightarrow \int_{r_0}^{r_1} \frac{dr}{r \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r)} = \int_0^{2 \cdot \pi} dt = 2 \cdot \pi.$

First we calculate the integral $\int_{r_0}^{r_1} \frac{dr}{r \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r)}$.

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{r \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r)} = \int_{r_0}^{r_1} \frac{\mathrm{d}r}{r \cdot (\mu_1 - r) \cdot (\mu_1 + r) \cdot (\mu_2 - r)}$$

$$= \int_{r_0}^{r_1} \left[\frac{A}{r} + \frac{B}{(\mu_1 - r)} + \frac{C}{(\mu_1 + r)} + \frac{D}{(\mu_2 - r)}\right] \cdot \mathrm{d}r$$

$$A = A(\mu_1, \mu_2); \quad B = B(\mu_1, \mu_2); \quad C = C(\mu_1, \mu_2); \quad D = D(\mu_1, \mu_2)$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\frac{\begin{array}{l}\{A\cdot(\mu_1-r)\cdot(\mu_1+r)\cdot(\mu_2-r)+B\cdot r\cdot(\mu_1+r)\cdot(\mu_2-r)\\ +C\cdot r\cdot(\mu_1-r)\cdot(\mu_2-r)+D\cdot r\cdot(\mu_1-r)\cdot(\mu_1+r)\}\end{array}}{r\cdot(\mu_1-r)\cdot(\mu_1+r)\cdot(\mu_2-r)}$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\frac{\begin{array}{l}\{A\cdot(\mu_1^2-r^2)\cdot(\mu_2-r)+B\cdot r\cdot(\mu_1+r)\cdot(\mu_2-r)\\ +C\cdot r\cdot(\mu_1-r)\cdot(\mu_2-r)+D\cdot r\cdot(\mu_1^2-r^2)\}\end{array}}{r\cdot(\mu_1-r)\cdot(\mu_1+r)\cdot(\mu_2-r)}$$

$$A \cdot (\mu_1^2 - r^2) \cdot (\mu_2 - r) = A \cdot \mu_1^2 \cdot \mu_2 - A \cdot \mu_1^2 \cdot r - A \cdot r^2 \cdot \mu_2 + A \cdot r^3$$

$$B \cdot r \cdot (\mu_1 + r) \cdot (\mu_2 - r) = B \cdot \mu_1\mu_2 r - B \cdot \mu_1 r^2 + B \cdot r^2\mu_2 - B \cdot r^3$$

$$C \cdot r \cdot (\mu_1 - r) \cdot (\mu_2 - r) = C \cdot r \cdot \mu_1 \cdot \mu_2 - C \cdot \mu_1 \cdot r^2 - C \cdot r^2 \cdot \mu_2 + C \cdot r^3$$

$$D \cdot r \cdot (\mu_1^2 - r^2) = D \cdot r \cdot \mu_1^2 - D \cdot r^3$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\frac{\begin{array}{l}\{A\cdot\mu_1^2\cdot\mu_2-A\cdot\mu_1^2\cdot r-A\cdot r^2\cdot\mu_2+A\cdot r^3\\ +B\cdot\mu_1\cdot\mu_2 r-B\cdot\mu_1\cdot r^2+B\cdot r^2\cdot\mu_2-B\cdot r^3\\ +C\cdot r\cdot\mu_1\cdot\mu_2-C\cdot\mu_1\cdot r^2-C\cdot r^2\cdot\mu_2+C\cdot r^3\\ +D\cdot r\cdot\mu_1^2-D\cdot r^3\}\end{array}}{r\cdot(\mu_1-r)\cdot(\mu_1+r)\cdot(\mu_2-r)}$$

$$A \cdot \mu_1^2 \cdot \mu_2 + r \cdot [B \cdot \mu_1 \cdot \mu_2 - A \cdot \mu_1^2 + C \cdot \mu_1 \cdot \mu_2 + D \cdot \mu_1^2]$$
$$+ r^2 \cdot [B \cdot (\mu_2 - \mu_1) - C \cdot (\mu_1 + \mu_2) - A \cdot \mu_2] \ + r^3 \cdot [A - B + C - D]$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\frac{\begin{array}{l}\{A\cdot\mu_1^2\cdot\mu_2+r\cdot[B\cdot\mu_1\cdot\mu_2-A\cdot\mu_1^2+C\cdot\mu_1\cdot\mu_2+D\cdot\mu_1^2]\\ +r^2\cdot[B\cdot(\mu_2-\mu_1)-C\cdot(\mu_1+\mu_2)-A\cdot\mu_2]\\ +r^3\cdot[A-B+C-D]\}\end{array}}{r\cdot(\mu_1-r)\cdot(\mu_1+r)\cdot(\mu_2-r)}$$

$$A\cdot\mu_1^2\cdot\mu_2=1\Rightarrow A=\frac{1}{\mu_1^2\cdot\mu_2};B\cdot\mu_1\cdot\mu_2-A\cdot\mu_1^2+C\cdot\mu_1\cdot\mu_2+D\cdot\mu_1^2=0$$

$$B \cdot (\mu_2 - \mu_1) - C \cdot (\mu_1 + \mu_2) - A \cdot \mu_2 = 0; \quad A - B + C - D = 0.$$

Then we get three equations with B, C, and D parameters.

$$B \cdot \mu_1 \cdot \mu_2 + C \cdot \mu_1 \cdot \mu_2 + D \cdot \mu_1^2 = \frac{1}{\mu_2};$$
$$B \cdot (\mu_2 - \mu_1) - C \cdot (\mu_1 + \mu_2) = \frac{1}{\mu_1^2}$$
$$\frac{1}{\mu_1^2 \cdot \mu_2} - B + C - D = 0 \Rightarrow D = \frac{1}{\mu_1^2 \cdot \mu_2} - B + C$$

$$B \cdot \mu_1 \cdot \mu_2 + C \cdot \mu_1 \cdot \mu_2 + \left[\frac{1}{\mu_1^2 \cdot \mu_2} - B + C\right] \cdot \mu_1^2 = \frac{1}{\mu_2}$$

$$B \cdot \mu_1 \cdot [\mu_2 - \mu_1] + C \cdot \mu_1 \cdot [\mu_2 + \mu_1] = 0 \Rightarrow B = C \cdot \frac{[\mu_1 + \mu_2]}{[\mu_1 - \mu_2]}$$

$$C \cdot \frac{[\mu_1 + \mu_2]}{[\mu_1 - \mu_2]} \cdot (\mu_2 - \mu_1) - C \cdot (\mu_1 + \mu_2) = \frac{1}{\mu_1^2};$$
$$- C \cdot \frac{[\mu_1 + \mu_2]}{[\mu_2 - \mu_1]} \cdot (\mu_2 - \mu_1) - C \cdot (\mu_1 + \mu_2) = \frac{1}{\mu_1^2}$$

$$-2 \cdot C \cdot (\mu_1 + \mu_2) = \frac{1}{\mu_1^2} \Rightarrow C = -\frac{1}{2 \cdot \mu_1^2 \cdot (\mu_1 + \mu_2)}; \quad B = -\frac{1}{2 \cdot \mu_1^2 \cdot [\mu_1 - \mu_2]}$$

$$D = \frac{1}{\mu_1^2 \cdot \mu_2} - B + C = \frac{1}{\mu_1^2 \cdot \mu_2} + \frac{1}{2 \cdot \mu_1^2 \cdot [\mu_1 - \mu_2]} - \frac{1}{2 \cdot \mu_1^2 \cdot (\mu_1 + \mu_2)}$$

$$D = \frac{2 \cdot [\mu_1 - \mu_2] \cdot [\mu_1 + \mu_2] + \mu_2 \cdot [\mu_1 + \mu_2] - \mu_2 \cdot [\mu_1 - \mu_2]}{2 \cdot \mu_1^2 \cdot \mu_2 \cdot [\mu_1 - \mu_2] \cdot [\mu_1 + \mu_2]}$$

$$D = \frac{1}{\mu_2 \cdot [\mu_1 - \mu_2] \cdot [\mu_1 + \mu_2]}; \quad D = \frac{1}{\mu_2 \cdot [\mu_1^2 - \mu_2^2]}.$$

We can summary our parameter functions:

$$A = \frac{1}{\mu_1^2 \cdot \mu_2}; \quad B = -\frac{1}{2 \cdot \mu_1^2 \cdot [\mu_1 - \mu_2]};$$
$$C = -\frac{1}{2 \cdot \mu_1^2 \cdot [\mu_1 + \mu_2]}; \quad D = \frac{1}{\mu_2 \cdot [\mu_1^2 - \mu_2^2]}$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\frac{1}{\mu_1^2\cdot\mu_2}\cdot\frac{1}{r}-\frac{1}{2\cdot\mu_1^2\cdot[\mu_1-\mu_2]}\cdot\frac{1}{(\mu_1-r)}$$
$$-\frac{1}{2\cdot\mu_1^2\cdot[\mu_1+\mu_2]}\cdot\frac{1}{(\mu_1+r)}+\frac{1}{\mu_2\cdot[\mu_1^2-\mu_2^2]}\cdot\frac{1}{(\mu_2-r)}$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\frac{1}{\mu_1^2\cdot\mu_2}\cdot\frac{1}{r}+\frac{1}{2\cdot\mu_1^2\cdot[\mu_1-\mu_2]}\cdot\frac{1}{(r-\mu_1)}$$
$$-\frac{1}{2\cdot\mu_1^2\cdot[\mu_1+\mu_2]}\cdot\frac{1}{(\mu_1+r)}-\frac{1}{\mu_2\cdot[\mu_1^2-\mu_2^2]}\cdot\frac{1}{(r-\mu_2)}.$$

We define for simplicity four functions: $\phi_k(\mu_1,\mu_2);\ k=1,2,3,4.$

$$\phi_1(\mu_1,\mu_2)=\frac{1}{\mu_1^2\cdot\mu_2};\quad \phi_2(\mu_1,\mu_2)=\frac{1}{2\cdot\mu_1^2\cdot[\mu_1-\mu_2]};$$
$$\phi_3(\mu_1,\mu_2)=\frac{1}{2\cdot\mu_1^2\cdot[\mu_1+\mu_2]};\quad \phi_4(\mu_1,\mu_2)=\frac{1}{\mu_2\cdot[\mu_1^2-\mu_2^2]}$$

$$\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}=\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}$$
$$-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}$$

$$\int_{r_0}^{r_1}\left[\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}\right]\cdot \mathrm{d}r=\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}\right.$$
$$\left.-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\cdot \mathrm{d}r\right]$$

$$\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\right]\cdot \mathrm{d}r$$
$$=\phi_1(\mu_1,\mu_2)\cdot\int_{r_0}^{r_1}\frac{1}{r}\cdot \mathrm{d}r+\phi_2(\mu_1,\mu_2)\cdot\int_{r_0}^{r_1}\frac{1}{(r-\mu_1)}\cdot \mathrm{d}r$$
$$-\phi_3(\mu_1,\mu_2)\cdot\int_{r_0}^{r_1}\frac{1}{(\mu_1+r)}\cdot \mathrm{d}r-\phi_4(\mu_1,\mu_2)\cdot\int_{r_0}^{r_1}\frac{1}{(r-\mu_2)}\cdot \mathrm{d}r$$

$$\int_{r_0}^{r_1} [\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}]\cdot \mathrm{d}r$$

$$=\phi_1(\mu_1,\mu_2)\cdot[\ln(r_1)-\ln(r_0)]+\phi_2(\mu_1,\mu_2)\cdot[\ln(r_1-\mu_1)-\ln(r_0-\mu_1)]$$
$$-\phi_3(\mu_1,\mu_2)\cdot[\ln(\mu_1+r_1)-\ln(\mu_1+r_0)]-\phi_4(\mu_1,\mu_2)\cdot[\ln(r_1-\mu_2)-\ln(r_0-\mu_2)]$$

$$\phi_1(\mu_1,\mu_2)\cdot[\ln(r_1)-\ln(r_0)]=\phi_1(\mu_1,\mu_2)\cdot\ln\left[\frac{r_1}{r_0}\right]$$
$$\Rightarrow \phi_1(\mu_1,\mu_2)\cdot[\ln(r_1)-\ln(r_0)]=\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}}\right]$$

$$\phi_2(\mu_1,\mu_2)\cdot[\ln(r_1-\mu_1)-\ln(r_0-\mu_1)]=\phi_2(\mu_1,\mu_2)\cdot\ \ln\left[\frac{(r_1-\mu_1)}{(r_0-\mu_1)}\right]$$

$$\phi_2(\mu_1,\mu_2)\cdot[\ln(r_1-\mu_1)-\ln(r_0-\mu_1)]=\ln\left[\frac{(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\right]$$

$$\phi_3(\mu_1,\mu_2)\cdot[\ln(\mu_1+r_1)-\ln(\mu_1+r_0)]=\phi_3(\mu_1,\mu_2)\cdot\ \ln\left[\frac{(\mu_1+r_1)}{(\mu_1+r_0)}\right]$$

$$\phi_3(\mu_1,\mu_2)\cdot[\ln(\mu_1+r_1)-\ln(\mu_1+r_0)]=\ln\left[\frac{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}}{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}}\right]$$

$$\phi_4(\mu_1,\mu_2)\cdot[\ln(r_1-\mu_2)-\ln(r_0-\mu_2)]=\phi_4(\mu_1,\mu_2)\cdot\ln\left[\frac{(r_1-\mu_2)}{(r_0-\mu_2)}\right]$$

$$\phi_4(\mu_1,\mu_2)\cdot[\ln(r_1-\mu_2)-\ln(r_0-\mu_2)]=\ln\left[\frac{(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\right]$$

$$\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\right]\cdot \mathrm{d}r$$
$$=\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}}\right]+\ln\left[\frac{(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\right]-\ln\left[\frac{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}}{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}}\right]-\ln\left[\frac{(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\right]$$

$$\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\right]\cdot \mathrm{d}r$$
$$=\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}}\cdot\frac{(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\right]-\ln\left[\frac{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}}{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}}\cdot\frac{(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\right]$$

$$\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\right]\cdot \mathrm{d}r$$
$$=\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}}\cdot\frac{(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\cdot\frac{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}\cdot(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}\cdot(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\right]$$

$$\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\right]\cdot dr$$
$$=\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}\cdot(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}\cdot(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\cdot\frac{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}\cdot(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}\cdot(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\right]$$

$$\int_{r_0}^{r_1}\frac{\mathrm{d}r}{r\cdot(\mu_1^2-r^2)\cdot(\mu_2-r)}=\int_{r_0}^{r_1}\left[\frac{A}{r}+\frac{B}{(\mu_1-r)}+\frac{C}{(\mu_1+r)}+\frac{D}{(\mu_2-r)}\right]\cdot\mathrm{d}r$$
$$=\int_{r_0}^{r_1}\left[\phi_1(\mu_1,\mu_2)\cdot\frac{1}{r}+\phi_2(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_1)}-\phi_3(\mu_1,\mu_2)\cdot\frac{1}{(\mu_1+r)}-\phi_4(\mu_1,\mu_2)\cdot\frac{1}{(r-\mu_2)}\right]\cdot \mathrm{d}r$$
$$=\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}\cdot(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}\cdot(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\cdot\frac{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}\cdot(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}\cdot(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\right]$$

$$\ln\left[\frac{r_1^{\phi_1(\mu_1,\mu_2)}\cdot(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}\cdot(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\cdot\frac{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}\cdot(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}\cdot(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}\right]$$
$$=2\cdot\pi$$

$$\exp(2\cdot\pi)=\frac{r_1^{\phi_1(\mu_1,\mu_2)}\cdot(r_1-\mu_1)^{\phi_2(\mu_1,\mu_2)}}{(\mu_1+r_1)^{\phi_3(\mu_1,\mu_2)}\cdot(r_1-\mu_2)^{\phi_4(\mu_1,\mu_2)}}\cdot\frac{(\mu_1+r_0)^{\phi_3(\mu_1,\mu_2)}\cdot(r_0-\mu_2)^{\phi_4(\mu_1,\mu_2)}}{r_0^{\phi_1(\mu_1,\mu_2)}\cdot(r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)}}$$

$$\phi_1=\phi_1(\mu_1,\mu_2);\quad \phi_2=\phi_2(\mu_1,\mu_2);\quad \phi_3=\phi_3(\mu_1,\mu_2);\quad \phi_4=\phi_4(\mu_1,\mu_2)$$

$$\exp(2\cdot\pi) = \frac{r_1^{\phi_1}\cdot(r_1-\mu_1)^{\phi_2}}{(\mu_1+r_1)^{\phi_3}\cdot(r_1-\mu_2)^{\phi_4}}\cdot\frac{(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{r_0^{\phi_1}\cdot(r_0-\mu_1)^{\phi_2}}.$$

For getting the Poincare map ψ; specific values of μ_1, μ_2 parameters:

$$\phi_1(\mu_1,\mu_2) = 1 \Rightarrow \frac{1}{\mu_1^2\cdot\mu_2} = 1; \quad \phi_2(\mu_1,\mu_2) = 1 \Rightarrow \frac{1}{2\cdot\mu_1^2\cdot[\mu_1-\mu_2]} = 1$$

$$\frac{1}{\mu_1^2\cdot\mu_2} = 1 \Rightarrow \mu_2 = \frac{1}{\mu_1^2}; \quad \frac{1}{2\cdot\mu_1^2\cdot[\mu_1-\mu_2]} = 1 \Rightarrow \frac{1}{2\cdot\mu_1^2\cdot[\mu_1-\frac{1}{\mu_1^2}]} = 1$$

$$\frac{1}{2\cdot\mu_1^2\cdot\left[\mu_1-\frac{1}{\mu_1^2}\right]} = 1 \Rightarrow \frac{1}{2\cdot[\mu_1^3-1]} = 1 \Rightarrow \mu_1^3 = \frac{3}{2} \Rightarrow \mu_1 = \sqrt[3]{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{1}{3}}$$

$$\mu_2 = \frac{1}{\mu_1^2} = \frac{1}{\left(\frac{3}{2}\right)^{\frac{2}{3}}} = \frac{1}{\left[\sqrt[3]{\frac{3}{2}}\right]^2} = \left(\frac{3}{2}\right)^{-\frac{2}{3}}; \phi_3(\mu_1,\mu_2) = \frac{1}{2\cdot\mu_1^2\cdot[\mu_1+\mu_2]} = \frac{1}{2\cdot\left(\frac{3}{2}\right)^{\frac{2}{3}}\cdot\left[\left(\frac{3}{2}\right)^{\frac{1}{3}}+\left(\frac{3}{2}\right)^{-\frac{2}{3}}\right]}$$

$$\phi_3(\mu_1,\mu_2) = \frac{1}{2\cdot\mu_1^2\cdot[\mu_1+\mu_2]} = \frac{1}{5}; \phi_4(\mu_1,\mu_2) = \frac{1}{\mu_2\cdot[\mu_1^2-\mu_2^2]} = \frac{1}{\left[\left(\frac{3}{2}\right)^{-\frac{2}{3}}\cdot\left(\frac{3}{2}\right)^{\frac{2}{3}}-\left(\frac{3}{2}\right)^{-\frac{4}{3}}\right]}$$

$$\phi_4(\mu_1,\mu_2) = \frac{1}{\mu_2\cdot[\mu_1^2-\mu_2^2]} = \frac{1}{\left[1-\left(\frac{3}{2}\right)^{-2}\right]} = \frac{1}{\left[1-\left(\frac{2}{3}\right)^2\right]} = \frac{1}{\left[1-\frac{4}{9}\right]} = \frac{9}{5}.$$

We can summarize our last results in Table 7.1.

$$r_1^{\phi_1=1} = r_1; \ (r_1-\mu_1)^{\phi_2=1} = r_1-\mu_1; \ r_0^{\phi_1(\mu_1,\mu_2)=1} = r_0; \ (r_0-\mu_1)^{\phi_2(\mu_1,\mu_2)=1} = r_0-\mu_1$$

$$(1+\beta\cdot r)^\alpha = 1+\frac{\alpha\cdot\beta\cdot r}{1!}+\frac{\alpha\cdot(\alpha-1)\cdot r^2\cdot\beta^2}{2!}+\frac{\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot r^3\cdot\beta^3}{3!}+\cdots$$

Table 7.1 Summary of our last results

k	$\phi_k(\mu_1,\mu_2)$	μ_1	μ_2	Remark
1	1	$\sqrt[3]{\frac{3}{2}}$	$\frac{1}{\left[\sqrt[3]{\frac{3}{2}}\right]^2}$	$\phi_1(\mu_1,\mu_2) = 1$
2	1	$\sqrt[3]{\frac{3}{2}}$	$\frac{1}{\left[\sqrt[3]{\frac{3}{2}}\right]^2}$	$\phi_2(\mu_1,\mu_2) = 1$
3	1/5	$\sqrt[3]{\frac{3}{2}}$	$\frac{1}{\left[\sqrt[3]{\frac{3}{2}}\right]^2}$	$\phi_3(\mu_1,\mu_2) < 1$
4	9/5	$\sqrt[3]{\frac{3}{2}}$	$\frac{1}{\left[\sqrt[3]{\frac{3}{2}}\right]^2}$	$\phi_4(\mu_1,\mu_2) > 1$

Binomial series: $(1+\beta\cdot r)^{\alpha}=\sum_{n=0}^{\infty}\frac{\alpha\cdot(\alpha-1)\cdots(\alpha-n+1)\cdot\beta^{n}\cdot r^{n}}{n!}$.

This series is valid for $|r|<1$, and sometimes also valid at one or both and points $r=\pm 1$ (in our case $r=1$), the function is a polynomial and the series terminates (finally many terms are zero).

$$(\mu_1+r_1)^{\phi_3=\frac{1}{5}}=\left(\mu_1\cdot\left[1+\frac{r_1}{\mu_1}\right]\right)^{\frac{1}{5}}=\mu_1^{\frac{1}{5}}\cdot\left[1+\frac{r_1}{\mu_1}\right]^{\frac{1}{5}}=\sqrt[5]{\mu_1}\cdot\sqrt[5]{1+\frac{r_1}{\mu_1}}$$

$$\sqrt[5]{1+\frac{r_1}{\mu_1}}=\left(1+\frac{r_1}{\mu_1}\right)^{\frac{1}{5}}=1+\frac{1}{5\cdot\mu_1}\cdot r+\frac{\frac{1}{5}\cdot\left(-\frac{4}{5}\right)\cdot r^2\cdot\frac{1}{\mu_1^2}}{2!}+\frac{\frac{1}{5}\cdot\left(-\frac{4}{5}\right)\cdot\left(-\frac{9}{5}\right)\cdot r^3\cdot\frac{1}{\mu_1^3}}{3!}+\cdots$$

$$\sqrt[5]{1+\frac{r_1}{\mu_1}}=\left(1+\frac{r_1}{\mu_1}\right)^{\frac{1}{5}}\approx 1+\frac{1}{5\cdot\mu_1}\cdot r_1;(\mu_1+r_1)^{\phi_3=\frac{1}{5}}=\sqrt[5]{\mu_1}\cdot\sqrt[5]{1+\frac{r_1}{\mu_1}}\approx\sqrt[5]{\mu_1}\cdot\left(1+\frac{1}{5\cdot\mu_1}\cdot r_1\right)$$

$$(r_1-\mu_2)^{\phi_4=\frac{9}{5}}=[\sqrt[5]{r_1-\mu_2}]^9=\left(-\mu_2\cdot\left[-\frac{r_1}{\mu_2}+1\right]\right)^{\frac{9}{5}}=(-\mu_2)^{\frac{9}{5}}\cdot\left(-\frac{r_1}{\mu_2}+1\right)^{\frac{9}{5}}$$

$$(1+\beta\cdot r)^{\alpha}=1+\frac{\alpha\cdot\beta\cdot r}{1!}+\frac{\alpha\cdot(\alpha-1)\cdot r^2\cdot\beta^2}{2!}+\frac{\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot r^3\cdot\beta^3}{3!}+\cdots$$

$$\left(-\frac{r_1}{\mu_2}+1\right)^{\frac{9}{5}};\ \beta=-\frac{1}{\mu_2};\alpha=\frac{9}{5}\left(-\frac{r_1}{\mu_2}+1\right)^{\frac{9}{5}}$$
$$=1+\frac{\frac{9}{5}\cdot\left(-\frac{1}{\mu_2}\right)\cdot r_1}{1!}+\frac{\frac{9}{5}\cdot\left(\frac{9}{5}-1\right)\cdot r_1^2\cdot\left(-\frac{1}{\mu_2}\right)^2}{2!}+\frac{\frac{9}{5}\cdot\left(\frac{9}{5}-1\right)\cdot\left(\frac{9}{5}-2\right)\cdot r_1^3\cdot\left(-\frac{1}{\mu_2}\right)^3}{3!}+\cdots$$

$$\left(-\frac{r_1}{\mu_2}+1\right)^{\frac{9}{5}}=1-\frac{9}{5}\cdot\frac{1}{\mu_2}\cdot r_1+\frac{1}{2}\cdot\frac{9}{5}\cdot\frac{4}{5}\cdot r_1^2\cdot\left(-\frac{1}{\mu_2}\right)^2-\frac{1}{3}\cdot\frac{9}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}\cdot r_1^3\cdot\left(-\frac{1}{\mu_2}\right)^3+\cdots$$

$$(r_1-\mu_2)^{\phi_4=\frac{9}{5}}=(-\mu_2)^{\frac{9}{5}}\cdot\left(-\frac{r_1}{\mu_2}+1\right)^{\frac{9}{5}}\approx(-\mu_2)^{\frac{9}{5}}\cdot\left(1-\frac{9}{5}\cdot\frac{1}{\mu_2}\cdot r_1\right)$$

$$\exp(2\cdot\pi)=\frac{r_1^{\phi_1}\cdot(r_1-\mu_1)^{\phi_2}}{(\mu_1+r_1)^{\phi_3}\cdot(r_1-\mu_2)^{\phi_4}}\cdot\frac{(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{r_0^{\phi_1}\cdot(r_0-\mu_1)^{\phi_2}}$$

$$\exp(2\cdot\pi)\approx\frac{r_1\cdot(r_1-\mu_1)}{\sqrt[5]{\mu_1}\cdot\left(1+\frac{1}{5\cdot\mu_1}\cdot r_1\right)\cdot(-\mu_2)^{\frac{9}{5}}\cdot\left(1-\frac{9}{5}\cdot\frac{1}{\mu_2}\cdot r_1\right)}\cdot\frac{(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{r_0^{\phi_1}\cdot(r_0-\mu_1)^{\phi_2}}$$

$$\exp(2\cdot\pi)\cdot\sqrt[5]{\mu_1}\cdot(-\mu_2)^{\frac{9}{5}}\cdot\left(1+\frac{1}{5\cdot\mu_1}\cdot r_1\right)\cdot\left(1-\frac{9}{5}\cdot\frac{1}{\mu_2}\cdot r_1\right) \approx \frac{r_1\cdot(r_1-\mu_1)\cdot(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{r_0^{\phi_1}\cdot(r_0-\mu_1)^{\phi_2}}$$

$$\exp(2\cdot\pi)\cdot\sqrt[5]{\mu_1}\cdot(-\mu_2)^{\frac{9}{5}}\cdot\left(1+\frac{1}{5\cdot\mu_1}\cdot r_1\right)\cdot\left(1-\frac{9}{5}\cdot\frac{1}{\mu_2}\cdot r\right)_1 \approx \frac{r_1\cdot(r_1-\mu_1)\cdot(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{r_0^{\phi_1}\cdot(r_0-\mu_1)^{\phi_2}}$$

$$\left(1+\frac{1}{5\cdot\mu_1}\cdot r_1\right)\cdot\left(1-\frac{9}{5}\cdot\frac{1}{\mu_2}\cdot r_1\right) = 1+\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\cdot r_1-\frac{9}{25\cdot\mu_1\cdot\mu_2}\cdot r_1^2;$$
$$r_1\cdot(r_1-\mu_1) = r_1^2 - r_1\cdot\mu_1$$

$$1+\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\cdot r_1-\frac{9}{25\cdot\mu_1\cdot\mu_2}\cdot r_1^2 \approx \frac{(r_1^2-r_1\cdot\mu_1)\cdot(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{\exp(2\cdot\pi)\cdot\sqrt[5]{\mu_1}\cdot(-\mu_2)^{\frac{9}{5}}\cdot r_0\cdot(r_0-\mu_1)}.$$

We define function: $\chi(r_0,\mu_1,\mu_2) = \frac{(\mu_1+r_0)^{\phi_3}\cdot(r_0-\mu_2)^{\phi_4}}{\exp(2\cdot\pi)\cdot\sqrt[5]{\mu_1}\cdot(-\mu_2)^{\frac{9}{5}}\cdot r_0\cdot(r_0-\mu_1)}$

$$1+\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\cdot r_1-\frac{9}{25\cdot\mu_1\cdot\mu_2}\cdot r_1^2 \approx (r_1^2-r_1\cdot\mu_1)\cdot\chi(r_0,\mu_1,\mu_2)$$

$$r_1^2\cdot\chi(r_0,\mu_1,\mu_2)+\frac{9}{25\cdot\mu_1\cdot\mu_2}\cdot r_1^2-r_1\cdot\mu_1\cdot\chi(r_0,\mu_1,\mu_2)-\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\cdot r_1-1 \approx 0$$

$$r_1^2\cdot\left[\chi(r_0,\mu_1,\mu_2)+\frac{9}{25\cdot\mu_1\cdot\mu_2}\right]-r_1\cdot\left[\mu_1\cdot\chi(r_0,\mu_1,\mu_2)+\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\right]-1 \approx 0$$

$$r_1 = \frac{\left[\mu_1\cdot\chi(r_0,\mu_1,\mu_2)+\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\right]\pm\sqrt{\begin{array}{l}\left[\mu_1\cdot\chi(r_0,\mu_1,\mu_2)+\left[\frac{1}{\mu_1}-\frac{9}{\mu_2}\right]\cdot\frac{1}{5}\right]^2\\ +4\cdot\left[\chi(r_0,\mu_1,\mu_2)+\frac{9}{25\cdot\mu_1\cdot\mu_2}\right]\end{array}}}{2\cdot\left[\chi(r_0,\mu_1,\mu_2)+\frac{9}{25\cdot\mu_1\cdot\mu_2}\right]}.$$

Hence Poincare map is ψ:

$$\psi(r) = \frac{\left\{\mu_1 \cdot \chi(r,\mu_1,\mu_2) + \left[\frac{1}{\mu_1} - \frac{9}{\mu_2}\right] \cdot \frac{1}{5}\right\} \pm \sqrt{\begin{array}{l}\left[\mu_1 \cdot \chi(r,\mu_1,\mu_2) + \left[\frac{1}{\mu_1} - \frac{9}{\mu_2}\right] \cdot \frac{1}{5}\right]^2 \\ +4 \cdot \left[\chi(r,\mu_1,\mu_2) + \frac{9}{25\cdot\mu_1\cdot\mu_2}\right]\end{array}}}{2 \cdot \left[\chi(r,\mu_1,\mu_2) + \frac{9}{25\cdot\mu_1\cdot\mu_2}\right]}$$

Results There are two possible Poincare maps.

$$\psi^{(1)}(r) = \frac{\left\{\mu_1 \cdot \chi(r,\mu_1,\mu_2) + [\frac{1}{\mu_1} - \frac{9}{\mu_2}] \cdot \frac{1}{5}\right\} + \sqrt{\begin{array}{l}\left[\mu_1 \cdot \chi(r,\mu_1,\mu_2) + \left[\frac{1}{\mu_1} - \frac{9}{\mu_2}\right] \cdot \frac{1}{5}\right]^2 \\ +4 \cdot \left[\chi(r,\mu_1,\mu_2) + \frac{9}{25\cdot\mu_1\cdot\mu_2}\right]\end{array}}}{2 \cdot \left[\chi(r,\mu_1,\mu_2) + \frac{9}{25\cdot\mu_1\cdot\mu_2}\right]}$$

$$\psi^{(2)}(r) = \frac{\left\{\mu_1 \cdot \chi(r,\mu_1,\mu_2) + \left[\frac{1}{\mu_1} - \frac{9}{\mu_2}\right] \cdot \frac{1}{5}\right\} - \sqrt{\begin{array}{l}\left[\mu_1 \cdot \chi(r,\mu_1,\mu_2) + \left[\frac{1}{\mu_1} - \frac{9}{\mu_2}\right] \cdot \frac{1}{5}\right]^2 \\ +4 \cdot \left[\chi(r,\mu_1,\mu_2) + \frac{9}{25\cdot\mu_1\cdot\mu_2}\right]\end{array}}}{2 \cdot \left[\chi(r,\mu_1,\mu_2) + \frac{9}{25\cdot\mu_1\cdot\mu_2}\right]}.$$

Remark It is reader exercise to plot Poincare maps and investigate occurrences of fixed point . Suppose that r^* is a fixed point of $\psi^{(1)}$ or $\psi^{(2)}$

Then $\psi^{(k)}(r^*) = r^* \; \forall \; k = 1, 2$. The trajectory starting at r^* returns to r^* after some time T [9].

7.2 Optoisolation van der pol Circuit Poincare Map and Periodic Orbit

We have a van der Pol oscillator circuit with parallel capacitor C_2. The active element of the circuit is semiconductor device (OptoNDR circuit device) with parallel capacitor C_2. Its acts like an ordinary nonlinear resistor when current $I(t)$ is high ($I(t) > I_{sat}$), but like nonlinear negative resistor (energy source) or negative differential resistance (NDR) when $I(t)$ is low $I(t) > I_{break}$ and $I(t) < I_{sat}$. Our circuit current voltage characteristic $V = f(I) \; \forall \; \frac{\mathrm{d}I}{\mathrm{d}t} = 0$ resembles a cubic function. We consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below I_{break} ($I(t) < I_{break}$).

$V_C > V_B$; $V = V_{CB} = V_C - V_B = -V_{BC}$, denote the voltage drop from point C to point B in the circuit. $V = V_{C_1}$; $I_{C_1} = C_1 \cdot \frac{dV}{dt}$; $I = -I_{C_1}$; $\frac{dV}{dt} = \frac{I_{C_1}}{C_1} = -\frac{I}{C_1}$; $V_{L_1} = V_{AB} = L_1 \cdot \frac{dI}{dt}$.

KVL (Kircoff's Voltage Law): $V_{CD} + V_{AB} - V = 0$; $V_{CD} = V_{D_1} + V_{CEQ_1} = f(I)$; $V_{AB} = V_{L_1}$

$$V_{CEQ_1} = V_{C_2};\quad V_{CD} = V_{D_1} + V_{C_2} = f(I);\ I_{C_2} = C_2 \cdot \frac{dV_{C_2}}{dt};\quad I_{C_2} = C_2 \cdot \frac{dV_{CEQ_1}}{dt}$$

$$f(I) + V_{L_1} - V = 0;\ V_{CD} = V_C - V_D = f(I); f(I) + L_1 \cdot \frac{dI}{dt} - V = 0;\ V_A = V_D.$$

Our OptoNDR element/circuit is constructed from LED and phototransistor in series (additional parallel capacitor C_2). The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant ($I_{BQ_1} = I_{LED} \cdot k = I_{D_1} \cdot k$; $I_{BQ_1} = (I_{CQ_1} + I_{C_2}) \cdot k$) and is the phototransistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations [7, 85, 86] (Fig. 7.2).

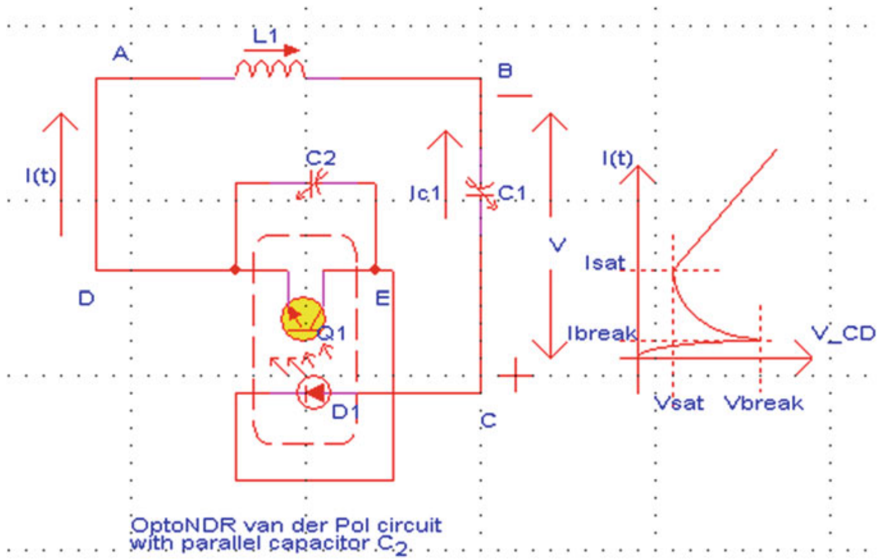


Fig. 7.2 OptoNDR van der Pol circuit with parallel capacitor C_2

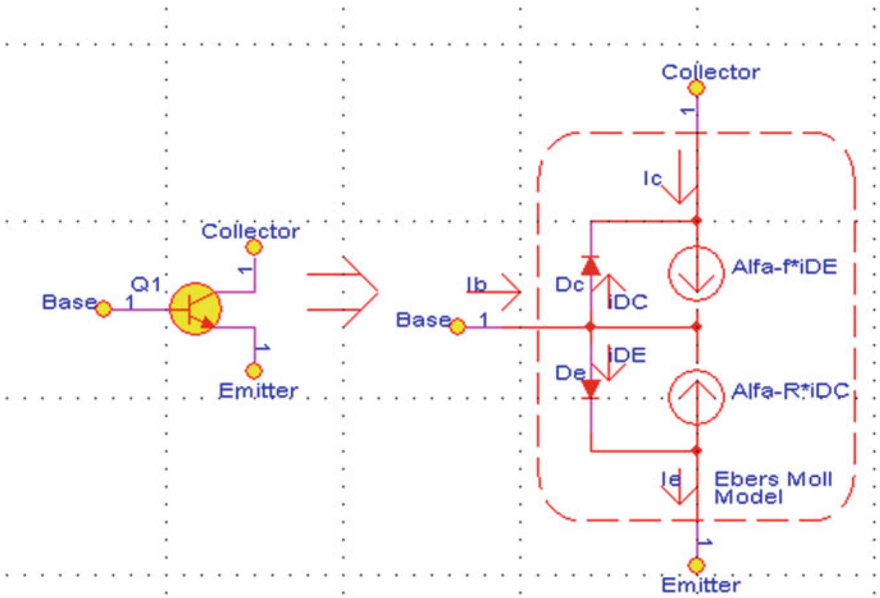


Fig. 7.3 Ebers-Moll schematics for NPN bipolar transistor

The basic Ebers–Moll schematic for NPN bipolar transistor is shown in the next figure. We need to implement the regular Ebers–Moll model to the opto coupler circuit (transistor Q_1 and LED D_1) and get a complete final expression for the Negative Differential Resistance (NDR) characteristics of that circuit [18] (Fig. 7.3).

$$i_{DE} + i_{DC} = i_{bQ_1} + \alpha_f \cdot i_{DEQ_1} + \alpha_r \cdot i_{DCQ_1};$$
$$i_{DCQ_1} + I_{CQ_1} = \alpha_f \cdot i_{DEQ_1};\; i_{DEQ_1} = \alpha_r \cdot i_{DCQ_1} + i_{EQ_1}$$

$$i_{DCQ_1} + I_{CQ_1} = \alpha_f \cdot i_{DEQ_1} \Rightarrow i_{DCQ_1} = \alpha_f \cdot i_{DEQ_1} - I_{CQ_1};$$
$$i_{DEQ_1} = \alpha_r \cdot (\alpha_f \cdot i_{DEQ_1} - I_{CQ_1}) + i_{EQ_1}$$

$$i_{DEQ_1} = \alpha_r \cdot \alpha_f \cdot i_{DEQ_1} - \alpha_r \cdot I_{CQ_1} + i_{EQ_1}; \quad i_{DEQ_1} - \alpha_r \cdot \alpha_f \cdot i_{DEQ_1} = i_{EQ_1} - \alpha_r \cdot I_{CQ_1}$$
$$i_{DEQ_1} = \frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}; \quad i_{DCQ_1} = \alpha_f \cdot i_{DEQ_1} - I_{CQ_1} = \alpha_f \cdot (\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}) - I_{CQ_1}$$
$$i_{DCQ_1} = \alpha_f \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) - I_{CQ_1} = \frac{\alpha_f \cdot (i_{EQ_1} - \alpha_r \cdot I_{CQ_1}) - I_{CQ_1} \cdot (1 - \alpha_r \cdot \alpha_f)}{1 - \alpha_r \cdot \alpha_f}$$
$$i_{DCQ_1} = \frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}; \quad V_{\text{Base}-\text{Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_1} + 1\right];$$
$$V_{\text{Base}-\text{Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot (\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}) + 1\right]$$

$$V_{\text{Base-Collector}} = V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot i_{DCQ_1} + 1\right]; \; V_{\text{Base-Collector}} = V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot \left(\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Collector-Emitter}} = V_{\text{Collector-Base}} + V_{\text{Base-Emitter}}; \; V_{\text{Collector-Base}} = -V_{\text{Base-Collector}}$$

$$V_{\text{Collector-Emitter}} = V_{\text{Collector-Base}} + V_{\text{Base-Emitter}} = V_{\text{Base-Emitter}} - V_{\text{Base-Collector}}$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right] - V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot \left(\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \ln\left[\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\left\{\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\} \cdot \left\{\frac{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\left\{\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\} \cdot \left\{\frac{I_{sc}}{I_{se}}\right\}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right];$$

$$I_{CQ_1} + I_{C_2} = I_{D_1}; \quad I_{EQ_1} + I_{C_2} = I(t) = I$$

$$V_{D_1} = V_{CE} = V_t \cdot \ln\left[\frac{I_{D_1}}{I_0} + 1\right]; \quad V_{D_1} = V_{CE} = V_t \cdot \ln\left[\frac{I_{CQ_1} + I_{C_2}}{I_0} + 1\right];$$

$$i_{EQ_1} \to I_{EQ_1}; \quad i_{CQ_1} \to I_{CQ_1}.$$

The optical coupling between the LED (D_1) to the phototransistor (Q_1) is represented as transistor dependent base current on LED (D_1) current.

$$I_{BQ_1} = I_{D_1} \cdot k; \; I_{D_1} = I_{CQ_1} + I_{C_2}; \; I_{BQ_1} = I_{D_1} \cdot k = (I_{CQ_1} + I_{C_2}) \cdot k; \; I_{EQ_1} + I_{C_2} = I$$

$$I_{EQ_1} = I_{CQ_1} + I_{BQ_1} = I_{CQ_1} + (I_{CQ_1} + I_{C_2}) \cdot k; \; I_{EQ_1} = I_{CQ_1} \cdot (1 + k) + I_{C_2} \cdot k$$

$$I_{EQ_1} + I_{C_2} = I \Rightarrow I_{CQ_1} \cdot (1 + k) + I_{C_2} \cdot k + I_{C_2} = I; \; I_{CQ_1} \cdot (1 + k) + I_{C_2} \cdot (k + 1) = I$$

$$I_{CQ_1} = \frac{I}{(1 + k)} - I_{C_2}; \; I_{BQ_1} = (I_{CQ_1} + I_{C_2}) \cdot k = \left(\frac{I}{(1 + k)}\right) \cdot k; \; I_{BQ_1} = \left(\frac{I}{(1 + k)}\right) \cdot k.$$

As long as the phototransistor (Q_1) is in cutoff region, the current I_{CQ_1}, I_{EQ_1} and I_{BQ_1} are very low and the impedance between Q_1's collector–emitter is very high. Capacitor C_2 is charged. When the phototransistor (Q_1) reaches breakover voltage it enters saturation region ($V_{\text{Collector-Emitter}}$ decreases and I_{CQ_1} increases). The impedance between Q_1's collector–emitter decrease and capacitor C_2 is started to discharge through Q_1's collector–emitter junctions. The region which $V_{\text{Collector-Emitter}}$ decreases and I_{CQ_1} increases is the Negative Differential Resistance area of $V_{\text{CD}}\_I_{\text{CQ1}}$ characteristics. The positive feedback in which the phototransistor collector current I_{CQ_1} increases and then I_{BQ_1} increases ($I_{BQ_1} = (I_{CQ_1} + I_{C_2}) \cdot k$) is repeated in increasing cycles [1]. The positive feedback ends when the phototransistor reaches saturation state and capacitor C_2 is completely discharge (disconnected). Finally, we arrive at an expression which is the voltage $V_{\text{Collector-Emitter}}$ as a function of the current (I_{CQ_1}) for NDR circuit ($V_{CD} = V_{\text{Collector-Emitter}} + V_{D_1}; \quad V_{CD} = V_{C_2} + V_{D_1}$).

$$V_{CD} = V_{\text{Collector-Emitter}} + V_{D_1} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1} + I_{C_2}}{I_0} + 1\right].$$

Assume: $I_{sc} \approx I_{se}; V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] \approx 0$

$$V_{CD} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1} + I_{C_2}}{I_0} + 1\right];$$

$$I_{C_2} = C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t}$$

$$I_{CQ_1} = \frac{I}{(1+k)} - I_{C_2} = \frac{I}{(1+k)} - C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t};$$

$$I_{BQ_1} = \left(\frac{I}{(1+k)}\right) \cdot k; \quad I_{EQ_1} = I_{BQ_1} + I_{CQ_1} = I - C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t}$$

$$V_{CD} = f(I); \quad V_{CD} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1} + I_{C_2}}{I_0} + 1\right]$$

$$f(I) = V_t \cdot \ln\left[\frac{I - C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} - \alpha_r \cdot \left(\frac{I}{(1+k)} - C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot \left(I - C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t}\right) - \left(\frac{I}{(1+k)} - C_2 \cdot \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right]$$

$$V_{CEQ_1} = V_t \cdot \ln\left[\frac{I - C_2 \cdot \frac{dV_{CEQ_1}}{dt} - \alpha_r \cdot \left(\frac{I}{(1+k)} - C_2 \cdot \frac{dV_{CEQ_1}}{dt}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot \left(I - C_2 \cdot \frac{dV_{CEQ_1}}{dt}\right) - \left(\frac{I}{(1+k)} - C_2 \cdot \frac{dV_{CEQ_1}}{dt}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$

$$\exp\left(\frac{V_{CEQ_1}}{V_t}\right) = \frac{I - C_2 \cdot \frac{dV_{CEQ_1}}{dt} - \alpha_r \cdot \left(\frac{I}{(1+k)} - C_2 \cdot \frac{dV_{CEQ_1}}{dt}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot \left(I - C_2 \cdot \frac{dV_{CEQ_1}}{dt}\right) - \left(\frac{I}{(1+k)} - C_2 \cdot \frac{dV_{CEQ_1}}{dt}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}$$

$$I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) + (1-\alpha_f) \cdot C_2 \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) \cdot \frac{dV_{CEQ_1}}{dt} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)$$
$$= I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + C_2 \cdot \frac{dV_{CEQ_1}}{dt} \cdot (\alpha_r - 1) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)$$

$$\left[(1-\alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2 \cdot \frac{dV_{CEQ_1}}{dt} = I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)$$
$$- \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)$$

$$(*)\,\frac{dV_{CEQ_1}}{dt} = \frac{\begin{array}{c} I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) \\ - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) \end{array}}{\left[(1-\alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

&&&

$$I_{C_1} = C_1 \cdot \frac{dV}{dt};$$
$$I = -I_{C_1} \Rightarrow -I = C_1 \cdot \frac{dV}{dt} \Rightarrow \frac{dV}{dt} = -\frac{1}{C_1} \cdot I;$$
$$V = -\frac{1}{C_1} \cdot \int I \cdot dt$$

$$f(I) + L_1 \cdot \frac{dI}{dt} - V = 0;$$
$$V_{CEQ_1} = V_{C_2};$$
$$V_{D_1} + V_{C_2} = f(I);$$
$$V_{D_1} + V_{CEQ_1} = f(I)$$

$$V_{D_1} = V_t \cdot \ln\left[\frac{I_{CQ_1} + I_{C_2}}{I_0}\right] + 1; \quad I_{CQ_1} = \frac{I}{(1+k)} - I_{C_2};$$
$$V_{D_1} = V_t \cdot \ln\left[\frac{I}{(1+k)\cdot I_0} + 1\right]$$

$$f(I) = V_t \cdot \ln\left[\frac{I}{(1+k)\cdot I_0} + 1\right] + V_{CEQ_1}; \quad f(I) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - V = 0$$

$$V_t \cdot \ln\left[\frac{I}{(1+k)\cdot I_0} + 1\right] + V_{CEQ_1} + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C_1} \cdot \int I \cdot \mathrm{d}t = 0.$$

Derivate the above differential equation (both sides):

$$V_t \cdot \frac{\frac{1}{(1+k)\cdot I_0}}{\left[\frac{I}{(1+k)\cdot I_0} + 1\right]} + \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{C_1} \cdot I = 0;$$
$$V_t \cdot \frac{1}{[I + (1+k)\cdot I_0]} + \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{1}{C_1} \cdot I = 0$$

$$\frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} = -\frac{1}{C_1} \cdot I - L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} - V_t \cdot \frac{1}{[I + (1+k)\cdot I_0]}.$$

We define for simplicity new variable: $\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \frac{\mathrm{d}X}{\mathrm{d}t}; \quad \frac{\mathrm{d}I}{\mathrm{d}t} = X$

$$(**)\ \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} = -\frac{1}{C_1} \cdot I - L_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} - V_t \cdot \frac{1}{[I + (1+k)\cdot I_0]}$$

(**) → (*)

$$-\frac{1}{C_1} \cdot I - L_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} - V_t \cdot \frac{1}{[I + (1+k)\cdot I_0]}$$
$$= \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

$$L_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{1}{C_1} \cdot I - V_t \cdot \frac{1}{[I + (1+k)\cdot I_0]}$$
$$- \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

$$\frac{dX}{dt} = -\frac{1}{C_1 \cdot L_1} \cdot I - \frac{V_t}{L_1} \cdot \frac{1}{[I + (1+k) \cdot I_0]} - \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2 \cdot L_1}$$

We can summarize our system three differential equations:

$$\frac{dX}{dt} = -\frac{1}{C_1 \cdot L_1} \cdot I - \frac{V_t}{L_1} \cdot \frac{1}{[I + (1+k) \cdot I_0]} - \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2 \cdot L_1}$$

$$\frac{dI}{dt} = X; \frac{dV_{CEQ_1}}{dt} = \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}.$$

Remark Our system variables are X, I, V_{CEQ1}.

We define three functions: $g_1(X, I, V_{CEQ_1}), g_2(X, I, V_{CEQ_1}),\ g_3(X, I, V_{CEQ_1})$

$$g_1(X, I, V_{CEQ_1}) = -\frac{1}{C_1 \cdot L_1} \cdot I - \frac{V_t}{L_1} \cdot \frac{1}{[I + (1+k) \cdot I_0]} - \frac{I \cdot (1 - \alpha_r \cdot \frac{1}{(1+k)}) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot (\alpha_f - \frac{1}{(1+k)}) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2 \cdot L_1}$$

$$g_2(X, I, V_{CEQ_1}) = X;$$

$$g_3(X, I, V_{CEQ_1}) = \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

$$g_1 = g_1(X, I, V_{CEQ_1}), g_2 = g_2(X, I, V_{CEQ_1}), g_3 = g_3(X, I, V_{CEQ_1})$$

$$\frac{dX}{dt} = g_1(X, I, V_{CEQ_1}); \quad \frac{dI}{dt} = g_2(X, I, V_{CEQ_1}); \quad \frac{dV_{CEQ_1}}{dt} = g_3(X, I, V_{CEQ_1}).$$

At fixed points: $\frac{\mathrm{d}X}{\mathrm{d}t}=0;\quad \frac{\mathrm{d}I}{\mathrm{d}t}=0;\frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t}=0;\quad X^*=0$

$$g_1(X^*,I^*,V^*_{CEQ_1}) = -\frac{1}{C_1\cdot L_1}\cdot I^* - \frac{V_t}{L_1}\cdot\frac{1}{[I^*+(1+k)\cdot I_0]} - \frac{I^*\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-\left[I^*\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]\cdot C_2\cdot L_1}$$

At fixed points: $g_1(X^*,I^*,V^*_{CEQ_1})=0$

$$-\frac{1}{C_1\cdot L_1}\cdot I^* - \frac{V_t}{L_1}\cdot\frac{1}{[I^*+(1+k)\cdot I_0]} - \frac{I^*\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-\left[I^*\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]\cdot C_2\cdot L_1}=0$$

$$g_3(X^*,I^*,V^*_{CEQ_1}) = \frac{I^*\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-[I^*\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)]\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]\cdot C_2}$$

At fixed points: $g_3(X^*,I^*,V^*_{CEQ_1})=0$

$$\frac{I^*\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-\left[I^*\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]\cdot C_2}=0$$

$$(1-\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\neq 0\Rightarrow(1-\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)\neq(\alpha_r-1);$$
$$\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)\neq\frac{(\alpha_r-1)}{(1-\alpha_f)}$$

$$I^*\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-\left[I^*\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)=0$$

$$I^*=\frac{I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)-\left(\alpha_f-\frac{1}{(1+k)}\right)\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}.$$

We define the following parameters for simplicity: $I^*=\frac{\Xi_1\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)-\Xi_2}{\Xi_3-\Xi_4\cdot\exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}$

$$\Xi_1 = I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f); \quad \Xi_2 = I_{se} \cdot (1 - \alpha_r \cdot \alpha_f); \quad \Xi_3 = \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right);$$
$$\Xi_4 = \left(\alpha_f - \frac{1}{(1+k)}\right)$$

$$-\frac{1}{C_1 \cdot L_1} \cdot I^* - \frac{V_t}{L_1} \cdot \frac{1}{[I^* + (1+k) \cdot I_0]} - \frac{I^* \cdot \Xi_3 + \Xi_2 - [I^* \cdot \Xi_4 + \Xi_1] \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f) \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2 \cdot L_1} = 0$$

$$-\frac{1}{C_1 \cdot L_1} \cdot \left[\frac{\Xi_1 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) - \Xi_2}{\Xi_3 - \Xi_4 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}\right] - \frac{V_t}{L_1} \cdot \frac{1}{\left[\frac{\Xi_1 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) - \Xi_2}{\Xi_3 - \Xi_4 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)} + (1+k) \cdot I_0\right]}$$

$$-\frac{\left[\frac{\Xi_1 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) - \Xi_2}{\Xi_3 - \Xi_4 \cdot \exp(\frac{V^*_{CEQ_1}}{V_t})}\right] \cdot \Xi_3 + \Xi_2 - \left(\left[\frac{\Xi_1 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) - \Xi_2}{\Xi_3 - \Xi_4 \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}\right] \cdot \Xi_4 + \Xi_1\right) \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f) \cdot \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2 \cdot L_1} = 0.$$

We define for simplicity the function: $\eta^* = \eta^*(V^*_{CEQ_1}) = \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right)$

$$-\frac{1}{C_1 \cdot L_1} \cdot \left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*}\right] - \frac{V_t}{L_1} \cdot \frac{1}{\left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*} + (1+k) \cdot I_0\right]} - \frac{\left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*}\right] \cdot \Xi_3 + \Xi_2 - \left(\left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*}\right] \cdot \Xi_4 + \Xi_1\right) \cdot \eta^*}{[(1-\alpha_f) \cdot \eta^* - (\alpha_r - 1)] \cdot C_2 \cdot L_1} = 0$$

$$-\frac{1}{C_1 \cdot L_1} \cdot \left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*}\right] - \frac{V_t}{L_1} \cdot \frac{(\Xi_3 - \Xi_4 \cdot \eta^*)}{[\Xi_1 \cdot \eta^* - \Xi_2 + (1+k) \cdot I_0 \cdot (\Xi_3 - \Xi_4 \cdot \eta^*)]}$$
$$-\frac{[\Xi_1 \cdot \eta^* - \Xi_2] \cdot \Xi_3 + \Xi_2 \cdot (\Xi_3 - \Xi_4 \cdot \eta^*) - ([\Xi_1 \cdot \eta^* - \Xi_2] \cdot \Xi_4 + \Xi_1 \cdot [\Xi_3 - \Xi_4 \cdot \eta^*]) \cdot \eta^*}{[(1-\alpha_f) \cdot \eta^* - (\alpha_r - 1)] \cdot C_2 \cdot L_1} = 0$$

$$-\frac{1}{C_1} \cdot \left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*}\right] - V_t \cdot \frac{(\Xi_3 - \Xi_4 \cdot \eta^*)}{[\Xi_1 \cdot \eta^* - \Xi_2 + (1+k) \cdot I_0 \cdot (\Xi_3 - \Xi_4 \cdot \eta^*)]}$$
$$-\frac{[\Xi_1 \cdot \eta^* - \Xi_2] \cdot \Xi_3 + \Xi_2 \cdot (\Xi_3 - \Xi_4 \cdot \eta^*) - ([\Xi_1 \cdot \eta^* - \Xi_2] \cdot \Xi_4 + \Xi_1 \cdot [\Xi_3 - \Xi_4 \cdot \eta^*]) \cdot \eta^*}{[(1-\alpha_f) \cdot \eta^* - (\alpha_r - 1)] \cdot C_2} = 0$$

$$-\frac{1}{C_1} \cdot \left[\frac{\Xi_1 \cdot \eta^* - \Xi_2}{\Xi_3 - \Xi_4 \cdot \eta^*}\right] - V_t \cdot \frac{(\Xi_3 - \Xi_4 \cdot \eta^*)}{[\{\Xi_1 - (1+k) \cdot I_0 \cdot \Xi_4\} \cdot \eta^* + (1+k) \cdot I_0 \cdot \Xi_3 - \Xi_2]} = 0.$$

We define for simplicity: $\Omega_1 = \Xi_1 - (1+k) \cdot I_0 \cdot \Xi_4$; $\Omega_2 = (1+k) \cdot I_0 \cdot \Xi_3 - \Xi_2$

$$-\left(\frac{1}{C_1} \cdot \frac{[\Xi_1 \cdot \eta^* - \Xi_2]}{[\Xi_3 - \Xi_4 \cdot \eta^*]} + V_t \cdot \frac{[\Xi_3 - \Xi_4 \cdot \eta^*]}{[\Omega_1 \cdot \eta^* + \Omega_2]}\right) = 0; \ \frac{[\Xi_1 \cdot \eta^* - \Xi_2]}{C_1 \cdot [\Xi_3 - \Xi_4 \cdot \eta^*]} + \frac{V_t \cdot [\Xi_3 - \Xi_4 \cdot \eta^*]}{[\Omega_1 \cdot \eta^* + \Omega_2]} = 0$$

$$\frac{[\Xi_1 \cdot \eta^* - \Xi_2] \cdot [\Omega_1 \cdot \eta^* + \Omega_2] + V_t \cdot C_1 \cdot [\Xi_3 - \Xi_4 \cdot \eta^*] \cdot [\Xi_3 - \Xi_4 \cdot \eta^*]}{C_1 \cdot [\Xi_3 - \Xi_4 \cdot \eta^*] \cdot [\Omega_1 \cdot \eta^* + \Omega_2]} = 0$$

$$\Xi_3 - \Xi_4 \cdot \eta^* \neq 0 \Rightarrow \eta^* \neq \frac{\Xi_3}{\Xi_4}; \quad \Omega_1 \cdot \eta^* + \Omega_2 \neq 0 \Rightarrow \eta^* \neq -\frac{\Omega_2}{\Omega_1}$$

$$[\Xi_1 \cdot \eta^* - \Xi_2] \cdot [\Omega_1 \cdot \eta^* + \Omega_2] + V_t \cdot C_1 \cdot [\Xi_3 - \Xi_4 \cdot \eta^*] \cdot [\Xi_3 - \Xi_4 \cdot \eta^*] = 0$$

$$[\Xi_1 \cdot \eta^* - \Xi_2] \cdot [\Omega_1 \cdot \eta^* + \Omega_2] = \Xi_1 \cdot \Omega_1 \cdot [\eta^*]^2 + (\Xi_1 \cdot \Omega_2 - \Xi_2 \cdot \Omega_1) \cdot \eta^* - \Xi_2 \cdot \Omega_2$$
$$[\Xi_3 - \Xi_4 \cdot \eta^*] \cdot [\Xi_3 - \Xi_4 \cdot \eta^*] = \Xi_3^2 - 2 \cdot \Xi_3 \cdot \Xi_4 \cdot \eta^* + \Xi_4^2 \cdot [\eta^*]^2$$

$$(\Xi_1 \cdot \Omega_1 + V_t \cdot C_1 \cdot \Xi_4^2) \cdot [\eta^*]^2 + [\Xi_1 \cdot \Omega_2 - \Xi_2 \cdot \Omega_1 - 2 \cdot V_t \cdot C_1 \cdot \Xi_3 \cdot \Xi_4] \cdot \eta^* - \Xi_2 \cdot \Omega_2 + V_t \cdot C_1 \cdot \Xi_3^2 = 0.$$

We define for simplicity: $\Pi_1 = \Xi_1 \cdot \Omega_1 + V_t \cdot C_1 \cdot \Xi_4^2$; $\Pi_2 = \Xi_1 \cdot \Omega_2 - \Xi_2 \cdot \Omega_1 - 2 \cdot V_t \cdot C_1 \cdot \Xi_3 \cdot \Xi_4 \Pi_3 = -\Xi_2 \cdot \Omega_2 + V_t \cdot C_1 \cdot \Xi_3^2$

$$\Pi_1 \cdot [\eta^*]^2 + \Pi_2 \cdot \eta^* + \Pi_3 = 0 \Rightarrow \eta^*_{1,2} = \frac{-\Pi_2 \pm \sqrt{\Pi_2^2 - 4 \cdot \Pi_1 \cdot \Pi_3}}{2 \cdot \Pi_1}$$

$$\eta^* = \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) \Rightarrow \exp\left(\frac{V^*_{CEQ_1}}{V_t}\right) = \frac{-\Pi_2 \pm \sqrt{\Pi_2^2 - 4 \cdot \Pi_1 \cdot \Pi_3}}{2 \cdot \Pi_1}$$

$$V^*_{CEQ_1} = V_t \cdot \ln\left(\frac{-\Pi_2 \pm \sqrt{\Pi_2^2 - 4 \cdot \Pi_1 \cdot \Pi_3}}{2 \cdot \Pi_1}\right);$$
$$I^* = \frac{\Xi_1 \cdot \left[\frac{-\Pi_2 \pm \sqrt{\Pi_2^2 - 4 \cdot \Pi_1 \cdot \Pi_3}}{2 \cdot \Pi_1}\right] - \Xi_2}{\Xi_3 - \Xi_4 \cdot \left[\frac{-\Pi_2 \pm \sqrt{\Pi_2^2 - 4 \cdot \Pi_1 \cdot \Pi_3}}{2 \cdot \Pi_1}\right]}.$$

Stability Analysis We need to implement linearization technique for our system. First, find system Jacobian matrix at the fixed point $(X^*, I^*, V^*_{CEQ_1})$ [5, 6].

$$A = \begin{pmatrix} \frac{\partial g_1(X,I,V_{CEQ_1})}{\partial X} & \frac{\partial g_1(X,I,V_{CEQ_1})}{\partial I} & \frac{\partial g_1(X,I,V_{CEQ_1})}{\partial V_{CEQ_1}} \\ \frac{\partial g_2(X,I,V_{CEQ_1})}{\partial X} & \frac{\partial g_2(X,I,V_{CEQ_1})}{\partial I} & \frac{\partial g_2(X,I,V_{CEQ_1})}{\partial V_{CEQ_1}} \\ \frac{\partial g_3(X,I,V_{CEQ_1})}{\partial X} & \frac{\partial g_3(X,I,V_{CEQ_1})}{\partial I} & \frac{\partial g_3(X,I,V_{CEQ_1})}{\partial V_{CEQ_1}} \end{pmatrix}_{(X^*,I^*,V^*_{CEQ_1})}.$$

We need to find the partial derivatives of our functions:

$$g_1 = g_1(X,I,V_{CEQ_1}), g_2=g_2(X,I,V_{CEQ_1}),\ g_3=g_3(X,I,V_{CEQ_1});$$
$$\frac{\partial g_1(X,I,V_{CEQ_1})}{\partial X} = 0$$

$$g_1(X,I,V_{CEQ_1}) = -\frac{1}{C_1 \cdot L_1} \cdot I - \frac{V_t}{L_1} \cdot \frac{1}{[I+(1+k)\cdot I_0]} - \frac{I\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-\left[I\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]\cdot C_2\cdot L_1}$$

$$\frac{\partial g_1(X,I,V_{CEQ_1})}{\partial I} = -\frac{1}{C_1\cdot L_1} + \frac{V_t}{L_1}\cdot\frac{1}{[I+(1+k)\cdot I_0]^2} - \frac{1}{C_2\cdot L_1}\cdot\left\{\frac{\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)-\left(\alpha_f-\frac{1}{(1+k)}\right)\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]}\right.$$

$$\begin{aligned}\frac{\partial g_1(X,I,V_{CEQ_1})}{\partial V_{CEQ_1}} = -\frac{1}{C_2\cdot L_1} \cdot \Bigg(& -\left[I\cdot(\alpha_f-\frac{1}{(1+k)})+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\frac{1}{V_t}\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right) \\ & \cdot\left[(1-\alpha_f)\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right] \\ & -\left\{I\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)\right. \\ & \left.-\left[I\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)\right\} \\ & \cdot(1-\alpha_f)\cdot\frac{1}{V_t}\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)\Bigg) \Bigg/ \left[(1-\alpha_f)\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]^2\end{aligned}$$

$$g_2(X,I,V_{CEQ_1}) = X;\quad \frac{\partial g_2(X,I,V_{CEQ_1})}{\partial X} = 1;$$
$$\frac{\partial g_2(X,I,V_{CEQ_1})}{\partial I} = 0;\quad \frac{\partial g_2(X,I,V_{CEQ_1})}{\partial V_{CEQ_1}} = 0$$

$$g_3(X, I, V_{CEQ_1}) = \frac{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

$$\frac{\partial g_3(X, I, V_{CEQ_1})}{\partial X} = 0$$

$$\frac{\partial g_3(X, I, V_{CEQ_1})}{\partial I} = \frac{\left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) - \left(\alpha_f - \frac{1}{(1+k)}\right) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

$$\frac{\partial g_3(X, I, V_{CEQ_1})}{\partial V_{CEQ_1}} = \frac{1}{C_2} \cdot \frac{\begin{array}{l} -\left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \frac{1}{V_t} \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) \\ \cdot\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \\ -\left\{I \cdot \left(1 - \alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) \right. \\ \left. -\left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)\right\} \\ \cdot (1 - \alpha_f) \cdot \frac{1}{V_t} \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) \end{array}}{\left[(1 - \alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right]^2}$$

$$g_2(X, I, V_{CEQ_1}) = X; \quad \frac{\partial g_2(X, I, V_{CEQ_1})}{\partial X} = 1;$$
$$\frac{\partial g_2(X, I, V_{CEQ_1})}{\partial I} = 0; \quad \frac{\partial g_2(X, I, V_{CEQ_1})}{\partial V_{CEQ_1}} = 0$$

$$\frac{\partial g_1}{\partial X} = \frac{\partial g_1(X, I, V_{CEQ_1})}{\partial X};$$
$$\frac{\partial g_1}{\partial I} = \frac{\partial g_1(X, I, V_{CEQ_1})}{\partial I};$$
$$\frac{\partial g_1}{\partial V_{CEQ_1}} = \frac{\partial g_1(X, I, V_{CEQ_1})}{\partial V_{CEQ_1}}$$

$$\frac{\partial g_3}{\partial X} = \frac{\partial g_3(X, I, V_{CEQ_1})}{\partial X};$$
$$\frac{\partial g_3}{\partial I} = \frac{\partial g_3(X, I, V_{CEQ_1})}{\partial I};$$
$$\frac{\partial g_3}{\partial V_{CEQ_1}} = \frac{\partial g_3(X, I, V_{CEQ_1})}{\partial V_{CEQ_1}}$$

$$A = \begin{pmatrix} 0 & \frac{\partial g_1}{\partial I} & \frac{\partial g_1}{\partial V_{CEQ_1}} \\ 1 & 0 & 0 \\ 0 & \frac{\partial g_3}{\partial I} & \frac{\partial g_3}{\partial V_{CEQ_1}} \end{pmatrix}_{(X^*, I^*, V^*_{CEQ_1})} ; \det(A - \lambda \cdot I) = 0; \det \begin{pmatrix} -\lambda & \frac{\partial g_1}{\partial I} & \frac{\partial g_1}{\partial V_{CEQ_1}} \\ 1 & -\lambda & 0 \\ 0 & \frac{\partial g_3}{\partial I} & \frac{\partial g_3}{\partial V_{CEQ_1}} - \lambda \end{pmatrix}_{(X^*, I^*, V^*_{CEQ_1})} = 0$$

$$-\lambda \cdot \det \begin{pmatrix} -\lambda & 0 \\ \frac{\partial g_3}{\partial I} & \frac{\partial g_3}{\partial V_{CEQ_1}} - \lambda \end{pmatrix}_{(X^*, I^*, V^*_{CEQ_1})} - \left.\frac{\partial g_1}{\partial I}\right|_{(X^*, I^*, V^*_{CEQ_1})} \cdot \det \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial g_3}{\partial V_{CEQ_1}} - \lambda \end{pmatrix}_{(X^*, I^*, V^*_{CEQ_1})}$$
$$+ \left.\frac{\partial g_1}{\partial V_{CEQ_1}}\right|_{(X^*, I^*, V^*_{CEQ_1})} \cdot \det \begin{pmatrix} 1 & -\lambda \\ 0 & \frac{\partial g_3}{\partial I} \end{pmatrix}_{(X^*, I^*, V^*_{CEQ_1})} = 0$$

$$\lambda^2 \cdot \left(\left.\frac{\partial g_3}{\partial V_{CEQ_1}}\right|_{(X^*, I^*, V^*_{CEQ_1})} - \lambda \right) - \left.\frac{\partial g_1}{\partial I}\right|_{(X^*, I^*, V^*_{CEQ_1})} \cdot \left(\left.\frac{\partial g_3}{\partial V_{CEQ_1}}\right|_{(X^*, I^*, V^*_{CEQ_1})} - \lambda \right)$$
$$+ \left.\left(\frac{\partial g_1}{\partial V_{CEQ_1}} \cdot \frac{\partial g_3}{\partial I} \right)\right|_{(X^*, I^*, V^*_{CEQ_1})} = 0$$

$$\lambda^2 \cdot \left.\frac{\partial g_3}{\partial V_{CEQ_1}}\right|_{(X^*, I^*, V^*_{CEQ_1})} - \lambda^3 - \left.\left(\frac{\partial g_1}{\partial I} \cdot \frac{\partial g_3}{\partial V_{CEQ_1}} \right)\right|_{(X^*, I^*, V^*_{CEQ_1})} + \left.\frac{\partial g_1}{\partial I}\right|_{(X^*, I^*, V^*_{CEQ_1})} \cdot \lambda$$
$$+ \left.\left(\frac{\partial g_1}{\partial V_{CEQ_1}} \cdot \frac{\partial g_3}{\partial I} \right)\right|_{(X^*, I^*, V^*_{CEQ_1})} = 0$$

$$\lambda^3 - \lambda^2$$
$$\cdot \left.\frac{\partial g_3}{\partial V_{CEQ_1}}\right|_{(X^*, I^*, V^*_{CEQ_1})} - \left.\frac{\partial g_1}{\partial I}\right|_{(X^*, I^*, V^*_{CEQ_1})} \cdot \lambda + \left.\left(\frac{\partial g_1}{\partial I} \cdot \frac{\partial g_3}{\partial V_{CEQ_1}} - \frac{\partial g_1}{\partial V_{CEQ_1}} \cdot \frac{\partial g_3}{\partial I} \right)\right|_{(X^*, I^*, V^*_{CEQ_1})}$$
$$= 0$$

$$\sum_{k=0}^{3} \lambda^k \cdot \iota_k = 0; \quad \iota_3 = 1; \quad \iota_2 = -\left.\frac{\partial g_3}{\partial V_{CEQ_1}}\right|_{(X^*, I^*, V^*_{CEQ_1})}; \quad \iota_1 = -\left.\frac{\partial g_1}{\partial I}\right|_{(X^*, I^*, V^*_{CEQ_1})}$$

$$l_0 = \left(\frac{\partial g_1}{\partial I} \cdot \frac{\partial g_3}{\partial V_{CEQ_1}} - \frac{\partial g_1}{\partial V_{CEQ_1}} \cdot \frac{\partial g_3}{\partial I} \right) \Bigg|_{(X^*, I^*, V^*_{CEQ_1})}.$$

Eigenvalues Stability Discussion Our optoisolation van der Pol circuit with parallel capacitor (C_2) involving N variables ($N > 2$, $N = 3$), the characteristic equation is of degree $N = 3$ and must often be solved numerically. Expect in some particular cases, such an equation has ($N = 3$) distinct roots that can be real or complex. These values are the eigenvalues of the 3x3 Jacobian matrix (A). The general rule is that the Steady State (SS) is stable if there is no eigenvalue with positive real part. It is sufficient that one eigenvalue is positive for the steady state to be unstable. Our 3-variables (X, I, V_{CEQ_1}) optoisolation van der Pol circuit with parallel capacitor (C_2) has three eigenvalues. The type of behavior can be characterized as a function of the position of these eigenvalues in the Re/Im plane. Five nondegenerated cases can be distinguished: (1) the three eigenvalues are real and negative (stable steady state), (2) the three eigenvalues are real, two of them are negative (unstable steady state), (3) and (4) two eigenvalues are complex conjugates with a negative real part and the other eigenvalues are real and negative (stable steady state), two cases can be distinguished depending on the relative value of the real part of the complex eigenvalues and of the real one, (5) two eigenvalues are complex conjugates with a negative real part and other eigenvalue is positive (unstable steady state) [7, 8].

Plotting optoisolation van der Pol circuit with parallel capacitor (C_2), phase planes and variables (X, I, V_{CEQ_1}) time behavior. First we choose our circuit parameters values [99] (Tables 7.2 and 7.3)

Table 7.2 Circuit parameters values

α_r	0.5	C_1	1 μF
α_f	0.98	C_2	0.4–4 μF
k	0.02	L_1	0.1 mH
V_t	0.026 V	I_{se}	1 μA
I_0	1 μA	I_{sc}	2 μA

Table 7.3 Variables functions and MATLAB variables

Variables/functions	MATLAB variables
$X \rightarrow X$	$x(1)$
$I \rightarrow Y$	$x(2)$
$V_{CEQ_1} \rightarrow Z$	$x(3)$
$g_1(X, I, V_{CEQ_1})$	$g(1)$
$g_2(X, I, V_{CEQ_1})$	$g(2)$
$g_3(X, I, V_{CEQ_1})$	$g(3)$

$$(1+k)\cdot I_0 = 1.02\times 10^{-6};$$
$$\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right) = 0.509;$$
$$I_{se}\cdot(1-\alpha_r\cdot\alpha_f) = 0.51\times 10^{-6}$$
$$\left(\alpha_f-\frac{1}{(1+k)}\right) = -0.00039;$$
$$(1-\alpha_f) = 0.02;$$
$$(\alpha_r-1) = -0.5;$$
$$I_{sc}\cdot(1-\alpha_r\cdot\alpha_f) = 1.02\times 10^{-6}.$$

MATLAB scripts

```
function h= optovanderpol1 (C1,C2,L1,X0,Y0,Z0)
[t,x]=ODE45(@optovanderpol,[0,10],[X0,Y0,Z0],[],C1,C2,L1);
plot3 (x(:,1),x(:,2),x(:,3));
xlabel ('dI/dt(A/sec)')%x-axis
ylabel ('I(A)')%y-axis
zlabel ('VceQ1(volt)')%z-axis
grid on
axis square
%plot(t,x);
%plot(x(:,1),x(:,2));%X=dI/dt (x-axis) and I (y-axis)
%plot(x(:,1),x(:,3));%X=dI/dt (x-axis) and VCEQ1 (y-axis)
%plot(x(:,2),x(:,3));%I (x-axis) and VCEQ1 (y-axis)

function g=optovanderpol(t,x,C1,C2,L1)
g=zeros(3,1);% the elements of the vector g represent the right hand
sided of the three DEs
g(1)=-1/(C1*L1)*x(2)-(0.026/L1)*(1/(x(2)+1.02e-6))-(x(2)*0.509+0.51e-
6-(x(2)*(-0.00039)+1.02e-6)*exp(x(3)/0.026))./((0.02*exp(x(3)/0.026)-
(-0.5))*C2*L1);
g(2)=x(1);
g(3)=(x(2)*0.509+0.51e-6-(x(2)*(-0.00039)+1.02e-
6)*exp(x(3)/0.026))./((0.02*exp(x(3)/0.026)-(-0.5))*C2);
```

First we plot 3D graph `dI/dt(A/sec)` x-axis ; `'I(A)` y-axis ; `'VceQ1(volt)` z-axis
C_1=1uF ; C_2=0.4uF ; L_1=0.1mH ; dI/dt(t=0)=1 ; I(t=0)=1 ; V_{CEQ1}(t=0)=1

optovanderpol1 (1e−6,0.4e−6,0.1e−3,1,1,1) (Fig. 7.4).

Optoisolation van der pol Circuit with Parallel Capacitor (C_2) *Poincare map*
Poincare map is an important tool for the investigation of dynamical systems in applications. They are used for models (usually in 3D) that exhibit periodic or quasi-periodic behavior. A 2D Poincare section through a periodic 3D flow is a planar cross section transverse to the flow such that a periodic orbit intersects it at its center. The corresponding Poincare map is defined as a map correlating

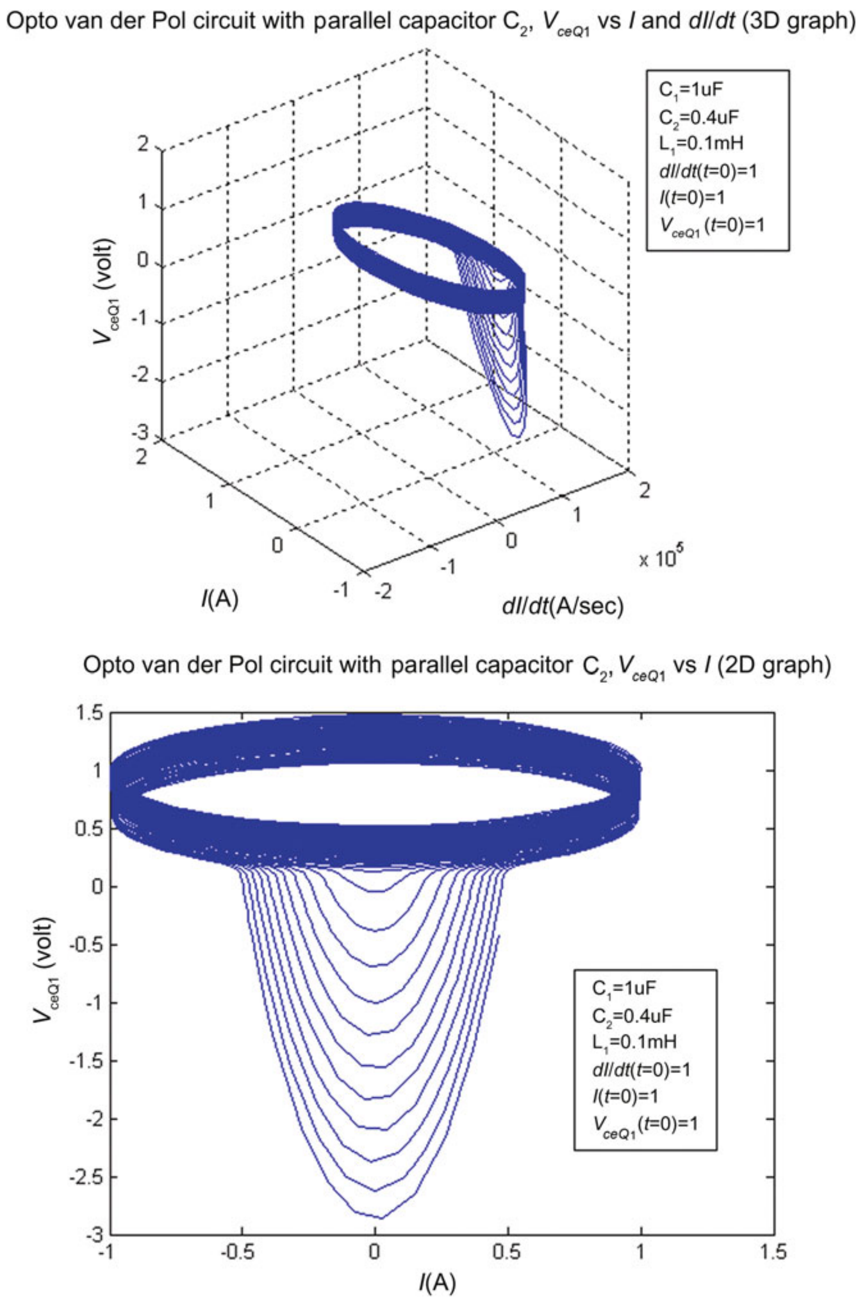
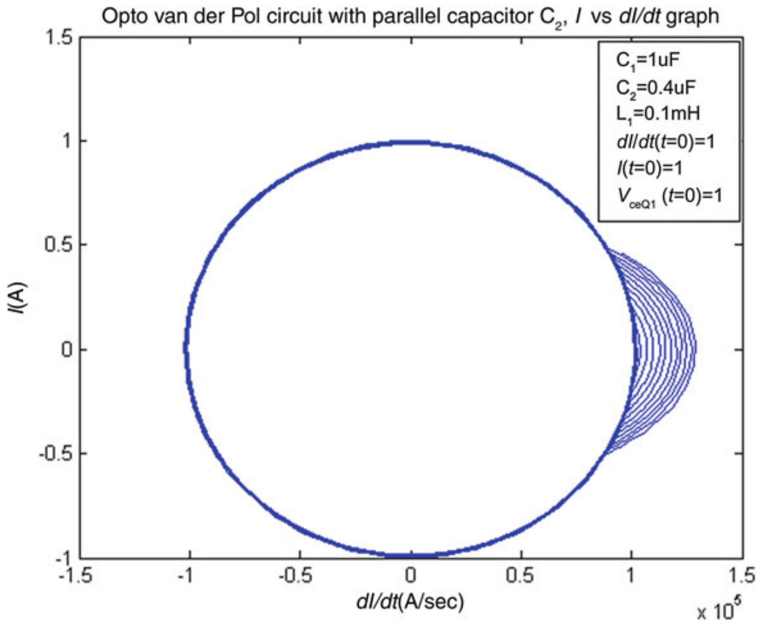


Fig. 7.4 Opto van der Pol circuit with parallel capacitor C_2 graphs



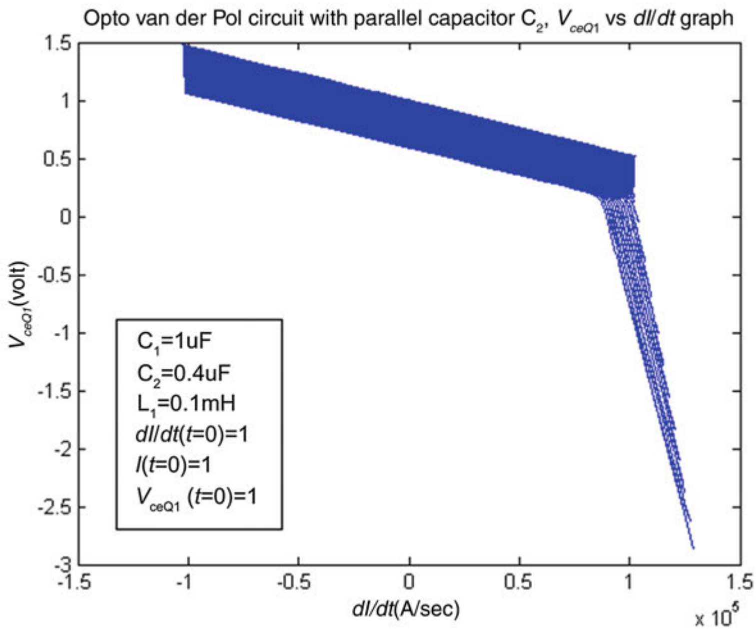


Fig. 7.4 (continued)

consecutive intersections of flow trajectories with the Poincare section. The Poincare map is a discrete dynamical system of one dimensional less than the continuous flow which it is constructed. Most important flow properties are inherited by the Poincare map and its analysis is usually simpler due to its reduced dimensionality. It is often used for analysis instead of the 3D flow. The investigation of dynamical system through Poincare map is done by taking intersections of the orbit of flow by a hyperplane parallel to one of the coordinate hyperplanes of co-dimension one. For a 3D-attractor, the Poincare map gives rise to 2D points, which can describe the dynamics of the attractor properly. In a special case, 2D points are considered as their 1D projection to obtain a 1D map. It is a practical way of reducing the 2D map by dropping one of the variables. The two coordinates of the points on the Poincare section are functionally related. In general, to describe the dynamics of the 3D chaotic attractor, the minimum dimension of the Poincare map must be two. It is always possible to obtain two-dimensional Poincare map for any three-dimensional dynamical system and in general it is not reducible to a one-dimensional Poincare map. Such 2D Poincare map is capable of studying the chaotic behavior of the system. Two-dimensional Poincare maps are the maps of minimum dimension, which are capable to explain the dynamics of three-dimensional attractors [5, 6]. It is important to emphasis that in Poincare maps; points on the Poincare section S get mapped back onto S by the flow. First we take (*) and (**) differential equations:

$$(*)\frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} = \frac{I\cdot\left(1-\alpha_r\cdot\frac{1}{(1+k)}\right)+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)-\left[I\cdot\left(\alpha_f-\frac{1}{(1+k)}\right)+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\right]\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f)\cdot\exp\left(\frac{V_{CEQ_1}}{V_t}\right)-(\alpha_r-1)\right]\cdot C_2}$$

$$(**)\frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} = -\frac{1}{C_1}\cdot I - L_1\cdot\frac{\mathrm{d}^2I}{\mathrm{d}t^2} - V_t\cdot\frac{1}{[I+(1+k)\cdot I_0]}.$$

First we transfer our system variables to Cartesian coordinates ($X(t)$, $Y(t)$) terminology ($I(t)\leftrightarrow X(t)$; $V_{CEQ_1}(t)\leftrightarrow Y(t)$). We need to prove that the system has periodic orbits and find Poincare map. It is done by changing system Cartesian coordinates ($X(t)$, $Y(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z-axis. In our system we refer to Cartesian X–Y plane (with equation $Z = 0$). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X, Y) are the same as for polar coordinates, namely $X(t) = r(t)\cdot\cos[\theta(t)]$; $Y(t) = r(t)\cdot\sin[\theta(t)]$; $r=\sqrt{X^2+Y^2}$. $\theta(t) = 0$ if $X = 0$ and $Y = 0$. $\theta(t) = \arcsin(Y/r)$ if $X >= 0$. $x\to X$, $y\to Y$. $I = r\cdot\cos\theta \Leftrightarrow X = r\cdot\cos\theta$; $V_{CEQ_1} = r\cdot\sin\theta \Leftrightarrow Y = r\cdot\sin\theta$.

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \cos\theta - r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \sin\theta; \quad \frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \sin\theta + r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \cos\theta$$

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \cdot \cos\theta - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \sin\theta - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \sin\theta - r \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \cdot \sin\theta - r \cdot \left[\frac{\mathrm{d}\theta}{\mathrm{d}t}\right]^2 \cdot \cos\theta.$$

First we represent the second differential equation (**) in cylindrical coordinates ($r(t)$, $\theta(t)$).

$$\frac{\mathrm{d}V_{CEQ_1}}{\mathrm{d}t} = -\frac{1}{C_1} \cdot I - L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} - V_t \cdot \frac{1}{[I + (1+k) \cdot I_0]}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} \cdot \sin\theta + r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \cos\theta = -\frac{1}{C_1} \cdot r \cdot \cos\theta - L_1 \cdot \left\{\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \cdot \cos\theta - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \sin\theta - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \sin\theta \right.$$
$$\left. - r \cdot \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \cdot \sin\theta - r \cdot \left[\frac{\mathrm{d}\theta}{\mathrm{d}t}\right]^2 \cdot \cos\theta \right\} - V_t \cdot \frac{1}{[r \cdot \cos\theta + (1+k) \cdot I_0]}$$

$$\theta = \omega \cdot \mathrm{t} = \frac{2 \cdot \pi}{T} \cdot \mathrm{t}; \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \Rightarrow \frac{2 \cdot \pi}{T} = \omega; \quad T = \frac{2 \cdot \pi}{\omega}; \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \Rightarrow \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 0$$

$\theta = \omega \cdot \mathrm{t}$ Which regard the optoisolation van der Pol circuit with parallel capacitor (C_2) system as a vector field on a cylinder $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$. Any vertical line on the cylinder is an appropriate section S: we choose $S = \{(\theta, r) : \theta = 0 \bmod 2 \cdot \pi\}$. Consider an initial condition on S given by $\theta(t = 0) = 0; r(t = 0) = r_0$ then the time of flight between successive intersections is $T = \frac{2 \cdot \pi}{\omega}$.

$$\frac{\mathrm{d}r}{\mathrm{d}t} \cdot \sin(\omega \cdot \mathrm{t}) + r \cdot \omega \cdot \cos(\omega \cdot \mathrm{t}) = -\frac{1}{C_1} \cdot r \cdot \cos(\omega \cdot \mathrm{t}) - L_1 \cdot \left\{\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \cdot \cos(\omega \cdot \mathrm{t}) - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \omega \cdot \sin(\omega \cdot \mathrm{t}) \right.$$
$$\left. - \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \omega \cdot \sin(\omega \cdot \mathrm{t}) - r \cdot \omega^2 \cdot \cos(\omega \cdot \mathrm{t}) \right\} - V_t \cdot \frac{1}{[r \cdot \cos(\omega \cdot \mathrm{t}) + (1+k) \cdot I_0]}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \cdot L_1 \cdot \cos(\omega \cdot \mathrm{t}) + \frac{\mathrm{d}r}{\mathrm{d}t} \cdot [1 - 2 \cdot L_1 \cdot \omega] \cdot \sin(\omega \cdot \mathrm{t}) + r \cdot \left[\omega \cdot \cos(\omega \cdot \mathrm{t}) + \frac{1}{C_1} \cdot \cos(\omega \cdot \mathrm{t}) \right.$$
$$\left. - \omega^2 \cdot L_1 \cdot \cos(\omega \cdot \mathrm{t})\right] + V_t \cdot \frac{1}{[r \cdot \cos(\omega \cdot \mathrm{t}) + (1+k) \cdot I_0]} = 0.$$

Second we represent the first differential equation (*) in cylindrical coordinates (r(t), θ(t)).

$$\frac{dV_{CEQ_1}}{dt} = \frac{I \cdot \left(1-\alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1-\alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1-\alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{\left[(1-\alpha_f) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) - (\alpha_r - 1)\right] \cdot C_2}$$

$$\frac{dV_{CEQ_1}}{dt} = \frac{I \cdot \left(1-\alpha_r \cdot \frac{1}{(1+k)}\right) + I_{se} \cdot (1-\alpha_r \cdot \alpha_f) - \left[I \cdot \left(\alpha_f - \frac{1}{(1+k)}\right) + I_{sc} \cdot (1-\alpha_r \cdot \alpha_f)\right] \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{(1-\alpha_f) \cdot C_2 \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) + (1-\alpha_r) \cdot C_2}$$

We define six parameters for simplicity: $H_1, H_2, \ldots, H_6$

$$H_1 = \left(1-\alpha_r \cdot \frac{1}{(1+k)}\right); \quad H_2 = I_{se} \cdot (1-\alpha_r \cdot \alpha_f); \quad H_3 = \left(\alpha_f - \frac{1}{(1+k)}\right); \quad H_4 = I_{sc} \cdot (1-\alpha_r \cdot \alpha_f)$$

$$H_5 = (1-\alpha_f) \cdot C_2; \quad H_6 = (1-\alpha_r) \cdot C_2$$

$$\frac{dV_{CEQ_1}}{dt} = \frac{I \cdot H_1 + H_2 - (I \cdot H_3 + H_4) \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right)}{H_5 \cdot \exp\left(\frac{V_{CEQ_1}}{V_t}\right) + H_6}$$

Moving to cylindrical coordinates ($r(t)$, $\theta(t)$).

$$\frac{dr}{dt} \cdot \sin\theta + r \cdot \frac{d\theta}{dt} \cdot \cos\theta = \frac{H_1 \cdot r \cdot \cos\theta + H_2 - (H_3 \cdot r \cdot \cos\theta + H_4) \cdot \exp\left(\frac{r \cdot \sin\theta}{V_t}\right)}{H_5 \cdot \exp\left(\frac{r \cdot \sin\theta}{V_t}\right) + H_6};$$

$$\theta = \omega \cdot \mathrm{t}; \quad \frac{d\theta}{dt} = \omega$$

$$\frac{dr}{dt} \cdot \sin(\omega \cdot \mathrm{t}) + r \cdot \omega \cdot \cos(\omega \cdot \mathrm{t}) = \frac{H_1 \cdot r \cdot \cos(\omega \cdot \mathrm{t}) + H_2 - (H_3 \cdot r \cdot \cos(\omega \cdot \mathrm{t}) + H_4) \cdot \exp\left(\frac{r \cdot \sin(\omega \cdot \mathrm{t})}{V_t}\right)}{H_5 \cdot \exp\left(\frac{r \cdot \sin(\omega \cdot \mathrm{t})}{V_t}\right) + H_6}$$

$$\frac{dr}{dt} = -r \cdot \omega \cdot \frac{\cos(\omega \cdot \mathrm{t})}{\sin(\omega \cdot \mathrm{t})} + \frac{H_1 \cdot r \cdot \cos(\omega \cdot \mathrm{t}) + H_2 - (H_3 \cdot r \cdot \cos(\omega \cdot \mathrm{t}) + H_4) \cdot \exp\left(\frac{r \cdot \sin(\omega \cdot \mathrm{t})}{V_t}\right)}{\left[H_5 \cdot \exp\left(\frac{r \cdot \sin(\omega \cdot \mathrm{t})}{V_t}\right) + H_6\right] \cdot \sin(\omega \cdot \mathrm{t})}$$

$$\frac{dr}{dt} = -r \cdot \omega \cdot \frac{1}{tg(\omega \cdot t)} + \frac{H_1 \cdot r \cdot \cos(\omega \cdot t) + H_2 - (H_3 \cdot r \cdot \cos(\omega \cdot t) + H_4) \cdot \exp\left(\frac{r \cdot \sin(\omega \cdot t)}{V_t}\right)}{\left[H_5 \cdot \exp\left(\frac{r \cdot \sin(\omega \cdot t)}{V_t}\right) + H_6\right] \cdot \sin(\omega \cdot t)}$$

Remark To compute Poincare map ψ(Poincare map) we need to solve the above differential equations $\frac{dr}{dt} = \cdots$; $\frac{d^2r}{dt^2} \cdot (\cdots) + \frac{dr}{dt} \cdot (\cdots) + r \cdot (\cdots) + \cdots = 0$ and find the $\psi(r)$ function. It is reader exercise.

$$r_1 = \psi(r_0); \quad r_2 = \psi(r_1); \quad r_3 = \psi(r_2) \ldots r_k = \psi(r_{k-1}) \quad \forall \quad k = 1, 2, 3, \ldots$$

7.3 Li Dynamical System Poincare Map and Periodic Orbit

One of the typical dynamical systems is Li autonomous system with toroidal chaotic attractors. The Li dynamical system is autonomous and the motion appears to occur on a surface with a toroidal structure. The Li system global Poincare surface of section has two disjoint components . A similarity transformation in the phase space emphasizes symmetry of the attractor. Poincare section located segments of a chaotic trajectory are good approximations to unstable periodic orbit. Li system can be described by three Ordinary Differential Equations (ODEs) [110, 111].

$$\frac{dX}{dt} = \mu_1 \cdot (Y - X) + \mu_2 \cdot X \cdot Z; \frac{dY}{dt} = \mu_3 \cdot X + \mu_4 \cdot Y - X \cdot Z;$$
$$\frac{dZ}{dt} = \mu_5 \cdot Z + X \cdot Y - \mu_6 \cdot X^2.$$

The system is invariant under the group of two fold rotations about the symmetry axis in the phase space $\mathbb{R}^3(X, Y, Z) : R_Z(\pi) : (X, Y, Z) \to (-X, -Y, +Z)$.

We would like to have measure of stability like rate of decay to a stable fixed point. It can be achieved by linearization about the fixed point. We define the following functions: $f_1(X, Y, Z) = \mu_1 \cdot (Y - X) + \mu_2 \cdot X \cdot Z$

$$f_2(X, Y, Z) = \mu_3 \cdot X + \mu_4 \cdot Y - X \cdot Z; \quad f_3(X, Y, Z) = \mu_5 \cdot Z + X \cdot Y - \mu_6 \cdot X^2$$

$$\frac{dX}{dt} = f_1(X, Y, Z); \quad \frac{dY}{dt} = f_2(X, Y, Z); \frac{dZ}{dt} = f_3(X, Y, Z).$$

We suppose that (X^*, Y^*, Z^*) is a fixed point. $f_1(X^*, Y^*, Z^*) = 0$; $f_2(X^*, Y^*, Z^*) = 0$

$f_3(X^*, Y^*, Z^*) = 0$. Let $u = X - X^*$; $v = Y - Y^*$; $w = Z - Z^*$ denote the components of a small disturbance from the fixed point. We derive differential equation for u, v, and w to inspect whether the disturbance grows or decays. Since X^*, Y^* and Z^* are constants: $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t}$; $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}Y}{\mathrm{d}t}$; $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}Z}{\mathrm{d}t}$.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} = f_1(X^* + u, Y^* + v, Z^* + w); \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}Y}{\mathrm{d}t} = f_2(X^* + u, Y^* + v, Z^* + w)$$

$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}Z}{\mathrm{d}t} = f_3(X^* + u, Y^* + v, Z^* + w)$. By using Taylor series expansion :

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} = f_1(X^*, Y^*, Z^*) + u \cdot \frac{\partial f_1}{\partial X} + v \cdot \frac{\partial f_1}{\partial Y} + w \cdot \frac{\partial f_1}{\partial Z} + O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots)$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}Y}{\mathrm{d}t} = f_2(X^*, Y^*, Z^*) + u \cdot \frac{\partial f_2}{\partial X} + v \cdot \frac{\partial f_2}{\partial Y} + w \cdot \frac{\partial f_2}{\partial Z} + O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots)$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}Z}{\mathrm{d}t} = f_3(X^*, Y^*, Z^*) + u \cdot \frac{\partial f_3}{\partial X} + v \cdot \frac{\partial f_3}{\partial Y} + w \cdot \frac{\partial f_3}{\partial Z} + O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots)$$

Since $f_k(X^*, Y^*, Z^*) = 0$; $k = 1, 2, 3$ then we can write the following expressions:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} = u \cdot \frac{\partial f_1}{\partial X} + v \cdot \frac{\partial f_1}{\partial Y} + w \cdot \frac{\partial f_1}{\partial Z} + O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots)$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}Y}{\mathrm{d}t} = u \cdot \frac{\partial f_2}{\partial X} + v \cdot \frac{\partial f_2}{\partial Y} + w \cdot \frac{\partial f_2}{\partial Z} + O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots)$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}Z}{\mathrm{d}t} = u \cdot \frac{\partial f_3}{\partial X} + v \cdot \frac{\partial f_3}{\partial Y} + w \cdot \frac{\partial f_3}{\partial Z} + O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots).$$

The above partial derivatives are evaluated at the fixed point (X^*, Y^*, Z^*) and they are numbers, not functions. $O(u^2, v^2, w^2, u \cdot v \cdot w, \ldots)$ denotes quadratic terms in u, v, and w. Since u, v, and w quadratic terms are small.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u \cdot \frac{\partial f_1}{\partial X} + v \cdot \frac{\partial f_1}{\partial Y} + w \cdot \frac{\partial f_1}{\partial Z};$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = u \cdot \frac{\partial f_2}{\partial X} + v \cdot \frac{\partial f_2}{\partial Y} + w \cdot \frac{\partial f_2}{\partial Z};$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = u \cdot \frac{\partial f_3}{\partial X} + v \cdot \frac{\partial f_3}{\partial Y}$$

$$\begin{pmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} \\ \frac{\mathrm{d}v}{\mathrm{d}t} \\ \frac{\mathrm{d}w}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \text{quadratic terms.}$$

The matrix $A = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}_{(X^*,Y^*,Z^*)}$ is called the Jacobian matrix at the fixed point X^*, Y^*, Z^*. Since the quadratic terms are very small we neglect them altogether and we get linearized Li system.

$$\frac{\partial f_1}{\partial X} = -\mu_1 + \mu_2 \cdot Z; \quad \frac{\partial f_1}{\partial Y} = \mu_1;$$
$$\frac{\partial f_1}{\partial Z} = \mu_2 \cdot X; \quad \frac{\partial f_2}{\partial X} = \mu_3 - Z;$$
$$\frac{\partial f_2}{\partial Y} = \mu_4; \quad \frac{\partial f_2}{\partial Z} = -X$$

$$\frac{\partial f_3}{\partial X} = Y - \mu_6 \cdot 2 \cdot X; \quad \frac{\partial f_3}{\partial Y} = X; \quad \frac{\partial f_3}{\partial Z} = \mu_5$$

Li system fixed points: $\frac{dX}{dt} = 0$; $\frac{dY}{dt} = 0$; $\frac{dZ}{dt} = 0$; $\mu_1 \cdot (Y^* - X^*) + \mu_2 \cdot X^* \cdot Z^* = 0$

$$\mu_3 \cdot X^* + \mu_4 \cdot Y^* - X^* \cdot Z^* = 0; \quad \mu_5 \cdot Z^* + X^* \cdot Y^* - \mu_6 \cdot [X^*]^2 = 0$$

$$\mu_3 \cdot X^* + \mu_4 \cdot Y^* - X^* \cdot Z^* = 0 \Rightarrow X^* \cdot Z^* = \mu_3 \cdot X^* + \mu_4 \cdot Y^*$$

$$\mu_1 \cdot (Y^* - X^*) + \mu_2 \cdot X^* \cdot Z^* = 0 \Rightarrow \mu_1 \cdot (Y^* - X^*) + \mu_2 \cdot (\mu_3 \cdot X^* + \mu_4 \cdot Y^*) = 0$$

$$(\mu_1 + \mu_2 \cdot \mu_4) \cdot Y^* + (\mu_2 \cdot \mu_3 - \mu_1) \cdot X^* = 0 \Rightarrow Y^* = \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} \cdot X^*$$

$$\mu_5 \cdot Z^* + X^* \cdot Y^* - \mu_6 \cdot [X^*]^2 = 0 \Rightarrow Z^* = \frac{\mu_6}{\mu_5} \cdot [X^*]^2 - \frac{1}{\mu_5} X^* \cdot Y^*$$
$$\mu_1 \cdot (Y^* - X^*) + \mu_2 \cdot X^* \cdot Z^* = 0 \Rightarrow \mu_1 \cdot (Y^* - X^*) + \mu_2 \cdot X^* \cdot \frac{\mu_6}{\mu_5} \cdot \left([X^*]^2 - \frac{1}{\mu_5} X^* \cdot Y^* \right) = 0$$
$$\mu_1 \cdot Y^* - \mu_1 \cdot X^* + \mu_2 \cdot \frac{\mu_6}{\mu_5} \cdot [X^*]^3 - \frac{\mu_2}{\mu_5} \cdot [X^*]^2 \cdot Y^* = 0$$

$$\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1 \right] \cdot \mu_1 \cdot X^* + \left(\mu_6 - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} \right) \cdot \frac{\mu_2}{\mu_5} \cdot [X^*]^3 = 0$$

$$\left\{ \left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1 \right] \cdot \mu_1 + \left(\mu_6 - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} \right) \cdot \frac{\mu_2}{\mu_5} \cdot [X^*]^2 \right\} \cdot X^* = 0$$

$$X^*_{(1)} = 0 \Rightarrow Y^*_{(1)} = 0 \Rightarrow Z^*_{(1)} = 0;$$

$$\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1\right] \cdot \mu_1 + \left(\mu_6 - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)}\right) \cdot \frac{\mu_2}{\mu_5} \cdot [X^*]^2 = 0$$

$$[X^*]^2 = \frac{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1\right] \cdot \mu_1}{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - \mu_6\right] \cdot \frac{\mu_2}{\mu_5}} \Rightarrow X^*_{(2,3)} = \pm\sqrt{\frac{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1\right] \cdot \mu_1}{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - \mu_6\right] \cdot \frac{\mu_2}{\mu_5}}}$$

$$\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1 = \frac{-\mu_2 \cdot (\mu_3 + \mu_4)}{\mu_1 + \mu_2 \cdot \mu_4};$$

$$\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - \mu_6 = \frac{\mu_1 - \mu_2 \cdot \mu_3 - \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4)}{\mu_1 + \mu_2 \cdot \mu_4}$$

$$X^*_{(2,3)} = \pm\sqrt{\frac{\frac{-\mu_2 \cdot (\mu_3 + \mu_4)}{\mu_1 + \mu_2 \cdot \mu_4} \cdot \mu_1}{\left[\frac{\mu_1 - \mu_2 \cdot \mu_3 - \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4)}{\mu_1 + \mu_2 \cdot \mu_4}\right] \cdot \frac{\mu_2}{\mu_5}}};$$

$$X^*_{(2,3)} = \pm\sqrt{\frac{\mu_1 \cdot \mu_5 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1}}$$

$$Y^*_{(2,3)} = \pm\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} \cdot \sqrt{\frac{\mu_1 \cdot \mu_5 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1}}$$

$$Z^*_{(2,3)} = \frac{\mu_6 \cdot \mu_1 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1} - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} \cdot \frac{\mu_1 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1}$$

$$Z^*_{(2,3)} = \frac{\mu_1 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1} \cdot \left[\mu_6 - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)}\right]$$

$$Z^*_{(2,3)} = \frac{\mu_6}{\mu_5} \cdot [X^*]^2 - \frac{1}{\mu_5} X^* \cdot Y^* = \frac{\mu_6}{\mu_5} \cdot \frac{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1\right] \cdot \mu_1}{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - \mu_6\right] \cdot \frac{\mu_2}{\mu_5}}$$

$$- \frac{1}{\mu_5} \cdot \left(\frac{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1\right] \cdot \mu_1}{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - \mu_6\right] \cdot \frac{\mu_2}{\mu_5}}\right) \cdot \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)}$$

$$Z^*_{(2,3)} = \left[\mu_6 - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)}\right] \cdot \left[\frac{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - 1\right] \cdot \mu_1}{\left[\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} - \mu_6\right] \cdot \mu_2}\right] = \left[1 - \frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)}\right] \cdot \frac{\mu_1}{\mu_2}$$

$$Z^*_{(2,3)} = \frac{(\mu_4 + \mu_3) \cdot \mu_1}{\mu_1 + \mu_2 \cdot \mu_4}.$$

The system has three fixed points (equilibrium points), one located on the symmetry axis at the origin $(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0, 0, 0)$. The other two fixed points are symmetry-related fixed points.

$$X^*_{(2,3)} = \pm\sqrt{\frac{\mu_1 \cdot \mu_5 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1}}$$

$$Y^*_{(2,3)} = \pm\frac{(\mu_1 - \mu_2 \cdot \mu_3)}{(\mu_1 + \mu_2 \cdot \mu_4)} \cdot \sqrt{\frac{\mu_1 \cdot \mu_5 \cdot (\mu_3 + \mu_4)}{\mu_2 \cdot \mu_3 + \mu_6 \cdot (\mu_1 + \mu_2 \cdot \mu_4) - \mu_1}};$$

$$Z^*_{(2,3)} = \frac{(\mu_4 + \mu_3) \cdot \mu_1}{\mu_1 + \mu_2 \cdot \mu_4}.$$

Stability classification of the first fixed point: $(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0, 0, 0)$

$$\left.\frac{\partial f_1}{\partial X}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = -\mu_1; \quad \left.\frac{\partial f_1}{\partial Y}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = \mu_1;$$

$$\left.\frac{\partial f_1}{\partial Z}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = \mu_2 \quad \left.\frac{\partial f_2}{\partial X}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = \mu_3;$$

$$\left.\frac{\partial f_2}{\partial Y}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = \mu_4; \quad \left.\frac{\partial f_2}{\partial Z}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = 0$$

$$\left.\frac{\partial f_3}{\partial X}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = 0;$$

$$\left.\frac{\partial f_3}{\partial Y}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = 0;$$

$$\left.\frac{\partial f_3}{\partial Z}\right|_{(X^*_{(1)}, Y^*_{(1)}, Z^*_{(1)}) = (0,0,0)} = \mu_5$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}_{(X^*_{(1)},Y^*_{(1)},Z^*_{(1)})=(0,0,0)} = \begin{pmatrix} -\mu_1 & \mu_1 & \mu_2 \\ \mu_3 & \mu_4 & 0 \\ 0 & 0 & \mu_5 \end{pmatrix};$$

$$A - \lambda \cdot I = \begin{pmatrix} -\mu_1 - \lambda & \mu_1 & \mu_2 \\ \mu_3 & \mu_4 - \lambda & 0 \\ 0 & 0 & \mu_5 - \lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \det \begin{pmatrix} -\mu_1 - \lambda & \mu_1 & \mu_2 \\ \mu_3 & \mu_4 - \lambda & 0 \\ 0 & 0 & \mu_5 - \lambda \end{pmatrix} = 0$$

$$\det(A - \lambda \cdot I) = -(\mu_1 + \lambda) \cdot \det \begin{pmatrix} \mu_4 - \lambda & 0 \\ 0 & \mu_5 - \lambda \end{pmatrix} - \mu_1 \cdot \det \begin{pmatrix} \mu_3 & 0 \\ 0 & \mu_5 - \lambda \end{pmatrix} + \mu_2 \cdot \det \begin{pmatrix} \mu_3 & \mu_4 - \lambda \\ 0 & 0 \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = -(\mu_1 + \lambda) \cdot (\mu_4 - \lambda) \cdot (\mu_5 - \lambda) - \mu_1 \cdot \mu_3 \cdot (\mu_5 - \lambda)$$

$$\det(A - \lambda \cdot I) = -\lambda^3 + \lambda^2 \cdot (\mu_4 + \mu_5 - \mu_1) + \lambda \cdot ([\mu_4 + \mu_5] \cdot \mu_1 - \mu_4 \cdot \mu_5 + \mu_1 \cdot \mu_3) - \mu_1 \cdot \mu_5 \cdot (\mu_4 + \mu_3)$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^3 + \lambda^2 \cdot (\mu_1 - \mu_4 - \mu_5) + \lambda \cdot (\mu_4 \cdot \mu_5 - [\mu_4 + \mu_5] \cdot \mu_1 - \mu_1 \cdot \mu_3) + \mu_1 \cdot \mu_5 \cdot (\mu_4 + \mu_3) = 0.$$

We define the following parameters: $\varsigma_3 = 1$; $\varsigma_2 = (\mu_1 - \mu_4 - \mu_5)$

$$\varsigma_1 = \mu_4 \cdot \mu_5 - [\mu_4 + \mu_5] \cdot \mu_1 - \mu_1 \cdot \mu_3; \quad \varsigma_0 = \mu_1 \cdot \mu_5 \cdot (\mu_4 + \mu_3)$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \sum_{k=0}^{3} \lambda^k \cdot \varsigma_k = 0.$$

Eigenvalues Stability Discussion Our Li dynamical system involving N variables ($N > 2$, $N = 3$), the characteristic equation is of degree $N = 3$ and must often be solved numerically. Expect in some particular cases, such an equation has ($N = 3$) distinct roots that can be real or complex. These values are the eigenvalues of the 3×3 Jacobian matrix (A). The general rule is that the Steady State (SS) is stable if there is no eigenvalue with positive real part. It is sufficient that one eigenvalue is positive for the steady state to be unstable. Our 3-variables (X, Y, Z) Li dynamical system has three eigenvalues. The type of behavior can be characterized as a function of the position of these eigenvalues in the Re/Im plane. Five non-degenerated cases can be distinguished: (1) the three eigenvalues are real and negative (stable steady state), (2) the three eigenvalues are real, two of them are negative (unstable steady state), (3) and (4) two eigenvalues are complex conjugates with a negative real part and the other eigenvalues are real and negative (stable steady state), two cases can be distinguished depending on the relative value of the real part of the complex eigenvalues and of the real one, (5) two eigenvalues are complex conjugates with a negative real part and other eigenvalue is positive (unstable steady state) [5, 6].

Stability classification of the second and third fixed points :

$$X^*_{(2,3)} = \pm\sqrt{\frac{\mu_1\cdot\mu_5\cdot(\mu_3+\mu_4)}{\mu_2\cdot\mu_3+\mu_6\cdot(\mu_1+\mu_2\cdot\mu_4)-\mu_1}}$$

$$Y^*_{(2,3)} = \pm\frac{(\mu_1-\mu_2\cdot\mu_3)}{(\mu_1+\mu_2\cdot\mu_4)}\cdot\sqrt{\frac{\mu_1\cdot\mu_5\cdot(\mu_3+\mu_4)}{\mu_2\cdot\mu_3+\mu_6\cdot(\mu_1+\mu_2\cdot\mu_4)-\mu_1}};$$

$$Z^*_{(2,3)} = \frac{(\mu_4+\mu_3)\cdot\mu_1}{\mu_1+\mu_2\cdot\mu_4}$$

Functions (f_1,f_2,f_3) partial derivatives at second and third fixed points:

$$\frac{\partial f_1}{\partial X}=-\mu_1+\mu_2\cdot Z^*_{(2,3)};\ \frac{\partial f_1}{\partial Y}=\mu_1;\ \frac{\partial f_1}{\partial Z}=\mu_2\cdot X^*_{(2,3)};\ \frac{\partial f_2}{\partial X}=\mu_3-Z^*_{(2,3)};\ \frac{\partial f_2}{\partial Y}=\mu_4$$

$$\frac{\partial f_2}{\partial Z}=-X^*_{(2,3)};\ \frac{\partial f_3}{\partial X}=Y^*_{(2,3)}-2\cdot\mu_6\cdot X^*_{(2,3)};\ \frac{\partial f_3}{\partial Y}=X^*_{(2,3)};\ \frac{\partial f_3}{\partial Z}=\mu_5$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{pmatrix}_{(X^*_{(1)},Y^*_{(1)},Z^*_{(1)})=(0,0,0)} = \begin{pmatrix} -\mu_1+\mu_2\cdot Z^*_{(2,3)} & \mu_1 & \mu_2\cdot X^*_{(2,3)} \\ \mu_3-Z^*_{(2,3)} & \mu_4 & -X^*_{(2,3)} \\ Y^*_{(2,3)}-2\cdot\mu_6\cdot X^*_{(2,3)} & X^*_{(2,3)} & \mu_5 \end{pmatrix}$$

$$A-\lambda\cdot I = \begin{pmatrix} -\mu_1+\mu_2\cdot Z^*_{(2,3)}-\lambda & \mu_1 & \mu_2\cdot X^*_{(2,3)} \\ \mu_3-Z^*_{(2,3)} & \mu_4-\lambda & -X^*_{(2,3)} \\ Y^*_{(2,3)}-2\cdot\mu_6\cdot X^*_{(2,3)} & X^*_{(2,3)} & \mu_5-\lambda \end{pmatrix};\quad \det(A-\lambda\cdot I)=0$$

$$A-\lambda\cdot I = (\mu_2\cdot Z^*_{(2,3)}-\mu_1-\lambda)\cdot\det\begin{pmatrix} \mu_4-\lambda & -X^*_{(2,3)} \\ X^*_{(2,3)} & \mu_5-\lambda \end{pmatrix} - \mu_1\cdot\det\begin{pmatrix} \mu_3-Z^*_{(2,3)} & -X^*_{(2,3)} \\ Y^*_{(2,3)}-2\cdot\mu_6\cdot X^*_{(2,3)} & \mu_5-\lambda \end{pmatrix}$$
$$+\mu_2\cdot X^*_{(2,3)}\cdot\det\begin{pmatrix} \mu_3-Z^*_{(2,3)} & \mu_4-\lambda \\ Y^*_{(2,3)}-2\cdot\mu_6\cdot X^*_{(2,3)} & X^*_{(2,3)} \end{pmatrix}$$

$$A - \lambda \cdot I = (\mu_2 \cdot Z^*_{(2,3)} - \mu_1 - \lambda) \cdot [(\mu_4 - \lambda) \cdot (\mu_5 - \lambda) + (X^*_{(2,3)})^2]$$
$$- \mu_1 \cdot [(\mu_3 - Z^*_{(2,3)}) \cdot (\mu_5 - \lambda) + (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot X^*_{(2,3)}]$$
$$+ \mu_2 \cdot X^*_{(2,3)} \cdot [(\mu_3 - Z^*_{(2,3)}) \cdot X^*_{(2,3)} - (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot (\mu_4 - \lambda)]$$

$$A - \lambda \cdot I = ([\mu_2 \cdot Z^*_{(2,3)} - \mu_1] - \lambda) \cdot [\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2 - \lambda \cdot (\mu_4 + \mu_5) + \lambda^2]$$
$$- \mu_1 \cdot [(\mu_3 - Z^*_{(2,3)}) \cdot \mu_5 - \lambda \cdot (\mu_3 - Z^*_{(2,3)}) + (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot X^*_{(2,3)}]$$
$$+ \mu_2 \cdot X^*_{(2,3)} \cdot [(\mu_3 - Z^*_{(2,3)}) \cdot X^*_{(2,3)} - (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot \mu_4 + (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot \lambda]$$

$$A - \lambda \cdot I = [\mu_2 \cdot Z^*_{(2,3)} - \mu_1] \cdot (\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2) - \lambda \cdot (\mu_4 + \mu_5) \cdot (\mu_2 \cdot Z^*_{(2,3)} - \mu_1)$$
$$+ \lambda^2 \cdot (\mu_2 \cdot Z^*_{(2,3)} - \mu_1) - \lambda \cdot (\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2) + \lambda^2 \cdot (\mu_4 + \mu_5) - \lambda^3 - \mu_1 \cdot (\mu_3 - Z^*_{(2,3)}) \cdot \mu_5$$
$$+ \lambda \cdot \mu_1 \cdot (\mu_3 - Z^*_{(2,3)}) - \mu_1 \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot X^*_{(2,3)} + \mu_2 \cdot X^*_{(2,3)} \cdot (\mu_3 - Z^*_{(2,3)}) \cdot X^*_{(2,3)}$$
$$- \mu_2 \cdot X^*_{(2,3)} \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot \mu_4 + \mu_2 \cdot X^*_{(2,3)} \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot \lambda$$

$$A - \lambda \cdot I = \Big\{[\mu_2 \cdot Z^*_{(2,3)} - \mu_1] \cdot (\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2) - \mu_1 \cdot (\mu_3 - Z^*_{(2,3)}) \cdot \mu_5$$
$$- \mu_1 \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot X^*_{(2,3)} + \mu_2 \cdot X^*_{(2,3)} \cdot (\mu_3 - Z^*_{(2,3)}) \cdot X^*_{(2,3)}$$
$$- \mu_2 \cdot X^*_{(2,3)} \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot \mu_4\Big\}$$
$$+ \lambda \cdot \Big\{\mu_1 \cdot (\mu_3 - Z^*_{(2,3)}) - (\mu_4 + \mu_5) \cdot (\mu_2 \cdot Z^*_{(2,3)} - \mu_1) - (\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2)$$
$$+ \mu_2 \cdot X^*_{(2,3)} \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)})\Big\} + \lambda^2 \cdot (\mu_2 \cdot Z^*_{(2,3)} - \mu_1) + \lambda^2 \cdot (\mu_4 + \mu_5) - \lambda^3$$

We define the following parameters:

$$\Pi_0 = [\mu_2 \cdot Z^*_{(2,3)} - \mu_1] \cdot (\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2) - \mu_1 \cdot (\mu_3 - Z^*_{(2,3)}) \cdot \mu_5 - \mu_1 \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot X^*_{(2,3)}$$
$$+ \mu_2 \cdot X^*_{(2,3)} \cdot (\mu_3 - Z^*_{(2,3)}) \cdot X^*_{(2,3)} - \mu_2 \cdot X^*_{(2,3)} \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)}) \cdot \mu_4$$

$$\Pi_1 = \mu_1 \cdot (\mu_3 - Z^*_{(2,3)}) - (\mu_4 + \mu_5) \cdot (\mu_2 \cdot Z^*_{(2,3)} - \mu_1) - (\mu_4 \cdot \mu_5 + (X^*_{(2,3)})^2)$$
$$+ \mu_2 \cdot X^*_{(2,3)} \cdot (Y^*_{(2,3)} - 2 \cdot \mu_6 \cdot X^*_{(2,3)})$$

$$\Pi_2 = \mu_2 \cdot Z^*_{(2,3)} - \mu_1; \Pi_3 = -1; \quad \det(A - \lambda \cdot I) = 0 \Rightarrow \sum_{k=0}^{3} \lambda^k \cdot \Pi_k = 0.$$

Eigenvalues Stability Discussion Our Li dynamical system involving N variables ($N > 2$, $N = 3$), the characteristic equation is of degree $N = 3$ and must often be solved numerically. Expect in some particular cases, such an equation has ($N = 3$) distinct roots that can be real or complex. These values are the eigenvalues of the 3×3 Jacobian matrix (A). The general rule is that the Steady State (SS) is stable if there is no eigenvalue with positive real part. It is sufficient that one eigenvalue is positive for the steady state to be unstable. Our 3-variables (X, Y, Z) Li dynamical system has three eigenvalues. The type of behavior can be characterized as a function of the position of these eigenvalues in the Re/Im plane. Five non-degenerated cases can be distinguished: (1) the three eigenvalues are real and negative (stable steady state), (2) the three eigenvalues are real, two of them are negative (unstable steady state), (3) and (4) two eigenvalues are complex conjugates with a negative real part and the other eigenvalues are real and negative (stable steady state), two cases can be distinguished depending on the relative value of the real part of the complex eigenvalues and of the real one, (5) two eigenvalues are complex conjugates with a negative real part and other eigenvalue is positive (unstable steady state).

Li dynamical system Poincare map discussion: The chaotic dynamics of Li system can be inspected by Poincare surface section. We need to choose a global Poincare surface of section for low dimensional attractors. The missing details are the bounding torus that contains the attractor. The Li system attractor is contained within a sphere and that this sphere is presented by two holes, one surrounding the z-axis, the other surrounding the x-axis. These two axes intersect at $(0, 0, Z^*_{(2,3)}) = \left(0, 0, \frac{(\mu_4 + \mu_3)\cdot\mu_1}{\mu_1 + \mu_2\cdot\mu_4}\right)$. The global Poincare surface of section has two disconnect components. We can trace these two components to the corresponding components of the Poincare surface in the original phase space $\mathbb{R}^3$. None of the intersections with the two components of the Poincare surface of section occurs with $Z < 50$. Intersections with the Poincare section locate segments of the chaotic trajectory very close to unstable periodic orbits. These segments were located by searching for close returns in the Poincare section. The topological period of these orbits is the number of distinct intersections with the Poincare section. Li system exhibits a chaotic attractor with an unusual topological structure. The Li system attractor is contained in a three dimensional space that is topological equivalent to a solid sphere pierced by two intersecting holes. The global Poincare surface of section consists of two disjoint components . Li dynamical system attractor exists in a genus—three bounding torus [8, 9].

Plotting Li dynamical system: phase planes and variables (X, Y, Z) time behavior. First we choose our Li system parameters values $(U1, U2, \ldots, U6)$ [98].

$$\mu_1 \to U1,\ \mu_2 \to U2,\ \mu_3 \to U3,\ \mu_4 \to U4,\ \mu_5 \to U5,\ \mu_6 \to U6$$

MATLAB scripts

```
function h=LiSystem1 (U1,U2,U3,U4,U5,U6,X0,Y0,Z0)
[t,x]=ODE45(@LiSystem,[0,1],[X0,Y0,Z0],[],U1,U2,U3,U4,U5,U6);
plot3 (x(:,1),x(:,2),x(:,3));
xlabel ('X')%x-axis
ylabel ('Y')%y-axis
zlabel ('Z')%z-axis
grid on
axis square
%plot(t,x);
%plot(x(:,1),x(:,2));%X=dI/dt (x-axis) and I (y-axis)
%plot(x(:,1),x(:,3));%X=dI/dt (x-axis) and VCEQ1 (y-axis)
%plot(x(:,2),x(:,3));%I (x-axis) and VCEQ1 (y-axis)

function g=LiSystem(t,x,U1,U2,U3,U4,U5,U6)
g=zeros(3,1);% the elements of the vector g represent the right hand
sided of the three DEs
g(1)=U1*(x(2)-x(1))+U2*x(1)*x(3);
g(2)=U3*x(1)+U4*x(2)-x(1)*x(3);
g(3)=U5*x(3)+x(1)*x(2)-U6*x(1)*x(1);
```

First Case $\mu_1 = 41,\ \mu_2 = 0.16,\ \mu_3 = 55,\ \mu_4 = 20,\ \mu_5 = 11,\ \mu_6 = 0.65$
Initial condition: $X_0 = 1;\ Y_0 = 1;\ Z_0 = 1$ (Fig. 7.5).

Second Case $\mu_1 = 41,\ \mu_2 = 0.16,\ \mu_3 = 55,\ \mu_4 = 20,\ \mu_5 = \frac{11}{6},\ \mu_6 = 0.65$
Initial condition: $X_0 = 1;\ Y_0 = 1;\ Z_0 = 1$ (Fig. 7.6).

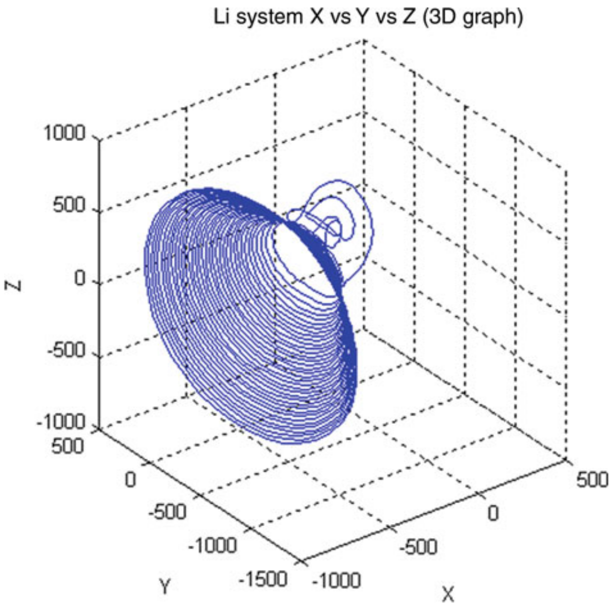
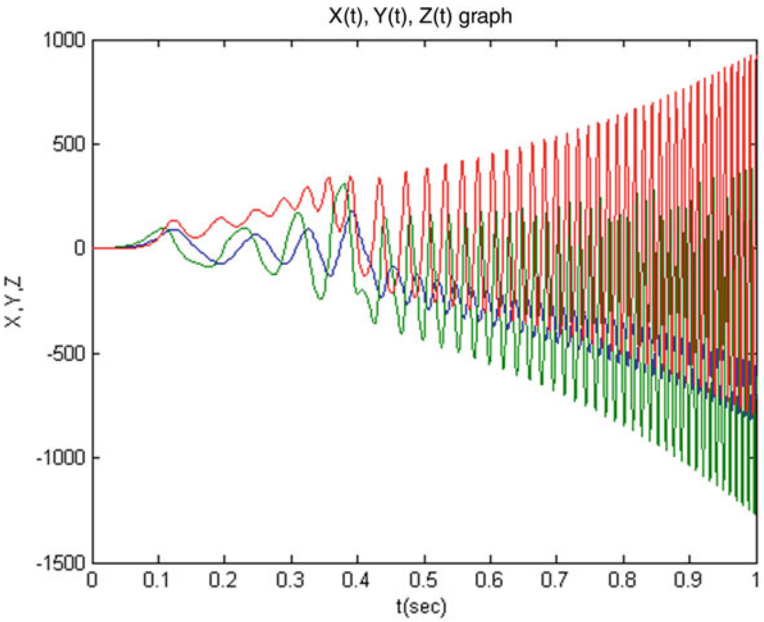


Fig. 7.5 Li system (2D) and (3D) graphs, $X(t)$, $Y(t)$ and $Z(t)$ graph



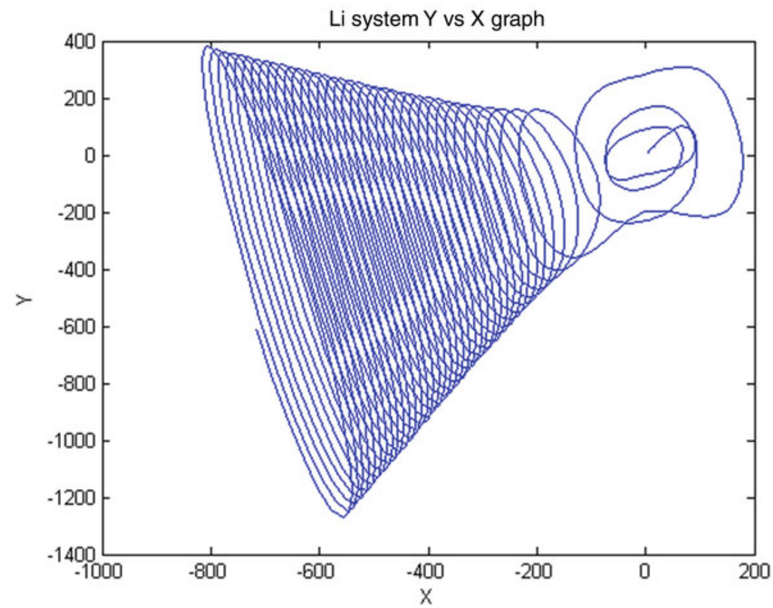


Fig. 7.5 (continued)

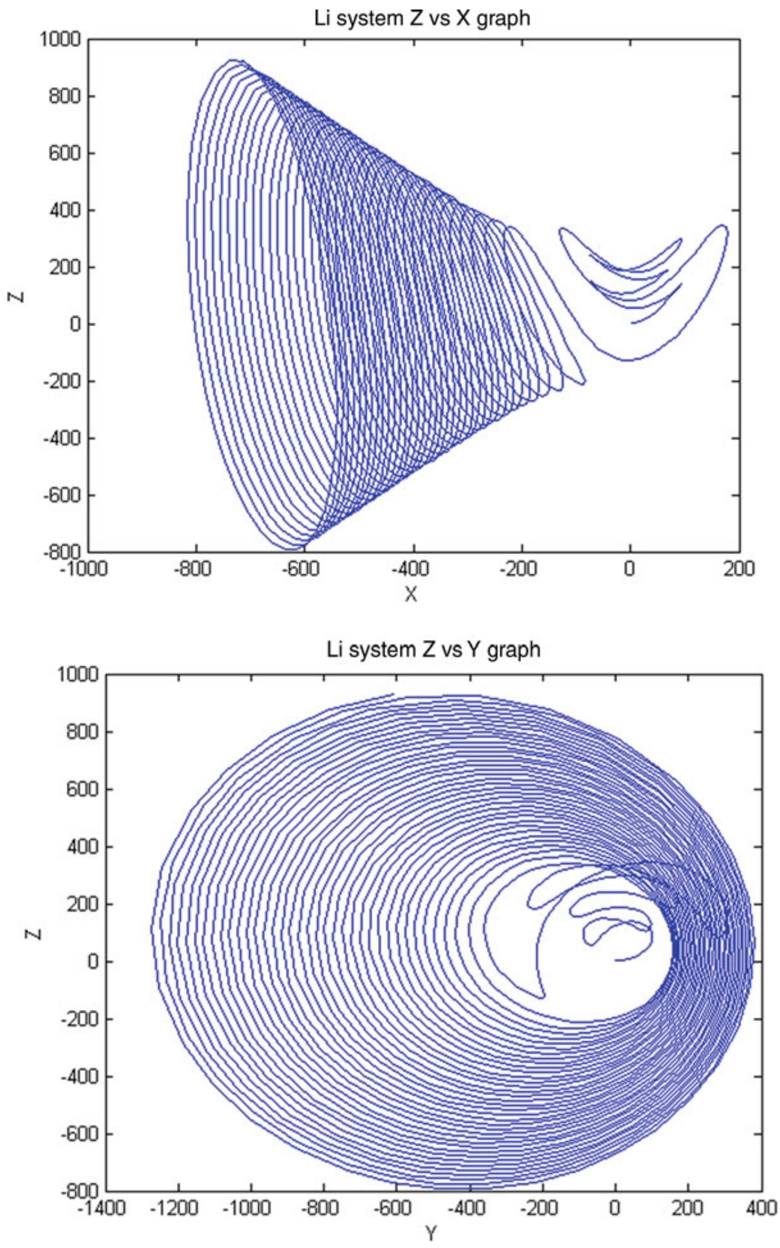
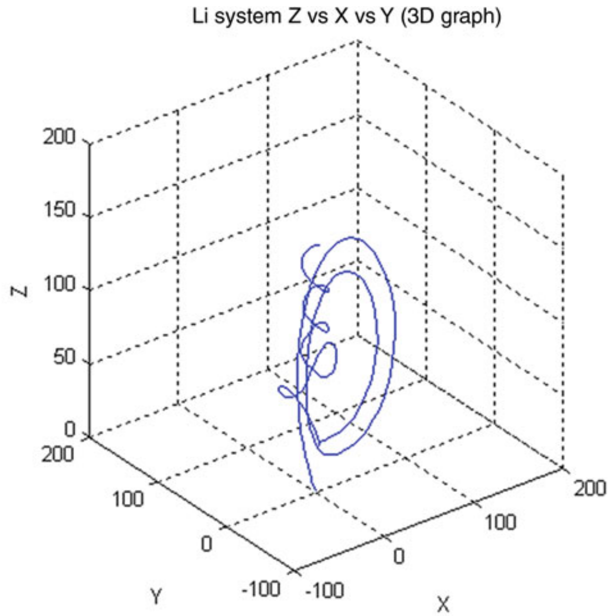


Fig. 7.5 (continued)



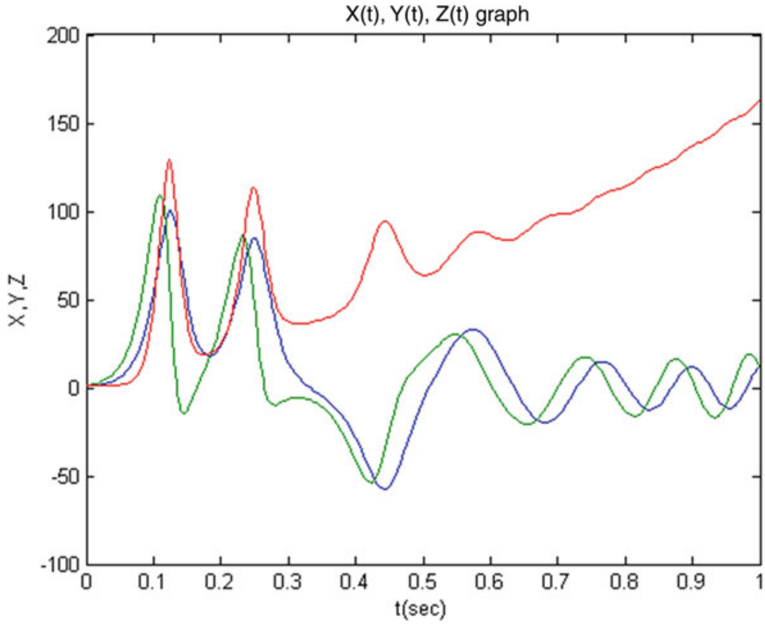


Fig. 7.6 Li system *Z* vs *X* vs *Y* (3D graph) and *X*(*t*), *Y*(*t*), and *Z*(*t*) graph and *Y* vs *X* graph and *Z* vs *X* graph and *Z* vs *Y* graph

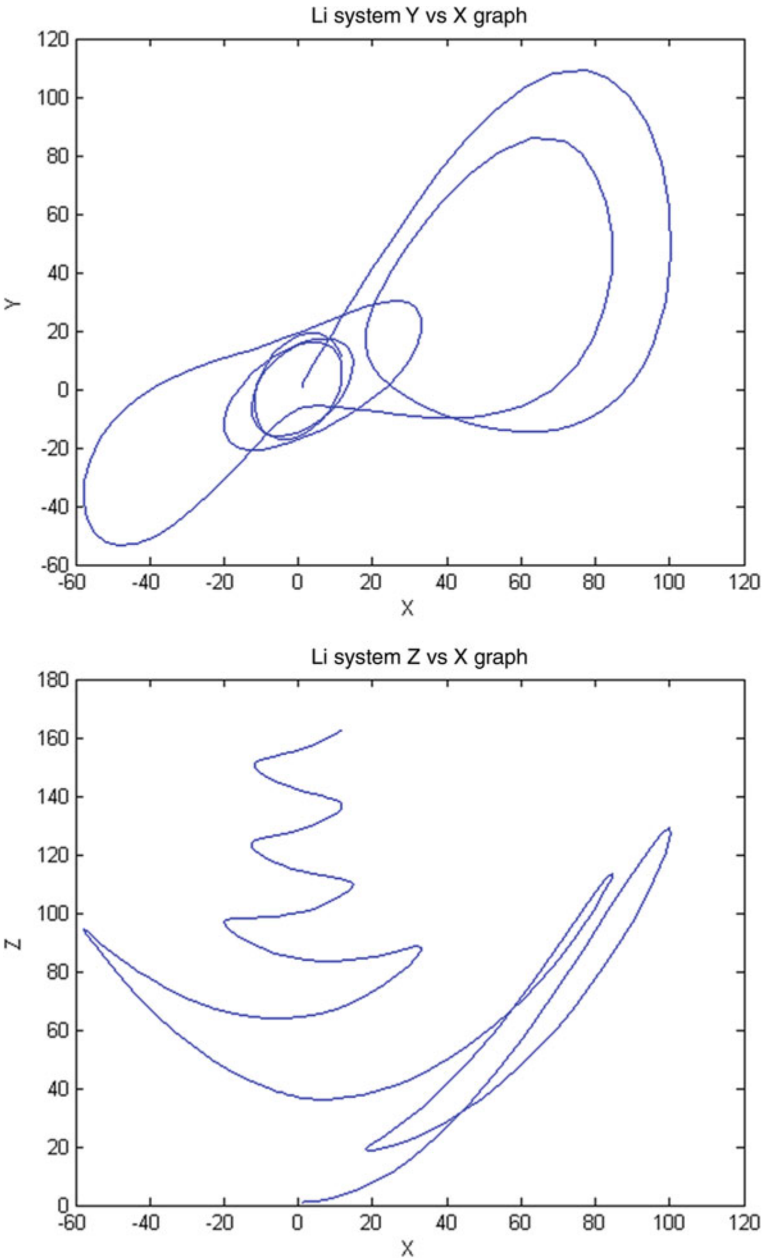


Fig. 7.6 (continued)

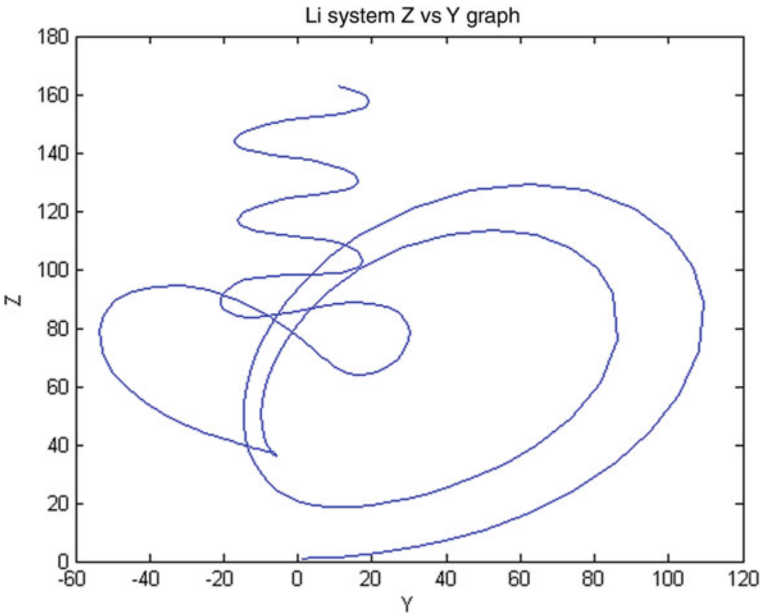


Fig. 7.6 (continued)

7.4 OptoNDR Negative Differential Resistance (NDR) Oscillator Circuit Poincare Map and Periodic Orbit

One way for the production of oscillations in L-C networks is to overcome circuit losses through the use of designed-in positive feedback or generation. Two alternatives exist for a potential oscillator. In the first alternative, the tuned active network may have excess loss. The second alternative is that of a successful oscillator where excess loss will have been compensated and a sustained oscillation is obtained at the circuit's output. We can distinguish two sine wave oscillator types. The feedback and the negative resistance oscillator, Opto NDR device has better performance characteristics cancelation in terms of resistive losses in oscillators than oscillators that are not based on NDR devices. All cancelation of resistive losses methods shown to be equivalent in the results and losses being effectively canceled out by the negative differential resistance contributed by the active device and associated reactive components. Two ways of carrying out this process are series and a parallel L-C circuit. In the case that the loss cancelation is incomplete, the loop gain of a sine wave oscillator will be less than unity and oscillations will not start building up. On the other hand, if the gain is close to unity the circuit will behave as a regenerative or high gain narrow band tuned amplifier. OptoNDR is one of compound active device that exhibit negative resistance regions

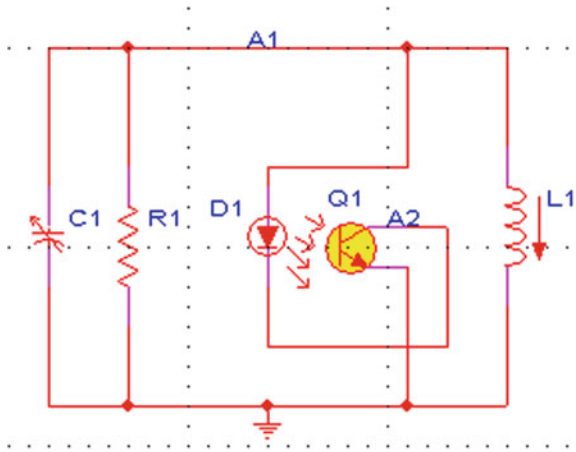


Fig. 7.7 Parallel circuit with OptoNDR circuit, C_1, L_1, and R_1 elements

on their static I–V characteristics curves. This device can be successfully employed in construction of L-C oscillators and regenerative amplifiers. We consider that a source of voltage is attached to the circuit's capacitor C_1 and then withdrawn. Initially, the electrical energy from the capacitor C_1 is transferred into the magnetic energy of the inductor. When the electrical energy of the capacitor becomes zero, the process is reserved. The magnetic energy from the inductor is transferred into the electrical energy of the capacitor. Electromagnetic oscillation occurs when energy is transferring between the capacitor and inductor. Parasitic resistance R_1 in the system causes the oscillation to damp and Opto NDR is the negative element which postpones the damping process [85, 86] (Fig. 7.7)

Our OptoNDR element/circuit is constructed from LED and photo transistor in series. The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant ($I_{BQ_1} = I_{LED} \cdot k = I_{D_1} \cdot k$; $I_{BQ_1} = I_{CQ_1} \cdot k$) and is the phototransistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations. The basic Ebers–Moll schematic for NPN bipolar transistor is shown in the next figure. We need to implement the regular Ebers-Moll model to the opto coupler circuit (transistor Q_1 and LED D_1) and get a complete final expression for the Negative Differential Resistance (NDR) characteristics of that circuit [18] (Fig. 7.8).

$$i_{DE} + i_{DC} = i_{bQ_1} + \alpha_f \cdot i_{DEQ_1} + \alpha_r \cdot i_{DCQ_1};$$
$$i_{DCQ_1} + I_{CQ_1} = \alpha_f \cdot i_{DEQ_1};$$
$$i_{DEQ_1} = \alpha_r \cdot i_{DCQ_1} + i_{EQ_1}$$

$$i_{DCQ_1} + I_{CQ_1} = \alpha_f \cdot i_{DEQ_1} \Rightarrow i_{DCQ_1} = \alpha_f \cdot i_{DEQ_1} - I_{CQ_1};$$
$$i_{DEQ_1} = \alpha_r \cdot (\alpha_f \cdot i_{DEQ_1} - I_{CQ_1}) + i_{EQ_1}$$

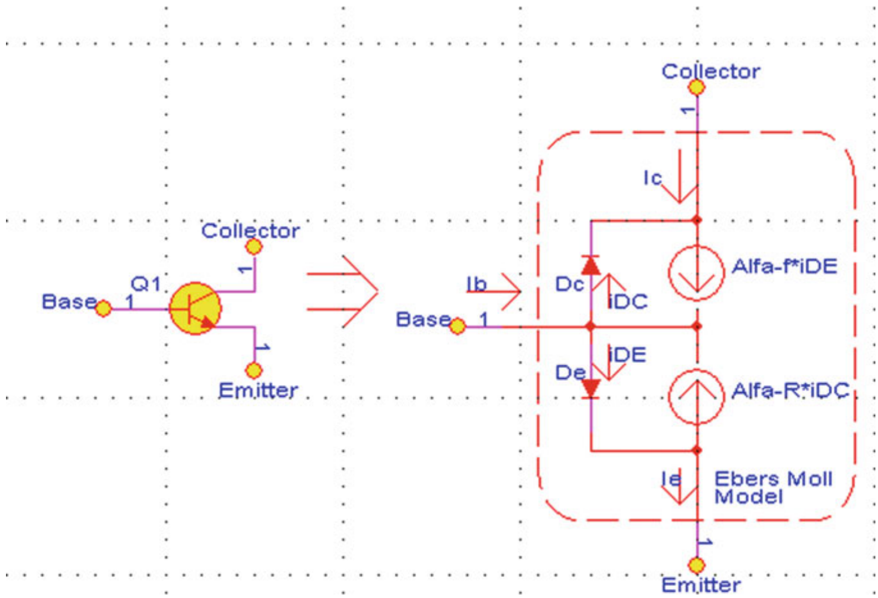


Fig. 7.8 Ebers-Moll schematic for NPN bipolar transistor

$$i_{DEQ_1} = \alpha_r \cdot \alpha_f \cdot i_{DEQ_1} - \alpha_r \cdot I_{CQ_1} + i_{EQ_1};$$
$$i_{DEQ_1} - \alpha_r \cdot \alpha_f \cdot i_{DEQ_1} = i_{EQ_1} - \alpha_r \cdot I_{CQ_1}$$

$$i_{DEQ_1} = \frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f};$$
$$i_{DCQ_1} = \alpha_f \cdot i_{DEQ_1} - I_{CQ_1} = \alpha_f \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) - I_{CQ_1}$$

$$\begin{aligned} i_{DCQ_1} &= \alpha_f \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) - I_{CQ_1} \\ &= \frac{\alpha_f \cdot (i_{EQ_1} - \alpha_r \cdot I_{CQ_1}) - I_{CQ_1} \cdot (1 - \alpha_r \cdot \alpha_f)}{1 - \alpha_r \cdot \alpha_f} \end{aligned}$$

$$i_{DCQ_1} = \frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f};$$
$$V_{\text{Base-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot i_{DEQ_1} + 1\right];$$
$$V_{\text{Base-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Base-Collector}} = V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot i_{DCQ_1} + 1\right];$$

$$V_{\text{Base-Collector}} = V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot \left(\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Collector-Emitter}} = V_{\text{Collector-Base}} + V_{\text{Base-Emitter}};$$

$$V_{\text{Collector-Base}} = -V_{\text{Base-Collector}}$$

$$V_{\text{Collector-Emitter}} = V_{\text{Collector-Base}} + V_{\text{Base-Emitter}} = V_{\text{Base-Emitter}} - V_{\text{Base-Collector}}$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{1}{I_{se}} \cdot \left(\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right] - V_t \cdot \ln\left[\frac{1}{I_{sc}} \cdot \left(\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1}}{1 - \alpha_r \cdot \alpha_f}\right) + 1\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] - V_t \cdot \ln\left[\frac{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\left\{\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\} \cdot \left\{\frac{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\left\{\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right\} \cdot \left\{\frac{I_{sc}}{I_{se}}\right\}\right]$$

$$V_{\text{Collector-Emitter}} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right];$$

$$I_{CQ_1} = I_{D_1}$$

$$V_{D_1} = V_t \cdot \ln\left[\frac{I_{D_1}}{I_0} + 1\right];$$

$$V_{D_1} = V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right];$$

$$i_{EQ_1} \to I_{EQ_1}; \quad i_{CQ_1} \to I_{CQ_1}.$$

The optical coupling between the LED (D_1) to the phototransistor (Q_1) is represented as transistor dependent base current on LED (D_1) current.

$$I_{BQ_1} = I_{D_1} \cdot k; \quad I_{D_1} = I_{CQ_1}; \quad I_{BQ_1} = I_{CQ_1} \cdot k; \quad I_{EQ_1} = I_{CQ_1} + I_{BQ_1} = I_{CQ_1} \cdot (1+k)$$

$$I_{EQ_1} = I_{CQ_1} \cdot (1+k) \Rightarrow I_{CQ_1} = \frac{I_{EQ_1}}{(1+k)}; \quad I_{BQ_1} = I_{CQ_1} \cdot k = \frac{I_{EQ_1} \cdot k}{(1+k)}.$$

As long as the phototransistor (Q_1) is in cut off region, the current I_{CQ_1}, I_{EQ_1} and I_{BQ_1} are very low. When the phototransistor (Q_1) reaches break over voltage it enters saturation region ($V_{\text{Collector-Emitter}}$ decreases and I_{CQ_1} increases). The region which $V_{\text{Collector-Emitter}}$ decreases and I_{CQ_1} increases is the Negative Differential Resistance area of V_{A1}-I_{CQ1} characteristics. The positive feedback in which the phototransistor collector current I_{CQ_1} increases and then I_{BQ_1} increases ($I_{BQ_1} = I_{CQ_1} \cdot k$) is repeated in increasing cycles [1]. The positive feedback ends when the phototransistor reaches saturation state. Finally, we arrive at an expression which is the voltage $V_{\text{Collector-Emitter}}$ as a function of the current (I_{CQ_1}) for NDR circuit ($V_{A_1} = V_{\text{Collector-Emitter}} + V_{D_1}$).

$$\begin{aligned} V_{A_1} &= V_{\text{Collector-Emitter}} + V_{D_1} \\ &= V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right] \end{aligned}$$

Assume: $I_{sc} \approx I_{se};\ V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] \approx 0$

$$V_{A_1} = V_t \cdot \ln\left[\frac{i_{EQ_1} - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot i_{EQ_1} - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]$$

$$I_{CQ_1} = \frac{I_{EQ_1}}{(1+k)}; \quad I_{BQ_1} = I_{CQ_1} \cdot k = \frac{I_{EQ_1} \cdot k}{(1+k)}; \quad I_{EQ_1} = I_{CQ_1} \cdot (1+k)$$

$$V_{A_1} = V_t \cdot \ln\left[\frac{I_{CQ_1} \cdot (1+k) - \alpha_r \cdot I_{CQ_1} + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\alpha_f \cdot I_{CQ_1} \cdot (1+k) - I_{CQ_1} + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]$$

$$V_{A_1} = V_t \cdot \ln\left[\frac{I_{CQ_1} \cdot (1+k-\alpha_r) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{CQ_1} \cdot (\alpha_f \cdot (1+k) - 1) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]$$

$$\begin{aligned} & I_{R_1} = \frac{V_{A_1}}{R_1}; \quad V_{L_1} = L_1 \cdot \frac{dI_{L_1}}{dt}; \\ & I_{C_1} = C_1 \cdot \frac{dV_{C_1}}{dt}; \quad V_{A_1} = V_{C_1} = V_{R_1} = V_{L_1}; \\ & I_{C_1} + I_{R_1} + I_{CQ_1} + I_{L_1} = 0 \end{aligned}$$

$$V_{L_1} = L_1 \cdot \frac{\mathrm{d}I_{L_1}}{\mathrm{d}t} \Rightarrow V_{A_1} = L_1 \cdot \frac{\mathrm{d}I_{L_1}}{\mathrm{d}t};$$
$$I_{L_1} = \frac{1}{L_1} \cdot \int V_{A_1} \cdot \mathrm{d}t; \quad I_{R_1} = \frac{V_{A_1}}{R_1}; \quad I_{C_1} = C_1 \cdot \frac{\mathrm{d}V_{A_1}}{\mathrm{d}t}$$

$$I_{C_1} + I_{R_1} + I_{CQ_1} + I_{L_1} = 0 \Rightarrow C_1 \cdot \frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} + \frac{V_{A_1}}{R_1} + I_{CQ_1} + \frac{1}{L_1} \cdot \int V_{A_1} \cdot \mathrm{d}t = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{C_1 \cdot \frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} + \frac{V_{A_1}}{R_1} + I_{CQ_1} + \frac{1}{L_1} \cdot \int V_{A_1} \cdot \mathrm{d}t = 0\right\}$$
$$\Rightarrow C_1 \cdot \frac{\mathrm{d}^2 V_{A_1}}{\mathrm{d}t^2} + \frac{1}{R_1} \cdot \frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \frac{1}{L_1} \cdot V_{A_1} = 0$$

We start from V_{A1} equation.

$$V_{A_1} = V_t \cdot \ln\left[\frac{I_{CQ_1} \cdot (1 + k - \alpha_r) + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I_{CQ_1} \cdot (\alpha_f \cdot (1 + k) - 1) + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]$$

We define some global parameters for simplicity:

$$\Omega_1 = 1 + k - \alpha_r; \quad \Omega_2 = I_{se} \cdot (1 - \alpha_r \cdot \alpha_f);$$
$$\Omega_3 = \alpha_f \cdot (1 + k) - 1; \quad \Omega_4 = I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)$$

$$V_{A_1} = V_t \cdot \ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + V_t \cdot \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]$$

Remark Natural logarithmic function derivative.

$$\frac{\partial}{\partial x}\ln(f(x)); \quad \frac{\partial}{\partial x}\ln(x) = \frac{1}{x};$$
$$\frac{\partial}{\partial x}\ln(x) = \lim_{\Delta x \to 0} \frac{\ln(x + \Delta x) - \ln(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln\left(\frac{x + \Delta x}{x}\right)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\ln\left(\frac{x + \Delta x}{x}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \ln\left(\frac{x + \Delta x}{x}\right) = \lim_{\Delta x \to 0} \ln\left(\frac{x + \Delta x}{x}\right)^{\frac{1}{\Delta x}};$$
$$n = \frac{x}{\Delta x} \Rightarrow x = n \cdot \Delta x; \quad \frac{1}{\Delta x} = \frac{n}{x}$$

$$\lim_{\Delta x \to 0} \ln\left(\frac{x + \Delta x}{x}\right)^{\frac{1}{\Delta x}} = \lim_{\Delta x \to 0} \ln\left(\frac{x + \Delta x}{x}\right)^{n \cdot \frac{1}{x}}$$
$$= \lim_{\Delta x \to 0} \frac{1}{x} \cdot \ln\left(\frac{x + \Delta x}{x}\right)^n = \lim_{\Delta x \to 0} \frac{1}{x} \cdot \ln\left(1 + \frac{\Delta x}{x}\right)^n$$

$$n = \frac{x}{\Delta x}; \quad \Delta x \to 0 \Rightarrow n \to \infty; \quad \lim_{n\to\infty} \frac{1}{x} \cdot \ln\left(1 + \frac{\Delta x}{x}\right)^n;$$
$$n = \frac{x}{\Delta x} \Rightarrow \frac{\Delta x}{x} = \frac{1}{n}; \quad \lim_{n\to\infty} \frac{1}{x} \cdot \ln\left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{1}{x} \cdot \ln\left(1 + \frac{1}{n}\right)^n = \frac{1}{x} \cdot \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n;$$
$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \mathrm{e}; \quad \frac{1}{x} \cdot \ln(e) = \frac{1}{x}; \quad \frac{\partial}{\partial x}\ln(x) = \frac{1}{x}$$

$$\ln(f(x)); \quad g(x) = \ln(x); \quad h(x) = f(x);$$
$$g(h(x)) = \ln(f(x)); \quad \frac{\partial \ln(f(x))}{\partial x} = g'(h(x)) \cdot h'(x)$$

$$g'(x) = \frac{1}{x}; \quad g'(h(x)) = \frac{1}{f(x)} \cdot f'(x); \quad \frac{\partial}{\partial x}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x)$$

$$f(x(t)) \Rightarrow \frac{\partial}{\partial t}\ln(f(x(t))) = \frac{1}{f(x(t))} \cdot \frac{\partial f(x(t))}{\partial t}$$
$$= \frac{1}{f(x(t))} \cdot \frac{\partial f(x(t))}{\partial x} \cdot \frac{\partial x}{\partial t}$$

$$\frac{\partial}{\partial t}\ln(f(x)) = \frac{1}{f} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t};$$
$$x \to I_{CQ_1} \longrightarrow [\partial \leftrightarrow d]\ \frac{\mathrm{d}}{\mathrm{d}t}\ln(f(I_{CQ_1})) = \frac{1}{f} \cdot \frac{\mathrm{d}f}{\mathrm{d}I_{CQ_1}} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t}$$

$$f(I_{CQ_1}) = \frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4} \quad \text{or} \quad f(I_{CQ_1}) = \frac{I_{CQ_1}}{I_0} + 1$$

&&&

$$\frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} = V_t \cdot \frac{I_{CQ_1} \cdot \Omega_3 + \Omega_4}{I_{CQ_1} \cdot \Omega_1 + \Omega_2} \cdot \left[\frac{\frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \cdot \Omega_1 \cdot (I_{CQ_1} \cdot \Omega_3 + \Omega_4) - (I_{CQ_1} \cdot \Omega_1 + \Omega_2) \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \cdot \Omega_3}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2}\right]$$
$$+ V_t \cdot \frac{1}{\frac{I_{CQ_1}}{I_0} + 1} \cdot \frac{1}{I_0} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} = V_t \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \cdot \left[\frac{\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right]$$

We define function $\xi(I_{CQ_1}) = \frac{\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}$

$$\frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} = V_t \cdot \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t}; \quad \frac{\mathrm{d}^2 V_{A_1}}{\mathrm{d}t^2} = V_t \cdot \frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}t} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + V_t \cdot \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}t} = \frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t};$$
$$\frac{\mathrm{d}^2 V_{A_1}}{\mathrm{d}t^2} = V_t \cdot \frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + V_t \cdot \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2}$$

$$\xi(I_{CQ_1}) = \frac{\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}$$

$$\frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} = \frac{-(\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3) \cdot [\Omega_3 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2) + \Omega_1 \cdot (I_{CQ_1} \cdot \Omega_3 + \Omega_4)]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} - \frac{1}{(I_{CQ_1} + I_0)^2}$$

$$\frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} = \frac{-(\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3) \cdot [2 \cdot I_{CQ_1} \cdot \Omega_1 \cdot \Omega_3 + \Omega_3 \cdot \Omega_2 + \Omega_1 \cdot \Omega_4]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} - \frac{1}{(I_{CQ_1} + I_0)^2}$$

$$\frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} = -\left\{\frac{(\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3) \cdot [2 \cdot I_{CQ_1} \cdot \Omega_1 \cdot \Omega_3 + \Omega_3 \cdot \Omega_2 + \Omega_1 \cdot \Omega_4]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2}\right\}$$

$$\begin{aligned}\frac{\mathrm{d}^2 V_{A_1}}{\mathrm{d}t^2} &= V_t \cdot \frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + V_t \cdot \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2} \\ &= V_t \cdot \left[\frac{\mathrm{d}\xi(I_{CQ_1})}{\mathrm{d}I_{CQ_1}} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2}\right]\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2 V_{A_1}}{\mathrm{d}t^2} = V_t \cdot \Bigg[-\Bigg\{&\frac{(\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3) \cdot [2 \cdot I_{CQ_1} \cdot \Omega_1 \cdot \Omega_3 + \Omega_3 \cdot \Omega_2 + \Omega_1 \cdot \Omega_4]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} \\ &+ \frac{1}{(I_{CQ_1} + I_0)^2}\Bigg\} \cdot \left[\frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t}\right]^2 + \left(\frac{\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2}\Bigg]\end{aligned}$$

Our system differential equation:

$$C_1 \cdot \frac{\mathrm{d}^2 V_{A_1}}{\mathrm{d}t^2} + \frac{1}{R_1} \cdot \frac{\mathrm{d}V_{A_1}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \frac{1}{L_1} \cdot V_{A_1} = 0$$

$$C_1 \cdot V_t \cdot \left[-\left\{ \frac{(\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3) \cdot [2 \cdot I_{CQ_1} \cdot \Omega_1 \cdot \Omega_3 + \Omega_3 \cdot \Omega_2 + \Omega_1 \cdot \Omega_4]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2} \right\} \cdot \left[\frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \right]^2 + \left(\frac{\Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)} \right) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2} \right] + \frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4} \right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1 \right] \right) = 0$$

$$\Xi_1 = \Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3; \quad \Xi_2 = 2 \cdot \Omega_1 \cdot \Omega_3; \quad \Xi_3 = \Omega_3 \cdot \Omega_2 + \Omega_1 \cdot \Omega_4$$

$$C_1 \cdot V_t \cdot \left[-\left\{ \frac{\Xi_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2} \right\} \cdot \left[\frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \right]^2 + \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)} \right) \cdot \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2} \right] + \frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) \cdot \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} + \frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4} \right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1 \right] \right) = 0$$

We define new variable $Y = \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}I_{CQ_1}}{\mathrm{d}t} = Y; \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\mathrm{d}^2 I_{CQ_1}}{\mathrm{d}t^2}$

$$C_1 \cdot V_t \cdot \left[-\left\{ \frac{\Xi_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2} \right\} \cdot Y^2 + \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)} \right) \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \right] + \frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) \cdot Y + Y + \frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4} \right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1 \right] \right) = 0$$

$$- C_1 \cdot V_t \cdot \left\{ \frac{\Xi_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2} \right\} \cdot Y^2 + C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)} \right) \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} + \frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) \cdot Y + Y + \frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4} \right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1 \right] \right) = 0$$

$$C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right) \cdot \frac{dY}{dt}$$
$$= C_1 \cdot V_t \cdot \left\{\frac{\Xi_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2}\right\} \cdot Y^2$$
$$- Y \cdot \left(\frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) + 1\right) - \frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]\right)$$

$$\frac{dY}{dt} = \frac{\left(\frac{\Xi_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2}\right) \cdot Y^2}{\left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)}$$
$$- Y \cdot \frac{\left(\frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) + 1\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)}$$
$$- \frac{\frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)}$$

System equilibrium points (fixed points): $\frac{dY}{dt} = 0; \quad \frac{dI_{CQ_1}}{dt} = 0 \Rightarrow Y^* = 0$

$$\frac{\frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I^*_{CQ_1} \cdot \Omega_1 + \Omega_2}{I^*_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{I^*_{CQ_1}}{I_0} + 1\right]\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I^*_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I^*_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)} = 0$$

$$\frac{\Xi_1}{(I^*_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I^*_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)} \neq 0$$

$$\ln\left[\frac{I^*_{CQ_1} \cdot \Omega_1 + \Omega_2}{I^*_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{I^*_{CQ_1}}{I_0} + 1\right] = 0 \Rightarrow \ln\left\{\left[\frac{I^*_{CQ_1} \cdot \Omega_1 + \Omega_2}{I^*_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] \cdot \left[\frac{I^*_{CQ_1}}{I_0} + 1\right]\right\} = 0$$

$$e^0 = \left[\frac{I^*_{CQ_1} \cdot \Omega_1 + \Omega_2}{I^*_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] \cdot \left[\frac{I^*_{CQ_1}}{I_0} + 1\right] \Rightarrow \left[\frac{I^*_{CQ_1} \cdot \Omega_1 + \Omega_2}{I^*_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] \cdot \left[\frac{I^*_{CQ_1}}{I_0} + 1\right] = 1$$

$$\left[\frac{I^*_{CQ_1}\cdot\Omega_1+\Omega_2}{I^*_{CQ_1}\cdot\Omega_3+\Omega_4}\right]\cdot\left[\frac{I^*_{CQ_1}+I_0}{I_0}\right]=1\Rightarrow(I^*_{CQ_1}\cdot\Omega_1+\Omega_2)\cdot(I^*_{CQ_1}+I_0)$$
$$=(I^*_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot I_0$$

$$[I^*_{CQ_1}]^2\cdot\Omega_1+I^*_{CQ_1}\cdot[I_0\cdot(\Omega_1-\Omega_3)+\Omega_2]+(\Omega_2-\Omega_4)\cdot I_0=0$$

$$I^*_{CQ_1(1,2)}=\frac{-[I_0\cdot(\Omega_1-\Omega_3)+\Omega_2]\pm\sqrt{[I_0\cdot(\Omega_1-\Omega_3)+\Omega_2]^2-4\cdot\Omega_1\cdot(\Omega_2-\Omega_4)\cdot I_0}}{2\cdot\Omega_1}$$

$$[I_0\cdot(\Omega_1-\Omega_3)+\Omega_2]^2-4\cdot\Omega_1\cdot(\Omega_2-\Omega_4)\cdot I_0>0$$
$$\Rightarrow[I_0\cdot(\Omega_1-\Omega_3)+\Omega_2]^2>4\cdot\Omega_1\cdot(\Omega_2-\Omega_4)\cdot I_0$$

We define two functions: $g_1(I_{CQ_1},Y);\ g_2(I_{CQ_1},Y)$

$$g_1(Y,I_{CQ_1})=\frac{\left(\frac{\Xi_1\cdot[I_{CQ_1}\cdot\Xi_2+\Xi_3]}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)^2\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(I_{CQ_1}+I_0)^2}\right)\cdot Y^2}{\left(\frac{\Xi_1}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I_{CQ_1}+I_0)}\right)}$$
$$-Y\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(I_{CQ_1})+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I_{CQ_1}+I_0)}\right)}$$
$$-\frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{I_{CQ_1}\cdot\Omega_1+\Omega_2}{I_{CQ_1}\cdot\Omega_3+\Omega_4}\right]+\ln\left[\frac{I_{CQ_1}}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I_{CQ_1}+I_0)}\right)}$$

$$g_2(Y,I_{CQ_1})=Y;\quad\frac{dY}{dt}=g_1(Y,I_{CQ_1});\quad\frac{dI_{CQ_1}}{dt}=g_2(Y,I_{CQ_1})$$

Stability Analysis We need to implement linearization technique for our system. First, find system Jacobian matrix at the fixed point $(Y^*,I^*_{CQ_1})$.

$$A=\begin{pmatrix}\frac{\partial g_1(Y,I_{CQ_1})}{\partial Y} & \frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}\\ \frac{\partial g_2(Y,I_{CQ_1})}{\partial Y} & \frac{\partial g_2(Y,I_{CQ_1})}{\partial I_{CQ_1}}\end{pmatrix}_{(Y^*,I^*_{CQ_1})}$$

We need to find the partial derivatives of our functions: $g_1=g_1(Y,I_{CQ_1})$ $g_2=g_2(Y,I_{CQ_1})$

$$\frac{\partial g_1(Y,I_{CQ_1})}{\partial Y} = \frac{\left(\frac{\Xi_1\cdot[I_{CQ_1}\cdot\Xi_2+\Xi_3]}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)^2\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(I_{CQ_1}+I_0)^2}\right)\cdot 2\cdot Y}{\left(\frac{\Xi_1}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I_{CQ_1}+I_0)}\right)} - \frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(I_{CQ_1})+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I_{CQ_1}+I_0)}\right)}$$

$$\left.\frac{\partial g_1(Y,I_{CQ_1})}{\partial Y}\right|_{(Y^*=0,I^*_{CQ_1})} = -\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(I^*_{CQ_1})+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I^*_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I^*_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I^*_{CQ_1}+I_0)}\right)}$$

Complex expression: $\left.\frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}\right|_{(Y^*=0,I^*_{CQ_1})}$.

$$\frac{\partial g_2(Y,I_{CQ_1})}{\partial Y}=1;\quad \frac{\partial g_2(Y,I_{CQ_1})}{\partial I_{CQ_1}}=0;\quad \frac{\partial g_2(Y,I_{CQ_1})}{\partial Y}\Big|_{(Y^*=0,I^*_{CQ_1})}=1;$$
$$\frac{\partial g_2(Y,I_{CQ_1})}{\partial I_{CQ_1}}\Big|_{(Y^*=0,I^*_{CQ_1})}=0$$

$$A=\begin{pmatrix}\frac{\partial g_1(Y,I_{CQ_1})}{\partial Y} & \frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}\\ \frac{\partial g_2(Y,I_{CQ_1})}{\partial Y} & \frac{\partial g_2(Y,I_{CQ_1})}{\partial I_{CQ_1}}\end{pmatrix}_{(Y^*,I^*_{CQ_1})} = \begin{pmatrix}-\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(I^*_{CQ_1})+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I^*_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I^*_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I^*_{CQ_1}+I_0)}\right)} & \left.\frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}\right|_{(Y^*=0,I^*_{CQ_1})}\\ 1 & 0\end{pmatrix}$$

$$\frac{\partial g_1}{\partial Y}=\frac{\partial g_1(Y,I_{CQ_1})}{\partial Y};\quad \frac{\partial g_1}{\partial I_{CQ_1}}=\frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}};$$
$$\frac{\partial g_2}{\partial Y}=\frac{\partial g_1(Y,I_{CQ_1})}{\partial Y};\quad \frac{\partial g_2}{\partial I_{CQ_1}}=\frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}$$

$$\det(A-\lambda\cdot I)=0$$

$$\det\begin{pmatrix}-\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(I^*_{CQ_1})+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I^*_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I^*_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I^*_{CQ_1}+I_0)}\right)}-\lambda & \left.\frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}\right|_{(Y^*=0,I^*_{CQ_1})}\\ 1 & -\lambda\end{pmatrix}=0$$

$$\lambda^2+\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(I^*_{CQ_1})+1\right)\cdot\lambda}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(I^*_{CQ_1}\cdot\Omega_3+\Omega_4)\cdot(I^*_{CQ_1}\cdot\Omega_1+\Omega_2)}+\frac{1}{(I^*_{CQ_1}+I_0)}\right)}-\left.\frac{\partial g_1(Y,I_{CQ_1})}{\partial I_{CQ_1}}\right|_{(Y^*=0,I^*_{CQ_1})}=0$$

Table 7.4 The stability based on circuit eigenvalues

Circuit eigenvalues λ_1, λ_2	Stability classification
$\lambda_1 = \lambda_2 < 0$	Attracting focus
$\lambda_1 < \lambda_2 < 0$	Attracting node
$\lambda_1 < \lambda_2 = 0$	Attracting line
$\lambda_1 < 0 < \lambda_2$	Saddle point
$\lambda_1 = 0 < \lambda_2$	Repelling line
$0 < \lambda_1 < \lambda_2$	Repelling node
$0 < \lambda_1 = \lambda_2$	Repelling focus

$$\sum_{k=0}^{2} \lambda^k \cdot \imath_k = 0; \quad \imath_2 = 1; \quad \imath_1 = \frac{\left(\frac{1}{R_1} \cdot V_t \cdot \xi(I^*_{CQ_1}) + 1\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I^*_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I^*_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I^*_{CQ_1} + I_0)}\right)}$$

$$\imath_0 = -\frac{\partial g_1(Y, I_{CQ_1})}{\partial I_{CQ_1}}\bigg|_{(Y^*=0, I^*_{CQ_1})}.$$

Eigenvalues Stability Discussion If $\lambda_2 < \lambda_1 < 0$ then both Eigen solutions decay exponentially. The fixed point is a stable node. Trajectories typically approach the origin tangent to the slow eigendirection, defined as the direction spanned by the eigenvector with smaller $|\lambda|$. In the backwards time ($t \to -\infty$), the trajectories become parallel to the fast eigendirection. If the eigenvalues are complex, the fixed point is either a center or a spiral. Example of center is when the origin is surrounded by a family of closed orbits. The centers are neutrally stable, since nearby trajectories are neither attracted to nor repelled from the fixed point. A spiral would occur if our circuit oscillator were lightly damped. Then the trajectory would fail to close, because our oscillator loses energy on every cycle. The below table describes the stability based on circuit eigenvalues [7, 8] (Table 7.4).

Plotting Opto NDR oscillator circuit phase plane (I_{CQ_1}, Y). First we choose our circuit parameters values (Table 7.5).

$$\Omega_1 = 1 + k - \alpha_r = 0.52; \quad \Omega_2 = I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) = 0.51 \times 10^{-6}; \quad \Omega_3 = \alpha_f \cdot (1 + k) - 1 = -0.0004$$
$$\Omega_4 = I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) = 1.02 \times 10^{-6}; \quad \Xi_1 = \Omega_1 \cdot \Omega_4 - \Omega_2 \cdot \Omega_3 = 0.53 \times 10^{-6}$$

$$\Xi_2 = 2 \cdot \Omega_1 \cdot \Omega_3 = -0.000416; \quad \Xi_3 = \Omega_3 \cdot \Omega_2 + \Omega_1 \cdot \Omega_4 = 0.529 \times 10^{-6}.$$

We define the following functions for simplicity (Table 7.6)

$$F1 = C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)$$

Table 7.5 Circuit parameters values

α_r	0.5	C_1	10 μF
α_f	0.98	R_1	1 kΩ
k	0.02	L_1	10 μH
V_t	0.026 V	I_{se}	1 μA
I_0	1 μA	I_{sc}	2 μA

Table 7.6 Variables/ Functions and MATLAB variables

Variables/functions	MATLAB variables
$Y \to Y$	$x(1)$
$I_{CQ_1} \to X$	$x(2)$
$g_1(I_{CQ_1}, Y)$	$g(1)$
$g_2(I_{CQ_1}, Y)$	$g(2)$

$$F2 = \left(\frac{\Omega_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2}\right);$$

$$F3 = \xi(I_{CQ_1})$$

$$F4 = \left(\frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) + 1\right);$$

$$F5 = \frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]\right).$$

MATLAB scripts

```
function h=NDRosc1 (C1,L1,R1,Y0,X0)
[t,x]=ODE45(@NDRosc,[0,.05],[Y0,X0],[],C1,L1,R1);
%plot(t,x);
plot(x(:,1),x(:,2));%Y=dICQ1/dt (x-axis) and ICQ1 (y-axis)

function g=NDRosc(t, x, C1,L1,R1)
g=zeros(2,1);% the elements of the vector g represent the right hand
sided of the three DEs
F1=((0.53*1e-6)/((x(2)*(-0.0004)+1.02*1e-6)*(x(2)*0.52+0.51*1e
6))+1/(x(2)+1e-6))*C1*0.026;
F2=(0.53*1e-6*(x(2)*(-0.000416)+0.529*1e-6))/((x(2)*(-
0.0004)+1.02*1e-6).^2*(x(2)*0.52+0.51*1e-6).^2)+1/((x(2)+1e-6).^2);
F3=(0.53*1e-6/((x(2)*(-0.0004)+1.02*1e-6)*(x(2)*0.52+0.51*1e-
6))+1/(x(2)+1e-6));
F4=(1/R1)*0.026*F3+1;
F5=(1/L1)*0.026*(log((x(2)*0.52+0.5*1e-6)/(x(2)*(-0.0004)+1.02*1e-
6))+log(x(2)/1e-6+1));
g(1)=(F2.*x(1).*x(1)*C1*0.026)./F1-(x(1).*F4)./F1-(F5./F1);
g(2)=x(1);
```

NDRosc1(1e−5,1e−5,1000,100,000,.1) (Fig. 7.9).

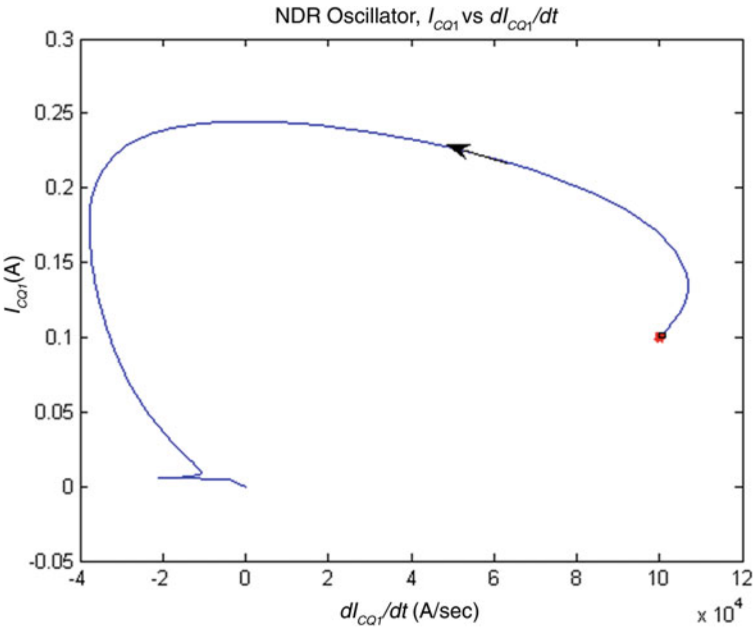


Fig. 7.9 NDR oscillator, I_{CQ_1} vs $\frac{dI_{CQ_1}}{dt}$

Opto NDR oscillator periodic orbit and Poincare map: Poincare map is an important tool for the investigation of dynamical systems in applications. Circuit dynamical system can be inspected through Poincare map. It is most beneficial tool in nonlinear dynamic. This is done by taking intersections of the orbit of flow by a hyper-plane parallel to one of the coordinate hyperplanes of co-dimension one. It is found that the two coordinates of the points on Poincare section are functionally related. First we write again our circuit differential equations: $\frac{dI_{CQ_1}}{dt} = Y$

$$\frac{dY}{dt} = \frac{\left(\frac{\Xi_1 \cdot [I_{CQ_1} \cdot \Xi_2 + \Xi_3]}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4)^2 \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(I_{CQ_1} + I_0)^2}\right) \cdot Y^2}{\left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)}$$
$$- Y \cdot \frac{\left(\frac{1}{R_1} \cdot V_t \cdot \xi(I_{CQ_1}) + 1\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)}$$
$$- \frac{\frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{I_{CQ_1} \cdot \Omega_1 + \Omega_2}{I_{CQ_1} \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{I_{CQ_1}}{I_0} + 1\right]\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(I_{CQ_1} \cdot \Omega_3 + \Omega_4) \cdot (I_{CQ_1} \cdot \Omega_1 + \Omega_2)} + \frac{1}{(I_{CQ_1} + I_0)}\right)}.$$

First we transfer our system variables to Cartesian coordinates ($X(t)$, $Y(t)$) terminology ($I_{CQ_1}(t) \leftrightarrow X(t)$; $Y(t) \leftrightarrow Y(t)$). We need to prove that the system has periodic orbits and find Poincare map. It is done by changing system Cartesian coordinates ($X(t)$, $Y(t)$) to cylindrical coordinates ($r(t)$, $\theta(t)$). Next we show that the cylinder is invariant. For the conversion between cylindrical and Cartesian systems and opposite, it is convenient to assume that the reference plane of the former is the Cartesian x–y plane (with equation $z = 0$), and the cylindrical axis is the Cartesian z-axis. In our system we refer to Cartesian X–Y plane (with equation $Z = 0$). Then the z coordinate is the same in both systems, and the correspondence between cylindrical (r, θ) and Cartesian (X, Y) are the same as for polar coordinates, namely $X(t) = r(t) \cdot \cos[\theta(t)]$; $Y(t) = r(t) \cdot \sin[\theta(t)]$; $r = \sqrt{X^2 + Y^2}$. $\theta(t) = 0$ if $X = 0$ and $Y = 0$. $\theta(t) = \arcsin(Y/r)$ if $X >= 0$. $x \to X$, $y \to Y$. $I_{CQ_1} = r \cdot \cos\theta \Leftrightarrow X = r \cdot \cos\theta$; $Y = r \cdot \sin\theta \Leftrightarrow Y = r \cdot \sin\theta$.

$$\frac{dI_{CQ_1}}{dt} = \frac{dr}{dt} \cdot \cos\theta - r \cdot \frac{d\theta}{dt} \cdot \sin\theta; \quad \frac{dY}{dt} = \frac{dr}{dt} \cdot \sin\theta + r \cdot \frac{d\theta}{dt} \cdot \cos\theta$$

$$\frac{dI_{CQ_1}}{dt} = Y \Rightarrow \frac{dr}{dt} \cdot \cos\theta - r \cdot \frac{d\theta}{dt} \cdot \sin\theta = r \cdot \sin\theta; \quad \frac{dr}{dt} = r \cdot tg\theta \cdot \left(1 + \frac{d\theta}{dt}\right)$$

$$\frac{dr}{dt} \cdot \cos\theta - r \cdot \frac{d\theta}{dt} \cdot \sin\theta = r \cdot \sin\theta \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{r \cdot tg\theta} - 1$$

$$\begin{aligned}
&\frac{dr}{dt} \cdot \sin\theta + r \cdot \frac{d\theta}{dt} \cdot \cos\theta \\
&= \frac{\left(\frac{\Xi_1 \cdot [r \cdot \cos\theta \cdot \Xi_2 + \Xi_3]}{(r \cdot \cos\theta \cdot \Omega_3 + \Omega_4)^2 \cdot (r \cdot \cos\theta \cdot \Omega_1 + \Omega_2)^2} + \frac{1}{(r \cdot \cos\theta + I_0)^2}\right) \cdot r^2 \cdot \sin^2\theta}{\left(\frac{\Xi_1}{(r \cdot \cos\theta \cdot \Omega_3 + \Omega_4) \cdot (r \cdot \cos\theta \cdot \Omega_1 + \Omega_2)} + \frac{1}{(r \cdot \cos\theta + I_0)}\right)} \\
&\quad - r \cdot \sin\theta \cdot \frac{\left(\frac{1}{R_1} \cdot V_t \cdot \xi(r \cdot \cos\theta) + 1\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(r \cdot \cos\theta \cdot \Omega_3 + \Omega_4) \cdot (r \cdot \cos\theta \cdot \Omega_1 + \Omega_2)} + \frac{1}{(r \cdot \cos\theta + I_0)}\right)} \\
&\quad - \frac{\frac{1}{L_1} \cdot V_t \cdot \left(\ln\left[\frac{r \cdot \cos\theta \cdot \Omega_1 + \Omega_2}{r \cdot \cos\theta \cdot \Omega_3 + \Omega_4}\right] + \ln\left[\frac{r \cdot \cos\theta}{I_0} + 1\right]\right)}{C_1 \cdot V_t \cdot \left(\frac{\Xi_1}{(r \cdot \cos\theta \cdot \Omega_3 + \Omega_4) \cdot (r \cdot \cos\theta \cdot \Omega_1 + \Omega_2)} + \frac{1}{(r \cdot \cos\theta + I_0)}\right)}
\end{aligned}$$

$$r\cdot tg\theta\cdot\left(1+\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\cdot\sin\theta+r\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\cos\theta$$
$$=\frac{\left(\frac{\Xi_1\cdot[r\cdot\cos\theta\cdot\Xi_2+\Xi_3]}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)^2\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(r\cdot\cos\theta+I_0)^2}\right)\cdot r^2\cdot\sin^2\theta}{\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$
$$-r\cdot\sin\theta\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(r\cdot\cos\theta)+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$
$$-\frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{r\cdot\cos\theta\cdot\Omega_1+\Omega_2}{r\cdot\cos\theta\cdot\Omega_3+\Omega_4}\right]+\ln\left[\frac{r\cdot\cos\theta}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{r\cdot(tg\theta\cdot\sin\theta+\cos\theta)}\cdot\left\{\frac{\left(\frac{\Xi_1\cdot[r\cdot\cos\theta\cdot\Xi_2+\Xi_3]}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)^2\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(r\cdot\cos\theta+I_0)^2}\right)\cdot r^2\cdot\sin^2\theta}{\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}\right.$$
$$-r\cdot\sin\theta\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(r\cdot\cos\theta)+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$
$$\left.-\frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{r\cdot\cos\theta\cdot\Omega_1+\Omega_2}{r\cdot\cos\theta\cdot\Omega_3+\Omega_4}\right]+\ln\left[\frac{r\cdot\cos\theta}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}-r\cdot tg\theta\cdot\sin\theta\right\}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\sin\theta+r\cdot\left[\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{1}{r\cdot tg\theta}-1\right]\cdot\cos\theta$$
$$=\frac{\left(\frac{\Xi_1\cdot[r\cdot\cos\theta\cdot\Xi_2+\Xi_3]}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)^2\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(r\cdot\cos\theta+I_0)^2}\right)\cdot r^2\cdot\sin^2\theta}{\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$
$$-r\cdot\sin\theta\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(r\cdot\cos\theta)+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$
$$-\frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{r\cdot\cos\theta\cdot\Omega_1+\Omega_2}{r\cdot\cos\theta\cdot\Omega_3+\Omega_4}\right]+\ln\left[\frac{r\cdot\cos\theta}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$

$$\frac{dr}{dt} = \frac{1}{\left(\sin\theta + \frac{1}{tg\theta}\cdot\cos\theta\right)} \cdot \left\{ \frac{\left(\frac{\Xi_1\cdot[r\cdot\cos\theta\cdot\Xi_2 + \Xi_3]}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)^2\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)^2} + \frac{1}{(r\cdot\cos\theta+I_0)^2}\right)\cdot r^2\cdot\sin^2\theta}{\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)} + \frac{1}{(r\cdot\cos\theta+I_0)}\right)} \right.$$

$$- r\cdot\sin\theta\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(r\cdot\cos\theta)+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)} + \frac{1}{(r\cdot\cos\theta+I_0)}\right)}$$

$$\left. - \frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{r\cdot\cos\theta\cdot\Omega_1+\Omega_2}{r\cdot\cos\theta\cdot\Omega_3+\Omega_4}\right] + \ln\left[\frac{r\cdot\cos\theta}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos\theta\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos\theta\cdot\Omega_1+\Omega_2)} + \frac{1}{(r\cdot\cos\theta+I_0)}\right)} + r\cdot\cos\theta \right\}.$$

If there is a stable limit cycle as specific value of radius r, $dr/dt = 0$ then

$$\frac{dr}{dt} = 0 \Rightarrow r\cdot tg\theta\cdot\left(1+\frac{d\theta}{dt}\right) = 0; \quad r \neq 0;$$
$$tg\theta = 0 \Rightarrow \theta = k\cdot\pi \quad \forall \quad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

$$1+\frac{d\theta}{dt} = 0 \Rightarrow d\theta = -dt \Rightarrow \theta = -t;$$
$$\theta = \omega\cdot t \Rightarrow \frac{d\theta}{dt} = \omega = \frac{2\cdot\pi}{T}; \frac{2\cdot\pi}{T} = -1 \Rightarrow T = -2\cdot\pi$$

$$\frac{d\theta}{dt} = \omega; \quad \frac{d\theta}{dt} = -1 \Rightarrow \omega = -1.$$

Case I $tg\theta = 0 \Rightarrow \theta = k\cdot\pi\ \forall\ k = \ldots, -2, -1, 0, 1, 2, \ldots$

Case II $1+\frac{d\theta}{dt} = 0 \Rightarrow \frac{d\theta}{dt} = -1$. It is reader exercise for each case to calculate and get the radius expression and value. It is easier to do it numerically than analytic [7, 8].

Poincare Map We need to compute Poincare map for our system $\psi(r)$. Angel $\theta = \omega\cdot t$ and regard the system as a vector field on a cylinder. Then $\frac{d\theta}{dt} = \omega$ Consider initial condition on S (it is $n-1$ dimensional surface of section and the Poincare map $\psi(r)$ is a mapping from S to itself), $\theta(t=0) = 0$; $r(t=0) = r_0$. Then the time of flight between successive intersections is $\frac{2\cdot\pi}{\omega}$. The analytical expressions we get for calculation of Poincare map are described below:

$$\omega = \frac{1}{r\cdot(tg(\omega\cdot t)\cdot\sin(\omega\cdot t)+\cos(\omega\cdot t))}\cdot\left\{\frac{\left(\frac{\Xi_1\cdot[r\cdot\cos(\omega\cdot t)\cdot\Xi_2+\Xi_3]}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)^2\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)^2}\right)\cdot r^2\cdot\sin^2(\omega\cdot t)}{\left(\frac{\Xi_1}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)}\right)}\right.$$

$$-r\cdot\sin(\omega\cdot t)\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(r\cdot\cos(\omega\cdot t))+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)}\right)}$$

$$-\frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2}{r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4}\right]+\ln\left[\frac{r\cdot\cos(\omega\cdot t)}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)}\right)}$$

$$\left.-r\cdot tg(\omega\cdot t)\cdot\sin(\omega\cdot t)\right\}$$

$$\frac{dr}{dt} = \frac{1}{\left(\sin(\omega\cdot t)+\frac{1}{tg\theta}\cdot\cos(\omega\cdot t)\right)}\cdot\left\{\frac{\left(\frac{\Xi_1\cdot[r\cdot\cos(\omega\cdot t)\cdot\Xi_2+\Xi_3]}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)^2\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)^2}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)^2}\right)\cdot r^2\cdot\sin^2(\omega\cdot t)}{\left(\frac{\Xi_1}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)}\right)}\right.$$

$$-r\cdot\sin(\omega\cdot t)\cdot\frac{\left(\frac{1}{R_1}\cdot V_t\cdot\xi(r\cdot\cos(\omega\cdot t))+1\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)}\right)}$$

$$\left.-\frac{\frac{1}{L_1}\cdot V_t\cdot\left(\ln\left[\frac{r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2}{r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4}\right]+\ln\left[\frac{r\cdot\cos(\omega\cdot t)}{I_0}+1\right]\right)}{C_1\cdot V_t\cdot\left(\frac{\Xi_1}{(r\cdot\cos(\omega\cdot t)\cdot\Omega_3+\Omega_4)\cdot(r\cdot\cos(\omega\cdot t)\cdot\Omega_1+\Omega_2)}+\frac{1}{(r\cdot\cos(\omega\cdot t)+I_0)}\right)}+r\cdot\cos(\omega\cdot t)\right\}$$

Remark It is very difficult to get the expression of Poincare map $\psi(r)$ analytically rather numerically. We recommend the reader to choose the easiest way.

7.5 Exercises

1. We have system which is characterized by the following two differential equations: $\frac{dX}{dt} = -\mu\cdot Y + X\cdot\left(1-\frac{X^2}{\mu}-Y^2\right)$; $\frac{dY}{dt} = X + Y\cdot\left(1-\frac{X^2}{\mu}-Y^2\right)$ μ is a parameter that establishes the dynamic of the system ($\mu\in\mathbb{R}$).

   1.1 Find system fixed points and investigate how they change for different values of μ parameter.

1.2 Find system Jacobian matrix after linearization. Discuss stability and stability switching for different values of μ parameter.
1.3 Show that there is a periodic solution of the system that lies on the ellipse $\left(\frac{X}{\sqrt{\mu}}\right)^2 + Y^2 = 1$.
1.4 Write system in cylindrical coordinates $r(t), \theta(t)$ and find differential equations $\frac{\mathrm{d}r}{\mathrm{d}t} = \xi_1(r, \theta)$; $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \xi_2(r, \theta)$.
1.5 Find if there is a limit cycle as a specific value of radius r ($\frac{\mathrm{d}r}{\mathrm{d}t} = 0$).
1.6 Consider the system vector field given in polar coordinates: $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$.
Find Poincare map (ψ). How Poincare map behavior is changed for different values of μ parameter?
1.7 Plot Poincare maps and investigate occurrences of fixed points.

2. We have two sinusoidal forced circuits which can be written in dimensionless form as $\frac{\mathrm{d}X}{\mathrm{d}t} + X = \mu_1 \cdot \sin(\omega_1 \cdot t)$; $\frac{\mathrm{d}Y}{\mathrm{d}t} + Y = \mu_2 \cdot \cos(\omega_2 \cdot t)$

$$\omega_1, \omega_2 \in \mathbb{R}_+;\ \omega_1, \omega_2 > 0;\ \mu_1, \mu_2 \in \mathbb{R}$$

2.1 Using Poincare map, show that these systems have a unique, globally stable limit cycles. How these limit cycles depend on μ_1, μ_2 parameters?
2.2 Plot the graphs of ψ_1, ψ_2 Poincare maps and intersection with the diagonal at unique points respectively.
2.3 Show that the deviation of X_k, Y_k from the fixed points is reduced by a constant factor with each iteration.
2.4 Try to draw the implementation optoisolation circuits for these two sinusoidal forced circuits. Investigate stability and stability switching for different values of μ_1, μ_2 parameters.
2.5 Show that the circuits always settle into the same forced oscillations, regardless of the interval conditions.

3. We have system model which characterize by two differential equations: $\frac{\mathrm{d}X}{\mathrm{d}t} = \mu_1 - X \cdot \sqrt{\mu_2} - \frac{X \cdot Y}{1 + \mu_3 \cdot X^2}$; $\frac{\mathrm{d}Y}{\mathrm{d}t} = \mu_3 - \frac{X \cdot Y}{1 + \mu_2 \cdot X^2}$
$\mu_1, \mu_2, \mu_3 \in \mathbb{R}_+;\ \mu_1, \mu_2, \mu_3 > 0$

3.1 Find all the system fixed points and classify them. Investigate the dynamics and behavior of the model.
3.2 Try to find conditions on μ_1, μ_2, μ_3 under which the system has a stable limit cycle.
3.3 Discuss stability and bifurcation for different values of μ_1, μ_2, μ_3 parameters.
3.4 Move to cylindrical coordinates $r(t), \theta(t)$ and find limit cycle as a specific value of radius ($\frac{\mathrm{d}r}{\mathrm{d}t} = 0$).

3.5 Consider the system vector field given in polar coordinates $\frac{d\theta}{d_t} = 1$. Find Poincare map (ψ). How Poincare map behavior is changed for different values of μ_1, μ_2, μ_3 parameters.

4. We have a van der Pol oscillator circuit with two parallel capacitors C_2 and C_3. The active element of the circuit is semiconductor device (OptoNDR circuit device) with two parallel capacitors C_2 and C_3. Capacitor C_2 is parallel to phototransistor Q_1 collector emitter ports and capacitor C_3 is parallel to LED D_1. OptoNDR circuit acts like an ordinary nonlinear resistor when current $I(t)$ is high ($I(t) > I_{sat}$), but like nonlinear negative resistor (energy source) or negative differential resistance (NDR) when $I(t)$ is low $I(t) > I_{break}$ and $I(t) < I_{sat}$. Our circuit current voltage characteristic $V = f(I)\,\forall\,\frac{dI}{d_t} = 0$ resembles a cubic function. We consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below I_{break} ($I(t) < I_{break}$) (Fig. 7.10).

4.1 Write circuit differential equations. *Hint*: The mathematical analysis is based on the basic transistor Ebers-Moll equations. The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant and is the phototransistor base current. Find circuit fixed points.

4.2 Write circuit Jacobian matrix at fixed points. Implement linearization technique for the system circuit. Discuss stability.

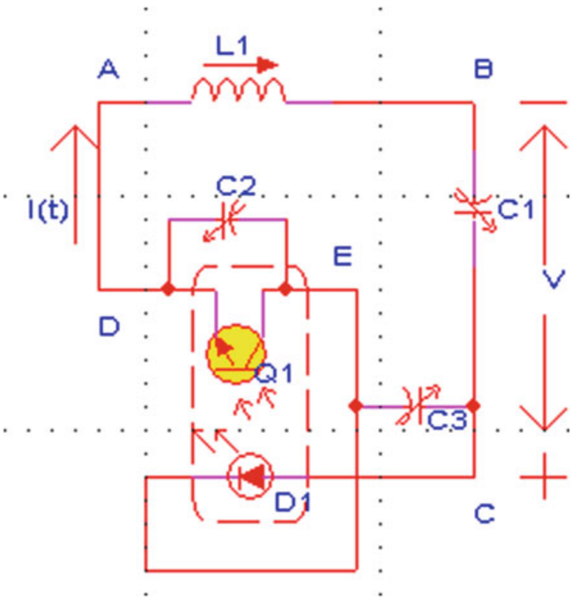


Fig. 7.10 Van der Pol oscillator circuit with two parallel capacitors C_2 and C_3

4.3 Plot optoisolation van der Pol circuit with two parallel capacitors (C_2, C_3)'s phase planes and variables time behavior.

4.4 Move to cylindrical coordinates and show that the cylinder is invariant. Compute Poincare map (ψ).

4.5 Capacitor C_2 is disconnected, How it influences our circuit system dynamics? Write circuit differential equations and find fixed points. Discuss stability and find Poincare map (ψ).

5. Consider an overdamped linear oscillator forced by Triangle wave. The system can be nondimensional to $\frac{dX}{dt} + X = g(t)$ where $g(t)$ is a Triangle wave of a period T. $k = \ldots, -2, -1, 0, 1, 2, \ldots$

$$g(t) = \begin{Bmatrix} \frac{2 \cdot A}{T} \cdot (t - k \cdot T) \quad \forall \quad k \cdot T < t < T \cdot \left(k + \frac{1}{2}\right) \\ A - \frac{2 \cdot A}{T} \cdot \left(t - T \cdot \left[k + \frac{1}{2}\right]\right) \quad \forall \quad \left(k + \frac{1}{2}\right) \cdot T < t < T \cdot (k+1) \\ A \quad \forall \quad t = \left(k + \frac{1}{2}\right) \cdot T \\ 0 \quad \forall \quad t = k \cdot T \end{Bmatrix}$$

Function $g(t)$ is periodically repeated for all other t. Consider that the system is strobed by Triangle wave (Fig. 7.11).

5.1 Show that all system trajectories approach a unique periodic solution. Let $X(t = 0) = X_0$ and find Poincare map. : Consider our system as a general *Hint* $\frac{dX}{dt} = a(t) \cdot X + b(t); \quad a(t) = -1; \quad b(t) = g(t)$. First consider the homogeneous problem $\frac{dX}{dt} = a(t) \cdot X; \quad X_H(t) = C \cdot e^{A(t)} \; A(t) = \int a(t) \cdot dt$. The unknown constant is found by setting $t = t_0$ and using initial data. The general system is solve by using integration factor. We can rewrite the equation as $\frac{dX}{dt} - a(t) \cdot X = b(t) = g(t)$ and we call $F(t) = e^{A(t)}$ the integration factor. Let $X(t) = y(t) \cdot F(t)$ and finally $X(t) = K \cdot e^{A(t)} + e^{A(t)} \cdot \int^t e^{-A(s)} \cdot b(s) \cdot ds$.

5.2 What are the limits of $X(T)$ if $T \to 0$ or $T \to \infty$?

5.3 Let $X_1 = X(T)$, and define the Poincare map ψ by the $X_1 = \psi(X_0)$ and more generally $X_{n+1} = \psi(X_n)$. Plot the graph of ψ.

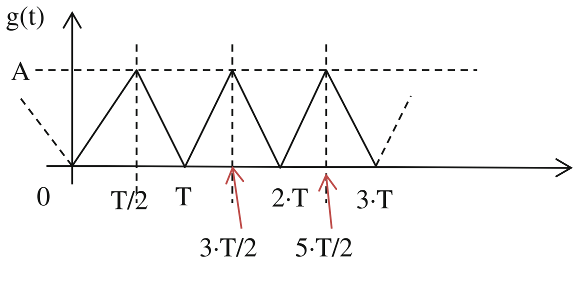


Fig. 7.11 Function $g(t)$ in time

5.4 Show that Poincare map ψ has a globally fixed point and the original system settles into a periodic response to the forcing.
5.5 Try to implement overdamped linear oscillator forced by Triangle Wave by using optoisolation devices and discrete components (capacitors, resistors, inductors, etc.).

6. The Brusselator (Brussels and oscillator) is a theoretical model for a type of autocatalytic reaction. The dynamics of the forced Brusselator reaction can be described by a system of two ODEs. In dimensional form they are presented below. Forced terms: $g_1(t) = A \cdot \sin(\theta);\ g_2(t) = B \cdot \cos(\theta)$

$$\frac{dX}{dt} = 1 - (1 + \mu_2) \cdot X + \mu_1 \cdot X^2 \cdot Y + g_1(t);$$
$$\frac{dY}{dt} = \mu_2 \cdot X - \mu_1 \cdot X^2 \cdot Y + g_2(t)$$

$$X, Y \in \mathbb{R}; \quad \mu_1, \mu_2 \in \mathbb{R}; \quad \mu_1, \mu_2 > 0; \quad \theta = \omega \cdot t; \quad \omega > 0; \quad \theta(t = 0) = 0$$

$$X(t = 0) = X_0.$$

6.1 Find the equilibria of the system when forced terms are equal to zero $g_1(t) = 0;\ g_2(t) = 0 (A = 0, B = 0)$. Find Jacobian matrix of the system and analyze the stability of the system.
6.2 Compute system $\frac{dX}{dt}$ and $\frac{dY}{dt}$ null clines. *Hint*: It is done by setting $\frac{dX}{dt} = 0$ and $\frac{dY}{dt} = 0$. Null cline is a good tool to analyze system phase plane, sketching and analyzing phase planes. The X - null cline is a set of points in the phase plane so that $\frac{dX}{dt} = 0$. Geometrically, these are the points where the vectors are either straight up or straight down. In the same manner, *Y*-null cline is a set of points in the phase plane so that $\frac{dY}{dt} = 0$. These are the points where the vectors are horizontal going either to the left or to the right.
6.3 Discuss Brusselator system bifurcation and trapping region.
6.4 Compute Poincare maps ψ_X, ψ_Y $(g_1(t) = A \cdot \sin(\theta);\ g_2(t) = B \cdot \cos(\theta))$. To compute ψ_X, ψ_Y, we need to solve differential equations. Its general solution is a sum of homogeneous and particular solution.
6.5 Plot the graphs of ψ_X, ψ_Y. Show its intersection with the diagonal at unique points and show that the system always settles into the some forced oscillator regardless of the initial conditions.

7. We have oscillator circuit which has two OptoNDR devices, capacitor C_1, resistor R_1 and inductor L_1. Our OptoNDRs elements/circuits are constructed from LEDs and phototransistors. The first OptoNDR circuit is constructed from LED D_1 in series with phototransistor Q_1. Accordingly, the second OptoNDR circuit is constructed from LED D_2 in series with phototransistor Q_2. We

consider that a source of voltage is attached to the circuit's capacitor C_1 and then withdrawn. Initially, the electrical energy from the capacitor C_1 is transferred into the magnetic energy of the inductor. When the electrical energy of the capacitor becomes zero, the process is reserved. The magnetic energy from the inductor is transferred into the electrical energy of the capacitor. Parasitic resistance R_1 in the system causes the oscillation to damp and OptoNDRs are the negative elements which supposed to postpone the damping process (Fig. 7.12).

The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k_1 constant ($I_{BQ_1} = I_{LED_1} \cdot k_1 = I_{D_1} \cdot k_1;\ I_{BQ_1} = I_{CQ_1} \cdot k_1$) and is the phototransistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations. In the same manner for the second OptoNDR circuit ($I_{BQ_2} = I_{LED_2} \cdot k_2 = I_{D_2} \cdot k_2;\ I_{BQ_2} = I_{CQ_2} \cdot k_2;\ k_1 \neq k_2$).

We need to implement the regular Ebers–Moll model to the opto coupler circuits (OptoNDR Q_1-D_1 and OptoNDR Q_2-D_2) and get a complete final expressions for the Negative Differential Resistances (NDRs) characteristics of that circuit. In your circuit analysis consider that the first and the second OptoNDRs are not identical.

7.1 Write circuit differential equations and find fixed points (equilibrium points).

7.2 Discuss stability and stability switching for different values of circuit parameters. *Hint*: Implement linearization technique for our system. You need to calculate system Jacobian matrix at fixed points.

7.3 Move to circuit cylindrical coordinates $r(t), \theta(t)$. Show that the cylinder is invariant. Find the differential equations: $\frac{dr}{dt} = \varsigma_1(r,\theta)$ $\frac{d\theta}{dt} = \varsigma_2(r,\theta)$.

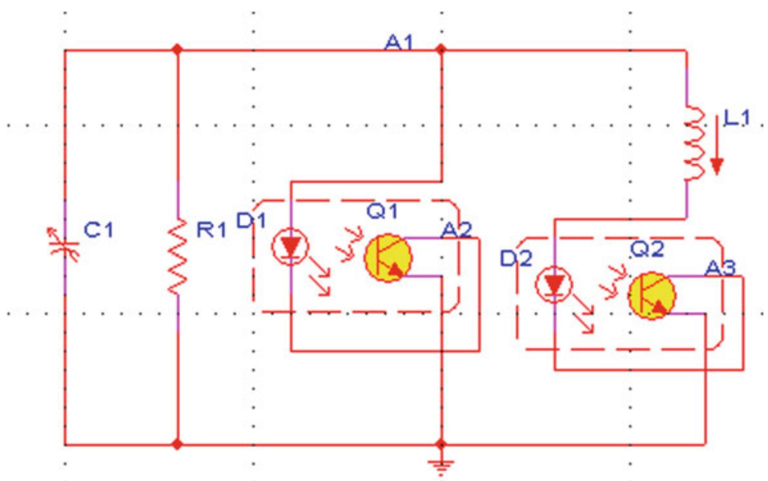


Fig. 7.12 Oscillator circuit with two OptoNDR devices, capacitors C_1 and C_2, resistor R_1, and inductor L_1

7.4 Show that there is a stable limit cycle as specific value of radius r; $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$.
7.5 Find Poincare map ψ. Draw ψ function and show it intersection with the diagonal at unique points.

8. We have systems that characterize by two differential equations: $\frac{\mathrm{d}X}{\mathrm{d}t} = -Y + X \cdot (1 - \mu_1 \cdot X^2 - \mu_2 \cdot Y^2)^3$; $\frac{\mathrm{d}Y}{\mathrm{d}t} = X + Y \cdot (-\mu_1 \cdot X^2 - \mu_2 \cdot Y^2)^3$ $X, Y \in \mathbb{R}$; $\mu_1, \mu_2 \in \mathbb{R}$.

8.1 Find fixed points (equilibrium points) and calculate system Jacobian matrix. Hint: Implement linearization technique for our system.
8.2 Discuss stability and stability switching for different values of μ_1, μ_2 parameters.
8.3 Move to system cylindrical coordinates $r(t), \theta(t)$. Find differential equations: $\frac{\mathrm{d}r}{\mathrm{d}t} = \varsigma_1(r, \theta)$; $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \varsigma_2(r, \theta)$ Show that there is a stable limit cycle as specific value of radius r $\left(\frac{\mathrm{d}r}{\mathrm{d}t} = 0\right)$
8.4 We move to system one parameter $\mu_1 = \mu$; $\mu_2 = \sqrt{\mu}$. How the behavior of the system is changed for different values of μ parameter? Discuss stability and stability switching for different values of μ parameter
8.5 8.5Find Poincare maps ψ_X, ψ_Y. Draw ψ_X, ψ_Y functions and show it intersection with the diagonal at unique points.

9. We have oscillator circuit which has two OptoNDR devices, capacitors C_1 and C_2, resistor R_1 and inductor L_1. Additionally, there is opto coupler (D_3 and Q_3). Our OptoNDRs elements/circuits are constructed from LEDs and phototransistors. The first OptoNDR circuit is constructed from LED D_1 in series with phototransistor Q_1. Accordingly, the second OptoNDR circuit is constructed from LED D_2 in series with phototransistor Q_2. We consider that a source of voltage is attached to the circuit's capacitors C_1 and C_2, and then withdrawn. Initially, the electrical energy from the capacitor C_1 and C_2 is transferred into the magnetic energy of the inductor. When the electrical energy of the capacitors becomes zero, the process is reserved. The magnetic energy from the inductor L_1 is transferred into the electrical energy of the capacitors C_1 and C_2. Parasitic resistance R_1 in the system causes the oscillation to damp and OptoNDRs are the negative elements, which supposed to postpone the damping process (Fig. 7.13).
The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k_1 constant ($I_{BQ_1} = I_{LED_1} \cdot k_1 = I_{D_1} \cdot k_1$; $I_{BQ_1} = I_{CQ_1} \cdot k_1$) and is the phototransistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations. In the same manner for the second OptoNDR circuit and third opto coupler Q_3–D_3 ($I_{BQ_2} = I_{LED_2} \cdot k_2$

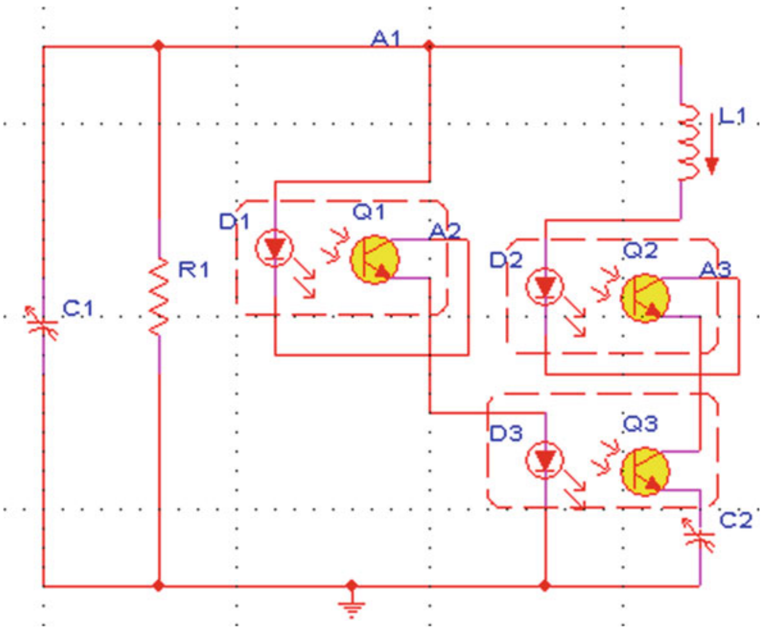


Fig. 7.13 Oscillator circuit with two OptoNDR elements, opto coupler, capacitors C_1 and C_2, resistor R_1, and inductor L_1

$I_{BQ_2} = I_{D_2} \cdot k_2;\ I_{BQ_2} = I_{CQ_2} \cdot k_2;\ I_{BQ_3} = I_{LED_3} \cdot k_3;\ I_{BQ_3} = I_{D_3} \cdot k_3;\ k_1 \neq k_2 \neq k_3).$

We need to implement the regular Ebers-Moll model to the opto coupler circuits (OptoNDR Q_1-D_1 and OptoNDR Q_2-D_2, opto coupler Q_3–D_3) and get a complete final expressions for the Negative Differential Resistances (NDRs) characteristics of that circuit. In your circuit analysis consider that the first and the second OptoNDRs are not identical.

9.1 Write circuit differential equations and find fixed points (equilibrium points).
9.2 Discuss stability and stability switching for different values of circuit parameters. *Hint*: Implement linearization technique for our system. You need to calculate system Jacobian matrix at fixed points.
9.3 Move to circuit cylindrical coordinates $r(t), \theta(t)$. Show that the cylinder is invariant. Find the differential equations: $\frac{\mathrm{d}r}{\mathrm{d}t} = \varsigma_1(r,\theta)\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \varsigma_2(r,\theta)$.
9.4 Show that there is a stable limit cycle as specific value of radius r; $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$.
9.5 Find Poincare map ψ. Draw ψ function and show it intersection with the diagonal at unique points.

10. We have forced system which can be written in dimensionless form as $\frac{\mathrm{d}X}{\mathrm{d}t} + |X| = A \cdot \sin(\omega_1 \cdot t) + B \cdot \cos(\omega_2 \cdot t);\qquad X \in \mathbb{R};\ \omega_1, \omega_2 > 0;\ \theta_1 =$

$\omega_1 \cdot t;\ \theta_2 = \omega_2 \cdot t\ \frac{d\theta_1}{dt} = \omega_1;\ \frac{d\theta_2}{dt} = \omega_2;\ \theta_1(t = 0) = 0;\ \theta_1(t = 0) = 0;\ X(t = 0) = X_0$ Consider that there are two possible time of flights between successive $(A, B \in \mathbb{R})$ intersections $(t_1 = \frac{2\cdot\pi}{\omega_1};\ t_2 = \frac{2\cdot\pi}{\omega_2})$. The system is strobed one per drive cycle. Differentiate two cases: $X > 0$ and $X < 0$. What happened for $X = 0$?

10.1 Show that all system trajectories approach a unique periodic solution. Let $X(t = 0) = X_0$ and find Poincare map.
10.2 What are the limits of $X(T)$ if $T \to 0$ or $T \to \infty$?
10.3 Let $X_1 = X(T)$, and define the Poincare map ψ by the $X_1 = \psi(X_0)$ and more generally $X_{n+1} = \psi(X_n)$. Plot the graph of ψ.
10.4 If $\omega_1 = \omega;\ \omega_2 = \omega^2$ find Poincare map and discuss periodic for different values of ω parameter $(\omega \in \mathbb{R})$.
10.5 Show that Poincare map ψ has a globally fixed point and the original system settles into a periodic response to the forcing.

Chapter 8
Optoisolation Circuits Averaging Analysis and Perturbation from Geometric Viewpoint

In many dynamical systems there are linear oscillators with small perturbations or weakly nonlinear sources. These systems are valid on semi-infinite time intervals under suitable conditions. In many perturbed systems, we start with a system which includes known solutions and add small perturbations of it. The solutions of unperturbed and perturbed systems are different and system with small perturbation has different structure of solutions. For finite times unperturbed and perturbed solutions are close. We study the asymptotic behavior of the solutions and structure. Generally perturbation theory has tools to solve problems with approximate solution by discussing of the exact solution of a simpler problem. An approximation of the full solution A, a series in the small parameters (ε), is like the following solution: $A = \sum_{k=0}^{\infty} A_k \cdot \varepsilon^k$. A_0 is the known solution to the exactly solvable initial problem and A_1, A_2, … represent the higher order terms. For small ε these higher order terms become successively smaller. The initial solution and "first-order" perturbation correction is $A \approx A_0 + \varepsilon \cdot A_1$. The method of averaging provides a useful means to study the behavior of nonlinear dynamical system under periodic forcing. Averaged equation of a time-dependent differential equation gave the Poincare map, stability analysis, and recover higher orders of averaging. It discusses higher order averaged expansions for periodic and quasiperiodic differential equations. Averaging can be implemented to systems of the form $\frac{dX}{dt} = \varepsilon \cdot f(X, t);\ X(t = 0) = X_0$, where f is T-periodic $f(X, t) = f(X, t + T);\ X \in \mathbb{R}^n$. The average of $f(X, t)$ is typically given as $\overline{f(X, t)} = \int_t^{t+T} f(X, \tau) \cdot d\tau$ where the evaluation point X is considered fixed. The average defines new autonomous equation $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y, t)};\ Y(t = 0) = X_0$. Averaging theory determines conditions under which the two flows coincide and to what degree they coincide. The parameter ε will provide a means to determine this coincide. Oscillators which include optoisolation elements are integral part of many engineering applications. The construction process of negative differential resistance (NDR) devices using analog optocouplers is used to make a highly stable,

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radio-frequency oscillator. Typical oscillator is van der Pol "Negative Resistance" (e.g., tunnel diode, optoisolation circuit, etc.) oscillators [5–8].

8.1 Poincare Maps and Averaging

In every dynamical system we need to use the global existence theorems. Some system's differential equations cannot be solved exactly and if there is exact solution, it exhibits an intricate dependency in the parameters that it is hard to use as such. We identified the parameter as ε, and the solution is available and reasonably simple for $\varepsilon = 0$. The system solution is altered for nonzero but small ε. We get systematic interpretation by the perturbation theory. We have system that characterized by the following differential equation: $\frac{dX}{dt} = \varepsilon \cdot f(X,t);\ X \in \mathbb{R}^n;\ \varepsilon \ll 1$ where f is periodic in t, $f(X,t) = f(X, t+T);\ X \in \mathbb{R}^n$.

The solution evolution is "SLOW" which characterized the T-periodic forcing term. We characterized a weakly nonlinear oscillator system by differential equation: $\frac{d^2X}{dt^2} + \omega^2 \cdot X = \varepsilon \cdot f\left(X, \frac{dX}{dt}, t\right)$. The linear oscillator $\frac{d^2X}{dt^2} + \omega^2 \cdot X = 0$ is perturbed. Basically, strongly nonlinear system is described as $\frac{dX}{dt} = f(X) X \in \mathbb{R}^n$ and we add weak dissipation and forcing term $\varepsilon \cdot g(X,t)$ and get the system differential equation $\frac{dX}{dt} = f(X) + \varepsilon \cdot g(X,t);\ X \in \mathbb{R}^n$. Poincare map in time periodic single degree of freedom is an integral part of our analysis. Our system is characterized by the differential equation: $\frac{dX}{dt} = \varepsilon \cdot f(X) X \in \mathbb{R}^n;\ 0 \le \varepsilon \ll 1;\ f\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$, bounded on bounded set and has T periodic in t ($T > 0$). We define our bounded set $U \in \mathbb{R}^n$ and define the associated autonomous averaged system: $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)};\ \overline{f(Y,t)} = \frac{1}{T} \cdot \int_0^T f(Y,t,0) \cdot dt$.

The Averaging Theorem

Under coordinate change $(X = Y + \varepsilon \cdot w(Y,t,\varepsilon))\ \frac{dX}{dt} = \varepsilon \cdot f(X,t);\ X \in \mathbb{R}^n;\ \varepsilon \ll 1$ becomes $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)} + \varepsilon^2 \cdot f_1(Y,t,\varepsilon)$. Function $f_1(Y,t,\varepsilon)$ is of period T in t.

(I) If $X(t)$ and $Y(t)$ are solutions of $\frac{dX}{dt} = \varepsilon \cdot f(X,t);\ X \in \mathbb{R}^n;\ \varepsilon \ll 1$ and $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)}$. Based on X_0, Y_0 respectively at $t = 0$ ($X_0 = X(t=0), Y_0 = Y(t=0)$) and $|X_0 - Y_0| = O(\varepsilon)$ then $|X(t) - Y(t)| = O(\varepsilon)$ on a timescale $t \sim \frac{1}{\varepsilon}$.

(II) We define Γ_0 as a hyperbolic fixed point of $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)};\ Y \in \mathbb{R}^n$ then there exist $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ exist $\frac{dX}{dt} = \varepsilon \cdot f(X,t)$ possesses a unique hyperbolic period orbit $\gamma_\varepsilon(t) = \Gamma_0 + O(\varepsilon)$ of the same stability type as Γ_0.

(III) If $X^S(t) \in W^S(\gamma_\varepsilon)$ is a solution of $\frac{dX}{dt} = \varepsilon \cdot f(X,t)$; $X \in \mathbb{R}^n$; $\varepsilon \ll 1$ lying in the stable manifold of the hyperbolic period orbit $\gamma_\varepsilon(t) = \Gamma_0 + O(\varepsilon)$ then $Y^S(t) \in W^S(\Gamma_0)$ is a solution of $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)}$ lying in the stable manifold of the hyperbolic fixed point Γ_0 and $|X^S(t=0) - Y^S(t=0)| = O(\varepsilon)$ then $|X^S(t) - Y^S(t)| = O(\varepsilon)$ for $t \in [0,\infty)$. In the same manner we implement it to solutions lying in the unstable manifolds on the time interval $t \in (-\infty, 0]$.

We need to show that according to averaging theorem, changing of coordinates $X = Y + \varepsilon \cdot w(Y,t,\varepsilon)$ yields to the differential equation: $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)} + \varepsilon^2 \cdot f_1(Y,t,\varepsilon)$; $Y \in \mathbb{R}^n$. We split the periodic function $f(X,t,\varepsilon)$ to two functions: $\overline{f(X,t)}$ part $(\overline{f(X,t)}|_{\varepsilon=0} = \frac{1}{T} \cdot \int_0^T f(X,t,0) \cdot dt$ and its oscillating part $\overline{f(X,t,\varepsilon)}$ $\left(f(X,t,\varepsilon) = \overline{f(X,t)} + \overline{f(X,t,\varepsilon)}\right)$. Let $X = Y + \varepsilon \cdot w(Y,t,\varepsilon)$ and differentiating respect to time gives $\frac{\partial X}{\partial t} = \frac{\partial Y}{\partial t} + \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial t}$.

$$f(X,t,\varepsilon) = \overline{f(X,t)} + \overline{f(X,t,\varepsilon)} \Rightarrow f(X,t,\varepsilon) = \overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t)} + \overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t,\varepsilon)}$$

$$\frac{dX}{dt} = \varepsilon \cdot f(X,t) \Rightarrow \frac{dX}{dt} = \varepsilon \cdot \left[\overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t)} + \overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t,\varepsilon)}\right]$$

We can subtract from the two sides of the equation $\varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial t}$ term [31, 45].

$$\frac{dX}{dt} - \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial t} = \varepsilon \cdot \left[\overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t)} + \overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t,\varepsilon)}\right] - \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial t}$$

Differentiate $X = Y + \varepsilon \cdot w(Y,t,\varepsilon)$ in Y $\left(D_Y = \frac{\partial}{\partial Y}\right)$ yields $\frac{\partial X}{\partial Y} = 1 + \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial Y}$. Multiplication by $\frac{\partial Y}{\partial t}$ term yields $\frac{\partial X}{\partial Y} \cdot \frac{\partial Y}{\partial t} = \left(1 + \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial Y}\right) \cdot \frac{\partial Y}{\partial t}$.

$$\frac{\partial X}{\partial t} = \left(1 + \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial Y}\right) \cdot \frac{\partial Y}{\partial t} \Rightarrow \frac{\partial X}{\partial t} = (1 + \varepsilon \cdot D_Y w(Y,t,\varepsilon)) \cdot \frac{\partial Y}{\partial t}$$

$$(1 + \varepsilon \cdot D_Y w(Y,t,\varepsilon)) \cdot \frac{\partial Y}{\partial t} = \frac{dX}{dt} - \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial t}$$

$$(1 + \varepsilon \cdot D_Y w(Y,t,\varepsilon)) \cdot \frac{\partial Y}{\partial t} = \varepsilon \cdot \left[\overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t)} + \overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t,\varepsilon)}\right] - \varepsilon \cdot \frac{\partial w(Y,t,\varepsilon)}{\partial t}$$

$$\frac{\partial Y}{\partial t} = \frac{1}{(1 + \varepsilon \cdot D_Y w(Y,t,\varepsilon))} \cdot \varepsilon \cdot \left\{\left[\overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t)} + \overline{f(Y+\varepsilon \cdot w(Y,t,\varepsilon),t,\varepsilon)}\right] - \frac{\partial w(Y,t,\varepsilon)}{\partial t}\right\}$$

Expanding the above expression in powers of ε and choosing w to be the anti-derivative of $\overline{f}$: $\frac{\partial w(Y,t,\varepsilon)}{\partial t} = \overline{f(Y,t,0)}$. We get the following expression:

$$\frac{\partial Y}{\partial t} = \varepsilon \cdot \overline{f(Y,t)} + \varepsilon^2 \cdot \left[\frac{\partial f(Y,t,0)}{\partial Y} \cdot w(Y,t,0) - \frac{\partial w(Y,t,0)}{\partial Y} \cdot f(Y,t,0) + \frac{\partial \overline{f(Y,t,0)}}{\partial \varepsilon}\right] + O(\varepsilon^3)$$

$$f_1(Y,t,\varepsilon) = \frac{\partial f(Y,t,0)}{\partial Y} \cdot w(Y,t,0) - \frac{\partial w(Y,t,0)}{\partial Y} \cdot f(Y,t,0) + \frac{\partial \overline{f(Y,t,0)}}{\partial \varepsilon}$$

$$\frac{\partial Y}{\partial t} = \varepsilon \cdot \overline{f(Y,t)} + \varepsilon^2 \cdot f_1(Y,t,\varepsilon)$$

One typical oscillator system is van der Pol oscillator. It includes nonlinear damping and energy is being dissipated at large amplitudes and generated at low amplitudes. The van der Pol system possesses limit cycles. At balance energy is generation and dissipation. The van der Pol system oscillators are sustained around a state. The basic van der Pol system can be written in the form $\frac{d^2X}{dt^2} + \alpha \cdot \phi(x) \cdot \frac{dX}{dt} + X = \beta \cdot P(t)$ where $\phi(x)$ an even function is and $\phi(x) < 0$ for $|X| < 1$, $\phi(x) > 0$ for $|X| > 1$. Function $P(t)$ is T-periodic ($f(t) = f(t+T)$). Parameters α and β are non-negative parameters ($\alpha, \beta \in \mathbb{R}_+$). If we want to move to autonomous system, we set $Y = \frac{dX}{dt}; \frac{dY}{dt} = \frac{d^2X}{dt^2}$ then we get the following system differential equations: $\frac{dX}{dt} = Y; \frac{dY}{dt} = -X - \alpha \cdot \phi(x) \cdot Y + \beta \cdot P(t)$.

If $T = 2 \cdot \pi \Rightarrow \theta = \omega \cdot t = \frac{2 \cdot \pi}{T} \cdot t|_{T=2\cdot\pi} = t$ then $\theta = t; \frac{d\theta}{dt} = 1$. Another way to write our autonomous system is as follows: $\frac{dX}{dt} = Y - \alpha \cdot \Phi(X); \frac{dY}{dt} = -X + \beta \cdot P(\theta); \frac{d\theta}{dt} = 1; X, Y, \theta \in \mathbb{R}$. Where $\Phi(X) = \int_0^X \phi(\xi) \cdot d\xi$ is odd function and $\Phi(X = 0) = \Phi(X = \pm a) = 0 \, \forall \, a > 0$ (Fig. 8.1).

$$\frac{dX}{dt} = Y - \alpha \cdot \Phi(X) \Rightarrow Y = \frac{dX}{dt} + \alpha \cdot \Phi(X); \frac{dY}{dt} = \frac{d^2X}{dt^1} + \alpha \cdot \frac{d\Phi(X)}{dt}$$

$$\frac{d\Phi(X)}{dt} = \frac{d}{dt}\int_0^X \phi(\xi) \cdot d\xi = \phi(X) \cdot \frac{dX}{dt} \Rightarrow \frac{dY}{dt} = \frac{d^2X}{dt^1} + \alpha \cdot \phi(X) \cdot \frac{dX}{dt}$$

$$\frac{dY}{dt} = -X + \beta \cdot P(\theta) \Rightarrow \frac{d^2X}{dt^2} + \alpha \cdot \phi(X) \cdot \frac{dX}{dt} = -X + \beta \cdot P(\theta)$$

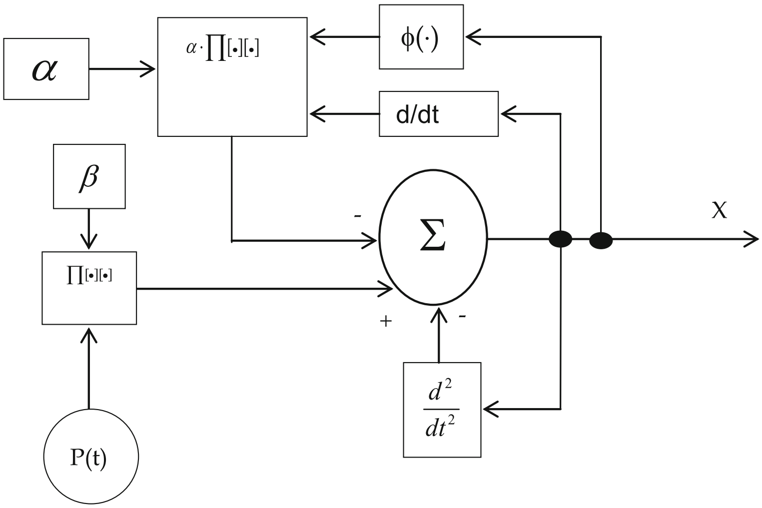

Fig. 8.1 Poincare maps and averaging block diagram

$$\frac{d^2X}{dt^2} + \alpha \cdot \phi(X) \cdot \frac{dX}{dt} = -X + \beta \cdot P(\theta) \Rightarrow \frac{d^2X}{dt^2} + \alpha \cdot \phi(X) \cdot \frac{dX}{dt} + X = \beta \cdot P(\theta); \frac{d\theta}{dt} = 1$$

When our van der Pol system is unforced system we need to set $\beta = 0$ then the differential equation becomes $\frac{d^2X}{dt^2} + \alpha \cdot \phi(x) \cdot \frac{dX}{dt} + X = 0$ [5–7]. The case parameter $\alpha \ll 1$ yields to $\alpha \cdot \phi(x) \cdot \frac{dX}{dt} \to \varepsilon$ and we get the differential equation: $\frac{d^2X}{dt^2} + X = 0$; $Y = \frac{dX}{dt}$; $\frac{dY}{dt} = \frac{d^2X}{dt^2}$; $\frac{dY}{dt} + X = 0 \Rightarrow \frac{dY}{dt} = -X$. At fixed points:

$$\frac{dX}{dt} = 0; \frac{dY}{dt} = 0; X^* = 0; Y^* = 0; \begin{pmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix}; A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}; \det(A - \lambda \cdot I) = 0 \Rightarrow \lambda^2 + 1 = 0; \lambda_{1,2} = \pm\sqrt{-1}; \lambda_{1,2} = \pm j$$

The eigenvalues are pure imaginary then all solutions are periodic with period $2 \cdot \pi$. The oscillations have fixed amplitude and the fixed point is a center. Practically, the system phase plane filled with circular periodic orbits each of period $2 \cdot \pi$. First we select the invertible transformation:

$$u = X \cdot \cos(t) - Y \cdot \sin(t);\ v = -X \cdot \sin(t) - Y\cos(t);\ \theta = \frac{2 \cdot \pi}{T} \cdot t|_{T=2\cdot\pi} = t$$

$$\frac{\mathrm{d}}{\mathrm{d}t} tg(t) = \sec^2(t) = \frac{1}{\cos^2(t)};\ u = X \cdot \cos(t) - Y \cdot \sin(t) \Rightarrow X = \frac{u}{\cos(t)} + Y \cdot tg(t)$$

$$v = -X \cdot \sin(t) - Y\cos(t) \Rightarrow v = -\left(\frac{u}{\cos(t)} + Y \cdot tg(t)\right) \cdot \sin(t) - Y\cos(t)$$

$$v = -u \cdot tg(t) - Y \cdot [tg(t) \cdot \sin(t) + \cos(t)];\ Y = \frac{-(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)}$$

$$X = \frac{u}{\cos(t)} + Y \cdot tg(t) = \frac{u}{\cos(t)} - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot tg(t)$$

Summary

$$X = \frac{u}{\cos(t)} - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot tg(t);, Y = \frac{-(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - tg(t) \cdot \left\{ \begin{matrix} \left(\frac{\mathrm{d}v}{\mathrm{d}t} \cdot tg(t) + u \cdot \frac{1}{\cos^2(t)}\right) \cdot [tg(t) \cdot \sin(t) + \cos(t)] \\ \frac{-(u \cdot tg(t) + v) \cdot \left(-\sin(t) + tg(t) \cdot \cos(t) + \sin(t) \cdot \frac{1}{\cos^2(t)}\right)}{[tg(t) \cdot \sin(t) + \cos(t)]^2} \end{matrix} \right\}$$

$$- \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - tg(t) \cdot \left\{ \frac{\left(\frac{\mathrm{d}v}{\mathrm{d}t} \cdot tg(t) + u \cdot \frac{1}{\cos^2(t)}\right)}{[tg(t) \cdot \sin(t) + \cos(t)]} - \frac{(u \cdot tg(t) + v) \cdot \left(tg(t) \cdot \cos(t) + \sin(t) \cdot \left[\frac{1}{\cos^2(t)} - 1\right]\right)}{[tg(t) \cdot \sin(t) + \cos(t)]^2} \right\} - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)}$$

Hint $\frac{1}{\cos^2(t)} - 1 = \frac{1-\cos^2(t)}{\cos^2(t)} = \frac{\sin^2(t)}{\cos^2(t)} = tg^2(t)$

$$\begin{aligned} tg(t) \cdot \cos(t) + \sin(t) \cdot \left[\frac{1}{\cos^2(t)} - 1\right] &= tg(t) \cdot \cos(t) + \sin(t) \cdot tg^2(t) \\ &= tg(t) \cdot [\cos(t) + \sin(t) \cdot tg(t)] \end{aligned}$$

$$\frac{dX}{dt} = \frac{du}{dt} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - tg(t) \cdot \left\{ \frac{\left(\frac{dv}{dt} \cdot tg(t) + u \cdot \frac{1}{\cos^2(t)}\right)}{[tg(t) \cdot \sin(t) + \cos(t)]} - \frac{(u \cdot tg(t) + v) \cdot tg(t) \cdot [\cos(t) + \sin(t) \cdot tg(t)]}{[tg(t) \cdot \sin(t) + \cos(t)]^2} \right\} - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)}$$

$$\frac{dX}{dt} = \frac{du}{dt} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - tg(t) \cdot \left\{ \left[\frac{\left(\frac{dv}{dt} \cdot tg(t) + u \cdot \frac{1}{\cos^2(t)}\right)}{[tg(t) \cdot \sin(t) + \cos(t)]} \right] - \left[\frac{(u \cdot tg(t) + v) \cdot tg(t)}{\cos(t) + \sin(t) \cdot tg(t)} \right] \right\} - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)}$$

$$\frac{dX}{dt} = \frac{du}{dt} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - \left[\frac{\left(\frac{dv}{dt} \cdot tg^2(t) + u \cdot \frac{1}{\cos^2(t)} \cdot tg(t)\right)}{[tg(t) \cdot \sin(t) + \cos(t)]} \right] + \left[\frac{(u \cdot tg(t) + v) \cdot tg^2(t)}{\cos(t) + \sin(t) \cdot tg(t)} \right] - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)}$$

$$\frac{dX}{dt} = \frac{du}{dt} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - \frac{dv}{dt} \cdot \frac{tg^2(t)}{[tg(t) \cdot \sin(t) + \cos(t)]} - \frac{u \cdot \frac{1}{\cos^2(t)} \cdot tg(t)}{[tg(t) \cdot \sin(t) + \cos(t)]} + \left[\frac{(u \cdot tg(t) + v) \cdot tg^2(t)}{\cos(t) + \sin(t) \cdot tg(t)} \right] - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)}$$

$$\frac{dX}{dt} = Y \Rightarrow \frac{du}{dt} \cdot \frac{1}{\cos(t)} + u \cdot tg(t) \cdot \frac{1}{\cos(t)} - \frac{dv}{dt} \cdot \frac{tg^2(t)}{[tg(t) \cdot \sin(t) + \cos(t)]} - \frac{u \cdot \frac{1}{\cos^2(t)} \cdot tg(t)}{[tg(t) \cdot \sin(t) + \cos(t)]} + \left[\frac{(u \cdot tg(t) + v) \cdot tg^2(t)}{\cos(t) + \sin(t) \cdot tg(t)} \right] - \frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)} \cdot \frac{1}{\cos^2(t)} = -\frac{(u \cdot tg(t) + v)}{tg(t) \cdot \sin(t) + \cos(t)}$$

$$\frac{dY}{dt} = -\frac{\left[\frac{du}{dt} \cdot tg(t) + u \cdot \frac{1}{\cos^2(t)} + \frac{dv}{dt}\right] \cdot [tg(t) \cdot \sin(t) + \cos(t)] - (u \cdot tg(t) + v) \cdot \left[-\sin(t) + tg(t) \cdot \cos(t) + \sin(t) \cdot \frac{1}{\cos^2(t)} \right]}{[tg(t) \cdot \sin(t) + \cos(t)]^2}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -X \Rightarrow \frac{\left[\frac{\mathrm{d}u}{\mathrm{d}t}\cdot tg(t) + u\cdot\frac{1}{\cos^2(t)} + \frac{\mathrm{d}v}{\mathrm{d}t}\right]\cdot[tg(t)\cdot\sin(t)+\cos(t)] - (u\cdot tg(t)+v)\cdot\left[-\sin(t)+tg(t)\cdot\cos(t)+\sin(t)\cdot\frac{1}{\cos^2(t)}\right]}{[tg(t)\cdot\sin(t)+\cos(t)]^2}$$

$$= \frac{u}{\cos(t)} - \frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot tg(t)$$

$$\frac{\left[\frac{\mathrm{d}u}{\mathrm{d}t}\cdot tg(t) + u\cdot\frac{1}{\cos^2(t)} + \frac{\mathrm{d}v}{\mathrm{d}t}\right]\cdot[tg(t)\cdot\sin(t)+\cos(t)] - (u\cdot tg(t)+v)\cdot\left[-\sin(t)+tg(t)\cdot\cos(t)+\sin(t)\cdot\frac{1}{\cos^2(t)}\right]}{[tg(t)\cdot\sin(t)+\cos(t)]^2} = \frac{u}{\cos(t)} - \frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot tg(t)$$

$$\frac{\left[\frac{\mathrm{d}u}{\mathrm{d}t}\cdot tg(t) + u\cdot\frac{1}{\cos^2(t)} + \frac{\mathrm{d}v}{\mathrm{d}t}\right]}{[tg(t)\cdot\sin(t)+\cos(t)]} - \frac{(u\cdot tg(t)+v)\cdot\left[tg(t)\cdot\cos(t)+\sin(t)\cdot\left(\frac{1}{\cos^2(t)}-1\right)\right]}{[tg(t)\cdot\sin(t)+\cos(t)]^2}$$

$$= \frac{u}{\cos(t)} - \frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot tg(t)$$

Hint $\frac{1}{\cos^2(t)} - 1 = \frac{1-\cos^2(t)}{\cos^2(t)} = \frac{\sin^2(t)}{\cos^2(t)} = tg^2(t)$

$$tg(t)\cdot\cos(t)+\sin(t)\cdot\left[\frac{1}{\cos^2(t)}-1\right] = tg(t)\cdot\cos(t)+\sin(t)\cdot tg^2(t)$$

$$= tg(t)\cdot[\cos(t)+\sin(t)\cdot tg(t)]$$

$$\frac{\left[\frac{\mathrm{d}u}{\mathrm{d}t}\cdot tg(t) + u\cdot\frac{1}{\cos^2(t)} + \frac{\mathrm{d}v}{\mathrm{d}t}\right]}{[tg(t)\cdot\sin(t)+\cos(t)]} - \frac{(u\cdot tg(t)+v)\cdot tg(t)}{\cos(t)+\sin(t)\cdot tg(t)}$$

$$= \frac{u}{\cos(t)} - \frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot tg(t)$$

$$\frac{tg(t)}{[tg(t)\cdot\sin(t)+\cos(t)]}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\frac{1}{\cos^2(t)}}{[tg(t)\cdot\sin(t)+\cos(t)]}\cdot u + \frac{1}{[tg(t)\cdot\sin(t)+\cos(t)]}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$-\frac{(u\cdot tg(t)+v)\cdot tg(t)}{\cos(t)+\sin(t)\cdot tg(t)} = \frac{u}{\cos(t)} - \frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot tg(t)$$

We can summarize our differential equations: $\frac{\mathrm{d}u}{\mathrm{d}t} = \ldots;\ \frac{\mathrm{d}v}{\mathrm{d}t} = \ldots$

$$\frac{du}{dt}\cdot\frac{1}{\cos(t)}+u\cdot tg(t)\cdot\frac{1}{\cos(t)}-\frac{dv}{dt}\cdot\frac{tg^2(t)}{[tg(t)\cdot\sin(t)+\cos(t)]}-\frac{u\cdot\frac{1}{\cos^2(t)}\cdot tg(t)}{[tg(t)\cdot\sin(t)+\cos(t)]}$$
$$+\left[\frac{(u\cdot tg(t)+v)\cdot tg^2(t)}{\cos(t)+\sin(t)\cdot tg(t)}\right]-\frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot\frac{1}{\cos^2(t)}=-\frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}$$

$$\frac{tg(t)}{[tg(t)\cdot\sin(t)+\cos(t)]}\cdot\frac{du}{dt}+\frac{\frac{1}{\cos^2(t)}}{[tg(t)\cdot\sin(t)+\cos(t)]}\cdot u+\frac{1}{[tg(t)\cdot\sin(t)+\cos(t)]}\cdot\frac{dv}{dt}$$
$$-\frac{(u\cdot tg(t)+v)\cdot tg(t)}{\cos(t)+\sin(t)\cdot tg(t)}=\frac{u}{\cos(t)}-\frac{(u\cdot tg(t)+v)}{tg(t)\cdot\sin(t)+\cos(t)}\cdot tg(t)$$

We need to find the functions: $\xi_1(u,v)$; $\xi_2(u,v)$. $\frac{du}{dt}=\xi_1(u,v)$; $\frac{dv}{dt}=\xi_2(u,v)$.

Averaging functions $\xi_1(u,v)$; $\xi_2(u,v)$ is done to approximate the functions u, v which vary slowly because $\frac{du}{dt}$ and $\frac{dv}{dt}$ are small. Integrating each function with respect to t from 0 to $T=2\cdot\pi$, holding u, v is fixed. When α is not small the averaging procedure no longer works. For $\alpha\ll 1$, and for trajectories close to the origin, the amplitude of oscillation grows very slowly, each oscillation with a different amplitude and period. This behavior gives rise to the concept of multiple timescale of oscillation. For $\alpha\gg 1$, the oscillator goes into important oscillations known as relaxation oscillations where a so-called crawl is followed by a sudden discharge. The method of average equations is as an alternative method to solving the van der Pol oscillator [29, 31, 45, 53].

Weakly Nonlinear Oscillator Perturbation Method

Weakly nonlinear oscillator can be presented by the differential equation: $\frac{d^2X}{dt^2}+\omega_0^2\cdot X=\varepsilon\cdot f\left(X,\frac{dX}{dt},t\right)$ where f is T periodic in t. If f is a sinusoidal with frequency $\omega\approx k\cdot\omega_0$, we have a system close to a resonance of order k. Where $0\le\varepsilon\ll 1$ and $f(X,\frac{dX}{dt},t)$ is called on arbitrary smoothing constant. Basically, the differential equation represents small perturbations of the linear oscillator $\frac{d^2X}{dt^2}+\omega_0^2\cdot X=0$ and are therefore called weakly nonlinear. Duffing equation and van der Pol equation are both weakly nonlinear oscillators. We can seek solution for differential equation $\frac{d^2X}{dt^2}+\omega_0^2\cdot X=\varepsilon\cdot f(X,\frac{dX}{dt},t)$ in the perturbation expansion form of $X(t,\varepsilon)=X_0(t)+\varepsilon\cdot X_1(t)+\varepsilon^2\cdot X_2(t)+\cdots$, where it is perturbation series. The equation becomes soluble when we set $\varepsilon=0$. We get linear oscillator for $\varepsilon=0$. Let us consider the following weakly nonlinear oscillator perturbed system: $\frac{d^2X}{dt^2}+X+\Gamma\cdot\varepsilon\cdot\frac{dX}{dt}=0$. When $\omega_0^2=1$ and $f(X,\frac{dX}{dt},t)=-\Gamma\cdot\frac{dX}{dt}$; $\Gamma\in\mathbb{R}$. If we set $Y=\frac{dX}{dt}$; $\frac{dY}{dt}+X+\Gamma\cdot\varepsilon\cdot Y=0$. At fixed point $\frac{dX}{dt}=0$; $\frac{dY}{dt}=0\Rightarrow X^*=0$; $Y^*=0$.

$$\begin{pmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\Gamma \cdot \varepsilon \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix}; A = \begin{pmatrix} 0 & 1 \\ -1 & -\Gamma \cdot \varepsilon \end{pmatrix}; A - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 \\ -1 & -\Gamma \cdot \varepsilon - \lambda \end{pmatrix}$$

$$\det(A - \lambda \cdot I) = \lambda \cdot (\Gamma \cdot \varepsilon + \lambda) + 1 = \lambda^2 + \lambda \cdot \Gamma \cdot \varepsilon + 1; \ \lambda_{1,2} = \frac{-\Gamma \cdot \varepsilon \pm \sqrt{\Gamma^2 \cdot \varepsilon^2 - 4}}{2}$$

$\lambda_1 = \frac{-\Gamma \cdot \varepsilon + \sqrt{\Gamma^2 \cdot \varepsilon^2 - 4}}{2}$; $\lambda_2 = \frac{-\Gamma \cdot \varepsilon - \sqrt{\Gamma^2 \cdot \varepsilon^2 - 4}}{2}$. We need to classify our system fixed points: $\Gamma > 0$; $\Gamma \in \mathbb{R}$; $0 \leq \varepsilon \ll 1$. (Table 8.1)

We assume that $X(t) = e^{r \cdot t}$ is a solution of $\frac{d^2X}{dt^2} + X + \Gamma \cdot \varepsilon \cdot \frac{dX}{dt} = 0$ where "r" is a constant. By finding the value of "r" and substitute $X(t) = e^{r \cdot t}$ into weakly nonlinear oscillator differential equation, it will be soluble. $\frac{dX(t)}{dt} = r \cdot e^{r \cdot t}$

$$\frac{d^2X(t)}{dt^2} = r^2 \cdot e^{r \cdot t}; \frac{d^2X}{dt^2} + X + \Gamma \cdot \varepsilon \cdot \frac{dX}{dt} = 0 \Rightarrow r^2 \cdot e^{r \cdot t} + e^{r \cdot t} + \Gamma \cdot \varepsilon \cdot r \cdot e^{r \cdot t} = 0$$

$[r^2 + \Gamma \cdot \varepsilon \cdot r + 1] \cdot e^{r \cdot t} = 0$. Now either $e^{r \cdot t} = 0$ or $r^2 + \Gamma \cdot \varepsilon \cdot r + 1 = 0$. We choose the latter as it is a quadratic equation and it allows us to solve for the value of the "r". Hence, we have $r^2 + \Gamma \cdot \varepsilon \cdot r + 1 = 0$ and this form of solution to differential equation is called the auxiliary equation: $r^2 + \Gamma \cdot \varepsilon \cdot r + 1 = 0 \Rightarrow r = \frac{-\Gamma \cdot \varepsilon \pm \sqrt{\Gamma^2 \cdot \varepsilon^2 - 4}}{2}$. $r = -\frac{1}{2} \cdot \Gamma \cdot \varepsilon \pm \frac{1}{2} \cdot \sqrt{\Gamma^2 \cdot \varepsilon^2 - 4}$. So the solution can be $X(t) = e^{\frac{1}{2} \cdot (-\Gamma \cdot \varepsilon \pm \sqrt{\Gamma^2 \cdot \varepsilon^2 - 4}) \cdot t}$.. We shall apply perturbation theory to solve the damped oscillator in equation:

$$\frac{d^2X}{dt^2} + X + \Gamma \cdot \varepsilon \cdot \frac{dX}{dt} = 0. \quad X(t, \varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \cdots$$

$$\frac{dX(t, \varepsilon)}{dt} = \frac{d}{dt}\left[X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \cdots\right] = \frac{dX_0(t)}{dt} + \varepsilon \cdot \frac{dX_1(t)}{dt} + \varepsilon^2 \cdot \frac{dX_2(t)}{dt} + \cdots$$

$$\frac{d^2X(t, \varepsilon)}{dt^2} = \frac{d^2}{dt^2}\left[X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \cdots\right] = \frac{d^2X_0(t)}{dt^2} + \varepsilon \cdot \frac{d^2X_1(t)}{dt^2} + \varepsilon^2 \cdot \frac{d^2X_2(t)}{dt^2} + \cdots$$

Table 8.1 Weakly nonlinear oscillator perturbed system

System parameter	Eigenvalues	Fixed points classification
$\Gamma = \frac{2}{\varepsilon}$	$\lambda_{1,2} = \frac{-\Gamma \cdot \varepsilon}{2} = -1$	Attracting focus
$\Gamma = -\frac{2}{\varepsilon}$	$\lambda_{1,2} = \frac{-\Gamma \cdot \varepsilon}{2} = 1$	Repelling focus
$\Gamma^2 \cdot \varepsilon^2 - 4 > 0 \Rightarrow \Gamma > \frac{2}{\varepsilon}$ or $\Gamma < -\frac{2}{\varepsilon}$ *Case A* $\Gamma > \frac{2}{\varepsilon} \Rightarrow \Gamma = \frac{2+\Gamma_0}{\varepsilon}\Gamma_0 > 0;\ \Gamma_0 \in \mathbb{R}$. If $\Gamma = \frac{2+\Gamma_0}{\varepsilon}$ then $\Gamma^2 \cdot \varepsilon^2 - 4 > 0 \Rightarrow \frac{(2+\Gamma_0)^2}{\varepsilon^2} \cdot \varepsilon^2 - 4 > 0$ $\frac{4+4\cdot\Gamma_0+\Gamma_0^2}{\varepsilon^2} \cdot \varepsilon^2 - 4 > 0;\ \Gamma_0 \cdot (4+\Gamma_0) > 0$	$\lambda_{1,2} = \frac{-\Gamma\cdot\varepsilon \pm \sqrt{\Gamma^2\cdot\varepsilon^2-4}}{2}$ $\lambda_{1,2} = \frac{-\left(\frac{2+\Gamma_0}{\varepsilon}\right)\cdot\varepsilon \pm \sqrt{\Gamma_0\cdot(4+\Gamma_0)}}{2}$ $\lambda_{1,2} = \frac{-(2+\Gamma_0) \pm \sqrt{\Gamma_0\cdot(4+\Gamma_0)}}{2}$ $\lambda_1 \cdot \lambda_2 < 0$	Saddle point
$\Gamma^2 \cdot \varepsilon^2 - 4 < 0 \Rightarrow -\frac{2}{\varepsilon} < \Gamma < \frac{2}{\varepsilon}$ $\Gamma < \frac{2}{\varepsilon};\ \Gamma = \frac{2-\Gamma_0}{\varepsilon};\ \Gamma_0 > 0;\ \Gamma_0 \in \mathbb{R}$ If $\Gamma = \frac{2-\Gamma_0}{\varepsilon}$ then $\Gamma^2 \cdot \varepsilon^2 - 4 < 0$ $\frac{(2-\Gamma_0)^2}{\varepsilon^2} \cdot \varepsilon^2 - 4 < 0;\ \Gamma_0 \cdot (\Gamma_0 - 4) < 0$ $\Gamma_0 > 0;\ \Gamma_0 - 4 < 0 \Rightarrow \Gamma_0 < 4$ *Summary* $0 < \Gamma_0 < 4$	$\lambda_{1,2} = \frac{-\Gamma\cdot\varepsilon \pm \sqrt{\Gamma^2\cdot\varepsilon^2-4}}{2}$ $\lambda_{1,2} = \frac{-\left(\frac{2-\Gamma_0}{\varepsilon}\right)\cdot\varepsilon \pm \sqrt{\Gamma_0\cdot(\Gamma_0-4)}}{2}\lambda_{1,2} = \frac{(\Gamma_0-2) \pm \sqrt{\Gamma_0\cdot(\Gamma_0-4)}}{2}$ $0 < \Gamma_0 < 2$ $\lambda_{1,2}$ are complex with negative real part	Attracting spiral
$\Gamma^2 \cdot \varepsilon^2 - 4 < 0 \Rightarrow -\frac{2}{\varepsilon} < \Gamma < \frac{2}{\varepsilon}$ $\Gamma < \frac{2}{\varepsilon};\ \Gamma = \frac{2-\Gamma_0}{\varepsilon};\ \Gamma_0 > 0;\ \Gamma_0 \in \mathbb{R}$ If $\Gamma = \frac{2-\Gamma_0}{\varepsilon}$ then $\Gamma^2 \cdot \varepsilon^2 - 4 < 0$ $\frac{(2-\Gamma_0)^2}{\varepsilon^2} \cdot \varepsilon^2 - 4 < 0;\ \Gamma_0 \cdot (\Gamma_0 - 4) < 0$ $\Gamma_0 > 0;\ \Gamma_0 - 4 < 0 \Rightarrow \Gamma_0 < 4$ Summary: $0 < \Gamma_0 < 4$	$\lambda_{1,2} = \frac{-\Gamma\cdot\varepsilon \pm \sqrt{\Gamma^2\cdot\varepsilon^2-4}}{2}$ $\lambda_{1,2} = \frac{-\left(\frac{2-\Gamma_0}{\varepsilon}\right)\cdot\varepsilon \pm \sqrt{\Gamma_0\cdot(\Gamma_0-4)}}{2}\lambda_{1,2} = \frac{(\Gamma_0-2) \pm \sqrt{\Gamma_0\cdot(\Gamma_0-4)}}{2}$ $2 < \Gamma_0 < 4$ $\lambda_{1,2}$ are complex with positive real part	Repelling spiral
$\Gamma = 0 \Rightarrow \Gamma^2 \cdot \varepsilon^2 - 4 = -4;\ \Gamma \cdot \varepsilon = 0$	$\lambda_{1,2} = \frac{\pm\sqrt{-4}}{2} = \pm\sqrt{-1} = \pm j$	Center

$$X(t,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k\cdot X_k(t);\ \frac{\mathrm{d}X(t,\varepsilon)}{\mathrm{d}t}=\sum_{k=0}^{\infty}\varepsilon^k\cdot\frac{\mathrm{d}X_k(t)}{\mathrm{d}t};\ \frac{\mathrm{d}^2X(t,\varepsilon)}{\mathrm{d}t^2}=\sum_{k=0}^{\infty}\varepsilon^k\cdot\frac{\mathrm{d}^2X_k(t)}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2}+X+\Gamma\cdot\varepsilon\cdot\frac{\mathrm{d}X}{\mathrm{d}t}=0\Rightarrow\sum_{k=0}^{\infty}\varepsilon^k\cdot\frac{\mathrm{d}^2X_k(t)}{\mathrm{d}t^2}\quad+\sum_{k=0}^{\infty}\varepsilon^k\cdot X_k(t)+\Gamma\cdot\varepsilon\cdot\sum_{k=0}^{\infty}\varepsilon^k\cdot\frac{\mathrm{d}X_k(t)}{\mathrm{d}t}=0.$$

We differentiate the above differential equation term by term with respect to t and we group terms according to the powers of ε, omitting all terms with coefficients of ε^2 and higher.

$$\left(\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}+\varepsilon\cdot\frac{\mathrm{d}^2X_1(t)}{\mathrm{d}t^2}+\varepsilon^2\cdot\frac{\mathrm{d}^2X_2(t)}{\mathrm{d}t^2}+\cdots\right)+\left(X_0(t)+\varepsilon\cdot X_1(t)+\varepsilon^2\cdot X_2(t)+\cdots\right)$$
$$+\Gamma\cdot\varepsilon\cdot\left(\frac{\mathrm{d}X_0(t)}{\mathrm{d}t}+\varepsilon\cdot\frac{\mathrm{d}X_1(t)}{\mathrm{d}t}+\varepsilon^2\cdot\frac{\mathrm{d}X_2(t)}{\mathrm{d}t}+\cdots\right)=0$$

$$\left[\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}+X_0(t)\right]\cdot\varepsilon^0+\left[\frac{\mathrm{d}^2X_1(t)}{\mathrm{d}t^2}+X_1(t)+\Gamma\cdot\frac{\mathrm{d}X_0(t)}{\mathrm{d}t}\right]\cdot\varepsilon+O\left(\varepsilon^2\right)=0.$$

Since $\varepsilon\neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero. The zeroth of ε gives us $\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}+X_0(t)=0;\ \frac{\mathrm{d}^2X_1(t)}{\mathrm{d}t^2}+X_1(t)+\Gamma\cdot\frac{\mathrm{d}X_0(t)}{\mathrm{d}t}=0.$

Next we need to solve the above differential equations. These equations are indeed homogenous linear differential equations with constant coefficients [133–135].

First we investigate the $\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}+X_0(t)=0$ differential equation. Let us assume that $X_0(t)=\mathrm{e}^{r\cdot t}$ is a solution to equation where r is a constant. The auxiliary equations are $\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}=r^2\cdot\mathrm{e}^{r\cdot t}\Rightarrow r^2\cdot\mathrm{e}^{r\cdot t}+\mathrm{e}^{r\cdot t}=0;\ (r^2+1)\cdot\mathrm{e}^{r\cdot t}=0;\ r=\pm j.$

Here the r real part is equal to zero and the imaginary part is equal to one. If as we assumed from above that the general solution for $X_0(t)$ is of the form $\cos(t)+\sin(t)$ and we double check for solution: $X_0(t)=\cos(t)+\sin(t)$. $\frac{\mathrm{d}X_0(t)}{\mathrm{d}t}=\cos(t)-\sin(t);\ \frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}=-\sin(t)-\cos(t)=-[\sin(t)+\cos(t)].$ Then $\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}+X_0(t)=0$. We can write our solution to $\frac{\mathrm{d}^2X_0(t)}{\mathrm{d}t^2}+X_0(t)=0$ in the convenient form: $X_0(t)=A_1\cdot\cos(t)+A_2\cdot\sin(t)$ where A_1 and A_2 are constants to be determined. If we utilized the initial conditions: $X_0(t=0)=0;\ \frac{\mathrm{d}X_0(t=0)}{\mathrm{d}t}=\Omega.$

Given to find the values of the constants A_1 and A_2, we have by substituting these values in equation: $X_0(t)=A_1\cdot\cos(t)+A_2\cdot\sin(t)$ that $A_1=0$ hence

$$X_0(t) = A_2 \cdot \sin(t);\ \frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = A_2 \cdot \cos(t);\ \frac{\mathrm{d}X_0(t=0)}{\mathrm{d}t} = A_2 = \Omega$$

then $X_0(t) = \Omega \cdot \sin(t)$.

The second differential equation is $\frac{\mathrm{d}^2 X_1(t)}{\mathrm{d}t^2} + X_1(t) + \Gamma \cdot \frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = 0$ and from our last solution $\frac{\mathrm{d}X_0(t)}{\mathrm{d}t} = \Omega \cdot \cos(t)$ then $\frac{\mathrm{d}^2 X_1(t)}{\mathrm{d}t^2} + X_1(t) = -\Gamma \cdot \Omega \cdot \cos(t)$. The last differential equation is a typical case of resonance. It is because of resonant interactions between consecutive orders that nonuniformity has appeared in the regular perturbation series. This is a simple harmonic oscillator with natural frequency one, driven by a periodic, external, forcing frequency which is equal to one ($\omega = 1$) on the right-hand side. The amplitude of oscillation for such a system is unbounded as $t \to \infty$ because the oscillator continually absorbs energy from the periodic external force, and thus, system is in resonance with the external force. The solution, therefore, to such a system, represents this fact in term "$t \cdot \sin(t)$" appeared in the solution in equation $X(t, \varepsilon)$ because the inhomogeneous term $-\Gamma \cdot \Omega \cdot \cos(t)$ is itself a solution of the associated homogeneous equation: $\frac{\mathrm{d}^2 X_1(t)}{\mathrm{d}t^2} + X_1(t) = -\Gamma \cdot \Omega \cdot \cos(t)$. The secular terms appear whenever the inhomogeneous term is itself a solution of the associated homogeneous differential equation. A secular term always grows faster than the corresponding solution of the homogeneous solution by at least a factor t. The appearance of secular terms demonstrates the nonuniform validity of the perturbation expansion for large t. The $X_1(t)$ solution is $-\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \sin(t)$ and then $X_1(t) = -\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \sin(t)$. Derivative $X_1(t)$ with time gives

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = -\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \sin(t) - \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \cos(t)$$

$$\frac{\mathrm{d}^2 X_1(t)}{\mathrm{d}t^2} = -\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \cos(t) - \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \cos(t) + \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \sin(t)$$

$$\frac{\mathrm{d}^2 X_1(t)}{\mathrm{d}t^2} + X_1(t) = -\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \cos(t) - \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \cos(t) + \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \sin(t) - \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \sin(t)$$

$$\frac{\mathrm{d}^2 X_1(t)}{\mathrm{d}t^2} + X_1(t) = -\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \cos(t) - \frac{1}{2} \cdot \Gamma \cdot \Omega \cdot \cos(t) = -\Gamma \cdot \Omega \cdot \cos(t).$$

It fulfils our differential equation solution.

Summary $X_0(t) = \Omega \cdot \sin(t)$; $X_1(t) = -\frac{1}{2} \cdot \Gamma \cdot \Omega \cdot t \cdot \sin(t)$ we only now have to substitute for $X_0(t)$ and $X_1(t)$ in equation $X(t, \varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \cdots$

$$X(t, \varepsilon) = \sum_{k=0}^{\infty} X_k(t) \cdot \varepsilon^k = X_0(t) + \varepsilon \cdot X_1(t) + O(\varepsilon^2).$$

Perturbation Theory Multiple Timescale

We can use multiple scale analysis for constructing uniform or global approximate solutions for both small and large values of independent variables. The dependent variables are uniformly expanded in terms of two or more independent variables, scales. The issue is the choice of ordering scheme and the form of the power series expansion. We can implement in many engineering systems, multiple scale perturbation theory (MSPT). In the perturbation series expansion itself, secular terms appear in all orders except O(1) which is Zeroth order and violate the boundedness of the solution. Basically, multiple scale analysis is a technique to construct uniformly valid approximations to the solutions of perturbation systems and problems, both for small as well as large values of the independent variables. It is done by fast-scale and low-scale variables for an independent variable, and subsequently treating these variables (fast and slow) as if they are independent. The resulting additional freedom introduced by the new independent variables which used to remove secular terms. The coordinate transforms and invariant manifolds provide a support for multiscale modelling. Practically, there are at least two timescales in weakly nonlinear oscillators. Two timing builds two timescales from the start and produces better approximations than the regular perturbation theory. We need to apply two timing to differential equation: $\frac{d^2x}{dt^2} + \omega_0^2 \cdot X = \varepsilon \cdot f\left(X, \frac{dx}{dt}, t\right)$. Let $t_A = t$ denote the fast O(1) time, and let $t_B = \varepsilon \cdot t$ denote the slow time. We take these times as independent variables. The functions of slow time t_B are constants on the fast timescale t_A. We expand the solution $\frac{d^2x}{dt^2} + \omega_0^2 \cdot X = \varepsilon \cdot f\left(X, \frac{dx}{dt}, t\right)$ as a series $X(t, \varepsilon) = X_0(t_A, t_B) + \varepsilon \cdot X_1(t_A, t_B) + O(\varepsilon^2)$. We need to implement time derivatives in systems differential equation and are transferred using the chain rule.

$$\partial t_A = \frac{\partial}{\partial t_A};\ Dt_A = \frac{d}{dt_A};\ \frac{\partial}{\partial t_A} \Leftrightarrow \frac{d}{dt_A};\ \partial t_B = \frac{\partial}{\partial t_B};\ Dt_B = \frac{d}{dt_B};$$
$$\frac{\partial}{\partial t_B} \Leftrightarrow \frac{d}{dt_B}, \frac{dX}{dt} = \frac{dX}{dt_A} + \frac{dX}{dt_B} \cdot \frac{dt_B}{dt} = \frac{\partial X}{\partial t_A} + \frac{\partial X}{\partial t_B} \cdot \frac{\partial t_B}{\partial t}.$$

It is time derivative of $X(t, \varepsilon)$ variable using the chain rule.

$$t_A = t;\ t_B = \varepsilon \cdot t;\ \frac{dt_B}{dt} = \frac{\partial t_B}{\partial t} = \varepsilon \Rightarrow \frac{dX}{dt} = \frac{dX}{dt_A} + \frac{dX}{dt_B} \cdot \varepsilon;$$
$$\frac{\partial X}{\partial t} = \frac{\partial X}{\partial t_A} + \frac{\partial X}{\partial t_B} \cdot \varepsilon, \frac{\partial X}{\partial t} = \frac{dX}{dt} = \partial t_A X + \varepsilon \cdot \partial t_B X.$$

We know that $X(t, \varepsilon) = X_0(t_A, t_B) + \varepsilon \cdot X_1(t_A, t_B) + O(\varepsilon^2)$

$$\frac{dX(t,\varepsilon)}{dt} = \frac{\partial X_0(t_A,t_B)}{\partial t_A} + \frac{\partial X_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon + \varepsilon\cdot\left[\frac{\partial X_1(t_A,t_B)}{\partial t_A} + \varepsilon\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B}\right] + O(\varepsilon^2)$$

$$\frac{dX(t,\varepsilon)}{dt} = \frac{\partial X_0(t_A,t_B)}{\partial t_A} + \left[\frac{\partial X_0(t_A,t_B)}{\partial t_B} + \frac{\partial X_1(t_A,t_B)}{\partial t_A}\right]\cdot\varepsilon + \varepsilon^2\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B} + O(\varepsilon^2)$$

$$\varepsilon^2\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B} + O(\varepsilon^2) \to O(\varepsilon^2) \Rightarrow \frac{dX(t,\varepsilon)}{dt} = \frac{\partial X_0(t_A,t_B)}{\partial t_A} + \left[\frac{\partial X_0(t_A,t_B)}{\partial t_B} + \frac{\partial X_1(t_A,t_B)}{\partial t_A}\right]\cdot\varepsilon + O(\varepsilon^2)$$

$$\frac{dX(t,\varepsilon)}{dt} = \frac{\partial X_0(t_A,t_B)}{\partial t_A} + \left[\frac{\partial X_0(t_A,t_B)}{\partial t_B} + \frac{\partial X_1(t_A,t_B)}{\partial t_A}\right]\cdot\varepsilon + \varepsilon^2\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B} + O(\varepsilon^2).$$

We already find the below differential equation:

$$\frac{dX(t,\varepsilon)}{dt} = \frac{\partial X_0(t_A,t_B)}{\partial t_A} + \frac{\partial X_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon + \varepsilon\cdot\left[\frac{\partial X_1(t_A,t_B)}{\partial t_A} + \varepsilon\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B}\right] + O(\varepsilon^2)$$

$$\frac{d^2X(t,\varepsilon)}{dt^2} = \frac{\partial}{\partial t_A}\left[\frac{\partial X_0(t_A,t_B)}{\partial t_A} + \frac{\partial X_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon\right] + \varepsilon\cdot\left\{\frac{\partial}{\partial t_B}\left[\frac{\partial X_0(t_A,t_B)}{\partial t_A} + \frac{\partial X_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon\right] + \frac{\partial}{\partial t_A}\left[\frac{\partial X_1(t_A,t_B)}{\partial t_A} + \varepsilon\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B}\right]\right\} + O(\varepsilon^2)$$

$$\frac{d^2X(t,\varepsilon)}{dt^2} = \frac{\partial^2 X_0(t_A,t_B)}{\partial t_A^2} + \frac{\partial^2 X_0(t_A,t_B)}{\partial t_A\partial t_B}\cdot\varepsilon + \varepsilon\cdot\frac{\partial^2 X_0(t_A,t_B)}{\partial t_B\partial t_A} + \frac{\partial^2 X_0(t_A,t_B)}{\partial t_B^2}\cdot\varepsilon^2 + \frac{\partial^2 X_1(t_A,t_B)}{\partial t_A^2}\cdot\varepsilon + \varepsilon^2\cdot\frac{\partial^2 X_1(t_A,t_B)}{\partial t_A\partial t_B} + O(\varepsilon^2)$$

$$\frac{d^2X(t,\varepsilon)}{dt^2} = \frac{\partial^2 X_0(t_A,t_B)}{\partial t_A^2} + \left[\frac{\partial^2 X_0(t_A,t_B)}{\partial t_A\partial t_B} + \frac{\partial^2 X_0(t_A,t_B)}{\partial t_B\partial t_A} + \frac{\partial^2 X_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon + \left[\frac{\partial^2 X_0(t_A,t_B)}{\partial t_B^2} + \frac{\partial^2 X_1(t_A,t_B)}{\partial t_A\partial t_B}\right]\cdot\varepsilon^2 + O(\varepsilon^2)$$

$$\left[\frac{\partial^2 X_0(t_A,t_B)}{\partial t_B^2} + \frac{\partial^2 X_1(t_A,t_B)}{\partial t_A\partial t_B}\right]\cdot\varepsilon^2 + O(\varepsilon^2) \to O(\varepsilon^2)$$

$$\frac{d^2X(t,\varepsilon)}{dt^2} = \frac{\partial^2 X_0(t_A,t_B)}{\partial t_A^2} + \left[2\cdot\frac{\partial^2 X_0(t_A,t_B)}{\partial t_A\partial t_B} + \frac{\partial^2 X_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon + O(\varepsilon^2).$$

We need to implement it to our weakly nonlinear oscillator perturbed system differential equation: $\frac{d^2X}{dt^2} + X + \Gamma\cdot\varepsilon\cdot\frac{dX}{dt} = 0;\ X(t=0,\varepsilon) = 0$

$$\frac{d^2X(t,\varepsilon)}{dt^2}+X(t,\varepsilon)+\Gamma\cdot\varepsilon\cdot\frac{dX(t,\varepsilon)}{dt}=0;\ X(t=0,\varepsilon)=0$$

$$\frac{\partial^2X_0(t_A,t_B)}{\partial t_A^2}+\left[2\cdot\frac{\partial^2X_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2X_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon+X_0(t_A,t_B)+\varepsilon\cdot X_1(t_A,t_B)$$
$$+\Gamma\cdot\varepsilon\cdot\left\{\frac{\partial X_0(t_A,t_B)}{\partial t_A}+\frac{\partial X_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon+\varepsilon\cdot\left[\frac{\partial X_1(t_A,t_B)}{\partial t_A}+\varepsilon\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B}\right]+O(\varepsilon^2)\right\}$$
$$+O(\varepsilon^2)=0$$

$$\frac{\partial^2X_0(t_A,t_B)}{\partial t_A^2}+\left[2\cdot\frac{\partial^2X_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2X_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon+X_0(t_A,t_B)+\varepsilon\cdot X_1(t_A,t_B)$$
$$+\Gamma\cdot\varepsilon\cdot\left\{\frac{\partial X_0(t_A,t_B)}{\partial t_A}+\left[\frac{\partial X_0(t_A,t_B)}{\partial t_B}+\frac{\partial X_1(t_A,t_B)}{\partial t_A}\right]\cdot\varepsilon\right\}+O(\varepsilon^2)=0$$

Since $\varepsilon^2\cdot\frac{\partial X_1(t_A,t_B)}{\partial t_B}+O(\varepsilon^2)\to O(\varepsilon^2)$

Since $\varepsilon\neq0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero [7, 8].

$$\frac{\partial^2X_0(t_A,t_B)}{\partial t_A^2}+X_0(t_A,t_B)+\left[2\cdot\frac{\partial^2X_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2X_1(t_A,t_B)}{\partial t_A^2}+X_1(t_A,t_B)+\Gamma\cdot\frac{\partial X_0(t_A,t_B)}{\partial t_A}\right]\cdot\varepsilon$$
$$+\left[\frac{\partial X_0(t_A,t_B)}{\partial t_B}+\frac{\partial X_1(t_A,t_B)}{\partial t_A}\right]\cdot\Gamma\cdot\varepsilon^2+O(\varepsilon^2)=0$$

$$\left[\frac{\partial X_0(t_A,t_B)}{\partial t_B}+\frac{\partial X_1(t_A,t_B)}{\partial t_A}\right]\cdot\Gamma\cdot\varepsilon^2+O(\varepsilon^2)\to O(\varepsilon^2)$$

$$\frac{\partial^2X_0(t_A,t_B)}{\partial t_A^2}+X_0(t_A,t_B)+\left[2\cdot\frac{\partial^2X_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2X_1(t_A,t_B)}{\partial t_A^2}+X_1(t_A,t_B)+\Gamma\cdot\frac{\partial X_0(t_A,t_B)}{\partial t_A}\right]\cdot\varepsilon=0.$$

We get pair of differential equations: $O(\varepsilon^0=1)\Rightarrow\frac{\partial^2X_0(t_A,t_B)}{\partial t_A^2}+X_0(t_A,t_B)=0$

$$O(\varepsilon)\Rightarrow\ 2\cdot\frac{\partial^2X_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2X_1(t_A,t_B)}{\partial t_A^2}+X_1(t_A,t_B)+\Gamma\cdot\frac{\partial X_0(t_A,t_B)}{\partial t_A}=0$$

Differential equation $\frac{\partial^2X_0(t_A,t_B)}{\partial t_A^2}+X_0(t_A,t_B)=0$ is for a simple harmonic oscillator. Its general solution is $X_0(t_A,t_B)=B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)$. The constants B_1 and B_2 are actually functions of slow time t_B. Times t_A and t_B should be regarded as independent variables and functions of t_B behaving like a constants on the fast timescale t_A. We need to determine the constants $B_1(t_B)$ and $B_2(t_B)$. It is done by going to the next order of ε substituting $X_0(t_A,t_B)=B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)$ into $O(\varepsilon)$ differential equation.

$$2\cdot\frac{\partial^2 X_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2 X_1(t_A,t_B)}{\partial t_A^2}+X_1(t_A,t_B)+\Gamma\cdot\frac{\partial X_0(t_A,t_B)}{\partial t_A}=0$$

$$\frac{\partial X_0(t_A,t_B)}{\partial t_A}=B_1\cdot\cos(t_A)-B_2\cdot\sin(t_A);\ \frac{\partial^2 X_0(t_A,t_B)}{\partial t_A^2}$$
$$=-B_1\cdot\sin(t_A)-B_2\cdot\cos(t_A)$$

$$\frac{\partial^2 X_0(t_A,t_B)}{\partial t_A\partial t_B}=\frac{\partial B_1}{\partial t_B}\cdot\cos(t_A)-\frac{\partial B_2}{\partial t_B}\cdot\sin(t_A)$$

$$2\cdot\left[\frac{\partial B_1}{\partial t_B}\cdot\cos(t_A)-\frac{\partial B_2}{\partial t_B}\cdot\sin(t_A)\right]+\frac{\partial^2 X_1(t_A,t_B)}{\partial t_A^2}+X_1(t_A,t_B)$$
$$+\Gamma\cdot[B_1\cdot\cos(t_A)-B_2\cdot\sin(t_A)]=0$$

$$\frac{\partial^2 X_1(t_A,t_B)}{\partial t_A^2}+X_1(t_A,t_B)=-\left[\Gamma\cdot B_1+2\cdot\frac{\partial B_1}{\partial t_B}\right]\cdot\cos(t_A)+\left[\Gamma\cdot B_2+2\cdot\frac{\partial B_2}{\partial t_B}\right]$$
$$\cdot\sin(t_A).$$

The right-hand side of the above equation is a resonant forcing that will produce secular terms like $t_A\cdot\sin(t_A)$ and $t_A\cdot\cos(t_A)$ in the solution of $X_1(t_A,t_B)$. These terms would lead to convergent. The approximation is done with no secular terms and need to set the coefficients of the resonant terms to zero.

$$\Gamma\cdot B_1+2\cdot\frac{\partial B_1}{\partial t_B}=0;\ \Gamma\cdot B_2+2\cdot\frac{\partial B_2}{\partial t_B}=0;\ \Gamma\cdot B_1+2\cdot\frac{\partial B_1}{\partial t_B}=0\Rightarrow\frac{\frac{\partial B_1}{\partial t_B}}{B_1}=-\frac{\Gamma}{2}$$

$$\frac{\frac{\partial B_1}{\partial t_B}}{B_1}=-\frac{\Gamma}{2}\Rightarrow\frac{\mathrm{d}(\ln(B_1))}{\mathrm{d}t_B}=-\frac{\Gamma}{2}\Rightarrow\int\frac{\mathrm{d}(\ln(B_1))}{\mathrm{d}t_B}\cdot\mathrm{d}t_B=-\int\frac{\Gamma}{2}\cdot\mathrm{d}t_B;$$
$$\ln(B_1)=-\frac{\Gamma}{2}\cdot t_B$$

$$\ln(B_1)=-\frac{\Gamma}{2}\cdot t_B\Rightarrow B_1=B_1(t_B=0)\cdot\mathrm{e}^{-\frac{\Gamma}{2}\cdot t_B};\quad \Gamma\cdot B_2+2\cdot\frac{\partial B_2}{\partial t_B}=0\Rightarrow\frac{\frac{\partial B_2}{\partial t_B}}{B_2}=-\frac{\Gamma}{2}$$

$$\frac{\frac{\partial B_2}{\partial t_B}}{B_2}=-\frac{\Gamma}{2}\Rightarrow\frac{\mathrm{d}(\ln(B_2))}{\mathrm{d}t_B}=-\frac{\Gamma}{2}\Rightarrow\int\frac{\mathrm{d}(\ln(B_2))}{\mathrm{d}t_B}\cdot\mathrm{d}t_B=-\int\frac{\Gamma}{2}\cdot\mathrm{d}t_B;$$
$$\ln(B_2)=-\frac{\Gamma}{2}\cdot t_B$$

$$\ln(B_2)=-\frac{\Gamma}{2}\cdot t_B\Rightarrow B_2=B_2(t_B=0)\cdot\mathrm{e}^{-\frac{\Gamma}{2}\cdot t_B}.$$

Next is to find initial values for $B_1(t_B = 0)$ and $B_2(t_B = 0)$; it is done by the following equations: $X(t, \varepsilon) = X_0(t_A, t_B) + \varepsilon \cdot X_1(t_A, t_B) + O(\varepsilon^2)$ and

$$X_0(t_A, t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A);\ X(t = 0, \varepsilon) = 0;\ \frac{\mathrm{d}X(t = 0, \varepsilon)}{\mathrm{d}t} = \Omega.$$

Equation: $X(t, \varepsilon) = X_0(t_A, t_B) + \varepsilon \cdot X_1(t_A, t_B) + O(\varepsilon^2)$ gives $X(t = 0, \varepsilon) = 0$ then

$$X(t, \varepsilon) = 0 \Rightarrow X_0(t_A = 0, t_B = 0) + \varepsilon \cdot X_1(t_A = 0, t_B = 0) + O(\varepsilon^2) = 0$$

$t = 0 \Rightarrow t_A = t;\ t_B = \varepsilon \cdot t \Rightarrow t_A = 0;\ t_B = 0$. To satisfy this equation for all sufficiently small ε, we must have $X_0(t_A = 0, t_B = 0) = 0$ and $X_1(t_A = 0, t_B = 0) = 0$.

Similarly $\frac{\mathrm{d}X(t=0,\varepsilon)}{\mathrm{d}t} = \Omega;\ \frac{\mathrm{d}X(t,\varepsilon)}{\mathrm{d}t} = \frac{\partial X_0(t_A,t_B)}{\partial t_A} + \varepsilon \cdot \left[\frac{\partial X_0(t_A,t_B)}{\partial t_B} + \frac{\partial X_1(t_A,t_B)}{\partial t_A}\right] + O(\varepsilon^2)$

$$\frac{\mathrm{d}X(t = 0, \varepsilon)}{\mathrm{d}t} = \frac{\partial X_0(t_A = 0, t_B = 0)}{\partial t_A} + \varepsilon \cdot \left[\frac{\partial X_0(t_A = 0, t_B = 0)}{\partial t_B} + \frac{\partial X_1(t_A = 0, t_B = 0)}{\partial t_A}\right] + O(\varepsilon^2) = \Omega.$$

Result $\frac{\partial X_0(t_A=0,t_B=0)}{\partial t_A} = \Omega;\ \frac{\partial X_0(t_A=0,t_B=0)}{\partial t_B} + \frac{\partial X_1(t_A=0,t_B=0)}{\partial t_A} = 0.$

Combining $X_0(t_A, t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A);\ X_0(t_A = 0, t_B = 0) = 0$ then $B_1 \cdot \sin(t_A) = 0;\ B_2 \cdot \cos(t_A) = 0$. Hence $B_2(t_B = 0) = 0;\ B_2(t_B) = 0$; similarly

$$X_0(t, \varepsilon) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A);\ \frac{\partial X_0(t_A = 0, t_B = 0)}{\partial t_A} = \Omega$$

Imply

$$\frac{\partial X_0(t_A, t_B)}{\partial t_A} = B_1 \cdot \cos(t_A) - B_2 \cdot \sin(t_A);\ B_2 = 0\,\&\,t_A = 0 \Rightarrow B_1(0) = \Omega;\ B_1(t_B) = \Omega \cdot \mathrm{e}^{-\frac{\Gamma}{2} \cdot t_B}.$$

$$X(t, \varepsilon) = X_0(t_A, t_B) + \varepsilon \cdot X_1(t_A, t_B) + O(\varepsilon^2);\ X_0(t_A, t_B) = \Omega \cdot \mathrm{e}^{-\frac{\Gamma}{2} \cdot t_B} \cdot \sin(t_A)$$

$$X(t, \varepsilon) = \Omega \cdot \mathrm{e}^{-\frac{\Gamma}{2} \cdot t_B} \cdot \sin(t_A) + \varepsilon \cdot X_1(t_A, t_B) + O(\varepsilon^2);\ \varepsilon \cdot X_1(t_A, t_B) = O(\varepsilon)$$

$$X(t, \varepsilon) = \Omega \cdot \mathrm{e}^{-\frac{\Gamma}{2} \cdot t_B} \cdot \sin(t_A) + O(\varepsilon) + O(\varepsilon^2).$$

It is approximate solution predicted by two timing.

8.2 OptoNDR Circuit van der Pol Perturbation Method

The van der pol oscillator can be given by the following equations: $\frac{\mathrm{d}X_1}{\mathrm{d}t} = X_2$ and $\frac{\mathrm{d}X_2}{\mathrm{d}t} = -X_1 - \mu \cdot (X_1^2 - 1) \cdot X_2$ (see Sect. 6.2). We can express it by second-degree differential equation: $\frac{\mathrm{d}^2 X_1}{\mathrm{d}t^2} + \mu \cdot (X_1^2 - 1) \cdot \frac{\mathrm{d}X_1}{\mathrm{d}t} + X_1 = 0$. The parameter μ is non-negative real number $\mu \geq 0$; $\mu \in \mathbb{R}$. The equation is related to nonlinear electrical circuits and related to simple harmonic oscillator which includes a nonlinear damping term $\mu \cdot (X_1^2 - 1) \cdot \frac{\mathrm{d}X_1}{\mathrm{d}t}$. Practically, this term is like ordinary positive damping for $|X_1| > 1$, but like negative damping for $|X_1| < 1$. The behavior, $\mu \cdot (X_1^2 - 1) \cdot \frac{\mathrm{d}X_1}{\mathrm{d}t}$ term, causes large amplitude oscillations to decay, but it pumps them back up if they become too small. Our system settles into a self-sustained oscillation and the special phenomena happened. The energy is dissipated over one cycle which balances the energy pumped in. The van der Pol equation has a unique, stable limit cycle for $\mu > 0$; $\mu \in \mathbb{R}$. We have a van der Pol oscillator circuit. The active element of the circuit is semiconductor device (OptoNDR circuit/device). It acts like an ordinary resistor when current $I(t)$ is high ($I(t) > I_{\mathrm{sat}}$), but like negative resistor (energy source) or negative differential resistance (NDR) when $I(t)$ is low $I(t) > I_{\mathrm{break}}$ and $I(t) < I_{\mathrm{sat}}$. Our circuit current–voltage characteristic $V = f(I) \forall \frac{\mathrm{d}I}{\mathrm{d}t} = 0$ resembles a cubic function. We consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below I_{break} ($I(t) < I_{\mathrm{breal}}$) [1, 2, 4] (Fig. 8.2).

Our OptoNDR element/circuit is constructed from LED and phototransistor in series. The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant ($I_{BQ_1} = I_{\mathrm{LED}} \cdot k = I_{CQ_1} \cdot k$) and is the phototransistor base current [85]. The mathematical analysis is based on the basic transistor Ebers–Moll equations. The basic Ebers–Moll schematic for NPN bipolar transistor is already discussed (see Sect. 6.2). We need to implement the regular Ebers–Moll model to the optocoupler circuit (transistor Q_1 and LED D_1) and get a complete final expression for the negative differential resistance (NDR) characteristics of that circuit (see Sect. 6.2).

$$f(I) = V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

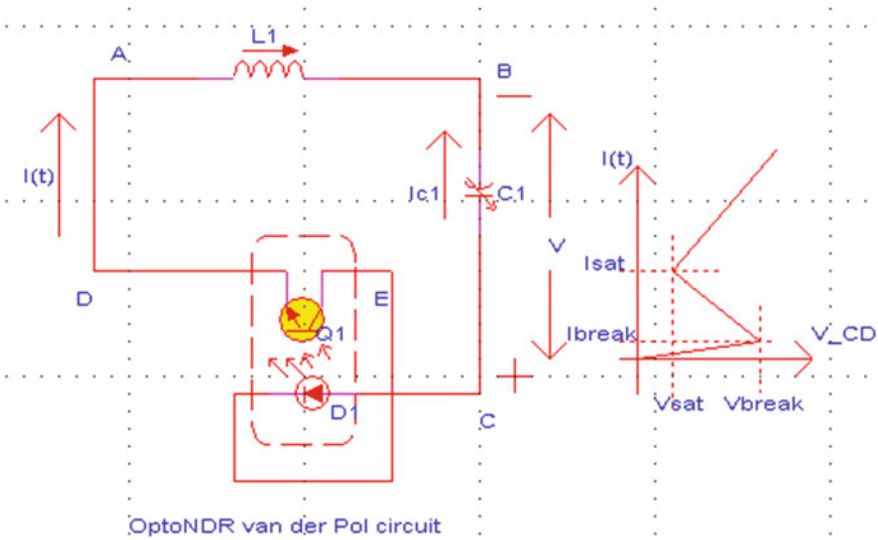

Fig. 8.2 OptoNDR circuit van der Pol perturbation

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t \cdot \frac{1}{I + (1+k) \cdot I_0} + V_t \cdot \left\{ \frac{(1-\alpha_r \cdot \alpha_f) \cdot \left\{ \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] \cdot I_{sc} - \left[\alpha_f - \frac{1}{(1+k)}\right] \cdot I_{se} \right\}}{\left\{ I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) \right\} \cdot \left\{ I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) \right\}} \right\}$$

We get the conditions in NDR region: $I \neq -(1+k) \cdot I_0$

$$I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) \neq 0 \Rightarrow I \neq -\frac{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right]}$$

$$I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) \neq 0 \Rightarrow I \neq -\frac{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{\left[\alpha_f - \frac{1}{(1+k)}\right]}.$$

We can demonstrate the $\frac{\mathrm{d}f(I)}{\mathrm{d}I}$ equation as a parametric function with some constant. Let us define the constants first (see Sect. 6.2).

$$\Gamma_1 = (1+k)\cdot I_0;\ \Gamma_2 = (1-\alpha_r\cdot\alpha_f)\cdot\left\{\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]\cdot I_{sc} - \left[\alpha_f - \frac{1}{(1+k)}\right]\cdot I_{se}\right\}$$

$$\Gamma_3 = \left[1-\alpha_r\cdot\frac{1}{(1+k)}\right];\ \Gamma_4 = I_{se}\cdot(1-\alpha_r\cdot\alpha_f);\ \Gamma_5 = \left[\alpha_f - \frac{1}{(1+k)}\right];\ \Gamma_6 = I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)$$

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t\cdot\frac{1}{I+\Gamma_1} + V_t\cdot\left\{\frac{\Gamma_2}{\{I\cdot\Gamma_3+\Gamma_4\}\cdot\{I\cdot\Gamma_5+\Gamma_6\}}\right\};\ \Gamma_1,\Gamma_2,\ldots,\Gamma_6\in\mathbb{R}.$$

We need to analyze the above equation for regions which are near the saturation region and cut-off region. For the region which is after the breakover voltage but near enough to the cut-off region:

$$Q_{1(\text{cutoff})}(\Delta f(I)\to\varepsilon\ll 1,\ \Delta I\to 0)\Rightarrow \text{NDR} = \frac{\Delta f(I)}{\Delta I}\to\infty.$$

For the region which is near and in the phototransistor saturation state:

$$Q_{1(\text{saturation})}(\Delta f(I)\to\varepsilon\ll 1,\ \Delta I\to \infty)\Rightarrow \text{NDR} = \frac{\Delta f(I)}{\Delta I}\to 0.$$

For the cut-off region before the breakover $k = 0$ then we get the expression $\frac{\mathrm{d}f(I)}{\mathrm{d}I}$.

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I}\Big|_{k=0} = V_t\cdot\frac{1}{I+I_0} + V_t\cdot\left\{\frac{(1-\alpha_r\cdot\alpha_f)\cdot\{[1-\alpha_r]\cdot I_{sc} - [\alpha_f-1]\cdot I_{se}\}}{\{I\cdot[1-\alpha_r]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)\}\cdot\{I\cdot[\alpha_f-1]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)\}}\right\}.$$

Back to our circuit van der Pol differential equations:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{I}{C_1};\ \ f(I) + L_1\cdot\frac{\mathrm{d}I}{\mathrm{d}t} - V = 0$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{I}{C_1};\ \ V_t\cdot\ln\left[\frac{I\cdot\left[1-\alpha_r\cdot\frac{1}{(1+k)}\right]+I_{se}\cdot(1-\alpha_r\cdot\alpha_f)}{I\cdot\left[\alpha_f-\frac{1}{(1+k)}\right]+I_{sc}\cdot(1-\alpha_r\cdot\alpha_f)}\right] + V_t\cdot\ln\left[\frac{I}{(1+k)\cdot I_0}+1\right] + V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right] + L_1\cdot\frac{\mathrm{d}I}{\mathrm{d}t} - V = 0.$$

We can investigate our OptoNDR van der Pol as a weakly nonlinear oscillator and analyze it using perturbation method.

$$f(I) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - V = 0 \Rightarrow V = f(I) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t}; \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}f(I)}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2}; \frac{\mathrm{d}f(I)}{\mathrm{d}t} = \frac{\mathrm{d}f(I)}{\mathrm{d}I} \cdot \frac{\mathrm{d}I}{\mathrm{d}t}$$

$$-\frac{I}{C_1} = \frac{\mathrm{d}f(I)}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \Rightarrow L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{C_1} = -\frac{\mathrm{d}f(I)}{\mathrm{d}t}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{I}{C_1}; \frac{\mathrm{d}f(I)}{\mathrm{d}t} = V_t \cdot \left(\frac{1}{I + \Gamma_1} + \frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}} \right) \cdot \frac{\mathrm{d}I}{\mathrm{d}t}; \quad \Gamma_1, \Gamma_2, \ldots, \Gamma_6 \in \mathbb{R}$$

$$L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{C_1} = -V_t \cdot \left(\frac{1}{I + \Gamma_1} + \frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}} \right) \cdot \frac{\mathrm{d}I}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{L_1 \cdot C_1} = -\frac{V_t}{L_1} \cdot \left(\frac{1}{I + \Gamma_1} + \frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}} \right) \cdot \frac{\mathrm{d}I}{\mathrm{d}t}.$$

We define $g\left(I, \frac{\mathrm{d}I}{\mathrm{d}t}, t\right) = -\left(\frac{1}{I + \Gamma_1} + \frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}} \right) \cdot \frac{\mathrm{d}I}{\mathrm{d}t}; \varepsilon = \frac{V_t}{L_1}$.

The OptoNDR van der Pol weakly nonlinear oscillator can be presented by the differential equation: $\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{L_1 \cdot C_1} = \frac{V_t}{L_1} \cdot g\left(I, \frac{\mathrm{d}I}{\mathrm{d}t}, t\right); \omega_0^2 = \frac{1}{L_1 \cdot C_1}$

$$\varepsilon = \frac{V_t}{L_1} \ll 1; \varepsilon = \frac{V_t}{L_1} > 0; \omega_0^2 = \frac{1}{L_1 \cdot C_1} \Rightarrow \omega_0 = \frac{1}{\sqrt{L_1 \cdot C_1}}; \omega \approx k \cdot \omega_0 = k \cdot \frac{1}{\sqrt{L_1 \cdot C_1}}.$$

We consider that function $g\left(I, \frac{\mathrm{d}I}{\mathrm{d}t}, t\right)$ is T periodic in time and is called on arbitrary smoothing in time. The differential equation represents small perturbations of the linear oscillator $\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{L_1 \cdot C_1} = 0$ and are therefore called weakly nonlinear. We can concern van der Pol equation as a weakly nonlinear oscillator and the same for duffing equation. We can seek solution for differential equation $\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{L_1 \cdot C_1} = \frac{V_t}{L_1} \cdot g\left(I, \frac{\mathrm{d}I}{\mathrm{d}t}, t\right); \omega_0^2 = \frac{1}{L_1 \cdot C_1}$ in the perturbation expansion form of $I(t, \varepsilon) = I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots$, where it is a perturbation series. The equation becomes soluble when we set $\varepsilon = 0$. We get linear oscillator for $\varepsilon = 0; \varepsilon = \frac{V_t}{L_1} \to 0 \Rightarrow L_1 \to \infty$. In our analysis we consider the following weakly nonlinear oscillator perturbed system: $\omega_0^2 = \frac{1}{L_1 \cdot C_1} \to 1; L_1 \cdot C_1 \to 1$

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + I \cdot \frac{1}{L_1 \cdot C_1} + \frac{V_t}{L_1} \cdot \left(\frac{1}{I + \Gamma_1} + \frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}} \right) \cdot \frac{\mathrm{d}I}{\mathrm{d}t} = 0; \varepsilon = \frac{V_t}{L_1}$$

$$\psi(I) = \frac{1}{I+\Gamma_1} + \frac{\Gamma_2}{\{I\cdot\Gamma_3+\Gamma_4\}\cdot\{I\cdot\Gamma_5+\Gamma_6\}}; \frac{d^2I}{dt^2} + I\cdot\frac{1}{L_1\cdot C_1} + \frac{V_t}{L_1}\cdot\psi(I)\cdot\frac{dI}{dt} = 0$$

$$Y = \frac{dI}{dt}; \frac{dY}{dt} + I\cdot\frac{1}{L_1\cdot C_1} + \frac{V_t}{L_1}\cdot\psi(I)\cdot Y = 0. \text{ At fixed points, } \frac{dI}{dt} = 0; \frac{dY}{dt} = 0$$

$$Y^* = 0; I\cdot\frac{1}{L_1\cdot C_1} + \frac{V_t}{L_1}\cdot\psi(I)\cdot Y^* = 0 \Rightarrow I^* = 0; \frac{1}{L_1\cdot C_1} \neq 0$$

$$\frac{dI}{dt} = Y; \frac{dY}{dt} = -I\cdot\frac{1}{L_1\cdot C_1} - \frac{V_t}{L_1}\cdot\psi(I)\cdot Y; \frac{dI}{dt} = \xi_1(I,Y); \frac{dY}{dt} = \xi_2(I,Y)$$

$$\xi_1(I,Y) = Y; \xi_2(I,Y) = -I\cdot\frac{1}{L_1\cdot C_1} - \frac{V_t}{L_1}\cdot\psi(I)\cdot Y; \frac{\partial\xi_1(I,Y)}{\partial I} = 0; \frac{\partial\xi_1(I,Y)}{\partial Y} = 1$$

$$\frac{\partial\xi_2(I,Y)}{\partial I} = -\frac{1}{L_1\cdot C_1} - \frac{V_t}{L_1}\cdot\frac{\partial\psi(I)}{\partial I}\cdot Y; \frac{\partial\xi_2(I,Y)}{\partial Y} = -\frac{V_t}{L_1}\cdot\psi(I)$$

The matrix A (Jacobian matrix) at the fixed point (I^*, Y^*) = (0, 0):

$$A = \begin{pmatrix} \frac{\partial\xi_1(I,Y)}{\partial I} & \frac{\partial\xi_1(I,Y)}{\partial Y} \\ \frac{\partial\xi_2(I,Y)}{\partial I} & \frac{\partial\xi_2(I,Y)}{\partial Y} \end{pmatrix}_{(I^*,Y^*)} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L_1\cdot C_1} - \frac{V_t}{L_1}\cdot\frac{\partial\psi(I)}{\partial I}\cdot Y & -\frac{V_t}{L_1}\cdot\psi(I) \end{pmatrix}_{(I^*=0,Y^*=0)}$$

$$\psi(I) = \frac{1}{I+\Gamma_1} + \frac{\Gamma_2}{\{I\cdot\Gamma_3+\Gamma_4\}\cdot\{I\cdot\Gamma_5+\Gamma_6\}} \Rightarrow \psi(I^*=0) = \frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4\cdot\Gamma_6}$$

$$\frac{\partial\psi(I)}{\partial I} = -\frac{1}{(I+\Gamma_1)^2} - \frac{\Gamma_2\cdot\{\Gamma_3\cdot(I\cdot\Gamma_5+\Gamma_6)+\Gamma_5\cdot(I\cdot\Gamma_3+\Gamma_4)\}}{(I\cdot\Gamma_3+\Gamma_4)^2\cdot(I\cdot\Gamma_5+\Gamma_6)^2}$$

$$\frac{\partial\psi(I)}{\partial I}\Big|_{I^*=0} = -\frac{1}{\Gamma_1^2} - \frac{\Gamma_2\cdot(\Gamma_3\cdot\Gamma_6+\Gamma_5\cdot\Gamma_4)}{\Gamma_4^2\cdot\Gamma_6^2} = -\left[\frac{1}{\Gamma_1^2} + \frac{\Gamma_2\cdot(\Gamma_3\cdot\Gamma_6+\Gamma_5\cdot\Gamma_4)}{\Gamma_4^2\cdot\Gamma_6^2}\right]$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L_1}\cdot\left[\frac{1}{C_1} + V_t\cdot\frac{\partial\psi(I)}{\partial I}\cdot Y\right] & -\frac{V_t}{L_1}\cdot\psi(I) \end{pmatrix}_{(I^*=0,Y^*=0)} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L_1\cdot C_1} & -\frac{V_t}{L_1}\cdot\psi(I^*=0) \end{pmatrix}$$

$$A|_{I^*=0,Y^*=0} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L_1\cdot C_1} & -\frac{V_t}{L_1}\cdot\left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4\cdot\Gamma_6}\right) \end{pmatrix}; \det(A - \lambda\cdot I) = 0$$

$$A - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 \\ -\frac{1}{L_1 \cdot C_1} & -\frac{V_t}{L_1} \cdot \left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4 \cdot \Gamma_6}\right) - \lambda \end{pmatrix}; \det(A - \lambda \cdot I)$$
$$= \lambda \cdot \left[\frac{V_t}{L_1} \cdot \left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4 \cdot \Gamma_6}\right) + \lambda\right] + \frac{1}{L_1 \cdot C_1}$$

$$\det(A - \lambda \cdot I) = 0 \Rightarrow \lambda \cdot \left[\frac{V_t}{L_1} \cdot \left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4 \cdot \Gamma_6}\right) + \lambda\right] + \frac{1}{L_1 \cdot C_1} = 0$$

$$\lambda^2 + \lambda \cdot \frac{V_t}{L_1} \cdot \left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4 \cdot \Gamma_6}\right) + \frac{1}{L_1 \cdot C_1} = 0;$$
$$\lambda_{1,2} = \frac{-\frac{V_t}{L_1} \cdot \left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4 \cdot \Gamma_6}\right) \pm \sqrt{\frac{V_t^2}{L_1^2} \cdot \left(\frac{1}{\Gamma_1} + \frac{\Gamma_2}{\Gamma_4 \cdot \Gamma_6}\right)^2 - 4 \cdot \frac{1}{L_1 \cdot C_1}}}{2}.$$

Classification of fixed point: We get two eigenvalues. Cases, $\lambda_1 = \lambda_2 < 0$ (attracting focus), $\lambda_1 < \lambda_2 < 0$ (attracting node), $\lambda_1 < \lambda_2 = 0$ (attracting line), $\lambda_1 < 0 < \lambda_2$; $\lambda_1 \cdot \lambda_2 < 0$ (saddle node), $\lambda_1 = 0 < \lambda_2$ (repelling line), $0 < \lambda_1 < \lambda_2$ (repelling node), $0 < \lambda_1 = \lambda_2$ (repelling focus).

Our system differential equation (second order): $\frac{d^2 I}{dt^2} + I \cdot \frac{1}{L_1 \cdot C_1} + \frac{V_t}{L_1} \cdot \psi(I) \cdot \frac{dI}{dt} = 0$

$$\omega_0^2 = \frac{1}{L_1 \cdot C_1} \to 1; \; L_1 \cdot C_1 \to 1; \; 0 < \varepsilon = \frac{V_t}{L_1} \ll 1; \; \frac{d^2 I}{dt^2} + I \cdot \omega_0^2 + \varepsilon \cdot \psi(I) \cdot \frac{dI}{dt} = 0.$$

We shall apply perturbation theory to solve the OptoNDR van der Pol oscillator in equation: $\frac{d^2 I}{dt^2} + I \cdot \omega_0^2 + \varepsilon \cdot \psi(I) \cdot \frac{dI}{dt} = 0$.

$$I(t, \varepsilon) = I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots; \; \frac{dI(t, \varepsilon)}{dt}$$
$$= \frac{dI_0(t)}{dt} + \varepsilon \cdot \frac{dI_1(t)}{dt} + \varepsilon^2 \cdot \frac{dI_2(t)}{dt} + \cdots$$

$$\frac{d^2 I(t, \varepsilon)}{dt^2} = \frac{d^2 I_0(t)}{dt^2} + \varepsilon \cdot \frac{d^2 I_1(t)}{dt^2} + \varepsilon^2 \cdot \frac{d^2 I_2(t)}{dt^2} + \cdots; \; I(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t); \; \frac{dI(t, \varepsilon)}{dt}$$
$$= \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{dI_k(t)}{dt}$$

$$\frac{d^2 I(t, \varepsilon)}{dt^2} = \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{d^2 I_k(t)}{dt^2}; \; \frac{d^2 I}{dt^2} + I \cdot \omega_0^2 + \varepsilon \cdot \psi(I) \cdot \frac{dI}{dt} = 0$$

$$\sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{d^2 I_k(t)}{dt^2} + (I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots) \cdot \omega_0^2 + \varepsilon \cdot \psi\left(I = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t)\right) \cdot \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{dI_k(t)}{dt} = 0.$$

We differentiate the above differential equation term by term with respect to t and we group terms according to the powers of ε, omitting all terms with coefficients of ε^2 and higher.

$$\frac{d^2 I_0(t)}{dt^2} + \varepsilon \cdot \frac{d^2 I_1(t)}{dt^2} + \varepsilon^2 \cdot \frac{d^2 I_2(t)}{dt^2} + \cdots + (I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots) \cdot \omega_0^2 + \varepsilon \cdot \psi\left(I = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t)\right) \cdot \left(\frac{dI_0(t)}{dt} + \varepsilon \cdot \frac{dI_1(t)}{dt} + \varepsilon^2 \cdot \frac{dI_2(t)}{dt} + \cdots\right) = 0$$

$$\psi\left(I = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t)\right) = \psi(I) = \frac{1}{\sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t) + \Gamma_1} + \frac{\Gamma_2}{\left\{\Gamma_3 \cdot \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t) + \Gamma_4\right\} \cdot \left\{\Gamma_5 \cdot \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t) + \Gamma_6\right\}}$$

$$\psi\left(I = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t)\right) = \psi(I) = \frac{1}{I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots + \Gamma_1} + \frac{\Gamma_2}{\{\Gamma_3 \cdot (I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots) + \Gamma_4\} \cdot \{\Gamma_5 \cdot (I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots) + \Gamma_6\}}.$$

We omit from the above expression the $O(\varepsilon^2)$ elements and get the expression:

$$\psi\left(I = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t)\right) = \psi(I) = \frac{1}{I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1} + \frac{\Gamma_2}{\{\Gamma_3 \cdot (I_0(t) + \varepsilon \cdot I_1(t)) + \Gamma_4\} \cdot \{\Gamma_5 \cdot (I_0(t) + \varepsilon \cdot I_1(t)) + \Gamma_6\}}$$

$$\psi\left(I = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(t)\right) = \psi(I) = \frac{1}{I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1} + \frac{\Gamma_2}{(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}$$

$$\psi(I) = \frac{(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) + \Gamma_2 \cdot (I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1)}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}$$

$$\psi(I) = \frac{(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) + \Gamma_2 \cdot I_0(t) + \varepsilon \cdot \Gamma_2 \cdot I_1(t) + \Gamma_2 \cdot \Gamma_1}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}$$

$$\begin{aligned}
&(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) = \Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + \Gamma_3 \cdot \Gamma_5 \cdot I_0(t) \cdot I_1(t) \cdot \varepsilon \\
&\quad + \Gamma_3 \cdot \Gamma_6 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot \Gamma_5 \cdot I_1(t) \cdot I_0(t) + \varepsilon^2 \cdot \Gamma_3 \cdot \Gamma_5 \cdot I_1(t) \cdot I_1(t) + \varepsilon \cdot \Gamma_3 \cdot \Gamma_6 \cdot I_1(t) + \Gamma_4 \cdot \Gamma_5 \cdot I_0(t) \\
&\quad + \Gamma_4 \cdot \Gamma_5 \cdot \varepsilon \cdot I_1(t) + \Gamma_4 \cdot \Gamma_6
\end{aligned}$$

$$\begin{aligned}
&(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) \\
&\quad = \Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + \Gamma_3 \cdot \Gamma_5 \cdot I_0(t) \cdot I_1(t) \cdot \varepsilon + \Gamma_3 \cdot \Gamma_6 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot \Gamma_5 \cdot I_1(t) \cdot I_0(t) \\
&\qquad + \varepsilon \cdot \Gamma_3 \cdot \Gamma_6 \cdot I_1(t) + \Gamma_4 \cdot \Gamma_5 \cdot I_0(t) + \Gamma_4 \cdot \Gamma_5 \cdot \varepsilon \cdot I_1(t) + \Gamma_4 \cdot \Gamma_6 + O(\varepsilon^2)
\end{aligned}$$

$$\begin{aligned}
&(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) \\
&\quad = \Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + \Gamma_3 \cdot \Gamma_6 \cdot I_0(t) + \Gamma_4 \cdot \Gamma_5 \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \varepsilon \cdot \{\Gamma_3 \cdot \Gamma_5 \cdot I_0(t) \cdot I_1(t) \\
&\quad + \Gamma_3 \cdot \Gamma_5 \cdot I_1(t) \cdot I_0(t) + \Gamma_3 \cdot \Gamma_6 \cdot I_1(t) + \Gamma_4 \cdot \Gamma_5 \cdot I_1(t)\} + O(\varepsilon^2)
\end{aligned}$$

$$\begin{aligned}
&(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) + \Gamma_2 \cdot I_0(t) + \varepsilon \cdot \Gamma_2 \cdot I_1(t) + \Gamma_2 \cdot \Gamma_1 \\
&\quad = \Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + \Gamma_3 \cdot \Gamma_6 \cdot I_0(t) + \Gamma_4 \cdot \Gamma_5 \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \varepsilon \cdot \{\Gamma_3 \cdot \Gamma_5 \cdot I_0(t) \cdot I_1(t) + \Gamma_3 \cdot \Gamma_5 \cdot I_1(t) \cdot I_0(t) . \\
&\qquad + \Gamma_3 \cdot \Gamma_6 \cdot I_1(t) + \Gamma_4 \cdot \Gamma_5 \cdot I_1(t)\} + \Gamma_2 \cdot I_0(t) + \varepsilon \cdot \Gamma_2 \cdot I_1(t) + \Gamma_2 \cdot \Gamma_1 + O(\varepsilon^2)
\end{aligned}$$

If we neglect the $O(\varepsilon^2)$ term in the above equation, we get the following expression:

$$\begin{aligned}
&(\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) + \Gamma_2 \cdot I_0(t) + \varepsilon \cdot \Gamma_2 \cdot I_1(t) + \Gamma_2 \cdot \Gamma_1 \\
&\quad = \Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1 + \varepsilon \cdot \{[\Gamma_3 \cdot \Gamma_5 + \Gamma_3 \cdot \Gamma_5] \cdot I_1(t) \cdot I_0(t) \\
&\qquad + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_1(t)\}
\end{aligned}$$

$$\psi(I) = \frac{\begin{array}{c}\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1 \\ + \varepsilon \cdot \{[\Gamma_3 \cdot \Gamma_5 + \Gamma_3 \cdot \Gamma_5] \cdot I_1(t) \cdot I_0(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_1(t)\}\end{array}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}$$

$$\varepsilon \cdot \psi(I) = \frac{\begin{array}{c}\varepsilon \cdot \{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\} \\ + \varepsilon^2 \cdot \{[\Gamma_3 \cdot \Gamma_5 + \Gamma_3 \cdot \Gamma_5] \cdot I_1(t) \cdot I_0(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_1(t)\}\end{array}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}$$

$$O(\varepsilon^2) = \varepsilon^2 \cdot \{[\Gamma_3 \cdot \Gamma_5 + \Gamma_3 \cdot \Gamma_5] \cdot I_1(t) \cdot I_0(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_1(t)\}$$

$$\varepsilon \cdot \psi(I) = \frac{\varepsilon \cdot \left\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\right\} + O(\varepsilon^2)}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}.$$

If we omit the $O(\varepsilon^2)$ term from the above expression, we get

$$\varepsilon \cdot \psi(I) = \frac{\varepsilon \cdot \left\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\right\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)}$$

$$\begin{aligned} &\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + \varepsilon \cdot \frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + \varepsilon^2 \cdot \frac{\mathrm{d}^2 I_2(t)}{\mathrm{d}t^2} + \cdots + (I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots) \cdot \omega_0^2 \\ &\quad + \varepsilon \cdot \frac{\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)} \\ &\quad \cdot \left(\frac{\mathrm{d}I_0(t)}{\mathrm{d}t} + \varepsilon \cdot \frac{\mathrm{d}I_1(t)}{\mathrm{d}t} + \varepsilon^2 \cdot \frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + \cdots\right) = 0 \end{aligned}$$

$$\begin{aligned} &\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + \varepsilon \cdot \frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + O(\varepsilon^2) + \left(I_0(t) + \varepsilon \cdot I_1(t) + O(\varepsilon^2)\right) \cdot \omega_0^2 \\ &\quad + \varepsilon \cdot \frac{\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)} \cdot \\ &\quad \cdot \left(\frac{\mathrm{d}I_0(t)}{\mathrm{d}t} + \varepsilon \cdot \frac{\mathrm{d}I_1(t)}{\mathrm{d}t} + O(\varepsilon^2)\right) = 0 \end{aligned}$$

We omit from the above expression the $O(\varepsilon^2)$ and get

$$\begin{aligned} &\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + \varepsilon \cdot \frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + (I_0(t) + \varepsilon \cdot I_1(t)) \cdot \omega_0^2 \\ &\quad + \varepsilon \cdot \frac{\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)} \\ &\quad \cdot \left(\frac{\mathrm{d}I_0(t)}{\mathrm{d}t} + \varepsilon \cdot \frac{\mathrm{d}I_1(t)}{\mathrm{d}t}\right) = 0 \end{aligned}$$

$$\begin{aligned} &\left[\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + I_0(t) \cdot \omega_0^2\right] + \varepsilon \cdot \left[\frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + I_1(t) \cdot \omega_0^2\right] \\ &+ \left(\varepsilon \cdot \frac{\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)} \cdot \frac{\mathrm{d}I_0(t)}{\mathrm{d}t}\right. \\ &\left.+ \varepsilon^2 \cdot \frac{\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)} \cdot \frac{\mathrm{d}I_1(t)}{\mathrm{d}t}\right) = 0 \end{aligned}$$

$$O(\varepsilon^2) = \varepsilon^2 \cdot \frac{\{\Gamma_3 \cdot \Gamma_5 \cdot I_0^2(t) + [\Gamma_3 \cdot \Gamma_6 + \Gamma_4 \cdot \Gamma_5 + \Gamma_2] \cdot I_0(t) + \Gamma_4 \cdot \Gamma_6 + \Gamma_2 \cdot \Gamma_1\}}{(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6)} \cdot \frac{\mathrm{d}I_1(t)}{\mathrm{d}t}.$$

We omit from the above expression the $O(\varepsilon^2)$ and get

$$\left[\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + I_0(t)\cdot\omega_0^2\right] + \varepsilon\cdot\left[\frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + I_1(t)\cdot\omega_0^2\right]$$
$$+\varepsilon\cdot\frac{\{\Gamma_3\cdot\Gamma_5\cdot I_0^2(t) + [\Gamma_3\cdot\Gamma_6 + \Gamma_4\cdot\Gamma_5 + \Gamma_2]\cdot I_0(t) + \Gamma_4\cdot\Gamma_6 + \Gamma_2\cdot\Gamma_1\}}{(I_0(t)+\varepsilon\cdot I_1(t)+\Gamma_1)\cdot(\Gamma_3\cdot I_0(t)+\varepsilon\cdot\Gamma_3\cdot I_1(t)+\Gamma_4)\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)}$$
$$\cdot\frac{\mathrm{d}I_0(t)}{\mathrm{d}t} = 0$$

$$h_4(I_0) = \Gamma_3\cdot\Gamma_5\cdot I_0^2(t) + [\Gamma_3\cdot\Gamma_6 + \Gamma_4\cdot\Gamma_5 + \Gamma_2]\cdot I_0(t) + \Gamma_4\cdot\Gamma_6 + \Gamma_2\cdot\Gamma_1;\ I_0 = I_0(t)$$

$$\left[\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + I_0(t)\cdot\omega_0^2\right] + \varepsilon\cdot\left[\frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + I_1(t)\cdot\omega_0^2\right]$$
$$+\varepsilon\cdot\frac{h_4(I_0)}{(I_0(t)+\varepsilon\cdot I_1(t)+\Gamma_1)\cdot(\Gamma_3\cdot I_0(t)+\varepsilon\cdot\Gamma_3\cdot I_1(t)+\Gamma_4)\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)}\cdot$$
$$\cdot\frac{\mathrm{d}I_0(t)}{\mathrm{d}t} = 0$$

Next step is to develop the denominator of the above expression.

$$(I_0(t)+\varepsilon\cdot I_1(t)+\Gamma_1)\cdot(\Gamma_3\cdot I_0(t)+\varepsilon\cdot\Gamma_3\cdot I_1(t)+\Gamma_4)\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)$$
$$= \{\Gamma_3\cdot I_0^2(t) + \varepsilon\cdot I_0(t)\cdot\Gamma_3\cdot I_1(t) + I_0(t)\cdot\Gamma_4 + \varepsilon\cdot\Gamma_3\cdot I_0(t)\cdot I_1(t) + \varepsilon^2\cdot\Gamma_3\cdot I_1^2(t) + \varepsilon\cdot I_1(t)\cdot\Gamma_4$$
$$+\Gamma_1\cdot\Gamma_3\cdot I_0(t) + \varepsilon\cdot\Gamma_1\cdot\Gamma_3\cdot I_1(t) + \Gamma_1\cdot\Gamma_4\}\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)$$

$$(I_0(t)+\varepsilon\cdot I_1(t)+\Gamma_1)\cdot(\Gamma_3\cdot I_0(t)+\varepsilon\cdot\Gamma_3\cdot I_1(t)+\Gamma_4)\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)$$
$$= \{\Gamma_3\cdot I_0^2(t) + \Gamma_1\cdot\Gamma_3\cdot I_0(t) + I_0(t)\cdot\Gamma_4 + \Gamma_1\cdot\Gamma_4 + \varepsilon\cdot[2\cdot\Gamma_3\cdot I_0(t)\cdot I_1(t) + I_1(t)\cdot(\Gamma_4 .$$
$$+\Gamma_1\cdot\Gamma_3)] + O(\varepsilon^2)\}\cdot(\Gamma_5\cdot I_0(t)+\Gamma_6+\varepsilon\cdot\Gamma_5\cdot I_1(t))$$

We omit from the above expression the $O(\varepsilon^2)$ and get

$$(I_0(t)+\varepsilon\cdot I_1(t)+\Gamma_1)\cdot(\Gamma_3\cdot I_0(t)+\varepsilon\cdot\Gamma_3\cdot I_1(t)+\Gamma_4)\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)$$
$$= \{\Gamma_3\cdot I_0^2(t) + (\Gamma_1\cdot\Gamma_3+\Gamma_4)\cdot I_0(t) + \Gamma_1\cdot\Gamma_4 + \varepsilon\cdot[2\cdot\Gamma_3\cdot I_0(t)\cdot I_1(t) + I_1(t)\cdot(\Gamma_4$$
$$+\Gamma_1\cdot\Gamma_3)]\}\cdot(\Gamma_5\cdot I_0(t)+\Gamma_6+\varepsilon\cdot\Gamma_5\cdot I_1(t))$$

$$(I_0(t)+\varepsilon\cdot I_1(t)+\Gamma_1)\cdot(\Gamma_3\cdot I_0(t)+\varepsilon\cdot\Gamma_3\cdot I_1(t)+\Gamma_4)\cdot(\Gamma_5\cdot I_0(t)+\varepsilon\cdot\Gamma_5\cdot I_1(t)+\Gamma_6)$$
$$= \Gamma_3\cdot\Gamma_5\cdot I_0^3(t) + \Gamma_3\cdot\Gamma_6\cdot I_0^2(t) + \varepsilon\cdot\Gamma_5\cdot\Gamma_3\cdot I_0^2(t)\cdot I_1(t)$$
$$+(\Gamma_1\cdot\Gamma_3+\Gamma_4)\cdot\Gamma_5\cdot I_0^2(t) + (\Gamma_1\cdot\Gamma_3+\Gamma_4)\cdot\Gamma_6\cdot I_0(t) + \varepsilon\cdot\Gamma_5\cdot(\Gamma_1\cdot\Gamma_3+\Gamma_4)\cdot I_0(t)\cdot I_1(t)$$
$$+\Gamma_1\cdot\Gamma_4\cdot\Gamma_5\cdot I_0(t) + \Gamma_1\cdot\Gamma_4\cdot\Gamma_6 + \varepsilon\cdot\Gamma_1\cdot\Gamma_4\cdot\Gamma_5\cdot I_1(t)$$
$$+\varepsilon\cdot[2\cdot\Gamma_3\cdot I_0(t)\cdot I_1(t) + I_1(t)\cdot(\Gamma_4\cdot+\Gamma_1\cdot\Gamma_3)]\cdot[\Gamma_5\cdot I_0(t)+\Gamma_6] + O(\varepsilon^2)$$

$$
\begin{aligned}
&(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) \\
&\quad = \Gamma_3 \cdot \Gamma_5 \cdot I_0^3(t) + \Gamma_3 \cdot \Gamma_6 \cdot I_0^2(t) + (\Gamma_1 \cdot \Gamma_3 + \Gamma_4) \cdot \Gamma_5 \cdot I_0^2(t) + (\Gamma_1 \cdot \Gamma_3 + \Gamma_4) \cdot \Gamma_6 \cdot I_0(t) \\
&\qquad + \Gamma_1 \cdot \Gamma_4 \cdot \Gamma_5 \cdot I_0(t) + \Gamma_1 \cdot \Gamma_4 \cdot \Gamma_6 + \varepsilon \cdot \{[2 \cdot \Gamma_3 \cdot I_0(t) \cdot I_1(t) + I_1(t) \cdot (\Gamma_4 \cdot + \Gamma_1 \cdot \Gamma_3)] \,. \\
&\qquad \cdot [\Gamma_5 \cdot I_0(t) + \Gamma_6] + \Gamma_5 \cdot \Gamma_3 \cdot I_0^2(t) \cdot I_1(t) + \Gamma_5 \cdot (\Gamma_1 \cdot \Gamma_3 + \Gamma_4) \cdot I_0(t) \cdot I_1(t) \\
&\qquad + \Gamma_1 \cdot \Gamma_4 \cdot \Gamma_5 \cdot I_1(t)\} + O(\varepsilon^2)
\end{aligned}
$$

We define the following functions:

$$
\begin{aligned}
h_1(I_0) = {}& \Gamma_3 \cdot \Gamma_5 \cdot I_0^3(t) + \Gamma_3 \cdot \Gamma_6 \cdot I_0^2(t) + (\Gamma_1 \cdot \Gamma_3 + \Gamma_4) \cdot \Gamma_5 \cdot I_0^2(t) \\
& + (\Gamma_1 \cdot \Gamma_3 + \Gamma_4) \cdot \Gamma_6 \cdot I_0(t) + \Gamma_1 \cdot \Gamma_4 \cdot \Gamma_5 \cdot I_0(t) + \Gamma_1 \cdot \Gamma_4 \cdot \Gamma_6
\end{aligned}
$$

$$
\begin{aligned}
h_2(I_0, I_1) = {}& [2 \cdot \Gamma_3 \cdot I_0(t) \cdot I_1(t) + I_1(t) \cdot (\Gamma_4 \cdot + \Gamma_1 \cdot \Gamma_3)] \cdot [\Gamma_5 \cdot I_0(t) + \Gamma_6] \\
& + \Gamma_5 \cdot \Gamma_3 \cdot I_0^2(t) \cdot I_1(t) + \Gamma_5 \cdot (\Gamma_1 \cdot \Gamma_3 + \Gamma_4) \cdot I_0(t) \cdot I_1(t)
\end{aligned}
$$

$$
h_3(I_1) = \Gamma_1 \cdot \Gamma_4 \cdot \Gamma_5 \cdot I_1(t);\; h_2(I_0, I_1) + h_3(I_1) = \sum_{k=2}^{3} h_k(I_0, I_1 \,\exists\, k = 2)
$$

$$
\begin{aligned}
&(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) \\
&= h_1(I_0) + \varepsilon \cdot [h_2(I_0, I_1) + h_3(I_1)] + O(\varepsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
&(I_0(t) + \varepsilon \cdot I_1(t) + \Gamma_1) \cdot (\Gamma_3 \cdot I_0(t) + \varepsilon \cdot \Gamma_3 \cdot I_1(t) + \Gamma_4) \cdot (\Gamma_5 \cdot I_0(t) + \varepsilon \cdot \Gamma_5 \cdot I_1(t) + \Gamma_6) \\
&= h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0, I_1 \,\exists\, k = 2) + O(\varepsilon^2)
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + I_0(t) \cdot \omega_0^2\right] + \varepsilon \cdot \left[\frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + I_1(t) \cdot \omega_0^2\right] + \varepsilon \\
&\quad \cdot \frac{h_4(I_0)}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0, I_1 \,\exists\, k = 2) + O(\varepsilon^2)} \cdot \frac{\mathrm{d}I_0(t)}{\mathrm{d}t} \\
&\quad = 0.
\end{aligned}
$$

We omit from the above equation the $O(\varepsilon^2)$ term and get the equation:

$$
\begin{aligned}
&\left[\frac{\mathrm{d}^2 I_0(t)}{\mathrm{d}t^2} + I_0(t) \cdot \omega_0^2\right] + \varepsilon \cdot \left[\frac{\mathrm{d}^2 I_1(t)}{\mathrm{d}t^2} + I_1(t) \cdot \omega_0^2\right] + \varepsilon \\
&\quad \cdot \frac{h_4(I_0)}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0, I_1 \,\exists\, k = 2)} \cdot \frac{\mathrm{d}I_0(t)}{\mathrm{d}t} \\
&\quad = 0.
\end{aligned}
$$

Assumption: $\omega_0^2 = \frac{1}{L_1 \cdot C_1} \to 1;\; L_1 \cdot C_1 \to 1$

$$\left[\frac{d^2 I_0(t)}{dt^2} + I_0(t)\right] + \varepsilon \cdot \left\{\frac{d^2 I_1(t)}{dt^2} + I_1(t) \quad + \frac{h_4(I_0)}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0, I_1 \ \exists \ k = 2)} \cdot \frac{dI_0(t)}{dt}\right\} = 0$$

Since $\varepsilon \neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero. The zeroth of ε gives us the following expressions:

$$\frac{d^2 I_0(t)}{dt^2} + I_0(t) = 0; \ \frac{d^2 I_1(t)}{dt^2} + I_1(t) + \frac{h_4(I_0)}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0, I_1 \ \exists \ k = 2)} \cdot \frac{dI_0(t)}{dt} = 0$$

Next we need to solve the above differential equations. These equations are indeed homogenous linear differential equations with functional coefficients.

First we investigate the $\frac{d^2 I_0(t)}{dt^2} + I_0(t) = 0$ differential equation. Let us assume that $I_0(t) = e^{r \cdot t}$ is a solution to equation where r is a constant. The auxiliary equations are $\frac{d^2 I_0(t)}{dt^2} = r^2 \cdot e^{r \cdot t} \Rightarrow r^2 \cdot e^{r \cdot t} + e^{r \cdot t} = 0; \ [r^2 + 1] \cdot e^{r \cdot t} = 0 \Rightarrow r = \pm j$.

Here the r real part is equal to zero and the imaginary part is equal to one. If we assumed from above that the general solution for $I_0(t)$ is of the form $\cos(t) + \sin(t)$. And we double check for solution: $I_0(t) = \cos(t) + \sin(t)$.

$$\frac{dI_0(t)}{dt} = -\sin(t) + \cos(t); \ \frac{d^2 I_0(t)}{dt^2} = -\cos(t) - \sin(t);$$
$$\frac{d^2 I_0(t)}{dt^2} = -[\sin(t) + \cos(t)]$$

We write our solution $\frac{d^2 I_0(t)}{dt^2} + I_0(t) = 0$ in the convenient form: $I_0(t) = A_1 \cdot \cos(t) + A_2 \cdot \sin(t)$ where A_1, A_2 are constants to be determined. If we utilized the initial conditions: $I_0(t = 0) = 0; \frac{dI_0(t=0)}{dt} = \Omega$. Given to find the values of the constants A_1, A_2, we have by substituting these values in equation: $I_0(t) = A_1 \cdot \cos(t) + A_2 \cdot \sin(t)$ that $A_1 = 0$ hence $I_0(t) = A_2 \cdot \sin(t)$ and $\frac{dI_0(t)}{dt} = A_2 \cdot \cos(t) \Rightarrow \frac{dI_0(t=0)}{dt} = \Omega = A_2$ then $I_0(t) = \Omega \cdot \sin(t)$. The second differential equation $\frac{d^2 I_1(t)}{dt^2} + I_1(t) + \frac{h_4(I_0)}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0, I_1 \exists k=2)} \cdot \frac{dI_0(t)}{dt} = 0$ and from our last solution $\frac{dI_0(t)}{dt} = \Omega \cdot \cos(t)$ then we get the following equation:

$$\frac{d^2 I_1(t)}{dt^2} + I_1(t) = -\frac{h_4(I_0(t) = \Omega \cdot \sin(t))}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0(t) = \Omega \cdot \sin(t), I_1 \exists k = 2)} \cdot \Omega \cdot \cos(t)$$

We need to check and approve the conditions which the above equation is a case of resonance [6, 7]. It is because of resonant interactions between consecutive orders that nonuniformity has appeared in the regular perturbation series. This is a type of harmonic oscillator with natural frequency one, driven by a periodic, external, forcing frequency is equal to one ($\omega = 1$) on the right-hand side. The amplitude of oscillation for such a system is unbounded as $t \to \infty$ because the oscillator continually absorbs energy from the periodic external force, thus, system in resonance with the external force. The solution, therefore, to such a system, represents this fact in term "$t \cdot \sin(t)$" appeared in the solution in equation $I(t, \varepsilon)$ because the inhomogeneous term $-\frac{h_4(I_0(t)=\Omega\cdot\sin(t))}{h_1(I_0)+\varepsilon\cdot\sum_{k=2}^{3} h_k(I_0(t)=\Omega\cdot\sin(t), I_1 \exists k=2)} \cdot \Omega \cdot \cos(t)$ is itself a solution of the associated homogeneous equation:

$$\frac{d^2 I_1(t)}{dt^2} + I_1(t) = -\frac{h_4(I_0(t) = \Omega \cdot \sin(t))}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0(t) = \Omega \cdot \sin(t), I_1 \exists k = 2)} \cdot \Omega \cdot \cos(t)$$

The secular terms appear whenever the inhomogeneous term is itself a solution of the associated homogeneous differential equation. A secular term always grows faster than the corresponding solution of the homogeneous solution by at least a factor t. The appearance of secular terms demonstrates the nonuniform validity of the perturbation expansion for large t. The $I_1(t)$ solution is quite complicated and is recommended to solve it using MATLAB and other simulation software. It is also possible to solve it numerically rather than analytic.

Summary We need to solve the differential equation for $I_1(t)$ and find the expression $I_1(t) = \ldots$

$$I_0(t) = \Omega \cdot \sin(t); \; \frac{d^2 I_1(t)}{dt^2} + I_1(t)$$
$$= -\frac{h_4(I_0(t) = \Omega \cdot \sin(t))}{h_1(I_0) + \varepsilon \cdot \sum_{k=2}^{3} h_k(I_0(t) = \Omega \cdot \sin(t), I_1 \; \exists \; k = 2)} \cdot \Omega \cdot \cos(t)$$

Now substitute for $I_0(t)$ and $I_1(t)$ in the $I(t, \varepsilon)$ equation

$$I(t, \varepsilon) = I_0(t) + \varepsilon \cdot I_1(t) + \varepsilon^2 \cdot I_2(t) + \cdots; \; I(t, \varepsilon) = \sum_{k=0}^{\infty} I_k(t) \cdot \varepsilon^k$$
$$= I_0(t) + \varepsilon \cdot I_1(t) + O(\varepsilon^2).$$

8.3 OptoNDR Circuit van der Pol Perturbation Method Multiple Timescale

We can use multiple scale analysis for construction uniform or global approximate solutions for both small and large values of independent variables of OptoNDR circuit van der Pol. The dependent variables are uniformly expanded in terms of two or more independent variables, scales. The issue is the choice of ordering scheme and the form of the power series expansion . We can implement in OptoNDR van der Pol system, multiple scale MSPT. The coordinate transforms and invariant manifolds provide a support for multiscale modeling. Practically, there are at least two timescales in weakly nonlinear OptoNDR circuit van der Pol oscillator. Two timing builds two timescales from the start and produces better approximations than the regular perturbation theory. We need to apply two timing to OptoNDR van der Pol differential equation $\left(\omega_0^2 = \frac{1}{L_1 \cdot C_1} \to 1\right)$:

$\frac{d^2 I}{dt^2} + I \cdot \omega_0^2 + \varepsilon \cdot \psi(I) \cdot \frac{dI}{dt} = 0$ (see Sect. 8.2). Let $t_A = t$ denote the fast O(1) time, and let $t_B = \varepsilon \cdot t$ denote the slow time. We take these times as independent variables. The functions of slow time t_B are constants on the fast timescale t_A, and we expand the solution $\frac{d^2 I}{dt^2} + I \cdot \omega_0^2 + \varepsilon \cdot \psi(I) \cdot \frac{dI}{dt} = 0$ as a series expression $I(t, \varepsilon) = I_0(t_A, t_B) + \varepsilon \cdot I_1(t_A, t_B) + O(\varepsilon^2)$. We need to implement time derivatives in systems differential equation and are transferred using the chain rule [5–7].

$$\partial t_A = \frac{\partial}{\partial t_A};\ Dt_A = \frac{d}{dt_A};\ \frac{\partial}{\partial t_A} \Leftrightarrow \frac{d}{dt_A};\ \partial t_B = \frac{\partial}{\partial t_B};\ Dt_B = \frac{d}{dt_B};\ \frac{\partial}{\partial t_B} \Leftrightarrow \frac{d}{dt_B}$$

$\frac{dI}{dt} = \frac{dI}{dt_A} + \frac{dI}{dt_B} \cdot \frac{dt_B}{dt} = \frac{\partial I}{\partial t_A} + \frac{\partial I}{\partial t_B} \cdot \frac{\partial t_B}{\partial t}$. It is time derivative of $I(t, \varepsilon)$ variable using the chain rule.

$$t_A = t;\ t_B = \varepsilon \cdot t;\ \frac{dt_B}{dt} = \frac{\partial t_B}{\partial t} = \varepsilon;\ \frac{dI}{dt} = \frac{dI}{dt_A} + \frac{dI}{dt_B} \cdot \varepsilon;$$
$$\frac{\partial I}{\partial t} = \frac{\partial I}{\partial t_A} + \frac{\partial I}{\partial t_B} \cdot \varepsilon \frac{\partial I}{\partial t} = \frac{dI}{dt} = \partial_{t_A} I + \varepsilon \cdot \partial_{t_B} I.$$

We know that $I(t, \varepsilon) = I_0(t_A, t_B) + \varepsilon \cdot I_1(t_A, t_B) + O(\varepsilon^2)$

$$\frac{dI(t,\varepsilon)}{dt} = \frac{\partial I_0(t_A, t_B)}{\partial t_A} + \frac{\partial I_0(t_A, t_B)}{\partial t_B} \cdot \varepsilon + \varepsilon \cdot \left[\frac{\partial I_1(t_A, t_B)}{\partial t_A} + \varepsilon \cdot \frac{\partial I_1(t_A, t_B)}{\partial t_B}\right] + O(\varepsilon^2)$$

$$\frac{dI(t,\varepsilon)}{dt} = \frac{\partial I_0(t_A, t_B)}{\partial t_A} + \left[\frac{\partial I_0(t_A, t_B)}{\partial t_B} + \frac{\partial I_1(t_A, t_B)}{\partial t_A}\right] \cdot \varepsilon + \varepsilon^2 \cdot \frac{\partial I_1(t_A, t_B)}{\partial t_B} + O(\varepsilon^2)$$

$$\varepsilon^2 \cdot \frac{\partial I_1(t_A, t_B)}{\partial t_B} + O(\varepsilon^2) \to O(\varepsilon^2) \Rightarrow \frac{dI(t,\varepsilon)}{dt} = \frac{\partial I_0(t_A, t_B)}{\partial t_A} + \left[\frac{\partial I_0(t_A, t_B)}{\partial t_B} + \frac{\partial I_1(t_A, t_B)}{\partial t_A}\right] \cdot \varepsilon + O(\varepsilon^2)$$

$$\frac{dI(t,\varepsilon)}{dt} = \frac{\partial I_0(t_A, t_B)}{\partial t_A} + \left[\frac{\partial I_0(t_A, t_B)}{\partial t_B} + \frac{\partial I_1(t_A, t_B)}{\partial t_A}\right] \cdot \varepsilon + \varepsilon^2 \cdot \frac{\partial I_1(t_A, t_B)}{\partial t_B} + O(\varepsilon^2).$$

We already find the below differential equation:

$$\frac{dI(t,\varepsilon)}{dt} = \frac{\partial I_0(t_A, t_B)}{\partial t_A} + \frac{\partial I_0(t_A, t_B)}{\partial t_B} \cdot \varepsilon + \varepsilon \cdot \left[\frac{\partial I_1(t_A, t_B)}{\partial t_A} + \varepsilon \cdot \frac{\partial I_1(t_A, t_B)}{\partial t_B}\right] + O(\varepsilon^2)$$

$$\frac{d^2 I(t,\varepsilon)}{dt^2} = \frac{\partial}{\partial t_A}\left[\frac{\partial I_0(t_A, t_B)}{\partial t_A} + \frac{\partial I_0(t_A, t_B)}{\partial t_B} \cdot \varepsilon\right] + \varepsilon \cdot \left\{\frac{\partial}{\partial t_B}\left[\frac{\partial I_0(t_A, t_B)}{\partial t_A} + \frac{\partial I_0(t_A, t_B)}{\partial t_B} \cdot \varepsilon\right] + \frac{\partial}{\partial t_A}\left[\frac{\partial I_1(t_A, t_B)}{\partial t_A} + \varepsilon \cdot \frac{\partial I_1(t_A, t_B)}{\partial t_B}\right]\right\} + O(\varepsilon^2)$$

$$\frac{d^2 I(t,\varepsilon)}{dt^2} = \frac{\partial^2 I_0(t_A, t_B)}{\partial t_A^2} + \frac{\partial^2 I_0(t_A, t_B)}{\partial t_A \partial t_B} \cdot \varepsilon + \varepsilon \cdot \frac{\partial^2 I_0(t_A, t_B)}{\partial t_B \partial t_A} + \frac{\partial^2 I_0(t_A, t_B)}{\partial t_B^2} \cdot \varepsilon^2 + \frac{\partial^2 I_1(t_A, t_B)}{\partial t_A^2} \cdot \varepsilon + \varepsilon^2 \cdot \frac{\partial^2 I_1(t_A, t_B)}{\partial t_A \partial t_B} + O(\varepsilon^2)$$

$$\frac{d^2 I(t,\varepsilon)}{dt^2} = \frac{\partial^2 I_0(t_A, t_B)}{\partial t_A^2} + \left[\frac{\partial^2 I_0(t_A, t_B)}{\partial t_A \partial t_B} + \frac{\partial^2 I_0(t_A, t_B)}{\partial t_B \partial t_A} + \frac{\partial^2 I_1(t_A, t_B)}{\partial t_A^2}\right] \cdot \varepsilon + \left[\frac{\partial^2 I_0(t_A, t_B)}{\partial t_B^2} + \frac{\partial^2 I_1(t_A, t_B)}{\partial t_A \partial t_B}\right] \cdot \varepsilon^2 + O(\varepsilon^2)$$

$$\left[\frac{\partial^2 I_0(t_A, t_B)}{\partial t_B^2} + \frac{\partial^2 I_1(t_A, t_B)}{\partial t_A \partial t_B}\right] \cdot \varepsilon^2 + O(\varepsilon^2) \to O(\varepsilon^2)$$

$$\frac{d^2 I(t,\varepsilon)}{dt^2} = \frac{\partial^2 I_0(t_A, t_B)}{\partial t_A^2} + \left[2 \cdot \frac{\partial^2 I_0(t_A, t_B)}{\partial t_A \partial t_B} + \frac{\partial^2 I_1(t_A, t_B)}{\partial t_A^2}\right] \cdot \varepsilon + O(\varepsilon^2).$$

We need to implement it to our weakly nonlinear OptoNDR van der Pol oscillator perturbed system differential equation:

$$\frac{d^2 I}{dt^2} + I \cdot \omega_0^2 + \varepsilon \cdot \psi(I) \cdot \frac{dI}{dt} = 0;\ I(t=0,\varepsilon) = 0$$

$$\frac{d^2 I(t,\varepsilon)}{dt^2} + I(t,\varepsilon) \cdot \omega_0^2 + \varepsilon \cdot \psi(I(t,\varepsilon)) \cdot \frac{dI(t,\varepsilon)}{dt} = 0;\ I(t=0,\varepsilon) = 0$$

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2}+\left[2\cdot\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon+O(\varepsilon^2)+[I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)]\cdot\omega_0^2$$
$$+\varepsilon\cdot\psi\left(I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)+O(\varepsilon^2)\right)\cdot\left[\frac{\partial I_0(t_A,t_B)}{\partial t_A}+\frac{\partial I_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon\right.$$
$$\left.+\varepsilon\cdot\left[\frac{\partial I_1(t_A,t_B)}{\partial t_A}+\varepsilon\cdot\frac{\partial I_1(t_A,t_B)}{\partial t_B}\right]+O(\varepsilon^2)\right]=0$$

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2}+\left[2\cdot\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon+O(\varepsilon^2)+[I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)]\cdot\omega_0^2$$
$$+\varepsilon\cdot\psi(I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)+O(\varepsilon^2))$$
$$\cdot\left[\frac{\partial I_0(t_A,t_B)}{\partial t_A}+\frac{\partial I_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon+\varepsilon\cdot\frac{\partial I_1(t_A,t_B)}{\partial t_A}+\varepsilon^2\cdot\frac{\partial I_1(t_A,t_B)}{\partial t_B}+O(\varepsilon^2)\right]=0$$

$$\varepsilon^2\cdot\frac{\partial I_1(t_A,t_B)}{\partial t_B}+O(\varepsilon^2)\to O(\varepsilon^2);\ \omega_0^2\to 1$$

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2}+\left[2\cdot\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon+O(\varepsilon^2)+[I_0(t_A,t_B)$$
$$+\varepsilon\cdot I_1(t_A,t_B)]+\varepsilon\cdot\psi\left(I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)+O(\varepsilon^2)\right)\cdot\left[\frac{\partial I_0(t_A,t_B)}{\partial t_A}\right.$$
$$\left.+\frac{\partial I_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon+\varepsilon\cdot\frac{\partial I_1(t_A,t_B)}{\partial t_A}+O(\varepsilon^2)\right]=0$$

Since $\varepsilon\neq 0$, for the right-hand side of the above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero. We ignore the $O(\varepsilon^2)$ elements.

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2}+\left[2\cdot\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2}\right]\cdot\varepsilon+I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)$$
$$+\varepsilon\cdot\psi(I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B))\cdot\left[\frac{\partial I_0(t_A,t_B)}{\partial t_A}+\frac{\partial I_0(t_A,t_B)}{\partial t_B}\cdot\varepsilon+\varepsilon\cdot\frac{\partial I_1(t_A,t_B)}{\partial t_A}\right]=0$$

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2}+I_0(t_A,t_B)+\varepsilon\cdot\left[2\cdot\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A\partial t_B}+\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2}\right]+\varepsilon\cdot I_1(t_A,t_B)$$
$$+\psi(I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B))\cdot\left[\varepsilon\cdot\frac{\partial I_0(t_A,t_B)}{\partial t_A}+\varepsilon^2\cdot\left(\frac{\partial I_0(t_A,t_B)}{\partial t_B}+\frac{\partial I_1(t_A,t_B)}{\partial t_A}\right)\right]=0$$

$$\varepsilon^2\cdot\left(\frac{\partial I_0(t_A,t_B)}{\partial t_B}+\frac{\partial I_1(t_A,t_B)}{\partial t_A}\right)\to O(\varepsilon^2);\ O(\varepsilon^2)\to 0$$

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2} + I_0(t_A,t_B) + \varepsilon \cdot \left\{ \left[2 \cdot \frac{\partial^2 I_0(t_A,t_B)}{\partial t_A \partial t_B} + \frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2} \right] + I_1(t_A,t_B) + \psi(I_0(t_A,t_B) + I_1(t_A,t_B)) \cdot \frac{\partial I_0(t_A,t_B)}{\partial t_A} \right\} = 0 .$$

We get pair of differential equations: $O(\varepsilon^0 = 1) \Rightarrow \frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2} + I_0(t_A,t_B) = 0$

$$O(\varepsilon) \Rightarrow \left[2 \cdot \frac{\partial^2 I_0(t_A,t_B)}{\partial t_A \partial t_B} + \frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2} \right] + I_1(t_A,t_B) + \frac{\partial I_0(t_A,t_B)}{\partial t_A} \cdot \psi(I_0(t_A,t_B) + I_1(t_A,t_B)) = 0$$

Differential equation $\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2} + I_0(t_A,t_B) = 0$ is a simple harmonic oscillator. The general solution is $I_0(t_A,t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A)$. The constants B_1 and B_2 are actually functions of slow time t_B. Times t_A, t_B should be regarded as independent variables and functions of t_B behaving like constants on the fast timescale t_A. We need to determine the constants $B_1(t_B)$ and $B_2(t_B)$. It is done by going to the next order ε substituting $I_0(t_A,t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A)$ into $O(\varepsilon)$ differential equation equation.

$$2 \cdot \frac{\partial^2 I_0(t_A,t_B)}{\partial t_A \partial t_B} + \frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2} + I_1(t_A,t_B) + \frac{\partial I_0(t_A,t_B)}{\partial t_A} \cdot \psi(I_0(t_A,t_B) + I_1(t_A,t_B)) = 0$$

$$\frac{\partial I_0(t_A,t_B)}{\partial t_A} = B_1 \cdot \cos(t_A) - B_2 \cdot \sin(t_A); \; \frac{\partial^2 I_0(t_A,t_B)}{\partial t_A^2} = -B_1 \cdot \sin(t_A) - B_2 \cdot \cos(t_A)$$

$$\frac{\partial^2 I_0(t_A,t_B)}{\partial t_A \partial t_B} = \frac{\partial B_1}{\partial t_B} \cdot \cos(t_A) - \frac{\partial B_2}{\partial t_B} \cdot \sin(t_A)$$

$$2 \cdot \left[\frac{\partial B_1}{\partial t_B} \cdot \cos(t_A) - \frac{\partial B_2}{\partial t_B} \cdot \sin(t_A) \right] + \frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2} + I_1(t_A,t_B) + [B_1 \cdot \cos(t_A) - B_2 \cdot \sin(t_A)] \cdot \psi(B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A) + I_1(t_A,t_B)) = 0$$

$$\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2} + I_1(t_A,t_B) = -[B_1 \cdot \cos(t_A) - B_2 \cdot \sin(t_A)] \cdot \psi(B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A) + I_1(t_A,t_B)) - 2 \cdot \left[\frac{\partial B_1}{\partial t_B} \cdot \cos(t_A) - \frac{\partial B_2}{\partial t_B} \cdot \sin(t_A) \right]$$

$$\frac{\partial^2 I_1(t_A,t_B)}{\partial t_A^2}+I_1(t_A,t_B)=-\left[B_1\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))+2\cdot\frac{\partial B_1}{\partial t_B}\right]\cdot\cos(t_A)$$
$$+\left[B_2\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))+2\cdot\frac{\partial B_2}{\partial t_B}\right]\cdot\sin(t_A).$$

The right-hand side of the above equation is a resonant forcing that will produce secular terms like $t_A\cdot\sin(t_A)$ and $t_A\cdot\cos(t_A)$ in the solution of $I_1(t_A,t_B)$.

These terms would lead to convergent. The approximation is done with no secular terms and need to set the coefficients of the resonant terms to zero.

$$B_1\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))+2\cdot\frac{\partial B_1}{\partial t_B}=0$$

$$B_2\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))+2\cdot\frac{\partial B_2}{\partial t_B}=0$$

$$\frac{\frac{\partial B_1}{\partial t_B}}{B_1}=-\frac{1}{2}\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))$$

$$\frac{\mathrm{d}(\ln(B_1))}{\mathrm{d}t_B}=-\frac{1}{2}\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))$$

$$\int\frac{\mathrm{d}(\ln(B_1))}{\mathrm{d}t_B}\cdot\mathrm{d}t_B=-\frac{1}{2}\cdot\int\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))\cdot\mathrm{d}t_B$$

$$\ln(B_1)=-\frac{1}{2}\cdot\int\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))\cdot\mathrm{d}t_B$$

$$B_1=B_1(t_B=0)\cdot\mathrm{e}^{-\frac{1}{2}\cdot\int\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))\cdot\mathrm{d}t_B}$$

$$\frac{\frac{\partial B_2}{\partial t_B}}{B_2}=-\frac{1}{2}\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))$$

$$\frac{\mathrm{d}(\ln(B_2))}{\mathrm{d}t_B}=-\frac{1}{2}\cdot\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))$$

$$\ln(B_2)=-\frac{1}{2}\cdot\int\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))\cdot\mathrm{d}t_B$$

$$B_2=B_2(t_B=0)\cdot\mathrm{e}^{-\frac{1}{2}\cdot\int\psi(B_1\cdot\sin(t_A)+B_2\cdot\cos(t_A)+I_1(t_A,t_B))\cdot\mathrm{d}t_B}$$

Next is to find initial values for $B_1(t_B=0)$ and $B_2(t_B=0)$, it is done by the following equation: $I(t,\varepsilon)=I_0(t_A,t_B)+\varepsilon\cdot I_1(t_A,t_B)+O(\varepsilon^2)$ and

$$I_0(t_A, t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A);\ I(t=0,\varepsilon) = 0;\ \frac{\mathrm{d}I(t=0,\varepsilon)}{\mathrm{d}t} = \Omega.$$

Equation $I(t,\varepsilon) = I_0(t_A,t_B) + \varepsilon \cdot I_1(t_A,t_B) + O(\varepsilon^2)$ gives $I(t=0,\varepsilon) = 0$ then

$$I(t=0,\varepsilon) = 0 \Rightarrow I_0(t_A=0, t_B=0) + \varepsilon \cdot I_1(t_A=0,t_B=0) + O(\varepsilon^2) = 0$$

$t = 0 \Rightarrow t_A = t;\ t_B = \varepsilon \cdot t \Rightarrow t_A = 0;\ t_B = 0$. To satisfy this equation for all sufficiently small ε, we must have $I_0(t_A=0,t_B=0) = 0$ and $I_1(t_A=0,t_B=0) = 0$.

Similarly, the following expression exists:

$$\frac{\mathrm{d}I(t=0,\varepsilon)}{\mathrm{d}t} = \Omega;\ \frac{\mathrm{d}I(t,\varepsilon)}{\mathrm{d}t} = \frac{\partial I_0(t_A,t_B)}{\partial t_A} + \left[\frac{\partial I_0(t_A,t_B)}{\partial t_B} + \frac{\partial I_1(t_A,t_B)}{\partial t_A}\right] \cdot \varepsilon + O(\varepsilon^2)$$

$$\frac{\mathrm{d}I(t=0,\varepsilon)}{\mathrm{d}t} = \frac{\partial I_0(t_A=0,t_B=0)}{\partial t_A} + \left[\frac{\partial I_0(t_A=0,t_B=0)}{\partial t_B} + \frac{\partial I_1(t_A=0,t_B=0)}{\partial t_A}\right] \cdot \varepsilon + O(\varepsilon^2) = \Omega$$

Result

$$\frac{\partial I_0(t_A=0,t_B=0)}{\partial t_A} = \Omega;\ \frac{\partial I_0(t_A=0,t_B=0)}{\partial t_B} + \frac{\partial I_1(t_A=0,t_B=0)}{\partial t_A} = 0.$$

Combining $I_0(t_A,t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A);\ I_0(t_A=0,t_B=0) = 0$ then $B_1 \cdot \sin(t_A) = 0;\ B_2 \cdot \cos(t_A) = 0$. Hence $B_2(t_B=0) = 0;\ B_2(t_B) = 0$ similarly

$$I_0(t,\varepsilon) = I_0(t_A,t_B) = B_1 \cdot \sin(t_A) + B_2 \cdot \cos(t_A);\ \frac{\partial I_0(t_A=0,t_B=0)}{\partial t_A} = \Omega$$

Imply

$$\frac{\partial I_0(t_A,t_B)}{\partial t_A} = B_1 \cdot \cos(t_A) - B_2 \cdot \sin(t_A);\ B_2 = 0\ \&\ t_A = 0 \Rightarrow B_1(0) = \Omega$$

$$\begin{aligned} B_1 &= B_1(t_B=0) \cdot \mathrm{e}^{-\frac{1}{2}\int \psi(B_1\cdot\sin(t_A) + B_2\cdot\cos(t_A) + I_1(t_A,t_B))\cdot \mathrm{d}t_B} \\ &= \Omega \cdot \mathrm{e}^{-\frac{1}{2}\cdot\int \psi(B_1\cdot\sin(t_A) + B_2\cdot\cos(t_A) + I_1(t_A,t_B))\cdot \mathrm{d}t_B} \end{aligned}$$

$$\begin{aligned} I(t,\varepsilon) &= I_0(t_A,t_B) + \varepsilon \cdot I_1(t_A,t_B) + O(\varepsilon^2);\ I_0(t_A,t_B) \\ &= \Omega \cdot \mathrm{e}^{-\frac{1}{2}\cdot\int \psi(B_1\cdot\sin(t_A) + B_2\cdot\cos(t_A) + I_1(t_A,t_B))\cdot \mathrm{d}t_B} \cdot \sin(t_A) \end{aligned}$$

$$I(t,\varepsilon) = \Omega \cdot \mathrm{e}^{-\frac{1}{2}\cdot\int \psi(B_1\cdot\sin(t_A) + B_2\cdot\cos(t_A) + I_1(t_A,t_B))\cdot \mathrm{d}t_B} \cdot \sin(t_A) + \varepsilon \cdot I_1(t_A,t_B) + O(\varepsilon^2)$$

$$O(\varepsilon) = \varepsilon \cdot I_1(t_A,t_B);$$

$$I(t,\varepsilon) = \Omega \cdot \mathrm{e}^{-\frac{1}{2}\cdot\int \psi(B_1\cdot\sin(t_A) + B_2\cdot\cos(t_A) + I_1(t_A,t_B))\cdot \mathrm{d}t_B} \cdot \sin(t_A) + O(\varepsilon) + O(\varepsilon^2)$$

It is approximate solution predicted by two timing and need to be solved numerically.

8.4 OptoNDR Circuit Forced van der Pol Perturbation Method Timescale

The unforced van der Pol equation has a limit cycle with radius approximately equal to two and period approximately 2π. The limit cycle is generated by the balance between internal energy loss and energy generation, and the forcing term will alter this balance. We consider forced van der Pol system with forcing term $F \cdot \cos(\omega \cdot t)$. If "$F$" is small (weak excitation), its effect depends on whether or not ω is close to the natural frequency. If it is, the oscillation might be generated which is a perturbation of the limit cycle. If "F" is not small (hard excitation) or if the natural and imposed frequency are not closely similar, we should expect that the "natural oscillation" might be extinguished, as occurs with the corresponding linear equation [7, 8]. The forced van der Pol system can be given by the following equations: $\frac{\mathrm{d}^2X_1}{\mathrm{d}t^2} + \varepsilon \cdot (X_1^2 - 1) \cdot \frac{\mathrm{d}X_1}{\mathrm{d}t} + X_1 = F \cdot \cos(\omega \cdot t)$ and can be expressed by first-degree differential equations $\frac{\mathrm{d}X_1}{\mathrm{d}t} = X_2$; $\frac{\mathrm{d}X_2}{\mathrm{d}t} + \varepsilon \cdot (X_1^2 - 1) \cdot X_2 + X_1 = F \cdot \cos(\omega \cdot t)$. By considering ω is close to the natural frequency, we set $\omega = 1$ and then define $\tau = \omega \cdot t$. If $\omega = 1$ then $\tau = t$ and we get the following forced van der Pol equation: $\frac{\mathrm{d}X_1}{\mathrm{d}t} = X_2$; $\frac{\mathrm{d}X_2}{\mathrm{d}t} + \varepsilon \cdot (X_1^2 - 1) \cdot X_2 + X_1 = F \cdot \cos(t)$. If $\omega \neq 1$ and $\tau = \omega \cdot t$ then $t = \frac{\tau}{\omega}$; $\frac{\mathrm{d}^2X_1}{\mathrm{d}(\frac{\tau}{\omega})^2} + \varepsilon \cdot (X_1^2 - 1) \cdot \frac{\mathrm{d}X_1}{\mathrm{d}(\frac{\tau}{\omega})} + X_1 = F \cdot \cos(\tau)$ and we get the following van der Pol equation: $\omega^2 \cdot \frac{\mathrm{d}^2X_1}{\mathrm{d}\tau^2} + \omega \cdot \varepsilon \cdot (X_1^2 - 1) \cdot \frac{\mathrm{d}X_1}{\mathrm{d}\tau} + X_1 = F \cdot \cos(\tau)$ where the differentiation with respect to τ. *First* we consider hard excitation, for from resonance and assume that ω is not close to an integer. Let call to variable X_1, X $(X_1 \to X)$, then our system second-order differential equation becomes $\omega^2 \cdot \frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} + \omega \cdot \varepsilon \cdot (X^2 - 1) \cdot \frac{\mathrm{d}X}{\mathrm{d}\tau} + X = F \cdot \cos(\tau)$. Let $X(\varepsilon,\tau) = \sum_{k=0}^{\infty} \varepsilon^k \cdot X_k(\tau)$ then $X(\varepsilon,\tau) = X_0(\tau) + \varepsilon \cdot X_1(\tau) + \cdots$. The first derivative of $X(\varepsilon,\tau)$, $\frac{\mathrm{d}X(\varepsilon,\tau)}{\mathrm{d}\tau} = \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{\mathrm{d}X_k(\tau)}{\mathrm{d}\tau}$ and the second derivative of $X(\varepsilon,\tau)$, $\frac{\mathrm{d}X(\varepsilon,\tau)}{\mathrm{d}\tau} = \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{\mathrm{d}X_k(\tau)}{\mathrm{d}\tau}$. Then we can define

$$\frac{dX(\varepsilon,\tau)}{d\tau} = \frac{dX_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dX_1(\tau)}{d\tau} + \varepsilon^2 \cdot \frac{dX_2(\tau)}{d\tau} + \cdots;$$
$$\frac{d^2X(\varepsilon,\tau)}{d\tau^2} = \frac{d^2X_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2} + \varepsilon^2 \cdot \frac{d^2X_2(\tau)}{d\tau^2} + \cdots$$

$$\omega^2 \cdot \left\{\frac{d^2X_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2} + \varepsilon^2 \cdot \frac{d^2X_2(\tau)}{d\tau^2} + \cdots\right\} + \omega \cdot \varepsilon \cdot \left(\{X_0(\tau) + \varepsilon \cdot X_1(\tau) + \varepsilon^2 \cdot X_2(\tau) + \cdots\}^2 - 1\right) \cdot \left\{\frac{dX_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dX_1(\tau)}{d\tau} + \varepsilon^2 \cdot \frac{dX_2(\tau)}{d\tau} + \cdots\right\} + X_0(\tau) + \varepsilon \cdot X_1(\tau) + \varepsilon^2 \cdot X_2(\tau) = F \cdot \cos(\tau)$$

Since $\varepsilon \neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε (except ε^0) should be equal to zero. We omit first all $O(\varepsilon^3)$ elements.

$$\left\{\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon^2 \cdot \frac{d^2X_2(\tau)}{d\tau^2}\right\} + \omega \cdot \varepsilon \cdot (X_0^2(\tau) + 2 \cdot \varepsilon \cdot X_0(\tau) \cdot X_1(\tau) + \varepsilon^2 \cdot X_1^2(\tau) - 1) \cdot \left\{\frac{dX_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dX_1(\tau)}{d\tau} + \varepsilon^2 \cdot \frac{dX_2(\tau)}{d\tau}\right\} + X_0(\tau) + \varepsilon \cdot X_1(\tau) + \varepsilon^2 \cdot X_2(\tau) = F \cdot \cos(\tau)$$

$$\left\{\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon^2 \cdot \frac{d^2X_2(\tau)}{d\tau^2}\right\} + (\omega \cdot \varepsilon \cdot X_0^2(\tau) + 2 \cdot \omega \cdot \varepsilon^2 \cdot X_0(\tau) \cdot X_1(\tau) + \omega \cdot \varepsilon^3 \cdot X_1^2(\tau) - \omega \cdot \varepsilon) \cdot \left\{\frac{dX_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dX_1(\tau)}{d\tau} + \varepsilon^2 \cdot \frac{dX_2(\tau)}{d\tau}\right\} + X_0(\tau) + \varepsilon \cdot X_1(\tau) + \varepsilon^2 \cdot X_2(\tau) = F \cdot \cos(\tau)$$

Next we omit all $O(\varepsilon^2)$ elements.

$$\left\{\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2}\right\} + \omega \cdot \varepsilon \cdot (X_0^2(\tau) - 1) \cdot \left\{\frac{dX_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dX_1(\tau)}{d\tau} + \varepsilon^2 \cdot \frac{dX_2(\tau)}{d\tau}\right\} + X_0(\tau) + \varepsilon \cdot X_1(\tau) = F \cdot \cos(\tau)$$

$$\left\{\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2}\right\} + (X_0^2(\tau) - 1) \cdot \left\{\omega \cdot \varepsilon \cdot \frac{dX_0(\tau)}{d\tau} + \omega \cdot \varepsilon^2 \cdot \frac{dX_1(\tau)}{d\tau} + \omega \cdot \varepsilon^3 \cdot \frac{dX_2(\tau)}{d\tau}\right\} + X_0(\tau) + \varepsilon \cdot X_1(\tau) = F \cdot \cos(\tau)$$

$$\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + \omega^2 \cdot \varepsilon \cdot \frac{d^2X_1(\tau)}{d\tau^2} + (X_0^2(\tau) - 1) \cdot \omega \cdot \varepsilon \cdot \frac{dX_0(\tau)}{d\tau} + X_0(\tau) + \varepsilon \cdot X_1(\tau) = F \cdot \cos(\tau)$$

$$\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + X_0(\tau) + \varepsilon \cdot \left\{\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} + (X_0^2(\tau) - 1) \cdot \omega \cdot \frac{dX_0(\tau)}{d\tau} + X_1(\tau)\right\}$$
$$= F \cdot \cos(\tau)$$

$$\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + X_0(\tau) = F \cdot \cos(\tau);$$
$$\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} + (X_0^2(\tau) - 1) \cdot \omega \cdot \frac{dX_0(\tau)}{d\tau} + X_1(\tau) = 0.$$

Functions $X_0(\tau)$ and $X_1(\tau)$ have period $2 \cdot \pi$. The only $X_0(\tau)$ solution having period is $X_0(\tau) = \frac{F}{1-\omega^2} \cdot \cos(\tau)$ and therefore $X(\varepsilon, \tau) = \frac{F}{1-\omega^2} \cdot \cos(\tau) + O(\varepsilon)$. The solution is therefore a perturbation of the ordinary linear response and the limit cycle is suppressed as expected. To get the exact solution of $X_1(\tau)$, we get $\frac{dX_0(\tau)}{d\tau} = -\frac{F}{1-\omega^2} \cdot \sin(\tau)$; $X_0^2(\tau) = \frac{F^2}{(1-\omega^2)^2} \cdot \cos^2(\tau)$.

$$\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} - \left(\frac{F^2}{(1-\omega^2)^2} \cdot \cos^2(\tau) - 1\right) \cdot \omega \cdot \frac{F}{(1-\omega^2)} \cdot \sin(\tau) + X_1(\tau) = 0$$

$$\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} + X_1(\tau) = \left(\frac{F^2}{(1-\omega^2)^2} \cdot \cos^2(\tau) - 1\right) \cdot \omega \cdot \frac{F}{(1-\omega^2)} \cdot \sin(\tau).$$

The above differential equation solution gives the $X_1(\tau)$ expression.

Second, we consider soft excitation, far from resonance. This case is similar to hard excitation above but with $F = \varepsilon \cdot F_0$ and this solution is unstable.

$$\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + X_0(\tau) + \varepsilon \cdot \{\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} + (X_0^2(\tau) - 1) \cdot \omega \cdot \frac{dX_0(\tau)}{d\tau} + X_1(\tau)\}$$
$$= \varepsilon \cdot F_0 \cdot \cos(\tau)$$

$$\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + X_0(\tau) + \varepsilon \cdot \left\{\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} + \left(X_0^2(\tau) - 1\right)\right.$$
$$\left. \cdot \omega \cdot \frac{dX_0(\tau)}{d\tau} + X_1(\tau) - F_0 \cdot \cos(\tau)\right\} = 0$$

Since $\varepsilon \neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero. The zeroth of ε gives us $\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + X_0(\tau) = 0$ and the second differential equation $\omega^2 \cdot \frac{d^2X_1(\tau)}{d\tau^2} + (X_0^2(\tau) - 1) \cdot \omega \cdot \frac{dX_0(\tau)}{d\tau} + X_1(\tau) - F_0 \cdot \cos(\tau) = 0$. First we investigate the $\omega^2 \cdot \frac{d^2X_0(\tau)}{d\tau^2} + X_0(\tau) = 0$ differential equation. Let us assume that $X_0(\tau) = e^{r \cdot \tau}$ is a solution to equation where r is a constant. The auxiliary equations are $\frac{d^2 X_0(\tau)}{d\tau^2} = r^2 \cdot e^{r \cdot \tau} \Rightarrow \omega^2 \cdot r^2 \cdot e^{r \cdot \tau} + e^{r \cdot \tau} = 0$; $[\omega^2 \cdot r^2 + 1] \cdot e^{r \cdot \tau} = 0$; $r^2 = \frac{-1}{\omega^2}$; $r = \frac{\pm j}{\omega}$. Here the r real part is equal to zero and the imaginary part is equal to $\frac{1}{\omega}$. If we assumed from above that the general solution for $X_0(\tau)$ is of the form $\sin(\frac{1}{\omega} \cdot \tau) + \cos(\frac{1}{\omega} \cdot \tau)$ and we double check for solution: $X_0(\tau) = \sin(\frac{1}{\omega} \cdot \tau) + \cos(\frac{1}{\omega} \cdot \tau)$

$$\frac{dX_0(\tau)}{d\tau} = \frac{1}{\omega} \cdot \cos\left(\frac{1}{\omega} \cdot \tau\right) - \frac{1}{\omega} \cdot \sin\left(\frac{1}{\omega} \cdot \tau\right); \frac{d^2 X_0(\tau)}{d\tau^2} = -\frac{1}{\omega^2} \cdot \left[\sin\left(\frac{1}{\omega} \cdot \tau\right) + \cos\left(\frac{1}{\omega} \cdot \tau\right)\right].$$

Then $\omega^2 \cdot \frac{d^2 X_0(\tau)}{d\tau^2} + X_0(\tau) = 0$. We can write our solution to $\omega^2 \cdot \frac{d^2 X_0(\tau)}{d\tau^2} + X_0(\tau) = 0$ in the convenient form: $X_0(\tau) = A_1 \cdot \sin(\frac{1}{\omega} \cdot \tau) + A_2 \cdot \cos(\frac{1}{\omega} \cdot \tau)$ where A_1, A_2 are constants to be determined. If we utilized the initial conditions: $X_0(\tau = 0) = 0, \frac{dX_0(\tau=0)}{d\tau} = \Omega$. Given to find the values of the constants A_1, A_2, we have by substituting these values in equation: $X_0(\tau) = A_1 \cdot \sin(\frac{1}{\omega} \cdot \tau) + A_2 \cdot \cos(\frac{1}{\omega} \cdot \tau)$ that $A_2 = 0$; hence $X_0(\tau) = A_1 \cdot \sin(\frac{1}{\omega} \cdot \tau)$; $\frac{dX_0(\tau)}{d\tau} = A_1 \cdot \frac{1}{\omega} \cdot \cos(\frac{1}{\omega} \cdot \tau)$; $\frac{dX_0(\tau=0)}{d\tau} = A_1 \cdot \frac{1}{\omega} = \Omega \Rightarrow A_1 = \omega \cdot \Omega$ then $X_0(\tau) = \omega \cdot \Omega \cdot \sin(\frac{1}{\omega} \cdot \tau)$. The second differential equation is $\omega^2 \cdot \frac{d^2 X_1(\tau)}{d\tau^2} + (X_0^2(\tau) - 1) \cdot \omega \cdot \frac{dX_0(\tau)}{d\tau} + X_1(\tau) - F_0 \cdot \cos(\tau) = 0$ and from our last solution $X_0(\tau) = \omega \cdot \Omega \cdot \sin(\frac{1}{\omega} \cdot \tau)$ then we get the following differential equation

$$\frac{dX_0(\tau)}{d\tau} = \Omega \cdot \cos\left(\frac{1}{\omega} \cdot \tau\right); \; X_0^2(\tau) = \omega^2 \cdot \Omega^2 \cdot \sin^2\left(\frac{1}{\omega} \cdot \tau\right)$$

$$\omega^2 \cdot \frac{d^2 X_1(\tau)}{d\tau^2} + \left[\omega^2 \cdot \Omega^2 \cdot \sin^2\left(\frac{1}{\omega} \cdot \tau\right) - 1\right] \cdot \omega \cdot \Omega \cdot \cos\left(\frac{1}{\omega} \cdot \tau\right) + X_1(\tau) - F_0 \cdot \cos(\tau) = 0$$

$$\omega^2 \cdot \frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = F_0 \cdot \cos(\tau) - \left[\omega^2 \cdot \Omega^2 \cdot \sin^2\left(\frac{1}{\omega} \cdot \tau\right) - 1\right] \cdot \omega \cdot \Omega \cdot \cos\left(\frac{1}{\omega} \cdot \tau\right).$$

The last differential equation is close to a typical case of resonance. It is because of resonant interactions between consecutive orders that nonuniformity has appeared in the regular perturbation series.

Third, we consider soft excitation, near resonance. In this case we define $F = \varepsilon \cdot \gamma$, and for near resonance $\omega = 1 + \varepsilon \cdot \omega_1$. The expansion is assumed to be $X(\varepsilon, \tau) = \sum_{k=0}^{\infty} \varepsilon^k \cdot X_k(\tau)$; $X(\varepsilon, \tau) = X_0(\tau) + \varepsilon \cdot X_1(\tau) + \cdots$ and we get our system van der Pol differential equation:

$$(1+\varepsilon\cdot\omega_1)^2\cdot\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2}+(1+\varepsilon\cdot\omega_1)\cdot\varepsilon\cdot(X^2-1)\cdot\frac{\mathrm{d}X}{\mathrm{d}\tau}+X=\varepsilon\cdot\gamma\cdot\cos(\tau)$$

$$\omega^2\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)+\varepsilon\cdot\left\{\omega^2\cdot\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}+(X_0^2(\tau)-1)\cdot\omega\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+X_1(\tau)\right\}$$
$$=\varepsilon\cdot\gamma\cdot\cos(\tau)$$

$$(1+\varepsilon\cdot\omega_1)^2\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)+\varepsilon\cdot\left\{(1+\varepsilon\cdot\omega_1)^2\cdot\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}\right.$$
$$\left.+(X_0^2(\tau)-1)\cdot(1+\varepsilon\cdot\omega_1)\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+X_1(\tau)\right\}=\varepsilon\cdot\gamma\cdot\cos(\tau)$$

$$(1+2\cdot\varepsilon\cdot\omega_1+\varepsilon^2\cdot\omega_1^2)\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)+\varepsilon\cdot\left\{(1+2\cdot\varepsilon\cdot\omega_1+\varepsilon^2\cdot\omega_1^2)\cdot\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}\right.$$
$$\left.+(X_0^2(\tau)-1)\cdot(1+\varepsilon\cdot\omega_1)\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+X_1(\tau)\right\}=\varepsilon\cdot\gamma\cdot\cos(\tau)$$

$$(1+2\cdot\varepsilon\cdot\omega_1+\varepsilon^2\cdot\omega_1^2)\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)+(\varepsilon+2\cdot\varepsilon^2\cdot\omega_1+\varepsilon^3\cdot\omega_1^2)\cdot\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}$$
$$+(X_0^2(\tau)-1)\cdot(\varepsilon+\varepsilon^2\cdot\omega_1)\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+\varepsilon\cdot X_1(\tau)=\varepsilon\cdot\gamma\cdot\cos(\tau).$$

Next we omit all $O(\varepsilon^2)$ and $O(\varepsilon^3)$ elements.

$$(1+2\cdot\varepsilon\cdot\omega_1)\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)+\varepsilon\cdot\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}$$
$$+(X_0^2(\tau)-1)\cdot\varepsilon\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+\varepsilon\cdot X_1(\tau)=\varepsilon\cdot\gamma\cdot\cos(\tau)$$

$$\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)+\varepsilon\cdot\left\{2\cdot\omega_1\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}+(X_0^2(\tau)-1)\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+X_1(\tau)-\gamma\cdot\cos(\tau)\right\}=0.$$

Since $\varepsilon\neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero. The zeroth of ε gives us $O(\varepsilon^0)\Rightarrow\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+X_0(\tau)=0$ and for $O(\varepsilon^1)$

$$2\cdot\omega_1\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}+\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}+(X_0^2(\tau)-1)\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+X_1(\tau)-\gamma\cdot\cos(\tau)=0$$

$$\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}+X_1(\tau)=-2\cdot\omega_1\cdot\frac{\mathrm{d}^2X_0(\tau)}{\mathrm{d}\tau^2}-(X_0^2(\tau)-1)\cdot\frac{\mathrm{d}X_0(\tau)}{\mathrm{d}\tau}+\gamma\cdot\cos(\tau).$$

We require solution with period $2 \cdot \pi$; then second-order differential equation $\frac{d^2 X_0(\tau)}{d\tau^2} + X_0(\tau) = 0$ has the solutions $X_0(\tau) = a_0 \cdot \cos(\tau) + b_0 \cdot \sin(\tau)$.

$$\frac{dX_0(\tau)}{d\tau} = -a_0 \cdot \sin(\tau) + b_0 \cdot \cos(\tau); \frac{d^2 X_0(\tau)}{d\tau^2} = -[a_0 \cdot \cos(\tau) + b_0 \cdot \sin(\tau)]$$

$$X_0(\tau) = a_0 \cdot \cos(\tau) + b_0 \cdot \sin(\tau) \Rightarrow X_0^2(\tau) \\ = a_0^2 \cdot \cos^2(\tau) + 2 \cdot a_0 \cdot b_0 \cdot \cos(\tau) \cdot \sin(\tau) + b_0^2 \cdot \sin^2(\tau)$$

$$2 \cdot a_0 \cdot b_0 \cdot \cos(\tau) \cdot \sin(\tau) = a_0 \cdot b_0 \cdot \sin(2 \cdot \tau); \\ X_0^2(\tau) = a_0^2 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin(2 \cdot \tau) + b_0^2 \cdot \sin^2(\tau)$$

$$\frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = 2 \cdot \omega_1 \cdot [a_0 \cdot \cos(\tau) + b_0 \cdot \sin(\tau)] - (a_0^2 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin(2 \cdot \tau) \\ + b_0^2 \cdot \sin^2(\tau) - 1) \cdot [-a_0 \cdot \sin(\tau) + b_0 \cdot \cos(\tau)] + \gamma \cdot \cos(\tau)$$

$$\frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = 2 \cdot \omega_1 \cdot a_0 \cdot \cos(\tau) + 2 \cdot \omega_1 \cdot b_0 \cdot \sin(\tau) - ([a_0^2 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin(2 \cdot \tau) \\ + b_0^2 \cdot \sin^2(\tau)] - 1) \cdot [-a_0 \cdot \sin(\tau) + b_0 \cdot \cos(\tau)] + \gamma \cdot \cos(\tau)$$

$$\frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = 2 \cdot \omega_1 \cdot a_0 \cdot \cos(\tau) + 2 \cdot \omega_1 \cdot b_0 \cdot \sin(\tau) - \{[a_0^2 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin(2 \cdot \tau) \\ + b_0^2 \cdot \sin^2(\tau)] \cdot [-a_0 \cdot \sin(\tau) + b_0 \cdot \cos(\tau)] + a_0 \cdot \sin(\tau) - b_0 \cdot \cos(\tau)\} + \gamma \cdot \cos(\tau)$$

$$\frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = 2 \cdot \omega_1 \cdot a_0 \cdot \cos(\tau) + b_0 \cdot \cos(\tau) + \gamma \cdot \cos(\tau) + 2 \cdot \omega_1 \cdot b_0 \cdot \sin(\tau) - a_0 \cdot \sin(\tau) \\ - [a_0^2 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin(2 \cdot \tau) + b_0^2 \cdot \sin^2(\tau)] \cdot [-a_0 \cdot \sin(\tau) + b_0 \cdot \cos(\tau)]$$

$$\frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = [2 \cdot \omega_1 \cdot a_0 + b_0 + \gamma] \cdot \cos(\tau) + [2 \cdot \omega_1 \cdot b_0 - a_0] \cdot \sin(\tau) \\ - [a_0^2 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin(2 \cdot \tau) + b_0^2 \cdot \sin^2(\tau)] \cdot [-a_0 \cdot \sin(\tau) + b_0 \cdot \cos(\tau)]$$

$$\frac{d^2 X_1(\tau)}{d\tau^2} + X_1(\tau) = [2 \cdot \omega_1 \cdot a_0 + \gamma + b_0 \cdot \sin^2(\tau) \cdot (2 \cdot a_0^2 - b_0)] \cdot \cos(\tau) \\ + [2 \cdot \omega_1 \cdot b_0 + a_0 \cdot \cos^2(\tau) \cdot (a_0^2 - 2 \cdot b_0^2)] \cdot \sin(\tau) - a_0^2 \cdot b_0 \cdot \cos^2(\tau) + a_0 \cdot b_0 \cdot \sin^2(\tau) .$$

We define a new parameter r_0 and it is equal to $r_0 = \sqrt{a_0^2 + b_0^2}$; $r_0 > 0$; $r_0, a_0, b_0 \in \mathbb{R}$.

$$\frac{\mathrm{d}^2X_1(\tau)}{\mathrm{d}\tau^2}+X_1(\tau)=\left[2\cdot\omega_1\cdot a_0+\gamma-b_0\cdot\left(\frac{1}{4}\cdot r_0^2-1\right)\right]\cdot\cos(\tau)$$
$$+\left[2\cdot\omega_1\cdot b_0+a_0\cdot\left(\frac{1}{4}\cdot r_0^2-1\right)\right]\cdot\sin(\tau)+\cdots$$

We ignore the higher harmonics elements in the above second-order differential equation. For periodic solution we require $2\cdot\omega_1\cdot a_0-b_0\cdot\left(\frac{1}{4}\cdot r_0^2-1\right)=-\gamma$ and $2\cdot\omega_1\cdot b_0-a_0\cdot\left(\frac{1}{4}\cdot r_0^2-1\right)=0$. By squaring and adding these two equations we obtain the following expression: $r_0^2\cdot\left[4\cdot\omega_1^2+\left(\frac{1}{4}\cdot r_0^2-1\right)^2\right]=\gamma^2$. We get possible amplitude r_0 of the response for given ω_1 and γ. Let us analyze van der Pol forced system $\omega^2\cdot\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2}+\omega\cdot\varepsilon\cdot(X^2-1)\cdot\frac{\mathrm{d}X}{\mathrm{d}\tau}+X=F\cdot\cos(\tau)$ by Fourier series. The solutions can emerge as series of sines and cosines with frequencies which are integer multiplies of the forcing frequency. These appeared as a result of reorganizing terms like X^3, but by making presentation by using Fourier series. We consider our van der Pol forced system as a more general forced equation: $\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2}+\varepsilon\cdot h(X,\frac{\mathrm{d}X}{\mathrm{d}t})+\Omega^2\cdot X=F(\tau)$. The transformation from forced van der Pol general differential equation to our van der Pol forced system is as follows: $\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2}+\frac{1}{\omega}\cdot\varepsilon\cdot(X^2-1)\cdot\frac{\mathrm{d}X}{\mathrm{d}\tau}+\frac{1}{\omega^2}\cdot X=F\cdot\frac{1}{\omega^2}\cdot\cos(\tau)$; $F(\tau)=F\cdot\frac{1}{\omega^2}\cdot\cos(\tau)$; $\Omega^2=\frac{1}{\omega^2}$ and $h(X,\frac{\mathrm{d}X}{\mathrm{d}\tau})=\frac{1}{\omega}\cdot(X^2-1)\cdot\frac{\mathrm{d}X}{\mathrm{d}\tau}$, where ε is a small parameter. Function $F(\tau)$ is periodic, with the time variable already scaled to give it $2\cdot\pi$, and its mean value is zero and there is zero constant term in its Fourier series representation, its time-averaged value of $F(\tau)$ is zero over a period and we expand $F(\tau)$ as a Fourier series: $F(\tau)=\sum_{n=1}^{\infty}(A_n\cdot\cos(n\cdot\tau)+B_n\cdot\sin(n\cdot\tau)$ [5–7]. The Fourier coefficients are given by

$$A_n=\frac{1}{\pi}\cdot\int\limits_0^{2\cdot\pi}F(\tau)\cdot\cos(n\cdot\tau)\cdot\mathrm{d}\tau$$

and

$$A_n=F\cdot\frac{1}{\omega^2}\cdot\frac{1}{\pi}\cdot\int\limits_0^{2\cdot\pi}\cos(\tau)\cdot\cos(n\cdot\tau)\cdot\mathrm{d}\tau;\ B_n=\frac{1}{\pi}\cdot\int\limits_0^{2\cdot\pi}F(\tau)\cdot\sin(n\cdot\tau)\cdot\mathrm{d}\tau$$

$$B_n=F\cdot\frac{1}{\omega^2}\cdot\frac{1}{\pi}\cdot\int\limits_0^{2\cdot\pi}\cos(\tau)\cdot\sin(n\cdot\tau)\cdot\mathrm{d}\tau.$$

We define Ω parameter which is close to some integer N by writing $\Omega^2=N^2+\varepsilon\cdot\beta$. The perturbation method requires that the periodic solutions emerge from periodic solutions of some appropriate linear equation. If $F(\tau) = \sum_{n=1}^{\infty} (A_n \cdot \cos(n \cdot \tau) + B_n \cdot \sin(n \cdot \tau)$ has a nonzero term of order N then $\frac{d^2X}{d\tau^2} + \varepsilon \cdot h\left(X, \frac{dX}{dt}\right) + \Omega^2 \cdot X = F(\tau)$, with $\varepsilon = 0$, is clearly not an appropriate linearization, since the forcing term has a component equal to the natural frequency N and there will be no periodic solutions. If we write $A_N = \varepsilon \cdot A$ and $B_N = \varepsilon \cdot B$ the term in $F(\tau)$ giving resonance is removed from the linearized equation and we have a possible family of generating solutions. By applying a mathematical manipulations to $\frac{d^2X}{d\tau^2} + \varepsilon \cdot h(X, \frac{dX}{dt}) + \Omega^2 \cdot X = F(\tau)$ and isolating the troublesome term in $F(\tau)$ by writing $f(\tau) = F(\tau) - \varepsilon \cdot A \cdot \cos(N \cdot \tau) - \varepsilon \cdot B \cdot \sin(N \cdot \tau)$

$f(\tau) = \sum_{n \neq N} (A_n \cdot \cos(n \cdot \tau) + B_n \cdot \sin(n \cdot \tau))$. Equation $\frac{d^2X}{d\tau^2} + \varepsilon \cdot h(X, \frac{dX}{dt}) + \Omega^2 \cdot X = F(\tau)$ becomes $F(\tau) = f(\tau) + \varepsilon \cdot A \cdot \cos(N \cdot \tau) + \varepsilon \cdot B \cdot \sin(N \cdot \tau)$ then $\Omega^2 = N^2 + \varepsilon \cdot \beta$

$$\frac{d^2X}{v\tau^2} + \Omega^2 \cdot X = f(\tau) + \varepsilon \cdot A \cdot \cos(N \cdot \tau) + \varepsilon \cdot B \cdot \sin(N \cdot \tau) - \varepsilon \cdot h\left(X, \frac{dX}{dt}\right)$$

$$\frac{d^2X}{d\tau^2} + (N^2 + \varepsilon \cdot \beta) \cdot X = f(\tau) + \varepsilon \cdot A \cdot \cos(N \cdot \tau) + \varepsilon \cdot B \cdot \sin(N \cdot \tau) - \varepsilon \cdot h\left(X, \frac{dX}{dt}\right)$$

$$\frac{d^2X}{d\tau^2} + N^2 \cdot X = f(\tau) + \varepsilon \cdot A \cdot \cos(N \cdot \tau) + \varepsilon \cdot B \cdot \sin(N \cdot \tau) - \varepsilon \cdot h\left(X, \frac{dX}{dt}\right) - \varepsilon \cdot \beta \cdot X$$

$$\frac{d^2X}{d\tau^2} + N^2 \cdot X = f(\tau) + \varepsilon \cdot \left[A \cdot \cos(N \cdot \tau) + B \cdot \sin(N \cdot \tau) - h\left(X, \frac{dX}{dt}\right) - \beta \cdot X\right].$$

We get the linearized equation $\frac{d^2X}{d\tau^2} + N^2 \cdot X = f(\tau)$, with no resonance. Now we write

$$X(\varepsilon, \tau) = \sum_{k=0}^{\infty} \varepsilon^k \cdot X_k(\tau); \; X(\varepsilon, \tau) = X_0(\tau) + \varepsilon \cdot X_1(\tau) + \cdots$$

where $X_0, X_1, \ldots$ have period of $2 \cdot \pi$. We expand $h(X, \frac{dX}{dt})$ in powers of ε and we have the following expression:

$$h\left(X, \frac{dX}{dt}\right) = h\left(X_0, \frac{dX_0}{dt}\right) + \varepsilon \cdot h_1\left(X_0, \frac{dX_0}{dt}, X_1, \frac{dX_1}{dt}\right) + \cdots$$

where h_1 function can be calculated, and by substituting $X(\varepsilon,\tau)=\sum_{k=0}^{\infty}\varepsilon^k\cdot X_k(\tau)$ and $h\left(X,\frac{dx}{dt}\right)=\ldots$ into $\frac{d^2x}{d\tau^2}+N^2\cdot X=f(\tau)+\varepsilon\cdot[A\cdot\cos(N\cdot\tau)+\cdots$ we obtain

$$\frac{d^2X}{d\tau^2}+N^2\cdot X=f(\tau)+\varepsilon\cdot\left[A\cdot\cos(N\cdot\tau)+B\cdot\sin(N\cdot\tau)-\left\{h\left(X_0,\frac{dX_0}{dt}\right)\right.\right.$$
$$\left.\left.+\varepsilon\cdot h_1\left(X_0,\frac{dX_0}{dt},X_1,\frac{dX_1}{dt}\right)\right\}-\beta\cdot X\right]$$

$$X(\varepsilon,\tau)=X_0(\tau)+\varepsilon\cdot X_1(\tau)+\varepsilon^2\cdot X_2(\tau);$$
$$\frac{d^2X(\varepsilon,\tau)}{d\tau^2}=\frac{d^2X_0(\tau)}{d\tau^2}+\varepsilon\cdot\frac{d^2X_1(\tau)}{d\tau^2}+\varepsilon^2\cdot\frac{d^2X_2(\tau)}{d\tau^2}$$

$$\frac{d^2X_0(\tau)}{d\tau^2}+\varepsilon\cdot\frac{d^2X_1(\tau)}{d\tau^2}+\varepsilon^2\cdot\frac{d^2X_2(\tau)}{d\tau^2}+N^2\cdot\{X_0(\tau)+\varepsilon\cdot X_1(\tau)+\varepsilon^2\cdot X_2(\tau)\}=f(\tau)$$
$$+\varepsilon\cdot\left[A\cdot\cos(N\cdot\tau)+B\cdot\sin(N\cdot\tau)-\left\{h\left(X_0,\frac{dX_0}{dt}\right)+\varepsilon\cdot h_1\left(X_0,\frac{dX_0}{dt},X_1,\frac{dX_1}{dt}\right)\right\}\right.$$
$$\left.-\beta\cdot\left\{X_0(\tau)+\varepsilon\cdot X_1(\tau)+\varepsilon^2\cdot X_2(\tau)\right\}\right]$$

$$\frac{d^2X_0(\tau)}{d\tau^2}+\varepsilon\cdot\frac{d^2X_1(\tau)}{d\tau^2}+\varepsilon^2\cdot\frac{d^2X_2(\tau)}{d\tau^2}+N^2\cdot X_0(\tau)+\varepsilon\cdot N^2\cdot X_1(\tau)+\varepsilon^2\cdot N^2\cdot X_2(\tau)=f(\tau)$$
$$+A\cdot\varepsilon\cdot\cos(N\cdot\tau)+B\cdot\varepsilon\cdot\sin(N\cdot\tau)-\left\{\varepsilon\cdot h\left(X_0,\frac{dX_0}{dt}\right)+\varepsilon^2\cdot h_1\left(X_0,\frac{dX_0}{dt},X_1,\frac{dX_1}{dt}\right)\right\}\cdot$$
$$-\beta\cdot\left\{\varepsilon\cdot X_0(\tau)+\varepsilon^2\cdot X_1(\tau)+\varepsilon^3\cdot X_2(\tau)\right\}$$

We ignore $O(\varepsilon^3)$ elements.

$$\frac{d^2X_0(\tau)}{d\tau^2}+\varepsilon\cdot\frac{d^2X_1(\tau)}{d\tau^2}+\varepsilon^2\cdot\frac{d^2X_2(\tau)}{d\tau^2}+N^2\cdot X_0(\tau)+\varepsilon\cdot N^2\cdot X_1(\tau)+\varepsilon^2\cdot N^2\cdot X_2(\tau)-f(\tau)$$
$$-A\cdot\varepsilon\cdot\cos(N\cdot\tau)-B\cdot\varepsilon\cdot\sin(N\cdot\tau)+\varepsilon\cdot h\left(X_0,\frac{dX_0}{dt}\right)+\varepsilon^2\cdot h_1\left(X_0,\frac{dX_0}{dt},X_1,\frac{dX_1}{dt}\right)$$
$$+\varepsilon\cdot\beta\cdot X_0(\tau)+\varepsilon^2\cdot\beta\cdot X_1(\tau)=0$$

$$\frac{d^2X_0(\tau)}{d\tau^2}+N^2\cdot X_0(\tau)-f(\tau)+\varepsilon\cdot\left\{\frac{d^2X_1(\tau)}{d\tau^2}+N^2\cdot X_1(\tau)-A\cdot\cos(N\cdot\tau)-B\cdot\sin(N\cdot\tau)\right.$$
$$\left.+h\left(X_0,\frac{dX_0}{dt}\right)+\beta\cdot X_0(\tau)\right\}+\varepsilon^2\cdot\left\{\frac{d^2X_2(\tau)}{d\tau^2}+N^2\cdot X_2(\tau)+h_1\left(X_0,\frac{dX_0}{dt},X_1,\frac{dX_1}{dt}\right)+\beta\cdot X_1(\tau)\right\}=0$$

Since $\varepsilon\neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero. The zeroth of ε gives us $\frac{d^2X_0(\tau)}{d\tau^2}+N^2\cdot X_0(\tau)-f(\tau)=0$

$$\frac{d^2X_0(\tau)}{d\tau^2} + N^2 \cdot X_0(\tau) = \sum_{n \neq N} (A_n \cdot \cos(n \cdot \tau) + B_n \cdot \sin(n \cdot \tau))$$

$$\frac{d^2X_1(\tau)}{d\tau^2} + N^2 \cdot X_1(\tau) = A \cdot \cos(N \cdot \tau) + B \cdot \sin(N \cdot \tau) - h\left(X_0, \frac{dX_0}{dt}\right) - \beta \cdot X_0(\tau)$$

$$\frac{d^2X_2(\tau)}{d\tau^2} + N^2 \cdot X_2(\tau) = -h_1\left(X_0, \frac{dX_0}{dt}, X_1, \frac{dX_1}{dt}\right) - \beta \cdot X_1(\tau).$$

The solution of $\frac{d^2X_0(\tau)}{d\tau^2} + N^2 \cdot X_0(\tau) = \sum_{n \neq N} (A_n \cdot \cos(n \cdot \tau) + B_n \cdot \sin(n \cdot \tau))$ is

$$X_0(\tau) = a_0 \cdot \cos(N \cdot \tau) + b_0 \cdot \sin(N \cdot \tau) + \sum_{n \neq N} \frac{A_n \cdot \cos(n \cdot \tau) + B_n \cdot \sin(n \cdot \tau)}{N^2 - n^2}$$

$$\phi(\tau) = \sum_{n \neq N} \frac{A_n \cdot \cos(n \cdot \tau) + B_n \cdot \sin(n \cdot \tau)}{N^2 - n^2}; \; X_0(\tau) = a_0 \cdot \cos(N \cdot \tau) + b_0 \cdot \sin(N \cdot \tau) + \phi(\tau).$$

Parameters a_0, b_0 are constants to be determined. We require X_0 has $2 \cdot \pi$ periodic solutions. This is equivalent to requiring that its right side has no Fourier term of order N, since such a term would lead to resonance. We required therefore that the below equations will exist [52–54]:

$$\beta \cdot a_0 = -\frac{1}{\pi} \cdot \int_0^{2\cdot\pi} h\Bigg(a_0 \cdot \cos(N \cdot \tau) + b_0 \cdot \sin(N \cdot \tau) + \phi(\tau), -a_0 \cdot N \cdot \sin(N \cdot \tau). + b_0 \cdot N \cdot \cos(N \cdot \tau) + \frac{d\phi(\tau)}{d\tau}\Bigg) \cdot \cos(N \cdot \tau) \cdot d\tau + A$$

$$\beta \cdot b_0 = -\frac{1}{\pi} \cdot \int_0^{2\cdot\pi} h\Bigg(a_0 \cdot \cos(N \cdot \tau) + b_0 \cdot \sin(N \cdot \tau) + \phi(\tau), -a_0 \cdot N \cdot \sin(N \cdot \tau). + b_0 \cdot N \cdot \cos(N \cdot \tau) + \frac{d\phi(\tau)}{d\tau}\Bigg) \cdot \sin(N \cdot \tau) \cdot d\tau + B .$$

Which constitute two equations for the unknowns? a_0, b_0 The resulting equations are the same as those for the first-order approximation, with $N = 1$. Each equation in the sequence has solutions containing constants $a_1, b_1, a_2, b_2, \ldots$

Whose values are similarly established at the succeeding step? The equations for subsequent constants are linear. The pairs $\beta \cdot a_0 = \ldots$ and $\beta \cdot b_0 = \ldots$ are the only ones which may have several solutions. We have a forced van der Pol circuit. We consider forced van der Pol circuit with forcing source $F \cdot \cos(\omega \cdot t)$. The active element of the circuit is semiconductor device (OptoNDR circuit/device). It acts like an ordinary resistor when current $I(t)$ is high ($I(t) > I_{\text{sat}}$), but like negative resistor (energy source) or negative differential resistance (NDR) when $I(t)$ is low $I_{\text{sat}} > I(t) > I_{\text{break}}$. Our circuit current–voltage characteristic $F \cdot \cos(\omega \cdot t) + V = f(I) \,\forall\, \frac{\mathrm{d}I}{\mathrm{d}t} = 0$ resembles a forced cubic function. Additionally, we consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below $I_{\text{break}}(I(t) < I_{\text{break}})$. Our OptoNDR element/circuit is constructed from LED and phototransistor in series. The LED (D_1) light strikes the phototransistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant ($I_{BQ_1} = I_{\text{LED}} \cdot k = I_{CQ_1} \cdot k$) and is the phototransistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations. We need to implement the regular Ebers–Moll model to the optocoupler circuit (transistor Q_1 and LED D_1) and get a complete final expression for the negative differential resistance (NDR) characteristics of that circuit (see Sect. 6.2) (Fig. 8.3).

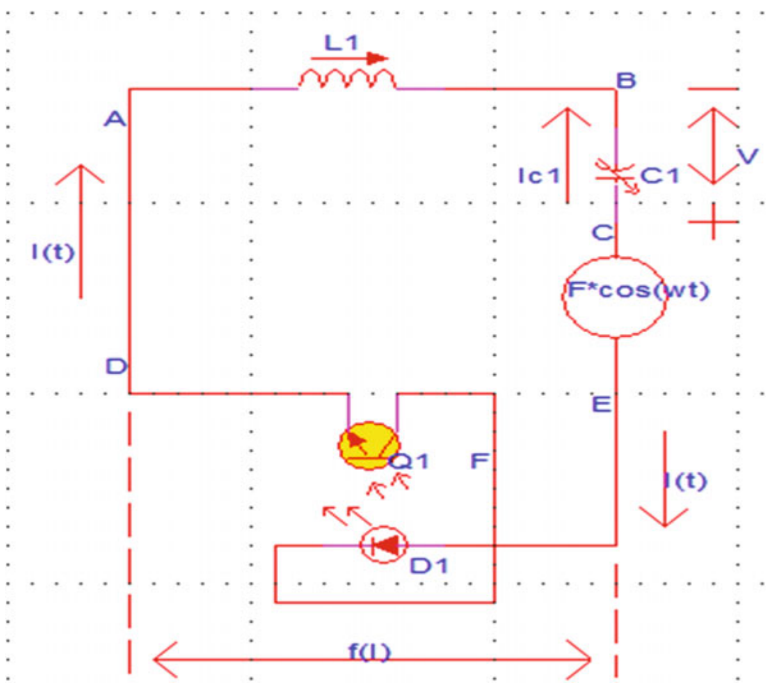

Fig. 8.3 OptoNDR forced van der Pol circuit

$$f(I) = V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right]$$

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t \cdot \frac{1}{I + (1+k) \cdot I_0} + V_t \cdot \left\{\frac{(1 - \alpha_r \cdot \alpha_f) \cdot \left\{\left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] \cdot I_{sc} - \left[\alpha_f - \frac{1}{(1+k)}\right] \cdot I_{se}\right\}}{\left\{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)\right\} \cdot \left\{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\right\}}\right\}.$$

We get the conditions in NDR region: $I \neq -(1+k) \cdot I_0$

$$I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f) \neq 0 \Rightarrow I \neq -\frac{I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{\left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right]}$$

$$I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f) \neq 0 \Rightarrow I \neq -\frac{I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}{\left[\alpha_f - \frac{1}{(1+k)}\right]}.$$

We can demonstrate the $\frac{\mathrm{d}f(I)}{\mathrm{d}I}$ equation as a parametric function with some constant. Let us define the constants first (see Sect. 6.2).

$$\Gamma_1 = (1+k) \cdot I_0;$$
$$\Gamma_2 = (1 - \alpha_r \cdot \alpha_f) \cdot \left\{\left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] \cdot I_{sc} - \left[\alpha_f - \frac{1}{(1+k)}\right] \cdot I_{se}\right\}$$
$$\Gamma_3 = \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right];$$
$$\Gamma_4 = I_{se} \cdot (1 - \alpha_r \cdot \alpha_f);\ \Gamma_5 = \left[\alpha_f - \frac{1}{(1+k)}\right];\ \Gamma_6 = I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)$$

$$\frac{\mathrm{d}f(I)}{\mathrm{d}I} = V_t \cdot \frac{1}{I + \Gamma_1} + V_t \cdot \left\{\frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}}\right\};\ \Gamma_1, \Gamma_2, \ldots, \Gamma_6 \in \mathbb{R}.$$

We need to analyze the above equation for regions which are near the saturation region and cut-off region. For the region which is after the breakover voltage but near enough to the cut-off region:

$$Q_{1(\text{cutoff})}(\Delta f(I) \to \varepsilon \ll 1,\ \Delta I \to 0) \Rightarrow \text{NDR} = \frac{\Delta f(I)}{\Delta I} \to \infty.$$

For the region which is near and in the phototransistor saturation state:

$$Q_{1(\text{saturation})}(\Delta f(I) \to \varepsilon \ll 1,\ \Delta I \to \infty) \Rightarrow \text{NDR} = \frac{\Delta f(I)}{\Delta I} \to 0.$$

For the cut-off region before the breakover $k = 0$ then we get the expression $\frac{\mathrm{d}f(I)}{\mathrm{d}I}$.

$$\begin{aligned}\frac{\mathrm{d}f(I)}{\mathrm{d}I}\Big|_{k=0} &= V_t \cdot \frac{1}{I + I_0} + V_t \\ &\cdot \left\{ \frac{(1 - \alpha_r \cdot \alpha_f) \cdot \{[1 - \alpha_r] \cdot I_{sc} - [\alpha_f - 1] \cdot I_{se}\}}{\{I \cdot [1 - \alpha_r] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)\} \cdot \{I \cdot [\alpha_f - 1] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)\}} \right\}.\end{aligned}$$

Back to our circuit van der Pol differential equations:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{I}{C_1}; \quad f(I) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - V = F \cdot \cos(\omega \cdot t)$$

$$\begin{aligned}&V_t \cdot \ln\left[\frac{I \cdot \left[1 - \alpha_r \cdot \frac{1}{(1+k)}\right] + I_{se} \cdot (1 - \alpha_r \cdot \alpha_f)}{I \cdot \left[\alpha_f - \frac{1}{(1+k)}\right] + I_{sc} \cdot (1 - \alpha_r \cdot \alpha_f)}\right] + V_t \cdot \ln\left[\frac{I}{(1+k) \cdot I_0} + 1\right] \\ &\quad + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - V = F \cdot \cos(\omega \cdot t)\end{aligned}.$$

We can investigate our OptoNDR van der Pol as a weakly nonlinear oscillator and analyze it using perturbation method [5, 6].

$$f(I) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - V = F \cdot \cos(\omega \cdot t) \Rightarrow V = f(I) + L_1 \cdot \frac{\mathrm{d}I}{\mathrm{d}t} - F \cdot \cos(\omega \cdot t)$$

$$\begin{aligned}&\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{I}{C_1}; \quad \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}f(I)}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + F \cdot \omega \cdot \sin(\omega \cdot t); \\ &-\frac{I}{C_1} = \frac{\mathrm{d}f(I)}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + F \cdot \omega \cdot \sin(\omega \cdot t)\end{aligned}$$

$$-\frac{I}{C_1} = \frac{\mathrm{d}f(I)}{\mathrm{d}I} \cdot \frac{\mathrm{d}I}{\mathrm{d}t} + L_1 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + F \cdot \omega \cdot \sin(\omega \cdot t)$$

$$-\frac{I}{C_1} = \left[V_t \cdot \frac{1}{I+\Gamma_1} + V_t \cdot \left\{\frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}}\right\}\right] \cdot \frac{dI}{dt} + L_1 \cdot \frac{d^2 I}{dt^2} + F \cdot \omega \cdot \sin(\omega \cdot t)$$

$$\frac{d^2 I}{dt^2} = -\frac{I}{L_1 \cdot C_1} - \frac{V_t}{L_1} \cdot \left[\frac{1}{I+\Gamma_1} + \left\{\frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}}\right\}\right] \cdot \frac{dI}{dt} - F \cdot \frac{1}{L_1}\omega \cdot \sin(\omega \cdot t).$$

We define $g\left(I, \frac{dI}{dt}, t\right) = -\left[\frac{1}{I+\Gamma_1} + \left\{\frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}}\right\}\right] \cdot \frac{dI}{dt}$; $\varepsilon = \frac{V_t}{L_1}$.

The OptoNDR forced van der Pol weakly nonlinear oscillator can be presented by the differential equation: $0 < \varepsilon \ll 1$

$$\frac{d^2 I}{dt^2} = -\frac{I}{L_1 \cdot C_1} + \frac{V_t}{L_1} \cdot g\left(I, \frac{dI}{dt}, t\right) - F \cdot \frac{1}{L_1}\omega \cdot \sin(\omega \cdot t);$$
$$\omega_0^2 = \frac{1}{L_1 \cdot C_1};\ \omega_0 = \frac{1}{\sqrt{L_1 \cdot C_1}}.$$

By considering ω is close to the natural frequency, we set $\omega = 1$ and then define $\tau = \omega \cdot t$. If $\omega = 1$ then $\tau = t$ and for $\tau = \omega \cdot t$ we get the following forced van der Pol equation ($\tau = \omega \cdot t \Rightarrow t = \frac{\tau}{\omega}$):

$$\omega^2 \cdot \frac{d^2 I}{d\tau^2} = -I \cdot \omega_0^2 + \varepsilon \cdot g\left(I, \omega \cdot \frac{dI}{d\tau}, \frac{\tau}{\omega}\right) - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

$$g\left(I, \frac{dI}{dt}, t\right) = h(I) \cdot \frac{dI}{dt};\quad h(I) = -\left[\frac{1}{I+\Gamma_1} + \left\{\frac{\Gamma_2}{\{I \cdot \Gamma_3 + \Gamma_4\} \cdot \{I \cdot \Gamma_5 + \Gamma_6\}}\right\}\right]$$

$$\omega^2 \cdot \frac{d^2 I}{d\tau^2} = -I \cdot \omega_0^2 + \varepsilon \cdot h(I) \cdot \frac{dI}{dt} - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

$$\omega^2 \cdot \frac{d^2 I}{d\tau^2} = -I \cdot \omega_0^2 + \omega \cdot \varepsilon \cdot h(I) \cdot \frac{dI}{d\tau} - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau).$$

First we consider hard excitation, for from resonance and assume that ω is not close to an integer. Our circuit system second-order differential equation:

$\omega^2 \cdot \frac{d^2 I}{d\tau^2} = -I \cdot \omega_0^2 + \omega \cdot \varepsilon \cdot h(I) \cdot \frac{dI}{d\tau} - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$. Let $I(\varepsilon, \tau) = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(\tau)$ and the first derivative $\frac{dI(\varepsilon,\tau)}{d\tau} = \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{dI_k(\tau)}{d\tau}$; $\frac{d^2 I(\varepsilon,\tau)}{d\tau^2} = \sum_{k=0}^{\infty} \varepsilon^k \cdot \frac{d^2 I_k(\tau)}{d\tau^2}$ then we can define $\frac{dI(\varepsilon,\tau)}{d\tau} = \frac{dI_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dI_1(\tau)}{d\tau} + \cdots$; $\frac{d^2 I(\varepsilon,\tau)}{d\tau^2} = \frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2} + \cdots$ (omitting $O(\varepsilon^2)$ and $O(\varepsilon^n)\,\forall\, n > 2$ elements) and we get $(I(\varepsilon, \tau) = I_0(\tau) + \varepsilon \cdot I_1(\tau))$:

$$\omega^2 \cdot \left[\frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + \varepsilon \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2}\right] = -[I_0(\tau) + \varepsilon \cdot I_1(\tau)] \cdot \omega_0^2 + \omega \cdot \varepsilon \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) \cdot \left[\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} + \varepsilon \cdot \frac{\mathrm{d}I_1(\tau)}{\mathrm{d}\tau}\right] - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + \varepsilon \cdot \omega^2 \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2} = -I_0(\tau) \cdot \omega_0^2 - \varepsilon \cdot I_1(\tau) \cdot \omega_0^2 + \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} \cdot \omega \cdot \varepsilon \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) + \varepsilon^2 \cdot \frac{\mathrm{d}I_1(\tau)}{\mathrm{d}\tau} \cdot \omega \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) \cdot - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

We omit from the above differential equation $O(\varepsilon^2)$ element:

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + \varepsilon \cdot \omega^2 \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2} = -I_0(\tau) \cdot \omega_0^2 - \varepsilon \cdot I_1(\tau) \cdot \omega_0^2 + \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} \cdot \omega \cdot \varepsilon \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) - F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + I_0(\tau) \cdot \omega_0^2 + \varepsilon \cdot \left[\omega^2 \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2} + I_1(\tau) \cdot \omega_0^2 - \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} \cdot \omega \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau))\right] = -F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau).$$

Since $\varepsilon \neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε (except ε^0) should be equal to zero.

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + I_0(\tau) \cdot \omega_0^2 = -F \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2} + I_1(\tau) \cdot \omega_0^2 - \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} \cdot \omega \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) = 0.$$

Remark It is a reader exercise to solve the above two differential equations and then to find the expressions for $I_0(\tau)$ and $I_1(\tau)$, then to get the solution $I(\varepsilon, \tau) = I_0(\tau) + \varepsilon \cdot I_1(\tau)$.

Second, we consider soft excitation, far from resonance. This case is similar to hard excitation above but with $F = \varepsilon \cdot F_0$ and this solution is unstable.

$$\omega^2 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}\tau^2} = -I \cdot \omega_0^2 + \omega \cdot \varepsilon \cdot h(I) \cdot \frac{\mathrm{d}I}{\mathrm{d}\tau} - \varepsilon \cdot F_0 \cdot \frac{1}{L_1}\omega \cdot \sin(\tau)$$

Let $I(\varepsilon,\tau)=\sum_{k=0}^{\infty}\varepsilon^k\cdot I_k(\tau)$ and the first derivative $\frac{\mathrm{d}I(\varepsilon,\tau)}{\mathrm{d}\tau}=\sum_{k=0}^{\infty}\varepsilon^k\cdot\frac{\mathrm{d}I_k(\tau)}{\mathrm{d}\tau}$; $\frac{\mathrm{d}^2I(\varepsilon,\tau)}{\mathrm{d}\tau^2}=\sum_{k=0}^{\infty}\varepsilon^k\cdot\frac{\mathrm{d}^2I_k(\tau)}{\mathrm{d}\tau^2}$, and then we can define $\frac{\mathrm{d}I(\varepsilon,\tau)}{\mathrm{d}\tau}=\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau}+\varepsilon\cdot\frac{\mathrm{d}I_1(\tau)}{\mathrm{d}\tau}+\cdots$; $\frac{\mathrm{d}^2I(\varepsilon,\tau)}{\mathrm{d}\tau^2}=\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+\varepsilon\cdot\frac{\mathrm{d}^2I_1(\tau)}{\mathrm{d}\tau^2}+\cdots$. (omitting $O(\varepsilon^2)$ and $O(\varepsilon^n)\,\forall\,n>2$ elements) and we get ($I(\varepsilon,\tau)=I_0(\tau)+\varepsilon\cdot I_1(\tau)$):

$$\omega^2\cdot\left[\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+\varepsilon\cdot\frac{\mathrm{d}^2I_1(\tau)}{\mathrm{d}\tau^2}\right]=-[I_0(\tau)+\varepsilon\cdot I_1(\tau)]\cdot\omega_0^2+\omega\cdot\varepsilon\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau))\cdot\left[\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau}+\varepsilon\cdot\frac{\mathrm{d}I_1(\tau)}{\mathrm{d}\tau}\right]-\varepsilon\cdot F_0\cdot\frac{1}{L_1}\omega\cdot\sin(\tau)$$

$$\omega^2\cdot\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+\varepsilon\cdot\omega^2\cdot\frac{\mathrm{d}^2I_1(\tau)}{\mathrm{d}\tau^2}=-I_0(\tau)\cdot\omega_0^2-\varepsilon\cdot I_1(\tau)\cdot\omega_0^2+\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau}\cdot\omega\cdot\varepsilon\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau))+\varepsilon^2\cdot\frac{\mathrm{d}I_1(\tau)}{\mathrm{d}\tau}\cdot\omega\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau))-F\cdot\frac{1}{L_1}\omega\cdot\sin(\tau)\,.$$

We omit from the above differential equation $O(\varepsilon^2)$ element.

$$\omega^2\cdot\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+\omega^2\cdot\varepsilon\cdot\frac{\mathrm{d}^2I_1(\tau)}{\mathrm{d}\tau^2}=-I_0(\tau)\cdot\omega_0^2-\varepsilon\cdot I_1(\tau)\cdot\omega_0^2+\omega\cdot\varepsilon\cdot\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau}\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau))-\varepsilon\cdot F_0\cdot\frac{1}{L_1}\omega\cdot\sin(\tau)$$

$$\omega^2\cdot\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+I_0(\tau)\cdot\omega_0^2+\varepsilon\cdot\left[\omega^2\cdot\frac{\mathrm{d}^2I_1(\tau)}{\mathrm{d}\tau^2}+I_1(\tau)\cdot\omega_0^2-\omega\cdot\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau}\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau))+F_0\cdot\frac{1}{L_1}\omega\cdot\sin(\tau)\right]=0\,.$$

Since $\varepsilon\neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero.

$$\omega^2\cdot\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+I_0(\tau)\cdot\omega_0^2=0\Rightarrow\left(\frac{\omega}{\omega_0}\right)^2\cdot\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+I_0(\tau)=0$$

$$\omega^2\cdot\frac{\mathrm{d}^2I_1(\tau)}{\mathrm{d}\tau^2}+I_1(\tau)\cdot\omega_0^2-\omega\cdot\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau}\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau))+F_0\cdot\frac{1}{L_1}\omega\cdot\sin(\tau)=0.$$

First, we investigate the $(\frac{\omega}{\omega_0})^2\cdot\frac{\mathrm{d}^2I_0(\tau)}{\mathrm{d}\tau^2}+I_0(\tau)=0$ differential equation. We can assume that the general solution for $I_0(\tau)$ is of the form $\sin(\frac{\omega_0}{\omega}\cdot\tau)+\cos(\frac{\omega_0}{\omega}\cdot\tau)$ and we double check for solution: $I_0(\tau)=\sin(\frac{\omega_0}{\omega}\cdot\tau)+\cos(\frac{\omega_0}{\omega}\cdot\tau)$

$$\frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} = \frac{\omega_0}{\omega} \cdot \cos\left(\frac{\omega_0}{\omega} \cdot \tau\right) - \frac{\omega_0}{\omega} \cdot \sin\left(\frac{\omega_0}{\omega} \cdot \tau\right); \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} = -\left(\frac{\omega_0}{\omega}\right)^2 \cdot \left[\sin(\frac{\omega_0}{\omega} \cdot \tau) + \cos\left(\frac{\omega_0}{\omega} \cdot \tau\right)\right],$$

then $(\frac{\omega}{\omega_0})^2 \cdot \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + I_0(\tau) = 0$. We can write our solution to $(\frac{\omega}{\omega_0})^2 \cdot \frac{\mathrm{d}^2 I_0(\tau)}{\mathrm{d}\tau^2} + I_0(\tau) = 0$ in a convenient form: $I_0(\tau) = A_1 \cdot \sin(\frac{\omega_0}{\omega} \cdot \tau) + A_2 \cdot \cos(\frac{\omega_0}{\omega} \cdot \tau)$ where A_1, A_2 are constants to be determined. If we utilized the initial conditions: $I_0(\tau = 0) = 0$, $\frac{\mathrm{d}I_0(\tau=0)}{\mathrm{d}\tau} = \Omega$. Given to find the values of the constants A_1, A_2, we have by substituting these values in equation: $I_0(\tau) = A_1 \cdot \sin(\frac{\omega_0}{\omega} \cdot \tau) + A_2 \cdot \cos(\frac{\omega_0}{\omega} \cdot \tau)$ that $A_2 = 0$ hence

$$I_0(\tau) = A_1 \cdot \sin(\frac{\omega_0}{\omega} \cdot \tau); \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} = A_1 \cdot \frac{\omega_0}{\omega} \cdot \cos(\frac{\omega_0}{\omega} \cdot \tau); \frac{\mathrm{d}I_0(\tau = 0)}{\mathrm{d}\tau} = \Omega = A_1 \cdot \frac{\omega_0}{\omega}$$

$A_1 = \Omega \cdot \frac{\omega}{\omega_0}$ then $I_0(\tau) = \Omega \cdot \frac{\omega}{\omega_0} \cdot \sin\left(\frac{\omega_0}{\omega} \cdot \tau\right)$. The second differential equation is

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2} + I_1(\tau) \cdot \omega_0^2 - \omega \cdot \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} \cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) + F_0 \cdot \frac{1}{L_1} \omega \cdot \sin(\tau) = 0$$

$$I_0(\tau) = \Omega \cdot \frac{\omega}{\omega_0} \cdot \sin\left(\frac{\omega_0}{\omega} \cdot \tau\right); \frac{\mathrm{d}I_0(\tau)}{\mathrm{d}\tau} = \Omega \cdot \cos\left(\frac{\omega_0}{\omega} \cdot \tau\right)$$

$$\omega^2 \cdot \frac{\mathrm{d}^2 I_1(\tau)}{\mathrm{d}\tau^2} + I_1(\tau) \cdot \omega_0^2 - \omega \cdot \Omega \cdot \cos\left(\frac{\omega_0}{\omega} \cdot \tau\right) \cdot h\left(\Omega \cdot \frac{\omega}{\omega_0} \cdot \sin\left(\frac{\omega_0}{\omega} \cdot \tau\right) + \varepsilon \cdot I_1(\tau)\right) + F_0 \cdot \frac{1}{L_1} \omega \cdot \sin(\tau) = 0.$$

Remark It is a reader exercise to solve the above differential equation.

Third, we consider soft excitation, near resonance. In this case we define $F = \varepsilon \cdot \gamma$, and for near resonance $\omega = 1 + \varepsilon \cdot \omega_1$ [8, 9]. The expansion is assumed to be $I(\varepsilon, \tau) = \sum_{k=0}^{\infty} \varepsilon^k \cdot I_k(\tau)$; $I(\varepsilon, \tau) = I_0(\tau) + \varepsilon \cdot I_1(\tau) + \cdots$ and get our circuit system van der Pol differential equation:

$$\omega^2 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}\tau^2} = -I \cdot \omega_0^2 + \omega \cdot \varepsilon \cdot h(I) \cdot \frac{\mathrm{d}I}{\mathrm{d}\tau} - F \cdot \frac{1}{L_1} \omega \cdot \sin(\tau)$$

$$(1 + \varepsilon \cdot \omega_1)^2 \cdot \frac{\mathrm{d}^2 I}{\mathrm{d}\tau^2} = -I \cdot \omega_0^2 + (1 + \varepsilon \cdot \omega_1) \cdot \varepsilon \cdot h(I) \cdot \frac{\mathrm{d}I}{\mathrm{d}\tau} - \varepsilon \cdot \gamma \cdot \frac{1}{L_1}(1 + \varepsilon \cdot \omega_1) \cdot \sin(\tau).$$

We concern the expansion $I(\varepsilon,\tau) = I_0(\tau) + \varepsilon \cdot I_1(\tau)$ and omit $O(\varepsilon^2)$ and $O(\varepsilon^n)\,\forall\, n > 2$ elements $\frac{dI(\varepsilon,\tau)}{d\tau} = \frac{dI_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dI_1(\tau)}{d\tau}; \frac{d^2 I(\varepsilon,\tau)}{d\tau^2} = \frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2}$.

$$(1+\varepsilon\cdot\omega_1)^2 \cdot \left[\frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2}\right] = -(I_0(\tau) + \varepsilon \cdot I_1(\tau)) \cdot \omega_0^2 + (1+\varepsilon\cdot\omega_1)\cdot\varepsilon\cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) \cdot \left[\frac{dI_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dI_1(\tau)}{d\tau}\right] - \varepsilon\cdot\gamma\cdot\frac{1}{L_1}(1+\varepsilon\cdot\omega_1)\cdot\sin(\tau)$$

$$(1+\varepsilon\cdot\omega_1)^2 = 1 + 2\cdot\varepsilon\cdot\omega_1 + \varepsilon^2\cdot\omega_1^2;\ (1+\varepsilon\cdot\omega_1)^2 = 1 + 2\cdot\varepsilon\cdot\omega_1 + O(\varepsilon^2).$$

We omit $O(\varepsilon^2)$ element from the above expression $(1+\varepsilon\cdot\omega_1)^2 = 1 + 2\cdot\varepsilon\cdot\omega_1$

$$(1+2\cdot\varepsilon\cdot\omega_1) \cdot \left[\frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2}\right] = -(I_0(\tau) + \varepsilon \cdot I_1(\tau)) \cdot \omega_0^2 + (1+\varepsilon\cdot\omega_1)\cdot\varepsilon\cdot h(I_0(\tau) + \varepsilon \cdot I_1(\tau)) \cdot \left[\frac{dI_0(\tau)}{d\tau} + \varepsilon \cdot \frac{dI_1(\tau)}{d\tau}\right] - \varepsilon\cdot\gamma\cdot\frac{1}{L_1}(1+\varepsilon\cdot\omega_1)\cdot\sin(\tau)$$

$$\left[\frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2}\right] + 2\cdot\varepsilon\cdot\omega_1\cdot\frac{d^2 I_0(\tau)}{d\tau^2} + 2\cdot\omega_1\cdot\varepsilon^2\cdot\frac{d^2 I_1(\tau)}{d\tau^2} = -I_0(\tau)\cdot\omega_0^2 - \varepsilon\cdot I_1(\tau)\cdot\omega_0^2 + (\varepsilon + \varepsilon^2\cdot\omega_1)\cdot h(I_0(\tau) + \varepsilon\cdot I_1(\tau))\cdot\left[\frac{dI_0(\tau)}{d\tau} + \varepsilon\cdot\frac{dI_1(\tau)}{d\tau}\right] - \varepsilon\cdot\gamma\cdot\frac{1}{L_1}(1+\varepsilon\cdot\omega_1)\cdot\sin(\tau).$$

We omit $O(\varepsilon^2)$ element from the above expression.

$$\frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2} + 2\cdot\varepsilon\cdot\omega_1\cdot\frac{d^2 I_0(\tau)}{d\tau^2} = -I_0(\tau)\cdot\omega_0^2 - \varepsilon\cdot I_1(\tau)\cdot\omega_0^2 + \varepsilon\cdot h(I_0(\tau) + \varepsilon\cdot I_1(\tau))\cdot\left[\frac{dI_0(\tau)}{d\tau} + \varepsilon\cdot\frac{dI_1(\tau)}{d\tau}\right] - \varepsilon\cdot\gamma\cdot\frac{1}{L_1}(1+\varepsilon\cdot\omega_1)\cdot\sin(\tau)$$

$$\frac{d^2 I_0(\tau)}{d\tau^2} + I_0(\tau)\cdot\omega_0^2 + \varepsilon \cdot \frac{d^2 I_1(\tau)}{d\tau^2} + 2\cdot\varepsilon\cdot\omega_1\cdot\frac{d^2 I_0(\tau)}{d\tau^2} + \varepsilon\cdot I_1(\tau)\cdot\omega_0^2 = h(I_0(\tau) + \varepsilon\cdot I_1(\tau))\cdot\left[\varepsilon\cdot\frac{dI_0(\tau)}{d\tau} + \varepsilon^2\cdot\frac{dI_1(\tau)}{d\tau}\right] - \gamma\cdot\frac{1}{L_1}(\varepsilon + \varepsilon^2\cdot\omega_1)\cdot\sin(\tau).$$

We omit $O(\varepsilon^2)$ element from the above expression.

$$\frac{d^2I_0(\tau)}{d\tau^2} + I_0(\tau)\cdot\omega_0^2 + \varepsilon\cdot\left[\frac{d^2I_1(\tau)}{d\tau^2} + 2\cdot\omega_1\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_1(\tau)\cdot\omega_0^2 - \frac{dI_0(\tau)}{d\tau}\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau)) + \gamma\cdot\frac{1}{L_1}\cdot\sin(\tau)\right] = 0$$

Since $\varepsilon \neq 0$, for the right-hand side of above differential equation to be equal to the left-hand side, the coefficient of each power of ε should be equal to zero.

$$\frac{d^2I_0(\tau)}{d\tau^2} + I_0(\tau)\cdot\omega_0^2 = 0 \Rightarrow \frac{1}{\omega_0^2}\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_0(\tau) = 0$$

$$\frac{d^2I_1(\tau)}{d\tau^2} + 2\cdot\omega_1\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_1(\tau)\cdot\omega_0^2 - \frac{dI_0(\tau)}{d\tau}\cdot h(I_0(\tau)+\varepsilon\cdot I_1(\tau)) + \gamma\cdot\frac{1}{L_1}\cdot\sin(\tau) = 0.$$

First, we investigate $\frac{1}{\omega_0^2}\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_0(\tau) = 0$ differential equation. Let us assume that $I_0(\tau) = e^{r\cdot\tau}$ is a solution to equation where r is a constant. The auxiliary equations are $\frac{d^2I_0(\tau)}{d\tau^2} = r^2\cdot e^{r\cdot\tau} \Rightarrow \frac{1}{\omega_0^2}\cdot r^2\cdot e^{r\cdot\tau} + e^{r\cdot\tau} = 0$; $\left[\frac{1}{\omega_0^2}\cdot r^2 + 1\right]\cdot e^{r\cdot\tau} = 0$ $\frac{1}{\omega_0^2}\cdot r^2 + 1 = 0$; $r = \sqrt{-\omega_0^2} = \pm j\cdot\omega_0$. Here the r real part is equal to zero and the imaginary part is equal to ω_0. If we assumed from above that the general solution for $I_0(\tau)$ is of the form $\sin(\omega_0\cdot\tau) + \cos(\omega_0\cdot\tau)$ and we double check for solution:

$$I_0(\tau) = \sin(\omega_0\cdot\tau) + \cos(\omega_0\cdot\tau);\ \frac{dI_0(\tau)}{d\tau} = \omega_0\cdot\cos(\omega_0\cdot\tau) - \omega_0\cdot\sin(\omega_0\cdot\tau)$$

$\frac{d^2I_0(\tau)}{d\tau^2} = -\omega_0^2\cdot[\sin(\omega_0\cdot\tau) + \cos(\omega_0\cdot\tau)]$ then $\frac{1}{\omega_0^2}\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_0(\tau) = 0$ [5–7]. We can write our solution to $\frac{1}{\omega_0^2}\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_0(\tau) = 0$ in the convenient form: $I_0(\tau) = A_1\cdot\sin(\omega_0\cdot\tau) + A_2\cdot\cos(\omega_0\cdot\tau)$ where A_1, A_2 are constants to be determined. If we utilized the initial conditions, $I_0(\tau = 0) = 0$; $\frac{dI_0(\tau=0)}{d\tau} = \Omega_1$. Given to find the values of the constants A_1, A_2, we have by substituting these values in equation: $I_0(\tau) = A_1\cdot\sin(\omega_0\cdot\tau) + A_2\cdot\cos(\omega_0\cdot\tau)$ that $A_2 = 0$ hence $I_0(\tau) = A_1\cdot\sin(\omega_0\cdot\tau)$ $\frac{dI_0(\tau)}{d\tau} = A_1\cdot\omega_0\cdot\cos(\omega_0\cdot\tau)$; $\frac{dI_0(\tau=0)}{d\tau} = A_1\cdot\omega_0 = \Omega_1 \Rightarrow A_1 = \frac{\Omega_1}{\omega_0}$ then $I_0(\tau) = \frac{\Omega_1}{\omega_0}\cdot\sin(\omega_0\cdot\tau)$. The second differential equation is as follows:

$$\frac{d^2I_1(\tau)}{d\tau^2} + 2\cdot\omega_1\cdot\frac{d^2I_0(\tau)}{d\tau^2} + I_1(\tau)\cdot\omega_0^2 - \frac{dI_0(\tau)}{d\tau}\cdot h(I_0(\tau) + \varepsilon\cdot I_1(\tau)) + \gamma\cdot\frac{1}{L_1}\cdot\sin(\tau) = 0$$

$$\frac{dI_0(\tau)}{d\tau} = \Omega_1 \cdot \cos(\omega_0 \cdot \tau); \frac{d^2 I_0(\tau)}{d\tau^2} = -\Omega_1 \cdot \omega_0 \cdot \sin(\omega_0 \cdot \tau)$$

$$\frac{d^2 I_1(\tau)}{d\tau^2} - 2 \cdot \omega_1 \cdot \Omega_1 \cdot \omega_0 \cdot \sin(\omega_0 \cdot \tau) + I_1(\tau) \cdot \omega_0^2$$
$$- \Omega_1 \cdot \cos(\omega_0 \cdot \tau) \cdot h\left(\frac{\Omega_1}{\omega_0} \cdot \sin(\omega_0 \cdot \tau\right) + \varepsilon \cdot I_1(\tau)) + \gamma \cdot \frac{1}{L_1} \cdot \sin(\tau) = 0$$

Remark It is a reader exercise to solve the above differential equation.

8.5 Exercises

1. We have system that can be characterization by the second-order differential equation: $\frac{d^2X}{dt^2} + X \cdot \Omega_1 + \varepsilon \cdot \Omega_2 \cdot \frac{dX}{dt} = 0$ with $X(t=0) = \Gamma_1$ $\frac{dX(t=0)}{dt} = \Gamma_2$; $\Gamma_1, \Gamma_2 \in \mathbb{R}$.

   1.1 Obtain the exact solution of the system second-order differential equation. How the exact solution is dependent on parameters Ω_1 and Ω_2 $(\Omega_1, \Omega_2 \in \mathbb{R})$.

   1.2 Using regular perturbation theory, find X_0, X_1 and X_2 in the series expansion $X(t,\varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + O(\varepsilon^3)$.

   1.3 Does the perturbation solution contain secular terms? How they are dependent on parameters Γ_1, Γ_2?

   1.4 We set the parameters Ω_1, Ω_2 as $\Omega_2 = \sqrt{1 + \Omega_1 \cdot \gamma}$; $\Omega = \Omega_1$; $\Omega, \gamma \in \mathbb{R}$. How the behavior of the system is changed for different values of Ω, γ? Draw the system series expansion terms (X_0, X_1, X_2) functions as a function of parameters Ω, γ.

   1.5 Try to implement the system second-order differential equation by using optoisolation elements and discrete components.

2. We have weakly nonlinear oscillator which is represented by the differential equation: $\frac{d^2X}{dt^2} + \omega_0^2 \cdot (X+1) = \varepsilon \cdot f\left(X, \frac{dX}{dt}, t\right)$ where f is T periodic it t. We set function $f\left(X, \frac{dX}{dt}, t\right)$ as $f\left(X, \frac{dX}{dt}, t\right) = \frac{dX}{dt} \cdot \sqrt{\Gamma_1 \cdot \gamma + 1}$ where $0 \le \varepsilon \ll 1$. We seek solution for differential equation in the perturbation expansion form $X(t,\varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \cdots$ where it is perturbation series $\left(X(t=0) = 1; \frac{dX(t=0)}{dt} = 0\right)$. The equation becomes soluble when we set $\varepsilon = 0$. We need to investigate the behavior of the system for $\varepsilon \neq 0$.

   2.1 Find the system fixed points and eigenvalues. How system fixed points and eigenvalues are dependent on γ parameter? Plot the graphs of system fixed points as a function of parameter γ.

2.2 Classify our system fixed points (attracting focus, repelling focus, saddle point, attracting spiral, repelling spiral, center, etc.). How our system fixed points classification is dependent on parameter γ?

2.3 Assume that $X(t) = e^{r \cdot t}$ is a solution of $\frac{d^2X}{dt^2} + \omega_0^2 \cdot (X+1) = \varepsilon \cdot f\left(X, \frac{dX}{dt}, t\right)$ where "r" is a constant, substitute $X(t) = e^{r \cdot t}$ into system differential equation and find solution $X(t)$.

2.4 Apply the perturbation theory to solve our system differential Equation $X(t,\varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \cdots$ groups terms according to the power of ε, omitting all terms with coefficients of ε^2 and higher ($\varepsilon \neq 0$). $X(t=0) = \Omega_1$; $\frac{dX(t=0)}{dt} = \Omega_2$.

2.5 Find the expressions for $X_0(t), X_1(t), X(t,\varepsilon)$ when $X(t,\varepsilon) = \sum_{k=0}^{\infty} X_k(t) \cdot \varepsilon^k = X_0(t) + \varepsilon \cdot X_1(t) + O(\varepsilon^2) + \cdots$

3. We have system that is characterized by differential equation $\frac{d^2X}{dt^2} + \Gamma_1 \cdot X + \varepsilon \cdot h\left(X, \frac{dX}{dt}\right) = 0$ with $0 < \varepsilon \ll 1$. Function $h\left(X, \frac{dX}{dt}\right)$ is $h\left(X, \frac{dX}{dt}\right) = (||X| - 1|) \cdot \left(\frac{dX}{dt}\right)^2$, (Hint: Absolute functions give some possibilities).

3.1 Find system fixed points and eigenvalues. How the system fixed points and eigenvalues are dependent on Γ_1 parameter? Plot the graphs of system fixed points as a function of Γ_1 parameter.

3.2 Classify our system fixed points (attracting focus, repelling focus, saddle point, attracting spiral, repelling spiral, center, etc.).

3.3 Solve the problem exactly and by using regular perturbation theory, find X_0, X_1 and X_2 in the series expansion $X(t,\varepsilon) = \sum_{k=0}^{\infty} X_k(t) \cdot \varepsilon^k$

$$X(t,\varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + O(\varepsilon^3).$$

3.4 Investigate the system if parameter $\Gamma_1 = 0$. Find fixed points and eigenvalues, classify system fixed points.

3.5 For parameter $\Gamma_1 = 0$, use regular perturbation theory and find X_0, X_1 and X_2 in series expansion $X(t,\varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + O(\varepsilon^3)$.

4. Consider system which is characterized by differential equation:

$$\frac{d^2X}{dt^2} \cdot \Gamma_1 + \varepsilon \cdot \Gamma_2 \cdot \left(\frac{dX}{dt}\right)^k + X = 0; \; k \in \mathbb{N}; \; k \geq 3; \; \Gamma_1, \Gamma_2 \in \mathbb{R}$$

4.1 Find system fixed points and eigenvalues for $k = 3$ and $k = 4$. How system fixed points and eigenvalues are dependent on Γ_1 and Γ_2 parameters?

4.2 Classify our system fixed points for $k = 3$ and $k = 4$. Discuss stability.

4.3 Solve the problem exactly (for the cases $k = 3$ and $k = 4$), use regular perturbation theory (for the cases $k = 3$ and $k = 4$), find X_0, X_1 and X_2 in the series expansion: $X(t,\varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + O(\varepsilon^3)$.

4.4 Investigate the system if $\Gamma_1 = 0$ (cases $k = 3$ and $k = 4$), find fixed points and eigenvalues investigate stability. Find $X(t,\varepsilon)$ by using regular perturbation theory.

4.5 Investigate the system if $\Gamma_2 = 0$ (cases $k = 3$ and $k = 4$), find fixed points and eigenvalues investigate stability. Find $X(t,\varepsilon)$ by using regular perturbation theory.

5. Fig. 8.4 van der Pol oscillator circuit with additional two capacitors C_a and C_b. The active element of the circuit is semiconductor device (OptoNDR/Device). It acts like an ordinary resistor when current $I(t)$ is high ($I(t) > I_{sat}$), but like negative resistor (energy source) or negative differential resistance (NDR) when $I(t)$ is low $I(t) > I_{\text{break}}$ and $I(t) < I_{\text{sat}}$. Our circuit current voltage characteristic $V = f(I) \forall \frac{dI}{dt} = 0$ and we consider that a source of current is attached to the

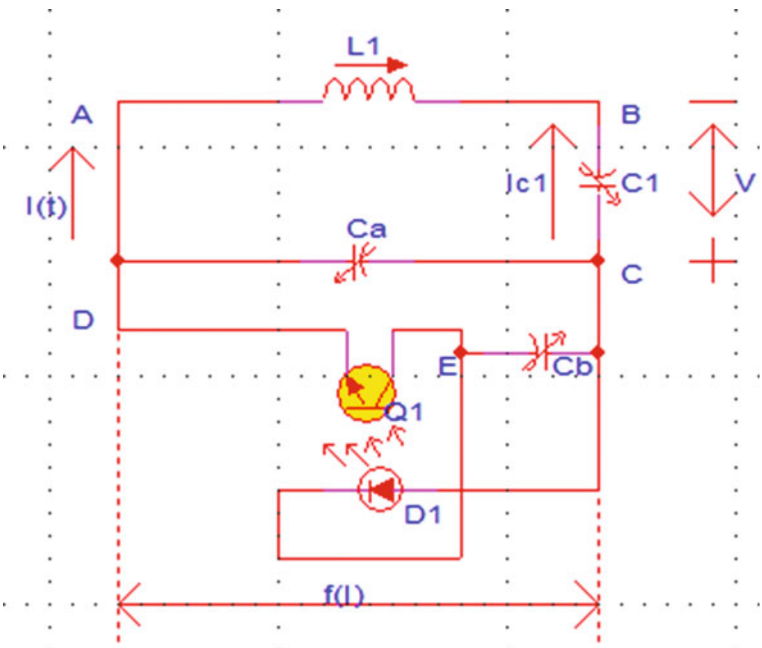

Fig. 8.4 OptoNDR element/circuit with additional two capacitors C_a and C_b

circuit and then withdrawn. We neglect in our analysis the current below $I_{\text{break}}(I(t) < I_{\text{break}})$.
Our OptoNDR element/circuit is constructed from LED and photo transistor in series. The LED (D_1) light strikes the photo transistor (Q_1) base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current, with proportional k constant ($I_{BQ_1} = I_{\text{LED}} \cdot k = I_{CQ_1} \cdot k$) and is the photo transistor base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations.

5.1 Find the mathematical expression for $f(I)$ and demonstrate the $\frac{\mathrm{d}f(I)}{\mathrm{d}I}$ equation as a parametric function with some constants.
5.2 Investigate OptoNDR van der Pol circuit with two additional capacitors (C_a and C_b) by using perturbation method (Hint: omit $O(\varepsilon^2)$ terms and higher terms).
5.3 Find circuit fixed points and eigenvalues. Classify circuit fixed points and investigate stability.
5.4 How the behavior of the circuit is changed if we disconnect capacitor C_b? Investigate the circuit (C_b is disconnected) by using perturbation method (Hint: omit $O(\varepsilon^2)$ terms and higher terms).
5.5 How the behavior of the circuit is changed if we disconnect capacitor C_a? Investigate the circuit (C_a is disconnected) by using perturbation method (Hint: omit $O(\varepsilon^2)$ terms and higher terms).

6. We have system which characterize by the differential equation: $\frac{\mathrm{d}^2X}{\mathrm{d}t^2} + \varepsilon \cdot \Gamma_1 \cdot \operatorname{sgn}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) \cdot \frac{\mathrm{d}X}{\mathrm{d}t} + \Gamma_2 \cdot X = 0;\ \Gamma_1, \Gamma_2 \in \mathbb{R}$. The Γ_1 parameter can be represent system frictional force, where sgn( ) is a Signum function. $\operatorname{sgn}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) = 1\,\forall\,\frac{\mathrm{d}X}{\mathrm{d}t} > 0;\ \operatorname{sgn}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) = -1\,\forall\,\frac{\mathrm{d}X}{\mathrm{d}t} < 0;\ \operatorname{sgn}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) = 0\,\forall\,\frac{\mathrm{d}X}{\mathrm{d}t} = 0$
$0 < \varepsilon \ll 1$. Hint: In your analysis differentiate different values of sgn( ) function cases.

6.1 Analyze the system behavior of the above oscillatory system by using phase plane technique. Discuss stability issue.
6.2 Solve the problem exactly, use regular perturbation theory, find X_0, X_1 and X_2 in the series expansion: $X(t,\varepsilon) = X_0 + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + O(\varepsilon^3)$
6.3 Investigate the system if $\Gamma_2 = 0$. Find fixed points and eigenvalues. Classify system fixed points. Find $X(t,\varepsilon)$ by using regular perturbation theory.
6.4 Solve the problem exactly, use regular perturbation theory, find X_0, X_1, X_2 and X_3 in the series expansion $X(t,\varepsilon) = \sum_{k=0}^{3} X_k \cdot \varepsilon^k + O(\varepsilon^4)$.

6.5 We replace $\text{sgn}\left(\frac{\text{d}X}{\text{d}t}\right)$ function in our system differential equation by $\left[\text{sgn}\left(\frac{\text{d}X}{\text{d}t}\right)\right]^2$ function. How the behavior of the system changed? Solve the problem exactly, use regular perturbation theory, find X_0, X_1 in the series expansion $X(t,\varepsilon) = X_0 + \varepsilon \cdot X_1(t) + O(\varepsilon^2)$.

7. Fig. 8.5 van der Pol oscillator circuit with additional capacitor *Ca* and coupling LED (D_2) photo transistor Q_1. The active element of the circuit is semiconductor device (OptoNDR circuit/Device). It acts like an ordinary resistor when current *I*(*t*) is high ($I(t) > I_{sat}$), but like negative resistor (energy source) or negative differential resistance (NDR) when *I*(*t*) is low $I(t) > I_{\text{break}}$ and $I(t) < I_{\text{sat}}$. Our circuit current–voltage characteristic $V = f(I)\,\forall\,\frac{\text{d}I}{\text{d}t} = 0$ and we consider that a source of current is attached to the circuit and then withdrawn. We neglect in our analysis the current below $I_{\text{break}}(I(t) < I_{\text{break}})$. Our OptoNDR element/circuit is constructed from LED D_1 and LED D_2, and photo transistor Q_1. The LED D_1 and LED D_2 light strikes the photo transistor Q_1 base window and can be represented as a dependent current source. The dependent current source depends on the LED (D_1) forward current and LED (D_2) forward current, with proportional k_1 and k_2 constants

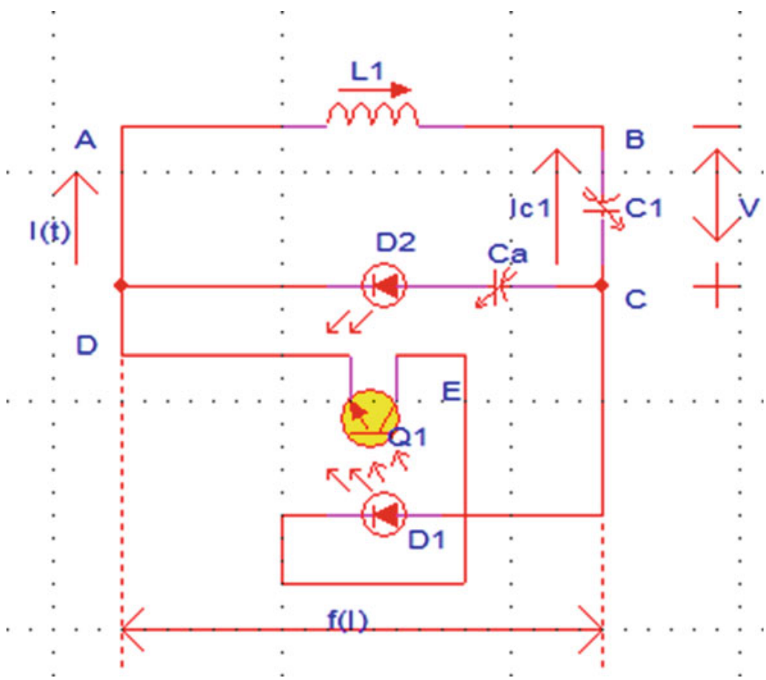

Fig. 8.5 Van der Pol oscillator circuit with additional capacitor C_a and coupling LED (D_2) photo transistor Q_1

($I_{BQ_1} = I_{\text{LED}_1} \cdot k_1 + I_{\text{LED}_2} \cdot k_2$; $I_{CQ_1} = I_{\text{LED}_1}$) and is the photo transistor base current. Constants $k_2 = k$; $k_1 = \sqrt{1 + k \cdot \gamma}$; $k_1, k_2, k, \gamma \in \mathbb{R}_+$. The mathematical analysis is based on the basic transistor Ebers–Moll equations.

7.1 Find the mathematical expression for *f*(*I*) and demonstrate the d*f*(*I*)/d*t* equation as a parametric function with some constants. How functions *f*(*I*) and d*f*(*I*)/d*t* are dependent on γ parameter?
7.2 Investigate OptoNDR van der Pol circuit with additional capacitor C_a and LED (D_2) by using perturbation method. Hint: omit $O(\varepsilon^2)$ terms and higher terms.
7.3 Find circuit fixed points and eigenvalues. Classify circuit fixed points and investigate stability for different values of γ parameter.
7.4 How the behavior of the circuit is changed if we short capacitor C_a? Investigate the circuit (C_a is short) using perturbation method. Hint: omit $O(\varepsilon^3)$ terms and higher terms.
7.5 Investigate OptoNDR van der Pol circuit with additional capacitor C_a and LED (D_2) by using perturbation method, omit $O(\varepsilon^3)$ terms and higher terms.

8. Consider a system which is characterized by the differential equation: $\frac{d^2X}{dt^2} + \omega_0^2 \cdot X + \varepsilon \cdot X^k + \varepsilon \cdot \left(\frac{dX}{dt}\right) \cdot \text{sgn}(X) = 0$; $k \in \mathbb{N}$; $k \geq 3$; $\omega_0 \in \mathbb{R}$
where sgn( ) is a Signum function: $\text{sgn}\left(\frac{dX}{dt}\right) = 1 \,\forall\, \frac{dX}{dt} > 0\, \text{sgn}\left(\frac{dX}{dt}\right) = -1 \,\forall\, \frac{dX}{dt} < 0$; $\text{sgn}\left(\frac{dX}{dt}\right) = 0 \,\forall\, \frac{dX}{dt} = 0$; $0 < \varepsilon \ll 1$.
Hint: In your analysis differentiate different values of sgn( ) function cases.

8.1 Find system fixed points and eigenvalues for *k* = 4 and *k* = 5. How the system fixed points and eigenvalues are dependent on ω_0 parameter?
8.2 Classify our system fixed points for *k* = 4 and *k* = 5. Discuss stability.
8.3 Solve the problem exactly (for the cases *k* = 4 and *k* = 5), use regular perturbation theory for these cases, find X_0, X_1 and X_2 in the series expansion: $X(t, \varepsilon) = X_0(t) + \varepsilon \cdot X_1(t) + \varepsilon^2 \cdot X_2(t) + \varepsilon^3 \cdot X_3(t) + O(\varepsilon^4)$.
8.4 Investigate the system if $\omega_0 = 0$ and $\omega_0 = 1$ (cases *k* = 3 and *k* = 4). Find fixed points and eigenvalues. Classify system fixed points. Find $X(t, \varepsilon)$ by using regular perturbation theory.
8.5 Investigate the system if $\omega_0 = \sqrt{1 + k^2}$; $\omega_0 = \xi_1(k)$ then we get the system differential equation:

$$\frac{d^2X}{dt^2} + \xi_1^2(k) \cdot X + \varepsilon \cdot X^k + \varepsilon \cdot \left(\frac{dX}{dt}\right) \cdot \mathrm{sgn}(X) = 0;\ k \in \mathbb{N};\ k \geq 3;\ \omega_0 \in \mathbb{R}$$

Find fixed points and eigenvalues; how they change for different values of k parameters? Classify system fixed points. Find $X(t, \varepsilon)$ by using regular perturbation theory.

9. Consider a modified van der Pol system which is characterized by the differential equation: $\frac{d^2X}{dt^2} - \varepsilon^2 \cdot (1 - X^2) \cdot \frac{dX}{dt} + X - \varepsilon \cdot X^3 = 0;\ 1 \gg \varepsilon > 0$

9.1 Find system fixed points and eigenvalues. Discuss stability of the system and limit cycle.
9.2 Find periodic solution to this equation to order ε^2 using perturbation theory.
9.3 Try to implement the modified van der Pol system with optoisolation elements and discrete components (resistors, capacitors, inductors, etc.).
9.4 We replace the system differential equation's term $(\varepsilon \cdot X^3)$ by $(\varepsilon \cdot X^5)$. How the behavior of the system is changed? Find fixed points, eigenvalues, and discuss stability.
9.5 In the case of $\varepsilon \cdot X^5$ term (9.4), find the periodic solution to this equation to order ε^3 using perturbation theory.

10. A capacitor of capacitance C is connected in series to a Non Linear Diode (NLD) which has current i. When e is the voltage across the capacitor C. The relation between e and i of the form: $i = \Gamma_1 \cdot e + \Gamma_2 \cdot e^2 + \Gamma_3 \cdot e^3$ with the voltage across the capacitor at $t = 0$ having the value $e = E$. The relevant circuit equation is $C \cdot \frac{de}{dt} + \Gamma_1 \cdot e + \Gamma_2 \cdot e^2 + \Gamma_3 \cdot e^3 = 0$. Introduction dimension less variable $x \equiv \frac{e}{E}$ and $\tau \equiv \frac{\Gamma_1 \cdot t}{C}$ then we get the system differential equation: $\frac{dx}{d\tau} = -x - \frac{\Gamma_2}{\Gamma_1} \cdot x^2 \cdot E - \frac{\Gamma_3}{\Gamma_1} \cdot x^3 \cdot E^2$. We choose $\varepsilon = \frac{\Gamma_2}{\Gamma_1} \cdot E$ when $\Gamma_3 = \sum_{i=1}^{2} \Gamma_i$ then $\frac{\Gamma_3}{\Gamma_1} = \frac{\sum_{i=1}^{2} \Gamma_i}{\Gamma_1} = 1 + \varepsilon \cdot \frac{1}{E}$. We get the system differential equation: $\frac{dx}{d\tau} = -x - x^2 \cdot E^2 - \varepsilon \cdot x^2 \cdot (1 + E)$.

10.1 Assume the standard expansion $x(\tau) = x_0 + \varepsilon \cdot x_1 + \varepsilon^2 \cdot x_2 + \varepsilon^3 \cdot x_3 + \cdots$ and since ε is arbitrary, equal power of ε can be equated. Write down the various orders of ε.
10.2 Solve the system exactly, use regular perturbation theory, find x_0, x_1, x_2 and x_3.
10.3 We change the relation between e and i to the form $i = \Gamma_1 \cdot e + \Gamma_2 \cdot e^2 + \Gamma_3 \cdot \ln(e)$ with the voltage across the capacitor at $t = 0$ having the value $e = E$. Assume the standard expansion $x(\tau) = x_0 + \varepsilon \cdot x_1 + \varepsilon^2 \cdot x_2 + \cdots$. Since ε is arbitrary, equal power of ε can be equated. Write down the various orders of ε and solve the system exactly (find $x(\tau)$).

10.4 We change the relation between e and i to the form $i = \Gamma_1 \cdot e + \Gamma_2 \cdot e^2 + \Gamma_3 \cdot \exp(\sqrt{e+1})$ with the voltage across the capacitor at $t = 0$ having the value $e = \gamma \cdot E$; $\gamma \in \mathbb{R}_+$. Assume standard expansion $x(\tau) = x_0 + \varepsilon \cdot x_1 + \varepsilon^2 \cdot x_2 + \varepsilon^3 \cdot x_3 + \cdots$ and since ε is arbitrary, equal power of ε can be equated. Write down the various orders of ε and solve the system exactly (find $x(\tau)$).

10.5 How the behavior of our system is changed for different values of γ parameter? Plot the graph of exact solution $x(\tau)$ as a function of γ parameter.

Chapter 9
Optoisolation Advance Circuits—Investigation, Comparison, and Conclusions

Optoisolation circuits in many topological structures represent many specific implementations which stand the target engineering features. Optoisolation circuits include photo transistor and light emitting diode (opto couplers), both coupled together. There are many other semiconductors which include optoisolation topologies like photo SCR, SSR, photo diode, etc. The basic optocoupler can be characterized by Ebers–Moll model and the associated equations. The optoisolation circuits include optocouplers and peripheral elements (capacitors, inductors, resistors, switches, operational amplifiers, etc.). In a phototransistor the photodetector is the collector-base junction so the capacitance impairs the collector rise time. Also, amplified photocurrent flows in the collector-base junction and modulated the photo response, thereby causing nonlinearity. One of the basic optocoupler structure is a Negative Differential Resistance (NDR) circuit. Negative resistance or negative differential resistance is a property of electrical circuit elements composed of certain materials in which, over certain voltage ranges, current is a decreasing function of voltage. This range of voltages is known as a negative resistance region. Such a circuit must contain an energy source , and can be used as a form of amplifier. However, the use of the term negative resistance to encompass negative differential resistance is more common. Absolute negative resistances without an external energy source cannot exist as they would violate the law of conservation of energy. In electrical circuits, static resistance is the ratio of the voltage across a circuit element to the current through it. However, the ratio of the voltage to the current may vary with either voltage or current. The ratio of the change in voltage to the change in current is known as dynamic resistance. It is more correct to say that a circuit element has a negative differential resistance region than to say that it exhibits negative resistance because even in this region the static resistance of the circuit element is positive, while it is the slope of the resistance curve which is negative. Electronic circuit elements displaying negative differential resistance (NDR) , such as tunnel diodes, have a wide variety of device applications, including oscillators, amplifiers, logic, and memory. We build negative differential resistance (NDR) devices using analog optocouplers which are positive differential resistance

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elements but specific connection gives the negative differential resistance behavior. The benefits are the negative differential parameters which can be changed and controlled by the specific connection of those optocouplers. We inspect circuit which involves optoisolation devices as a dynamical system where a fixed rule describes the time dependence of a specific circuit voltage in a geometrical space.

Among advance optoisolation circuits many exhibit limit-cycle behavior. A limit cycle is a closed trajectory, this means that its neighboring trajectories are not closed—they spiral either toward or away from the limit cycle. Limit cycles can only occur in those optoisolation circuits which exhibit nonlinearity. In contrast a linear system exhibiting oscillations closed trajectories are neighbored by other closed trajectories. A stable limit cycle is one which attracts all neighboring trajectories. A system with a stable limit cycle can exhibit self-sustained oscillations. Half-stable limit cycles are ones which attract trajectories from one side and repel those on other. If all neighboring trajectories approach the limit cycle, the limit cycle is stable or attracting otherwise the limit cycle is unstable. An optoisolation circuits which exhibits stable limit cycles oscillate even in the absence of external periodic force. Since limit cycles are nonlinear phenomena; they can not occur in linear system. We use in nonlinear dynamics some acronyms like: Stable Limit Cycle (SLC), Half-stable Limit Cycle (HLC), Unstable Limit Cycle (ULC); Unstable Equilibrium Point (UEP), Stable Equilibrium Point (SEP), and Half-stable Equilibrium Point (HEP). One of the system which demonstrate limit cycle is van der Pol oscillator and it can be implemented using optoisolation devices. Oscillation of van der Pol are called relaxation oscillations because the charge that builds up slowly is relaxed during a sudden discharge in a sudden discharge in strongly nonlinear limit ($\mu \gg 1$), $\frac{d^2X}{dt^2} + \mu \cdot \frac{dX}{dt} \cdot (X^2 - 1) + X = 0$. The next step is to analyze van der Pol system with optoisolation feedback loop and find fixed points and stability under parameters values change. We use Poincare–Bendixson theorem in those optoisolation systems and establish which parameters values the closed orbits exist in an optoisolation particular system. Poincare–Bendixon theorem supports that our optoisolation system resides in the plane. One crucial issue is to implement optoisoaltion oscillation circuit which can be modeled by second-order differential equation: $\frac{d^2V}{dt^2} + f(V) \cdot \frac{dV}{dt} + g(V) = 0$. This is a Lienard's equation. The equation is a generalization of the van der Pol oscillator system. The term $f(V) \cdot \frac{dV}{dt}$ is the nonlinear damping force and $g(V)$ is the nonlinear restoring force. Other implementation is the optoisolation circuit with weakly nonlinear oscillations. The general equation of nonlinear oscillator is of the form $\frac{d^2V}{dt^2} + V + \varepsilon \cdot h(V, \frac{dV}{dt}) = 0; V = V(t)$, V(t) is our system main variable where $0 < \varepsilon \ll 1$ and $h(V, \frac{dV}{dt})$ is a smooth function. If we have small perturbation of the linear oscillator then it is called weakly nonlinear oscillator. Various bifurcations can be exhibited by optoisolation circuits. Bifurcation describes the qualitative alterations that occur in the orbit structure of a dynamical system as the parameters on which the system depends are varied. An opto isolator can be implemented as a circuit which demonstrate cusp bifurcation for specific system voltage band and parameters values. Cusp-catastrophe bifurcation occurs in a one-dimensional state space ($n = 1$) and two-dimensional parameter space ($p = 2$). $\frac{\mathrm{d}V}{\mathrm{d}t} = f(V, \Gamma_1, \Gamma_2); V \in \mathbb{R}^{n=1}$

$\{\Gamma_1, \Gamma_2\} \in \mathbb{R}^{p=2}; M = \{(\Gamma_1, \Gamma_2, V) | \Gamma_1 + \Gamma_2 \cdot V - V^3 = 0\}$. Where Γ_1, Γ_2 parameters are two control parameters and V is the system state variable. A two parameters system near a triple equilibrium point is known as cusp bifurcation (equilibrium structure). This is the simplest degenerate case of the fold bifurcation related to the cusp catastrophe theory with hysteresis bifurcation. We implement cusp system using optoisolation elements, Op amps, resistors, capacitors, etc. Another bifurcation which can be implemented by optoisolation elements is Bautin bifurcation. The Bautin bifurcation is a bifurcation of an equilibrium in a two parameter family of autonomous ODEs at which the critical equilibrium has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov–Hopf bifurcation vanishes (generalized Hopf (GH) bifurcation). The bifurcation points separates branches of sub- and super-critical Andronov Hopf bifurcations in the parameter plane. For nearby parameter values, the system has two limit cycles which collide and disappear via a saddle-node bifurcation of periodic orbit. Another bifurcation system is Bogdanov–Takens (double-zero) bifurcation. The Bogdanov–Taken (BT) bifurcation is a bifurcation of an equilibrium point in a two parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue of multiplicity two. The system has two equilibria (a saddle and a non-saddle) which collide and disappear via saddle-node bifurcation. The non-saddle equilibrium undergoes an Andronov Hopf bifurcation generating a limit cycle. This cycle degenerates into an orbit homoclinic to the saddle and disappear via a saddle homoclinic bifurcation. The Fold–Hopf bifurcation is a bifurcation of an equilibrium point in a two parameter family of autonomous ODEs at which the critical equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues. Another names for Fold-Hopf bifurcation are Zero-Hopf (ZH) bifurcation, saddle-node Hopf bifurcation and Gavrilov–Guckenheimer bifurcation. We discuss the autonomous system of ODEs $\frac{\mathrm{d}V}{\mathrm{d}t} = f(V, \alpha)\, \forall\, \alpha \in \mathbb{R}^n$ depending on two parameters $\alpha \in \mathbb{R}^2$, where $f(V, \alpha)$ is a smooth function. We implement Rossler's prototype chaotic system using optoisolation circuits. We consider a continuous-time dynamical system depending on parameters, given by $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), \alpha); f \in C^k(\Omega \times \Lambda, \mathbb{R}^n)$ with open sets $0 \in \Omega \subset \mathbb{R}^N; 0 \in \Lambda \subset \mathbb{R}^2; k > 1$ sufficiently large, $N \geq 4$. The first used tool for exploring the dynamical behavior of the system is numerical time integration. We can employ one step methods, which consists in approximating the evolution operator by a discrete-time system $x \to g(x, \alpha)$ with $g \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$, where the step sizes were assumed to be previously chosen. Hopf–Hopf bifurcations occurs when the system presents equilibrium with two pairs of Hopf eigenvalues. The local bifurcation diagram near Hopf–Hopf point is known to present other phenomena such as Neimark–Sacker bifurcation of cycles, and homoclinic bifurcations. A point $(x_0, \alpha_0) \in \Omega \times \Lambda$ is referred to as a Hopf–Hopf bifurcation (HH point). It is also called double Hopf, Hopf/Hopf mode interaction, and multiple Hopf in case that moves than two pairs of Hopf eigenvalues are present. The Hopf–Hopf bifurcation appears in many parameter-dependent systems (optoisolation circuits) which describe the dynamic of various phenomena. We implement Hopf–Hopf bifurcation system using optoisolation elements, Op amps, resistors, capacitors, diodes, etc. We have a discrete-time dynamical system depending on a parameter $x \to f(x, \alpha)$, $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^1$ where, the map f is smooth respect to both x and α. We can write this system as $\tilde{x} = f(x, \alpha)$, $\tilde{x}, x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^1$ where $\tilde{x}$ denotes the image of x under the action of the map. Let $x = x_0$ be a hyperbolic fixed point of the system for $\alpha = \alpha_0$. We monitor this fixed point and its multipliers while the parameter varies. There are three ways in which the hyperbolicity condition can be violated. The first: a simple positive multiplier approaches the unit cycle $\mu_1 = 1$, the second: a simple negative multiplier approaches the unit circle $\mu_1 = -1$, the third: a pair of simple complex multipliers reaches the unit circle $\mu_{1,2} = e_0^{\pm i \cdot \theta}$, $0 < \theta_0 < \pi$ for some value of parameter. The bifurcation corresponding to the presence of $\mu_{1,2} = e_0^{\pm i \cdot \theta}$, $0 < \theta_0 < \pi$ is called a Neimark–Sacker (or torus) bifurcation. The fold and flip bifurcations are possible if $n \geq 1$, but for the Neimark–Sacker bifurcation $n \geq 2$. Neimark–Sacker bifurcation is the birth of a closed invariant curve from a fixed point in dynamical systems with discrete time (iterated maps), when the fixed point changes stability via a pair of complex eigenvalues with unit modules. The bifurcation can be supercritical or subcritical, resulting in a stable or unstable within an invariant two-dimensional manifold closed invariant curve, respectively. We implement multi-folded torus chaotic attractors system using optoisolation circuits. Many optoisolation systems are periodic and continuous in time therefore we implement Floquet theory and investigate behavior and stability. There are many optoisolation systems which contain periodic forcing source (voltage or current source) and can be analyze by Floquet theory. Floquet theory is the study of the stability of linear periodic systems in continuous time. Floquet exponents/multipliers are analogous to the eigenvalues of Jacobian matrices of equilibrium points. Basically it is the study of linear systems of differential equations with periodic coefficients.

Floquet theory is a linear theory, nonlinear models can be linearized near limit cycle solutions to enable the use of Floquet theory. Floquet theory deals with continuous-time systems and the theory of periodic discrete-time systems is closely analogous. We multiply the T transition matrices together to determine how a perturbation changes over a period which is similar to finding the fundamental matrix. One limitation of Floquet theory is that it applies only to periodic systems. The more general Lyapunov exponents plays the role of Floquet exponents. Lyapunov exponents are more challenging to compute numerically because, instead of calculating how a perturbation grows or shrinks over one period, this must be done in the limit at $T \to \infty$. Firstly we consider a set of linear, homogeneous, time periodic differential equations; $\frac{dx}{dt} = A(t) \cdot x$ where x is a n-dimensional vector and $A(t)$ is an $n \times n$ matrix with minimal period$T$. $A(t)$ vary periodically and the solutions are typically not periodic, and despite its linearity, closed from the solutions of $\frac{dx}{dt} = A(t) \cdot x$ typically can not be found. The general solution of $\frac{dx}{dt} = A(t) \cdot x$ takes the form $x(t) = \sum_i^m c_i \cdot e^{\mu_i \cdot t} \cdot p_i(t)$ where c_i are constants that depend on initial conditions, $p_i(t)$ are vector-values functions with periodic T, and μ_i are complex numbers called characteristic or Floquet exponents. It is true to state that $\rho_i = e^{\mu_i \cdot T}$ (μ_i—Floquet exponents, ρ_i—Floquet multipliers). We implement optoisolation circuit which has two variables with periodic sources. Analysis for optoisolation circuit's two variables with periodic sources is done and we investigate limit cycle stability. We prove that the system has periodic orbits by changing system Cartesian coordinates $V_1(t)$, $V_2(t)$ to cylindrical coordinates $r(t)$, $\theta(t)$. Next is to show that the cylinder is invariant. In fact there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers. Possible optoisolation circuit is second-order ODE with periodic source. We analyze the stability of a limit cycle of the optoisolation circuit second-order ODE with periodic source. Some optoisolation circuits can be characterized by Hills equations. The optoisolation circuit dynamic system has one main variable $U(t)$. System Hill's equation: $\frac{d^2 U(t)}{dt^2} + [\delta + a(t)] \cdot U(t) = 0$, δ is a constant and $a(t)$ is a π periodic function. We can write this equation in different way,

$$x = \begin{pmatrix} U(t) \\ \frac{dU(t)}{dt} \end{pmatrix}; \ \frac{dx}{dt} = \begin{pmatrix} \frac{dU(t)}{dt} \\ \frac{d^2U(t)}{dt^2} \end{pmatrix}$$

And we get system $\frac{dx}{dt} = A(t) \cdot x; A(t) = \begin{pmatrix} 0 & 1 \\ -[\delta + a(t)] & 0 \end{pmatrix}$. Floquet theory can be implemented to OptoNDR circuits, linear stability analysis as well, when dealing with an optoisolation periodic system. OptoNDR circuits nonlinear models can be linearized near limit cycle solutions to enable the use of Floquet theory. OptoNDR circuits' periodic study has many engineering implementations like oscillators, amplifiers, logic, and memory. Electrical negative resistance is often used to design oscillators. Many topologies are possible, such as Colpitts oscillator, Hartley oscillator, Wien bridge oscillator, and some types of relaxation oscillators. Another engineering implementation is to use OptoNDR as a Chua's element in Chua's circuit. Autonomous Chua's circuit is containing three energy storage elements, a linear resistor and a nolinear resistor N_R (Chua's element), and its discrete circuitry design and implementations. Since Chua's circuit is an extremely simple system, and yet it exhibits a rich variety of bifurcations and chaos among the chaos-producing mechanisms, it has a special significance. The term Chua's element is a general description for a two-terminal nonlinear resistor with piecewise-linear characteristic. There are two forms of Chua's element, the first type is a voltage-controlled nonlinear element characterized by $i_R = f(v_R)$ and the other type is a current-controlled nonlinear element characterized by $v_R = g(i_R)$. Chaotic oscillators designed with Chua's element are generally based on a single three-segment, add-symmetric, voltage-controlled piecewise-linear nonlinear resistor structure. OptoNDR systems can be periodic and continuous in time therefore Floquet theory is an ideal way for behavior and stability analysis. Autonomous Chua's circuit has many implementations in various electrical circuits. Since Chua's circuit is an extremely simple system and it exhibits a rich variety of bifurcations and chaos among the chaos production mechanism. Chua's circuit fixed points and stability analysis is a crucial step toward understanding circuits and systems behavior which include Chua's circuits. Several realizations of Chua's circuit are possible. The methodologies used in these realizations can be divided into two basic categories. In the first approach, a variety of circuit topologies have been considered for realizing the nonlinear resistor N_R in Chua's circuit. The second approach is related to the implementation of Chua's circuit and is an inductor less realization of Chua's circuit. We replace in our circuit the Chua's diode by OptoNDR element in Chua's circuit. The circuit contains three energy storage elements (inductance, and two capacitors). Chua's circuit exhibits different and unique variety of bifurcations and chaos among the chaos-producing mechanism with OptoNDR element. We need to check in which conditions the OptoNDR circuit's to variables system has periodic orbits and it is done by changing system Cartesian coordinates to cylindrical coordinates. We show that the cylinder is invariant and it approves that there is a stable periodic orbit and use the fact that a stable periodic orbit has two/three Floquet multipliers. One of them is unity. It is also important to analyze and model stability of a limit cycle for OptoNDR circuit's second-order ODE with periodic source. Optoisolation systems periodic orbits are frequently encountered as trajectories. We use solutions of initial value problems, as a mean of finding stable periodic orbits of vector fields. We can represent our system as a vector fields in Euclidean space , systems of differential equations: $\frac{dX}{dt} = f(X, \Omega); X \in \mathbb{R}^n; \Omega \in \mathbb{R}^k$ with Ω a vector of optoisolation circuit parameters. Periodic orbits are non-equilibrium trajectories $X(t)$ that satisfy $X(T) = X(0)$ for some $T > 0$. The smallest such T is the period of the orbit. The local dynamics near a periodic orbit are typically determined by return map. The return map has a fixed point at its intersection with a periodic orbit. If the Jacobian of the return map at this fixed point has eigenvalues inside the unit circle, the orbit is asymptotically stable. Initial conditions in the Basin Of Attraction (BOA) of the periodic orbit have trajectories whose limit set is the periodic orbit. It is the right circuit design issue, by choosing the optoisolation elements and discrete components, topological circuit structure which fulfil system periodic orbits. We can represent our planar cubic vector field system as two differential equations: $\frac{dX_1}{dt} = X_2; \frac{dX_2}{dt} = -(X_1^3 + r \cdot X_1^2 + n \cdot X_1 + m) + (\Gamma - X_1^2) \cdot X_2$. The system contains the unfolding of a co-dimension two bifurcation of an equilibrium point with a double eigenvalue zero in the presence of a rotational symmetry of the plane. Additional oscillation we investigate is Glycolytic oscillation. Glycolytic oscillation is the repetitive fluctuation of in the concentrations of metabolites. The problem of modeling glycolytic oscillation has been studied in control theory and dynamical systems. The behavior depends on the rate of substrate injection. Early models used two variables (X, Y), but the most complex behavior they could demonstrate was period oscillations due to the Poincaré–Bendixson theorem. The Poincaré–

Bendixson theorem is a statement about the long-term behavior of orbits of continuous dynamical systems on the plane, cylinder, or two-sphere [7, 8]. The condition that the dynamical system is on the plane is necessary to the theorem. Chaotic behavior can only arise in continuous dynamical systems whose phase space has three or more dimensions. However, the theorem does not apply to discrete dynamical systems, where chaotic behavior can arise in two- or even one-dimensional systems. A two-dimensional continuous dynamical system cannot give rise to a strange attractor. Periodic oscillations in biochemical systems are receiving much attention. The existence of sustained oscillations in yeast cell extracts and in the whole yeast cell as well as in heart muscle cell extracts has been proved. The model explains sustained oscillations in the yeast Glycolytic system. The model has no limit cycle for those values of its parameters with which self-oscillations are observed. The model represents an enzyme reaction with substrate inhibition and product activation. Practically the dynamical process called Glycolysis and can proceed in an oscillatory fashion. The differential equations model of Glycolytic oscillator is presented:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -X + \mu_1 \cdot Y + X^2 \cdot Y; \frac{\mathrm{d}Y}{\mathrm{d}t} = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y;$$

$$f_1(X, Y) = -X + \mu_1 \cdot Y + X^2 \cdot Y$$

$$f_2(X, Y) = \mu_2 - \mu_1 \cdot Y - X^2 \cdot Y; \frac{\mathrm{d}X}{\mathrm{d}t} = f_1(X, Y); \frac{\mathrm{d}Y}{dt} = f_2(X, Y)$$

We implement Glycolytic system limit cycle solution using optoisolation elements. Poincare maps is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower dimensional subspace, called the Poincaré section, transversal to the flow of the system. Poincare map is a discrete dynamical system with a state space that is one dimension smaller than the original continuous dynamical system. We use it for analyzing the original system. We use Poincare maps to study the flow near a periodic orbit, the flow in some chaotic system. We define n-dimensional system $\frac{\mathrm{d}X}{\mathrm{d}t} = f(X)$. Let "$S$" be an $(n-1)$ dimensional surface of section. "S" is required to be transverse to the flow and all trajectories starting on "S" flow through it, not parallel to it. The Poincare map ψ is a mapping from "S" to itself and trajectories from one intersection with "S" to the next. If $X_k \in S$ denotes the kth intersection, then Poincare map is defined by $X_{k+1} = \psi(X_k)$. If X^* is a fixed point then $\psi(X^*) = X^*$. Then a trajectory starting at X^* returns to X^* after time T and a closed orbit for the original system $\frac{\mathrm{d}X}{\mathrm{d}t} = f(X)$. Many systems are presented by set of differential equations and analysis is done by using Poincare maps. We implement these dynamical systems by using opto couplers, Op-amps, and discrete components (resistors, capacitors, etc.) and investigate is done using Poincare maps. In every dynamical system we need to use the global existence theorems. The dynamical system describes the physical behavior in time. Many dynamical systems described by its position and velocity as functions of time and the initial conditions. A dynamical system is a function $\phi(t, X)$, defined for all $t \in \mathbb{R}$ and $X \in \mathrm{E} \in \mathbb{R}^n$. It describes how points $X \in \mathrm{E}$ move with respect to time. The families of maps $\phi_t(X) = \phi(t, X)$ have the properties of a flow. Integral part of our analysis is periodic orbits or cycles, limit cycles, and separatrix cycles of a dynamical system $\phi(t, X)$ defined by $\frac{\mathrm{d}X}{\mathrm{d}t} = f(X)$. Periodic orbits have stable and unstable manifold just as equilibrium points do. A finite union of compatibly oriented separatrix cycles is called a compound separatrix cycle or graphic. We need to discuss the stability and bifurcations of periodic orbits by the Poincare map or first return map. A separatrix is the boundary separating two modes of behavior in a differential equation and an equation to determine the borders of a system. A phase curve meets a hyperbolic fixed point or connects the unstable and stable manifolds of a pair of hyperbolic or parabolic fixed points. A separatrix marks a boundary between phase curves with different properties. Poincare map is very useful tool to study the stability and bifurcations of periodic orbit. If Γ is periodic orbit of the system $\frac{\mathrm{d}X}{\mathrm{d}t} = f(X)$ through the point X_0 and Σ is a hyperplane perpendicular to Γ at X_0, then any point $X \in \Sigma$ sufficiently near X_0, the solution $\frac{dX}{dt} = f(X)$ through X at $t = 0$, $\phi_t(X)$ will cross the Σ again at a point $\psi(X)$ near X_0. The mapping $X \to \psi(X)$ is called the Poincare map. When the surface Σ intersects the curve Γ transversally at X_0, Σ is a smooth surface, through a point $X_0 \in \Gamma$, which is not tangent to Γ at X_0. Additionally, we can discuss the existence and continuity of the Poincare map $\psi(X)$ and of its first derivative $D\psi(X)$. We analyze the behavior of optoisolation van der Pol circuit Poincare map and periodic orbit. One of the typical dynamical systems is Li autonomous system with toroidal chaotic attractors. The Li dynamical system is autonomous and the motion appears to occur on a surface with a toroidal structure. The Li system global Poincare surface of section has two disjoint components. A similarity transformation in the phase space emphasizes symmetry of the attractor. Poincare section located segments of a chaotic trajectory are good approximations to unstable periodic orbit. Li system can be described by three Ordinary Differential Equations (ODEs).

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mu_1 \cdot (Y - X) + \mu_2 \cdot X \cdot Z; \frac{\mathrm{d}Y}{\mathrm{d}t} = \mu_3 \cdot X + \mu_4 \cdot Y - X \cdot Z;$$
$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \mu_5 \cdot Z + X \cdot Y - \mu_6 \cdot X^2$$

The system is invariant under the group of two fold rotations about the symmetry axis in the phase space $\mathbb{R}^3(X, Y, Z) : R_Z(\pi) : (X, Y, Z) \to (-X, -Y, +Z)$.

We would like to have measure of stability like rate of decay to a stable fixed point. One way for the production of oscillations in L-C networks is to overcome circuit losses through the use of designed-in positive feedback or generation. Two alternatives exist for a potential oscillator. In the first alternative, the tuned active network may have excess loss. The second alternative is that of a successful oscillator where excess loss will have been compensated and a sustained oscillation is obtained at the circuit's output. We can distinguish two sine wave oscillator types.

The feedback and the negative resistance oscillator, Opto NDR device has better performance characteristics cancelation in terms of resistive losses in oscillators than oscillators that are not based on NDR devices. All cancelation of resistive losses methods shown to be equivalent in the results and losses being effectively canceled out by the negative differential resistance contributed by the active device and associated reactive components. Two ways of carrying out this process are series and a parallel L-C circuit. In the case that the loss cancelation is incomplete, the loop gain of a sine wave oscillator will be less than unity and oscillations will not start building up. On the other hand, if the gain is close to unity the circuit will behave as a regenerative or high gain narrow band tuned amplifier. OptoNDR is one of compound active device that exhibit negative resistance regions on their static I-V characteristics curves. This device can be successfully employed in construction of L-C oscillators and regenerative amplifiers. In many dynamical systems there are linear oscillators with small perturbations or weakly nonlinear sources. These systems are valid on semi-infinite time intervals under suitable conditions. In many perturbed systems, we start with a system which includes known solutions and add small perturbations of it. The solutions of unperturbed and perturbed systems are different and system with small perturbation has different structure of solutions. For finite times unperturbed and perturbed solutions are close. We study the asymptotic behavior of the solutions and structure. Generally, perturbation theory has tools to solve problems with approximate solution by discussing of the exact solution of a simpler problem. An approximation of the full solution A, a series in the small parameters (ε), like the following solution $A = \sum_{k=0}^{\infty} A_k \cdot \varepsilon^k$. A_0 is the known solution to the exactly solvable initial problem and A_1, A_2, … represent the higher order terms. For small ε these higher order terms become successively smaller. The initial solution and "first-order" perturbation correction is $A \approx A_0 + \varepsilon \cdot A_1$. The method of averaging provides a useful means to study the behavior of nonlinear dynamical system under periodic forcing. Averaged equation of a time-dependent differential equation gave the Poincare map, stability analysis, and recover higher orders of averaging. It discusses higher order averaged expansions for periodic and quasi-periodic differential equations. Averaging can be implemented to systems of the form $\frac{dX}{dt} = \varepsilon \cdot f(X,t); X(t=0) = X_0$ where f is T-periodic

$f(X,t) = f(X,t+T); X \in \mathbb{R}^n$. The average of $f(X,t)$ is typically given as $\overline{f(X,t)} = \int_t^{t+T} f(X,\tau) \cdot d\tau$ where the evaluation point X is considered fixed. The average defines new autonomous equation $\frac{dY}{dt} = \varepsilon \cdot \overline{f(Y,t)}; Y(t=0) = X_0$. Averaging theory determines conditions under which the two flows coincide and to what degree they coincide. The parameter ε will provide a means to determine this coincide. Oscillators which include optoisolation elements are integral part of many engineering applications. The construction process of negative differential resistance (NDR) devices using analog opto couplers is used to make a highly stable, radio-frequency oscillator. Typical oscillator is van der Pol "Negative Resistance" (e.g., tunnel diode, opto isolation circuit, etc.) oscillators. In every dynamical system we need to use the global existence theorems. Some system's differential equations cannot be solving exactly and if there is exact solution, it exhibits an intricate dependency in the parameters that it is hard to use as such. We identified the parameter as ε, and the solution is available and reasonably simple for $\varepsilon = 0$. The system solution is altered for non-zero but small ε. We get systematic interpretation by the perturbation theory. We have system that characterized by the differential equation: $\frac{dX}{dt} = \varepsilon \cdot f(X,t); X \in \mathbb{R}^n; \varepsilon \ll 1$ where f is periodic in t, $f(X,t) = f(X,t+T)$; $X \in \mathbb{R}^n$. The solution evolution is "SLOW," which characterized the T-periodic forcing term. We characterized a weakly nonlinear oscillator system by differential equation: $\frac{d^2X}{dt^2} + \omega^2 \cdot X = \varepsilon \cdot f(X, \frac{dX}{dt}, t)$. OptoNDR van der Pol is analyzed by perturbation method. We can use multiple scale analysis for construction uniform or global approximate solutions for both small and large values of independent variables of OptoNDR circuit van der Pol. The dependent variables are uniformly expanded in terms of two or more independent variables, scales. The issue is the choice of ordering scheme and the form of the power series expansion. We can implement in OptoNDR van der Pol system, Multiple Scale Perturbation Theory (MSPT). The coordinate transforms and invariant manifolds provide a support for multiscale modeling. Practical there are at least two time scales in weakly nonlinear OptoNDR circuit van der Pol oscillator. Two timing builds two time scales from the start and produces better approximations than the regular perturbation theory. The unforced van der Pol equation has a limit cycle with radius approximately equal to two and period approximately 2π. The limit cycle is generated by the balance between internal energy loss and energy generation, and the forcing term will alter this balance. We consider forced van der Pol system with forcing term $F \cdot \cos(\omega \cdot t)$. If "$F$" is small (weak excitation), its effect depends on whether or not ω is close to the natural frequency. If it is, the oscillation might be generated which is a perturbation of the limit cycle. If "F" is not small (hard excitation) or if the natural and imposed frequency are not closely similar, we should expect that the "natural oscillation" might be extinguished, as occurs with the corresponding linear equation. The forced van der Pol system can be given by the following equations: $\frac{d^2X_1}{dt^2} + \varepsilon \cdot (X_1^2 - 1) \cdot \frac{dX_1}{dt} + X_1 = F \cdot \cos(\omega \cdot t)$ and can be express by first degree differential equation $\frac{dX_1}{dt} = X_2; \frac{dX_2}{dt} + \varepsilon \cdot (X_1^2 - 1) \cdot X_2 + X_1 = F \cdot \cos(\omega \cdot t)$.

Appendix A
Stability and Bifurcation Behaviors

A.1 Cusp Bifurcation

The cusp bifurcation is a bifurcation of equilibria in a two-parameter family of autonomous ODEs at which the critical equilibrium has one zero eigenvalue and the quadratic coefficient for the saddle–node bifurcation vanishes.

At the cusp bifurcation point, two branches of saddle–node bifurcation curve meet tangentially, forming a semicubic parabola. For nearby parameter values, the system can have three equilibria which collide and disappear pairwise via the saddle–node bifurcations. The cusp bifurcation implies the presence of a hysteresis phenomenon and observed in two-parameter space which associated to the presence of bistability and to hysteresis phenomenon. Because the cusp bifurcation can only be seen in a 2-parameter plane, it is called a co-dimension 2 bifurcation. Suppose the system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\alpha)$, $x \in \mathbb{R}^1$, $\alpha \in \mathbb{R}^2$ with a smooth function f, has at $\alpha = 0$ the equilibrium $x = 0$ for which the cusp bifurcation conditions are satisfied, namely $\lambda = f_x(0,0) = 0; \frac{\partial f}{\partial x} = f_x$ and $a = \frac{1}{2} \cdot f_{xx}(0,0) = 0; \frac{\partial^2 f}{\partial x^2} = f_{xx}$. If we expand of $f(x,\alpha)$ as a Taylor series with respect to x at $x = 0$ yields to expression:

$f(x,\alpha) = \sum_{j=0}^{3} f_j(\alpha) \cdot x^j + O(x^4)$ and since $x = 0$ is an equilibrium, we have $f_0(0) = 0$, $f(0, 0) = 0$. The cusp bifurcation conditions yield $f_1(0) = f_x(0,0) = 0$ and $f_2(0) = \frac{1}{2} \cdot f_{xx}(0,0) = 0$. With a parameter-dependent shift of the coordinate $\xi = x + \delta(\alpha) \Rightarrow x = \xi - \delta(\alpha) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\xi}{\mathrm{d}t}$ and taking into account the f(x,α) expansion (Taylor series with respect to x at $x = 0$) yields

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = [f_0(\alpha) - f_1(\alpha) \cdot \delta + \delta^2 \cdot \varphi(\alpha,\delta)] + [f_1(\alpha) - 2 \cdot f_2(\alpha) \cdot \delta + \delta^2 \cdot \phi(\alpha,\delta)] \cdot \xi$$
$$+ [f_2(\alpha) - 3 \cdot f_3(\alpha) \cdot \delta + \delta^2 \cdot \psi(\alpha,\delta)] \cdot \xi^2 + [f_3(\alpha) + \delta \cdot \theta(\alpha,\delta)] \cdot \xi^3 + O(\xi^4)$$

Let us consider smooth functions φ, Φ, ψ, θ. Since $f_2(0) = 0$, we cannot use the implicit function theorem to select a function $\delta(\alpha)$ to eliminate the linear term in ξ in

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the above equation. There is a smooth shift function $\delta(\alpha)$, $\delta(0) = 0$, which annihilates the quadratic term in the equation for all sufficiently small $||\alpha||$, provided that $f_3(\alpha = 0) = \frac{1}{6} \cdot f_{xxx}(0,0) \neq 0$; $\frac{\partial^3 f}{\partial x^3} = f_{xxx}$. We denote the coefficient in the front of ξ^2 by $F(\alpha,\delta)$: $F(\alpha,\delta) = f_2(\alpha) - 3f_3(\alpha) \cdot \delta + \delta^2 \cdot \psi(\alpha,\delta)$, additionally

$F(0,0) = 0; \frac{\partial F}{\partial \delta}|_{(0,0)} = -3 \cdot f_3(\alpha = 0) \neq 0$. Then the implicit function theorem gives the local existence and uniqueness of a smooth scalar function $\delta = \delta(\alpha)$, such that $\delta(0) = 0$ and $F(\alpha,\delta(\alpha)) \equiv 0$ for $||\alpha||$ small enough. The equation of ξ, with $\delta(\alpha)$ constructed as above contains no quadratic terms. We define parameters.

$\mu = (\mu_1, \mu_2)$ by the following settings:

$$\mu_1(\alpha) = f_0(\alpha) - f_1(\alpha) \cdot \delta(\alpha) + \delta^2(\alpha) \cdot \varphi(\alpha,\delta(\alpha));\ \mu_2(\alpha) \\ = f_1(\alpha) - 2 \cdot f_2(\alpha) \cdot \delta(\alpha) + \delta^2(\alpha) \cdot \phi(\alpha,\delta(\alpha))$$

μ_1 is the ξ independent term in the equation, while μ_2 is the coefficient in front of ξ and $\mu(0) = 0$. The parameters are well-defined if the Jacobian matrix of the map $\mu = \mu(\alpha)$ is nonsingular at $\alpha_1 = \alpha_2 = 0$.

$\det\left(\frac{\partial \mu}{\partial \alpha}\right)|_{\alpha=0} = \det\begin{pmatrix} f_{\alpha_1} & f_{\alpha_2} \\ f_{x\alpha_1} & f_{x\alpha_2} \end{pmatrix}|_{\alpha=0} \neq 0$. Then the inverse function theorem implies the local existence and uniqueness of a smooth inverse function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$. The equation for ξ is $\frac{d\xi}{dt} = \mu_1 + \mu_2 \cdot \xi + c(\mu) \cdot \xi^3 + O(\xi^4)$.

And $c(\mu) = f_3(\alpha(\mu)) + \delta(\alpha(\mu)) \cdot \theta(\alpha(\mu),\delta(\alpha(\mu)))$ is a smooth function of μ and $c(\mu = 0) = f_3(\alpha = 0) = \frac{1}{6} \cdot f_{xxx}(0,0) \neq 0$. Preform a linear scaling $\eta = \frac{\xi}{|c(\mu)|}$ $\beta_1 = \frac{\mu_1}{|c(\mu)|}; \beta_2 = \mu_2 \Rightarrow \frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta + s \cdot \eta^3 + O(\eta^4);\ s = signc(0) = \pm 1$ and the $O(\eta^4)$ terms can depend smoothly on β. The following lemma exists:

Suppose that a one-dimensional system $\frac{dx}{dt} = f(x,\alpha)$, $x \in \mathbb{R}^1$, $\alpha \in \mathbb{R}^2$ with smooth f, has at $\alpha = 0$ the equilibrium $x = 0$, and let the cusp bifurcation conditions hold $\lambda = f_x(0,0) = 0; \alpha = \frac{1}{2} \cdot f_{xx}(0,0) = 0$. Assuming the following generality conditions are satisfied: $f_{xxx}(0, 0) \neq 0$; $(f_{\alpha 1} \cdot f_{x\alpha 2} - f_{\alpha 2} \cdot f_{x\alpha 1})(0, 0) \neq 0$ and then there are smooth invertible coordinate and parameter changes transforming the system into $\frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta \pm \cdot \eta^3 + O(\eta^4)$. We define a unique and smooth fold bifurcation curve Γ, defined by $f(x, \alpha) = 0$; $f_x(x, \alpha) = 0$.

Fixed points fulfill $\frac{dx}{dt} = 0 \Rightarrow f(x,\alpha) = 0$ and system turning points fulfill $f_x(x,\alpha) = \frac{\partial f(x,\alpha)}{\partial x} = 0$, passes through $(x, \alpha) = (0, 0)$ in R^3 space with coordinates (x, α) and can be locally parameterized by x. Given $f_x(0, 0) = f_{xx}(0, 0) = 0$, the non-degeneracy condition and the transversally condition together are equivalent to the regularity (non-singularity of the Jacobian matrix) of the map $F: R^3 \to R^3$ which defined by $F: (x,\alpha) \to (f(x,\alpha), f_x(x,\alpha), f_{xx}(x,\alpha))$ at the point $(x, \alpha) = (0, 0)$. System $\frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta \pm \cdot \eta^3 + O(\eta^4)$ with the $O(\eta^4)$ terms truncated is called the approximate normal form for the cusp bifurcation. The bifurcation diagram show that higher order terms do not actually change them.

$\frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta \pm \cdot \eta^3 + O(\eta^4) \Rightarrow \frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta \pm \cdot \eta^3$. It is the topological normal form for the cusp bifurcation. By corresponding $s = -1$, $\frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta - \cdot \eta^3$ and the system can have form one to three equilibria. A fold bifurcation occurs at a bifurcation curve T on the (β_1, β_2) plane that is given by the projection of the curve $\Gamma : \beta_1 + \beta_2 \cdot \eta - \eta^3 = 0; \beta_2 - 3 \cdot \eta^2 = 0$ onto the parameter plane. Eliminating η from these equations gives the projection $T = \{(\beta_1, \beta_2) : 4 \cdot \beta_2^3 - 27 \cdot \beta_1^2 = 0\}$.

$$\beta_2 - 3 \cdot \eta^2 = 0 \Rightarrow \eta = \pm\sqrt{\frac{\beta_2}{3}} \Rightarrow \beta_1 \pm \beta_2 \cdot \sqrt{\frac{\beta_2}{3}} \mp \frac{\beta_2}{3} \cdot \sqrt{\frac{\beta_2}{3}} = 0$$
$$\Rightarrow 4 \cdot \beta_2^3 - 27 \cdot \beta_1^2 = 0.$$

It is called a semi cubic parabola. The curve T has two branches, T_1 and T_2, which meet tangentially at the cusp point (0, 0). The resulting wedge divides the parameter plane into two regions. First region inside the wedge, there are three equilibria, two stable and one unstable. The second region, outside the wedge, there is a single equilibrium, which is stable. A non-degenerate fold bifurcation with respect to the parameter β_1 takes place if we cross either T_1 or T_2 at any point other than the origin. If the T_1 curve is crossed from the first region to the second region, the right stable equilibrium collides with the unstable one and both disappear. The same happens to the left stable equilibrium and the unstable equilibrium at T_2. If we approach the cusp point from inside first region, all three equilibria merge together into a triple root of the right-hand side of $\frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta - \cdot \eta^3$. This bifurcation is presented by plotting the equilibrium manifold of $\frac{d\eta}{dt} = \beta_1 + \beta_2 \cdot \eta - \cdot \eta^3$. Equilibrium manifold $M = \{(\eta, \beta_1, \beta_2) : \beta_1 + \beta_2 \cdot \eta - \eta^3 = 0\}$ in R^3. The standard projection of M onto the (β_1, β_2) plane has singularities of the fold type along Γ except the origin, where a cusp singularity shows up. The curve Γ is smooth everywhere and has no geometrical singularity at the cusp point. It is the projection that makes the fold parametric boundary non-smooth. The cusp bifurcation implies the presence of the phenomenon known as hysteresis. It is a catastrophic "jump" to a different stable equilibrium which caused by the disappearance of a traced stable equilibrium via a fold bifurcation as the parameters vary, happens at branch T_1 or T_2 depending on whether the equilibrium being traced belongs initially to the upper or lower sheet of M. If we make a roundtrip in the parameter plane, crossing the wedge twice, a jump occurs on each branch of T. The case $s = 1$ can be treated similarly or reduced to the considered case using the substitutions $t \to -t$, $\beta_1 \to -\beta_1$, $\beta_2 \to -\beta_2$. In this case, the truncated system typically has either one unstable equilibrium or one stable and two unstable equilibria that can pairwise collide and disappear through fold bifurcations [5–9, 71, 72].

A.2 Bautin (Generalized Hopf) Bifurcation

We have a system $\frac{\mathrm{d}V}{\mathrm{d}_t} = f(V, \alpha)$, $V \in \mathbb{R}^2$, $\alpha \in \mathbb{R}^2$. $f(V, \alpha)$ is smooth and has at $\alpha = 0$ the equilibrium $V = 0$, which satisfies the Bautin bifurcation conditions. The equilibrium has purely imaginary eigenvalues $\lambda_{1,2} = \pm\omega_0 \, \forall \, \omega_0 > 0$ and the first Lyapunov coefficient vanishes ($l_1 = 0$). Since $\lambda = 0$ is not an eigenvalue, the equilibrium in general moves as α varies but remains isolated and close to the origin for all sufficiently small $\|\alpha\|$. We perform a shift of coordinates that puts this equilibrium at $V = 0$ for all α with $\|\alpha\|$ small enough, and assume that $f(0, \alpha) \equiv 0$. We can write $\frac{\mathrm{d}V}{\mathrm{d}_t} = f(V, \alpha)$ in a complex form $\frac{\mathrm{d}z}{\mathrm{d}_t} = [\mu(\alpha) + i \cdot \omega(\alpha)] \cdot z + g(z, \bar{z}, \alpha)$; $\alpha \in \mathbb{C}^1$ where μ, ω, g are smooth functions of there arguments, $\mu(0) = 0$, $\omega(0) = 0$, and $g(z, \bar{z}, \alpha)$ can be formally by $g(z, \bar{z}, \alpha) = \sum_{k+l \geq 2} \frac{1}{k! \cdot l!} \cdot g_{kl}(\alpha) \cdot z^k \cdot \bar{z}^l$; $g_{kl}(\alpha)$ is a smooth function.

$$\lambda(\alpha) = \mu(\alpha) + i \cdot \omega(\alpha) \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t} = \lambda(\alpha) \cdot z + g(z, \bar{z}, \alpha) \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t}$$
$$= \lambda(\alpha) \cdot z + \sum_{k+l \geq 2} \frac{1}{k! \cdot l!} \cdot g_{kl}(\alpha) \cdot z^k \cdot \bar{z}^l + O(|z|^6)$$

$\mu(0) = 0$, $\omega(0) = \omega_0 > 0$ can be transformed by invertible parameter-dependent change of the complex coordinate, smoothly depending on the parameters: $z = \omega + \sum_{2 \leq k+l \leq 5} \frac{1}{k! \cdot l!} \cdot h_{kl}(\alpha) \cdot \omega^k \cdot \bar{\omega}^l; h_{21}(\alpha) = h_{32}(\alpha) = 0$ for all sufficiently small $\|\alpha\|$ into the equation: $\frac{\mathrm{d}\omega}{\mathrm{d}_t} = \lambda(\alpha) \cdot \omega + c_1(\alpha) \cdot \omega \cdot |\omega|^2 + c_2(\alpha) \cdot \omega \cdot |\omega|^4 + O(|\omega|^6)$. The coefficients $c_1(\alpha)$ and $c_2(\alpha)$ are complex and they can be made simultaneously real by a time re-parameterization. The system $\frac{\mathrm{d}\omega}{\mathrm{d}_t} = \lambda(\alpha) \cdot \omega + c_1(\alpha) \cdot \omega \cdot |\omega|^2 + \ldots$

Is locally orbit ally equivalent to the system: $\lambda(\alpha) = v(\alpha) + i$; $c_1(\alpha) = l_1(\alpha)$; $c_2(\alpha) = l_2(\alpha)$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = [v(\alpha) + i] \cdot \omega + l_1(\alpha) \cdot \omega \cdot |\omega|^2 + l_2(\alpha) \cdot \omega \cdot |\omega|^4 + O(|\omega|^6)$$

where $v(\alpha)$, $l_1(\alpha)$, $l_2(\alpha)$ are real functions, $v(\alpha = 0) = 0$. The real function $l_2(\alpha)$ is called the second Lyapunov coefficient. Suppose that at a Bautin point $l_1(\alpha = 0) \neq 0$ (B.1). A neighborhood of the point $\alpha = 0$ can be parametrized by two new parameters, the zero locus of the first one corresponding to the Hopf bifurcation condition, while the simultaneous vanishing of both specifies the Bautin point. $v(\alpha)$ is the first parameter and $l_1(\alpha)$ is the second parameter. Both are defined for all sufficiently small $\|\alpha\|$ and vanish at $\alpha = 0$. We define our new parameters $\mu_1 = v(\alpha)$ and $\mu_2 = l_1(\alpha)$ assuming its regularity at $\alpha = 0$ (B.2)

$$v(\alpha) = \frac{\mu(\alpha)}{\omega(\alpha)}|_{\alpha=0,\omega(\alpha=0)=\omega_0} = \frac{\mu(\alpha=0)}{\omega_0}; d_1(\alpha) = \frac{c_1(\alpha)}{\omega(\alpha)}|_{\alpha=0,\omega(\alpha=0)=\omega_0} = \frac{c_1(\alpha=0)}{\omega_0}$$

$d_2(\alpha) = \frac{c_2(\alpha)}{\omega(\alpha)}|_{\alpha=0,\omega(\alpha=0)=\omega_0} = \frac{c_2(\alpha=0)}{\omega_0}$; v, d_1, d_2 are complex values.

$$\det\begin{pmatrix} \frac{\partial v}{\partial \alpha_1} & \frac{\partial v}{\partial \alpha_2} \\ \frac{\partial l_1}{\partial \alpha_1} & \frac{\partial l_1}{\partial \alpha_2} \end{pmatrix}|_{\alpha=0} = \frac{1}{\omega_0} \cdot \det\begin{pmatrix} \frac{\partial \mu}{\partial \alpha_1} & \frac{\partial \mu}{\partial \alpha_2} \\ \frac{\partial l_1}{\partial \alpha_1} & \frac{\partial l_1}{\partial \alpha_2} \end{pmatrix}|_{\alpha=0} \neq 0$$

We can write α in terms of μ, thus obtaining the equation:

$$\frac{d\omega}{dt} = [\mu_1 + i] \cdot \omega + \mu_2 \cdot \omega \cdot |\omega|^2 + L_2(\mu) \cdot \omega \cdot |\omega|^4 + O\left(|\omega|^6\right); L_2(\mu) = l_2(\alpha(\mu))$$

L_2 is a smooth function of μ, such that $L_2(0) = l_2(0) \neq 0$ due to (B.1). then rescaling $\omega = \sqrt{4}|L_2(\mu)|u, u \in \mathbb{C}^1$; $\beta_1 = \mu_1$; $\beta_2 = \sqrt{|L_2(\mu)|}\mu_2$ and we get the normal form $\frac{du}{dt} = (\beta_1 + i) \cdot u + \beta_2 \cdot u \cdot |u|^2 + s \cdot u \cdot |u|^4 + O(|u|^6)$; $s = \text{sign} l_2(0) = \pm 1$.

The results of the last discussion: The planar system, f is smooth with equilibrium $V = 0$; $\frac{dV}{dt} = f(V, \alpha)$, $V \in \mathbb{R}^2$, $\alpha \in \mathbb{R}^2$. The eigenvalues are $\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i \cdot \omega(\alpha)$ for all $\|\alpha\|$ sufficiently small, where $\omega(0) = \omega_0 > 0$. For $\alpha = 0$, let the Bautin bifurcation conditions hold: $\mu(0) = 0$, $l_1(0) = 0$ and the system can be reduced to $\frac{dz}{dt} = (\beta_1 + i) \cdot z + \beta_2 \cdot z \cdot |z|^2 + s \cdot z \cdot |z|^4 + O(|z|^6)$ where $s = \text{sign} l_2(0) = \pm 1$. $l_1(\alpha)$ is the first Lyapunov coefficient and the following conditions hold: (B.1) $l_2(0) \neq 0$, where $l_2(0)$ is the second Lyapunov coefficient, (B.2). The map $\alpha \to (\mu(\alpha), l_1(\alpha))^T$ is regular at $\alpha = 0$. Bautin bifurcation diagram of the normal form: $s = -1$ and we write the system $\frac{dz}{dt} = (\beta_1 + i) \cdot z + \beta_2 \cdot z \cdot |z|^2 + \ldots$

Without $O(|z|^6)$ term in polar coordinates (r, φ), where $z = r \cdot e^{i \cdot \varphi} \to r = z \cdot e^{-i \cdot \varphi}$ $\cdot \frac{dr}{dt} = r \cdot (\beta_1 + \beta_2 \cdot r^2 - r^4)$; $\frac{d\varphi}{dt} = 1$. To find our fixed planar radius we set $dr/dt = 0$ and get $\frac{dr}{dt} = 0 \Rightarrow r \cdot (\beta_1 + \beta_2 \cdot r^2 - r^4) = 0$; $\frac{d\varphi}{dt} = 1$ which gives some results $r^{(0)} = 0$; $(\beta_1 + \beta_2 \cdot [r^{(i)}]^2 - [r^{(i)}]^4) = 0 \, \forall \, i \geq 1$ and we get $r^{(i)}$ values:

$$[r^{(i)}]^2 = \frac{\beta_2 \pm \sqrt{\beta_2^2 + 4 \cdot \beta_1}}{2} \Rightarrow r^{(i)} = \pm\sqrt{\frac{\beta_2 \pm \sqrt{\beta_2^2 + 4 \cdot \beta_1}}{2}}$$
$$= \pm \frac{1}{\sqrt{2}} \cdot \sqrt{\beta_2 \pm \sqrt{\beta_2^2 + 4 \cdot \beta_1}}$$

We have some possibilities which need to be inspected deeply. We move to differential equation with complex z variable $r \in \mathbb{R}^1 \to z \in \mathbb{C}^1$.

$$r = z \cdot e^{-i\cdot\varphi} \Rightarrow \frac{dr}{dt} = \left[\frac{dz}{dt} - i \cdot z \cdot \frac{d\varphi}{dt}\right] \cdot e^{-i\cdot\varphi}; r = z \cdot e^{-i\cdot\varphi};$$
$$r^2 = z^2 \cdot e^{-2\cdot i\cdot\varphi}; r^4 = z^4 \cdot e^{-4\cdot i\cdot\varphi}$$

$$\left[\frac{dz}{dt} - i \cdot z \cdot \frac{d\varphi}{dt}\right] \cdot e^{-i\cdot\varphi} = z \cdot e^{-i\cdot\varphi} \cdot (\beta_1 + \beta_2 \cdot z^2 \cdot e^{-2\cdot i\cdot\varphi} - z^4 \cdot e^{-4\cdot i\cdot\varphi})$$

$$\frac{dz}{dt} - i \cdot z \cdot \frac{d\varphi}{dt} = z \cdot \left(\beta_1 + \beta_2 \cdot z^2 \cdot e^{-2\cdot i\cdot\varphi} - z^4 \cdot e^{-4\cdot i\cdot\varphi}\right)$$
$$\Rightarrow \frac{dz}{dt} = i \cdot z \cdot \frac{d\varphi}{dt} + z \cdot \beta_1 + \beta_2 \cdot z^3 \cdot e^{-2\cdot i\cdot\varphi} - z^5 \cdot e^{-4\cdot i\cdot\varphi}$$

$$\frac{dz}{dt}\Big|_{\frac{d\varphi}{dt}=1} = z \cdot [i + \beta_1] + \beta_2 \cdot z^3 \cdot e^{-2\cdot i\cdot\varphi} - z^5 \cdot e^{-4\cdot i\cdot\varphi}$$
$$\Rightarrow \frac{dz}{dt}\Big|_{\frac{d\varphi}{dt}=1} = z \cdot [i + \beta_1] + \beta_2 \cdot z \cdot \{z^2 \cdot e^{-2\cdot i\cdot\varphi}\} - z \cdot \{z^4 \cdot e^{-4\cdot i\cdot\varphi}\}$$

$$z^2 \cdot e^{-2\cdot i\cdot\varphi} = [z \cdot e^{-i\cdot\varphi}]^2|_{r=z\cdot e^{-i\cdot\varphi}} = |z|^2; z^4 \cdot e^{-4\cdot i\cdot\varphi} = [z \cdot e^{-i\cdot\varphi}]^4|_{r=z\cdot e^{-i\cdot\varphi}} = |z|^4$$

We get the equation: $\frac{dz}{dt}|_{\frac{d\varphi}{dt}=1, O(|z|^6)=0, s=-1} = z \cdot [i + \beta_1] + \beta_2 \cdot z \cdot |z|^2 - z \cdot |z|^4$

We can see the effect of high-order terms which the system:

$$\frac{dz}{dt} = (\beta_1 + i) \cdot z + \beta_2 \cdot z \cdot |z|^2 \pm z \cdot |z|^4 + O\left(|z|^6\right)$$

Is locally topologically equivalent near the origin to the system: $\frac{dz}{dt} = (\beta_1 + i) \cdot z + \beta_2 \cdot z \cdot |z|^2 \pm z \cdot |z|^4$ and any generic two-parameter system. $\frac{dV}{dt} = f(V, \alpha)$, $V \in \mathbb{R}^2$, $\alpha \in \mathbb{R}^2$ having at $\alpha = 0$ an equilibrium $V = 0$ that exhibits the Bautin bifurcation, is locally topologically equivalent near the origin to one of the following complex normal form: $\frac{dz}{dt} = (\beta_1 + i) \cdot z + \beta_2 \cdot z \cdot |z|^2 \pm z \cdot |z|^4$. The multidimensional case of the Bautin bifurcation can be treated by the center manifold reduction to the studied planar case. The topological normal form for Bautin bifurcation in R^n:

$\frac{dz}{dt} = (\beta_1 + i) \cdot z + \beta_2 \cdot z \cdot |z|^2 + s \cdot z \cdot |z|^4$; $\frac{d\zeta_-}{dt} = -\zeta_-$; $\frac{d\zeta_+}{dt} = \zeta_+$. To find fixed points we set $\frac{dz}{dt} = 0; \frac{d\zeta_-}{dt} = 0; \frac{d\zeta_+}{dt} = 0 \Rightarrow z^{(0)} \cdot \{(\beta_1 + i) + \beta_2 \cdot |z^{(i)}|^2 + s \cdot |z^{(i)}|^4\} = 0$.

$z^{(0)} = 0; (\beta_1 + i) + \beta_2 \cdot |z^{(i)}|^2 + s \cdot |z^{(i)}|^4 = 0 \, \forall \, i \geq 1; \; \zeta_-^{(i)} = 0; \; \zeta_+^{(i)} = 0 \, \forall \, i \geq 0$
where $s = \mathrm{sign} l_2(0) = \pm 1$, $z \in \mathbb{C}^1$, $\zeta_\pm \in \mathbb{R}^{n_\pm}$, and n_- and n_+ are the numbers of eigenvalues of the critical equilibrium with $Re\lambda > 0$ and $Re\lambda < 0$, respectively, $n_- + n_+ + 2 = n$. For a system with planar Bautin bifurcation control functions, we consider $\frac{\mathrm{d}V}{\mathrm{d}t} = F(V) + G(V) \cdot u$; $V = (V_1, V_2)^{\mathrm{T}} \in \mathbb{R}^2$ is the state vector F, G: $\mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields, $u = u(V; \mu, \beta)$, $\mu \in \mathbb{R}^2$, $\beta \in \mathbb{R}^3$, and $G(V) \cdot u$ is the control input. Assume that $F(0) = 0$, $G(0) = 0$ and $J = \mathrm{d}F(0)$ has purely imaginary eigenvalues $\lambda_{1,2} = \pm\omega_0 \cdot I$, $\omega_0 \geq 0$, and $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ is Bautin bifurcation parameter, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ and tr(M) are the control parameters. β_1 determines the stability of the equilibrium point, β_2 determines the orientation and the stability of the limit cycle emerging at a Hopf bifurcation, β_3 and tr (M) establish the orientation and stability of the limit cycles emerging at the Bautin bifurcation.

$$M = dG(0) = \begin{pmatrix} \frac{\partial G_1}{\partial V_1} & \frac{\partial G_1}{\partial V_2} \\ \frac{\partial G_2}{\partial V_1} & \frac{\partial G_2}{\partial V_2} \end{pmatrix} |_{V=0} = \left(m_{ij}\right)_{ij=1,2}$$

We consider the following expression for the scalar function (u):$u(V, \mu, \beta) = \beta_1 \cdot \mu_1 + (\beta_2 + \mu_2) \cdot (V_1^2 + V_2^2) + \beta_3 \cdot (V_1^2 + V_2^2)^2$. We need to find values of the control parameters $\beta_1, \beta_2, \beta_3$ and tr(M) such that our system $\frac{\mathrm{d}V}{\mathrm{d}t} = F(V) + G(V) \cdot u$; $V = (V_1, V_2)^{\mathrm{T}} \in \mathbb{R}^2$ undergoes controllable Bautin bifurcations [5–9, 71, 72].

A.3 Bogdanov–Takens (Double-Zero) Bifurcation

We consider a two-parameter system $\frac{\mathrm{d}V}{\mathrm{d}t} = f(V, \alpha), V = (V_1, V_2, \ldots, V_n)^{\mathrm{T}} \in \mathbb{R}^n$

$\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}} \in \mathbb{R}^2$ and f is a sufficiently smooth function of (V, α). Suppose that at $\alpha = \alpha^0$, the system has an equilibrium $V = V^0$ for which either the fold or Hopf bifurcation conditions are satisfied. There is a bifurcation curve B in the (α_1, α_2) plane along which the system has an equilibrium exhibiting the same bifurcation. Γ is bifurcation curve in R^3 ($\Gamma \subset \mathbb{R}^3$). Let the parameters (α_1, α_2) be varied simultaneously to track a bifurcation curve Γ (or B). Then, the following events might happen to the monitored non-hyperbolic equilibrium at some parameters values: extra eigenvalues can approach the imaginary axis, thus changing the dimension of the center manifold W^c, some of the generality conditions for the codim 1 bifurcation can be violated. For nearby parameter values we can expect the appearance of new phase portraits of the system, implying that a codim 2 bifurcation has occurred. Only violating a non-degeneracy condition can produce new phase portraits. If $\frac{\partial}{\partial \alpha_i} \mathrm{Re}\lambda_{1,2}(\alpha) = 0$ at the Hopf bifurcation point, then, generically, the eigenvalues do not cross the imaginary axis as α_i passes the critical value. If we follow the fold bifurcation curve B_t, a typical point in this curve defines an equilibrium with a simple zero eigenvalue $\lambda_1 = 0$ and no other eigenvalues on the imaginary axis. The restriction of $\frac{dV}{d_t} = f(V, \alpha), V = (V_1, V_2, \ldots, V_n)^T \in \mathbb{R}^n$ to a center manifold W^c has the form $\frac{d\xi}{d_t} = b \cdot \xi^2 + O(\xi^3)$. An additional real eigenvalue λ_2 approaches the imaginary axis, and W^c becomes two-dimensional: $\lambda_{1,2} = 0$. These are the conditions for Bogdanov–Takens or double-zero bifurcation. To have this bifurcation we need $n \geq 2$. The Bogdanov-Takens bifurcation can also be located along Hopf bifurcation curve, as ω_0 approaches zero. At this point, two purely imaginary eigenvalues collide and we have double-zero eigenvalue ($\lambda_1 = \lambda_2 = 0$). Topological normal form for Bogdanov–Takens (BT) bifurcation: Any generic planar two-parameter system $\frac{dV}{d_t} = f(V, \alpha)$ having, at $\alpha = 0$, an equilibrium that exhibits the Bogdanov–Takens (BT) bifurcation, is locally topologically equivalent near equilibrium to one of the following normal forms: $\frac{d\eta_1}{d_t} = \eta_2; \frac{d\eta_2}{d_t} = \beta_1 + \beta_2 \cdot \eta_1 + \eta_1^2 \pm \eta_1 \cdot \eta_2$. The Bogdanov–Takens bifurcation gives rise to a limit cycle bifurcation, namely, the appearance of the homoclinic orbit, for nearby parameter values. There is a global bifurcation in the system. The following topological normal form for the Bogdanov–Takens bifurcation in R^n: $\frac{d\eta_1}{d_t} = \eta_2; \frac{d\eta_2}{d_t} = \beta_1 + \beta_2 \cdot \eta_1 + \eta_1^2 + s \cdot \eta_1 \cdot \eta_2; \frac{d\zeta_-}{d_t} = -\zeta_-; \frac{d\zeta_+}{d_t} = \zeta_+$

We need to find our system fixed points: $\frac{d\eta_1}{d_t} = 0; \frac{d\eta_2}{d_t} = 0; \frac{d\zeta_-}{d_t} = 0; \frac{d\zeta_+}{d_t} = 0$

And we get $\eta_2^{(i)} = 0$; $[\eta_1^{(i)}]^2 + \beta_2 \cdot \eta_1^{(i)} + \beta_1 = 0 \Rightarrow \eta_1^{(i)} = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4 \cdot \beta_1}}{2}$; $\zeta_-^{(i)} = 0$; $\zeta_+^{(i)} = 0$. Where $s = \text{sign}\{A(0) \cdot B(0)\} = \pm 1$, $(\eta_1, \eta_2)^T \in \mathbb{R}^2$; $\zeta_\pm \in \mathbb{R}^{n_\pm}$, and n_- and n_+ are the numbers of eigenvalues of the critical equilibrium with $\text{Re}\lambda > 0$ and $Re\lambda < 0$, $n_- + n_+ + 2 = n$ [5–9, 71, 72].

A.4 Fold–Hopf Bifurcation

We have a two-parameter system $\frac{dV}{d_t} = f(V, \alpha); V = (V_1, \ldots, V_n)^T \in \mathbb{R}^n; \alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2$.

And f is a sufficiently smooth function of (V, α). The restriction of $dV/dt = f(V, \alpha)$ to a center manifold W^C has the form $\frac{d\xi}{d_t} = b \cdot \xi^2 + O(\xi^2)$. When two extra complex eigenvalues $\lambda_{2,3}$ arrive at the imaginary axis, and W^C becomes three-dimensional: $\lambda_1 = 0$; $\lambda_{2,3} = \pm i \cdot \omega_0$, $\omega_0 > 0$. These conditions correspond to the fold–Hopf bifurcation (Gavrilov–Guckenheimer/Zero–pair bifurcation). We obviously need $n \geq 3$ for these bifurcations to occur. Now we consider a smooth three-dimensional system depending on two parameters: $\frac{dV}{d_t} = f(V, \alpha); V \in \mathbb{R}^{n=3}; \alpha \in \mathbb{R}^2$.

Suppose that at $\alpha = 0$ the system has the equilibrium $V = 0$ with one zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \lambda_{2,3} = \pm i \cdot \omega_0$, $\omega_0 > 0$. The following is the deviation of the normal form. We can expand $f(V, \alpha)$ with respect to V at $V = 0$, $\frac{dV}{dt} = f(V, \alpha) \Rightarrow \frac{dV}{dt} = a(\alpha) + A(\alpha) \cdot V + F(V, \alpha)$ where $a(0) = 0$, $F(V, \alpha) = O(\|V\|^2)$. Since the eigenvalues $\lambda_1 = 0$; $\lambda_{2,3} = \pm i \cdot \omega_0$, $\omega_0 > 0$ of the matrix $A(0)$ are simple, the matrix $A(\alpha)$ has simple eigenvalues $\lambda_1(\alpha) = v(\alpha)$ and $\lambda_{2,3}(\alpha) = \mu(\alpha) \pm i \cdot \omega(\alpha)$ for all sufficiently small $\|\alpha\|$, where v, μ and ω are smooth functions of α such that $v(\alpha = 0) = \mu(\alpha = 0) = 0$, $\omega(0) = \omega_0 > 0$. These eigenvalues are the eigenvalues of the equilibrium $V = 0$ at $\alpha = 0$ but typically $a(\alpha) \neq 0$ for nearby parameter values and the matrix $A(\alpha)$ is not the Jacobian matrix of any equilibrium point of $\frac{dV}{dt} = a(\alpha) + A(\alpha) \cdot V + F(V, \alpha)$. The matrix $A(\alpha)$ is well defined and has two smoothly parameter-dependent eigenvalues:

$q_0(\alpha) \in \mathbb{R}^3; q_1(\alpha) \in \mathbb{C}^3$ corresponding to the eigenvalues $v(\alpha)$ and $\lambda(\alpha) = \mu(\alpha) + i \cdot \omega(\alpha)$, respectively $A(\alpha) \cdot q_0(\alpha) = v(\alpha) \cdot q_0(\alpha)$ an$A(\alpha) \cdot q_1(\alpha) = \lambda(\alpha) \cdot q_1(\alpha)$. The ad joint eigenvectors $P_0(\alpha) \in \mathbb{R}^3 \& P_1(\alpha) \in \mathbb{C}^3$ can be defined by $A^T(\alpha) \cdot P_0(\alpha) = v(\alpha) \cdot P_0(\alpha); A^T(\alpha) \cdot P_1(\alpha) = \overline{\lambda}(\alpha) \cdot P_1(\alpha)$. Normalize the eigenvectors such that $\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1$ for all $\|\alpha\|$ small. The following orthogonally properties follow from the Fredholm alternative theorem:

$\langle p_1, q_0 \rangle = \langle p_0, q_1 \rangle = 0$. Any real vector V can be represented as $V = u \cdot q_0(\alpha) + z \cdot q_1(\alpha) + \overline{z} \cdot \overline{q_1}(\alpha)$ with $u = \langle P_0(\alpha), V \rangle; z = \langle P_1(\alpha), V \rangle$ and in coordinates $u \in \mathbb{R}^1; z \in \mathbb{C}^1$ we get the following system differential equations:

$\frac{du}{dt} = \Gamma(\alpha) + v(\alpha) \cdot u + g(u, z, \overline{z}, \alpha)$; $\frac{dz}{dt} = \Omega(\alpha) + \lambda(\alpha) \cdot z + h(u, z, \overline{z}, \alpha)$, and $\Gamma(\alpha) = \langle P_0(\alpha), a(\alpha) \rangle; \Omega(\alpha) = \langle P_1(\alpha), a(\alpha) \rangle$ are smooth functions of α, $\Gamma(0) = 0$, $\Omega(0) = 0$ and the following notations:

(*) $g(u, z, \overline{z}, \alpha) = \langle P_0(\alpha), F(u \cdot q_0(\alpha) + z \cdot q_1(\alpha) + \overline{z} \cdot \overline{q_1}(\alpha), \alpha) \rangle$

(**) $h(u, z, \overline{z}, \alpha) = \langle P_1(\alpha), F(u \cdot q_0(\alpha) + z \cdot q_1(\alpha) + \overline{z} \cdot \overline{q_1}(\alpha), \alpha) \rangle$

Are smooth functions of $u, z, \overline{z}, \alpha$ whose Taylor expansions in the first three arguments start with quadratic terms:

$$g(u, z, \overline{z}, \alpha) = \sum_{j+k+l \geq 2} \frac{1}{j! \cdot k! \cdot l!} \cdot g_{jkl}(\alpha) \cdot u^j \cdot z^k \cdot \overline{z}^l$$

$$h(u, z, \overline{z}, \alpha) = \sum_{j+k+l \geq 2} \frac{1}{j! \cdot k! \cdot l!} \cdot h_{jkl}(\alpha) \cdot u^j \cdot z^k \cdot \overline{z}^l$$

$\gamma(\alpha)$ is real, and since g must be real, we have $g_{jkl}(\alpha) = \overline{g_{jkl}}(\alpha)$. Functions g_{jkl} and h_{jkl} can be computed by formal differentiation of expressions (*) and (**) with respect to $u, z, \overline{z}$. In the three-dimensional case we already describe the fold–Hopf bifurcation analytically by system $\frac{dV}{dt} = f(V, \alpha) \forall \alpha \in \mathbb{R}^n$, $n = 3 \Rightarrow V \in \mathbb{R}^3$. If the non-degeneracy conditions hold: (ZH.1) $B(0) \cdot C(0) \cdot E(0) \neq 0$ and (ZH.2) map $(V, \alpha) \rightarrow (f(V, \alpha), Tr(f_V(V, \alpha)), \det(f_V(V, \alpha)))$ is regular at $(V, \alpha) = (0, 0)$. This system is locally orbit ally smoothly equivalent near the origin to the complex normal form:

$\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot |\zeta|^2 + O(||(\xi, \zeta)||^4)$ and $\frac{d\zeta}{dt} = (\beta_2 + i \cdot \omega) \cdot \zeta + (\theta(\beta) + i \cdot \theta_1(\beta)) \cdot \xi \cdot \zeta + \xi^2 \cdot \zeta + O(||(\xi, \zeta)||^4)$.

Where $\xi \in \mathbb{R}$, $\zeta \in \mathbb{C}$, $\beta \in \mathbb{R}^2$, $s = \mathrm{sign}B(0) \cdot C(0) = \pm 1$, $\theta(0) = \frac{ReH_{110}}{B(0)}$.

When $E(0) < 0$, the orbital equivalence includes reversal of time.

The bifurcation diagram of the normal form depends on the O-terms, and its essential features are determined by the form:

$$\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot r^2; \; \frac{dr}{dt} = r \cdot (\beta_2 + \theta(\beta) \cdot \xi + \xi^2); \; \frac{d\varphi}{dt} = \omega + \theta_1(\beta) \cdot \xi.$$

The first two equations are independent of the third one, which describes a monotone rotation. The local bifurcation diagrams of the planar system:

$\frac{d\xi}{dt} = \beta_1 + \xi^2 + s \cdot r^2; \frac{dr}{dt} = r \cdot (\beta_2 + \theta(\beta) \cdot \xi + \xi^2)$ with (ZH.3) $\theta(0) \neq 0$.

We can distinguish four cases:

(1) $s = 1$, $\theta(0) > 0$ (subcritical Hopf bifurcations and no tori).
(2) $s = -1$, $\theta(0) < 0$ (subcritical Hopf bifurcations and no tori).
(3) $s = 1$, $\theta(0) < 0$ (sub and supercritical Hopf bifurcations and torus heteroclinic destruction).
(4) $s = -1$, $\theta(0) > 0$ (sub and supercritical Hopf bifurcations and torus blow up).

The normal form coefficients which are involved in the non-degeneracy conditions (ZH.1) and (ZH.3) can be computed for $n \geq 3$ and we write the Taylor expansion of $f(V, 0)$ at $V = 0$. We move from $V \to x$ and get

$f(x, \alpha) = A \cdot x + \frac{1}{2} \cdot B(x, x) + \frac{1}{6} \cdot C(x, x, x) + O(||x||^4)$ where $B(x, y)$ and $C(x, y, z)$ are the multi linear functions with components:

$$B_j(x, y) = \sum_{k,l=1}^{n} \frac{\partial^2 f_j(\xi, 0)}{\partial \xi_k \partial \xi_l}|_{\xi=0} \cdot x_k \cdot y_l; \; C_j(x, y, z) = \sum_{k,l,m=1}^{n} \frac{\partial^3 f_j(\xi, 0)}{\partial \xi_k \partial \xi_l \partial \xi_m}|_{\xi=0} \cdot x_k \cdot y_l \cdot z_m$$

$j = 1, 2, \ldots, n$. We have two eigenvectors $q_0 \in \mathbb{R}^n; q_1 \in \mathbb{C}^n, A \cdot q_0 = 0, A \cdot q_1 = i \cdot \omega \cdot q_1$ and two ad joint eigenvectors $p_0 \in \mathbb{R}^n; p_1 \in \mathbb{C}^n$, $A^T \cdot p_0 = 0$, $A^T \cdot p_1 = -i \cdot \omega \cdot p_1$. Normalize them such that $\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1$. $\langle n, m \rangle$ is the inner product of two vectors. $G_{200} = \frac{1}{2} \cdot \langle p_0, B(q_0, q_0) \rangle$; $H_{110} = \langle p_1, B(q_0, q_1) \rangle$; $G_{011} = \langle p_0, B(q_1, \overline{q}_1) \rangle$.

$$G_{300} = \frac{1}{6} \cdot \langle p_0, C(q_0, q_0, q_0) + 3 \cdot B(q_0, h_{200}) \rangle$$

$$G_{111} = \langle p_0, C(q_0, q_1, \overline{q}_1) + B(q_1, \overline{h}_{110}) + B(\overline{q}_1, h_{110}) + B(q_0, h_{011}) \rangle$$

$$H_{210} = \frac{1}{2} \cdot \langle p_1, C(q_0, q_0, q_1) + 2 \cdot B(q_0, h_{110}) + B(q_1, h_{200}) \rangle$$

$$H_{021} = \frac{1}{2} \cdot \langle p_1, C(q_1, q_1, \overline{q}_1) + 2 \cdot B(q_1, h_{011}) + B(\overline{q}_1, h_{020}) \rangle;\ h_{020} = (2 \cdot i \cdot \omega \cdot I_n - A)^{-1} \cdot B(q_1, q_1)$$

While the vectors h_{200}, h_{011}, h_{110} are the solutions of the following non-singular systems:

$$\begin{pmatrix} A & q_0 \\ p_0^{\mathrm{T}} & 0 \end{pmatrix} \cdot \begin{pmatrix} h_{200} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_0, q_0) + \langle p_0, B(q_0, q_0) \rangle \cdot q_0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A & q_0 \\ p_0^{\mathrm{T}} & 0 \end{pmatrix} \cdot \begin{pmatrix} h_{011} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_1, \overline{q}_1) + \langle p_0, B(q_1, \overline{q}_1) \rangle \cdot q_0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i\omega I_n - A & q_1 \\ \overline{p}_1^{\mathrm{T}} & 0 \end{pmatrix} \cdot \begin{pmatrix} h_{110} \\ s \end{pmatrix} = \begin{pmatrix} B(q_0, q_1) - \langle p_1, B(q_0, q_1) \rangle \cdot q_1 \\ 0 \end{pmatrix}$$

$B(0) = G_{200}$; $C(0) = G_{011}$

$$E(0) = \mathrm{Re}\left[H_{210}(0) + H_{110}(0) \cdot \left(\frac{\mathrm{Re} H_{021}(0)}{G_{011}(0)} - \frac{3 \cdot G_{300}(0)}{2 \cdot G_{200}(0)} + \frac{G_{111}(0)}{2 G_{011}(0)} \right) - \frac{H_{021}(0) \cdot G_{200}(0)}{G_{011}(0)} \right]$$

The following theorem is for simple fold-Hopf bifurcation. We consider the system $V \to x \Rightarrow \frac{dx}{dt} = f(x, \alpha)$, $x \in \mathbb{R}^3$, $\alpha \in \mathbb{R}^2$ with smooth f function, has at $\alpha = 0$ the equilibrium at $x = 0$ with eigenvalues: $\lambda_1(0) = 0$, $\lambda_{2,3} = \pm i \cdot \omega_0$, $\omega_0 > 0$.

Let the following generality conditions hold: (ZH0.1) $G_{200}(0) \cdot G_{011}(0) > 0$, (ZH0.2) $\theta_0 = \frac{\mathrm{Re} H_{110}(0)}{G_{200}(0)} > 0$. (ZH0.3) the map $\alpha \to (\gamma(\alpha), \mu(\alpha))^{\mathrm{T}}$ is regular at $\alpha = 0$. Then the system is locally topologically equivalent near the origin to the system: $\frac{d\rho}{dt} = \beta_2 \cdot \rho + \theta_0 \cdot \zeta \cdot \rho$; $\frac{d\varphi}{dt} = 1$; $\frac{d\zeta}{dt} = \beta_1 + \zeta^2 + \rho^2$, where (ρ, φ, ζ) are cylindrical coordinates. ρ is the radial distance, Euclidean distance from the z axis to the point P. φ azimuth which is the angle between the reference direction on the chosen plane and the line from the origin to the projection of P on the plane. ζ is the height, signed distance from the chosen plane to the point P. For the conversion between cylindrical and Cartesian coordinate systems, it is convenient to assume that the reference plane of the former is the Cartesian X–Y plane with equation $Z = 0$, and the cylindrical axis is the Cartesian z-axis. Then the z coordinate is the same in both systems, and the correspondence between cylindrical (ρ, φ) and Cartesian (X, Y) are the same as for polar coordinates $(\rho, \varphi) \to (X, Y)$. $X = \rho \cdot \cos\varphi$; $Y = \rho \cdot \sin\varphi$; $\rho = \sqrt{X^2 + Y^2}$.

$X = 0 \,\&\, Y = 0 \Rightarrow \varphi = 0;\; X \geq 0 \Rightarrow \varphi = \arcsin(Y/\rho);\; X < 0 \Rightarrow \varphi = -\arcsin(Y/\rho) + \pi.$

$$\frac{d\rho}{dt} = \beta_2 \cdot \rho + \theta_0 \cdot \zeta \cdot \rho \Rightarrow \zeta = \left[\frac{d\rho/dt}{\rho} - \beta_2\right] \cdot \frac{1}{\theta_0} \Rightarrow \frac{d\zeta}{dt} = \left\{\frac{d^2\rho/dt^2 \cdot \rho - [d\rho/dt]^2}{\rho^2}\right\} \cdot \frac{1}{\theta_0}$$

$$\frac{d\zeta}{dt} = \beta_1 + \zeta^2 + \rho^2 \Rightarrow \left\{\frac{d^2\rho/dt^2 \cdot \rho - [d\rho/dt]^2}{\rho^2}\right\} \cdot \frac{1}{\theta_0} = \beta_1 + \left[\frac{d\rho/dt}{\rho} - \beta_2\right]^2 \cdot \frac{1}{\theta_0^2} + \rho^2$$

$$\left\{\frac{d^2\rho/dt^2 \cdot \rho - [d\rho/dt]^2}{\rho^2}\right\} \cdot \theta_0 = \left[\frac{d\rho/dt}{\rho} - \beta_2\right]^2 + \left(\beta_1 + \rho^2\right) \cdot \theta_0^2$$

$$\ddot{\rho} = \frac{d^2\rho}{dt^2} \,\&\, \dot{\rho} = \frac{d\rho}{dt} \Rightarrow \left\{\frac{\ddot{\rho} \cdot \rho - [\dot{\rho}]^2}{\rho^2}\right\} \cdot \theta_0 = \left[\frac{\dot{\rho}}{\rho} - \beta_2\right]^2 + \left(\beta_1 + \rho^2\right) \cdot \theta_0^2$$

$$\rho = \sqrt{X^2 + Y^2} \Rightarrow \frac{d\rho}{dt} = \left(X^2 + Y^2\right)^{-\frac{1}{2}} \cdot \left(X \cdot \dot{X} + Y \cdot \dot{Y}\right) = \frac{X \cdot \dot{X} + Y \cdot \dot{Y}}{\sqrt{X^2 + Y^2}}$$

$$\frac{d^2\rho}{dt^2} = \frac{\left\{\left[\dot{X}\right]^2 + X \cdot \ddot{X} + \left[\dot{Y}\right]^2 + Y \cdot \ddot{Y}\right\} \cdot \sqrt{X^2 + Y^2} - \frac{\{X \cdot \dot{X} + Y \cdot \dot{Y}\}}{\sqrt{X^2 + Y^2}} \cdot \left\{X \cdot \dot{X} + Y \cdot \dot{Y}\right\}}{X^2 + Y^2}$$

$$\frac{d^2\rho}{dt^2} = \frac{\left\{\left[\dot{X}\right]^2 + X \cdot \ddot{X} + \left[\dot{Y}\right]^2 + Y \cdot \ddot{Y}\right\} \cdot (X^2 + Y^2) - \left\{X \cdot \dot{X} + Y \cdot \dot{Y}\right\}^2}{(X^2 + Y^2) \cdot \sqrt{X^2 + Y^2}}$$

$$\ddot{\rho} \cdot \rho = \frac{\left\{\left[\dot{X}\right]^2 + X \cdot \ddot{X} + \left[\dot{Y}\right]^2 + Y \cdot \ddot{Y}\right\} \cdot (X^2 + Y^2) - \left\{X \cdot \dot{X} + Y \cdot \dot{Y}\right\}^2}{(X^2 + Y^2)}$$

$$[\dot{\rho}]^2 = \frac{\left[X \cdot \dot{X} + Y \cdot \dot{Y}\right]^2}{X^2 + Y^2};\; \frac{\dot{\rho}}{\rho} = \left[\frac{X \cdot \dot{X} + Y \cdot \dot{Y}}{\sqrt{X^2 + Y^2}}\right] \cdot \frac{1}{\sqrt{X^2 + Y^2}} = \frac{X \cdot \dot{X} + Y \cdot \dot{Y}}{X^2 + Y^2}$$

We get the first system Cartesian (X, Y) differential equation:

$$\left\{\frac{\frac{\{[\dot{X}]^2+X\cdot\ddot{X}+[\dot{Y}]^2+Y\cdot\ddot{Y}\}\cdot(X^2+Y^2)-\{X\cdot\dot{X}+Y\cdot\dot{Y}\}^2}{(X^2+Y^2)}-\frac{[X\cdot\dot{X}+Y\cdot\dot{Y}]^2}{X^2+Y^2}}{(X^2+Y^2)}\right\}\cdot\theta_0$$
$$=\left[\frac{X\cdot\dot{X}+Y\cdot\dot{Y}}{X^2+Y^2}-\beta_2\right]^2+\left(\beta_1+X^2+Y^2\right)\cdot\theta_0^2$$

$$\left\{\frac{\left\{[\dot{X}]^2+X\cdot\ddot{X}+[\dot{Y}]^2+Y\cdot\ddot{Y}\right\}\cdot(X^2+Y^2)-2\cdot\left\{X\cdot\dot{X}+Y\cdot\dot{Y}\right\}^2}{(X^2+Y^2)^2}\right\}\cdot\theta_0$$
$$=\left[\frac{X\cdot\dot{X}+Y\cdot\dot{Y}}{X^2+Y^2}-\beta_2\right]^2+\left(\beta_1+X^2+Y^2\right)\cdot\theta_0^2$$

The next step is to get the second system Cartesian (X, Y) differential equation:

$$\varphi=\arcsin\left(\frac{Y}{\rho}\right)=\arcsin\left(\frac{Y}{\sqrt{X^2+Y^2}}\right);g_1(X,Y)=\frac{Y}{\sqrt{X^2+Y^2}}\Rightarrow\varphi$$
$$=\arcsin g_1(X,Y)$$

$$\frac{d\varphi}{dt}=\frac{d\varphi}{dg_1}\cdot\frac{dg_1}{dt};\frac{d\varphi}{dg_1}=\frac{1}{\sqrt{1-g_1^2}}=\frac{1}{\sqrt{1-\frac{Y^2}{X^2+Y^2}}};\frac{dg_1}{dt}$$
$$=\frac{\dot{Y}\cdot(X^2+Y^2)-Y\cdot\left(X\cdot\dot{X}+Y\cdot\dot{Y}\right)}{(X^2+Y^2)\cdot\sqrt{X^2+Y^2}}$$

$$\frac{dg_1}{dt}=\frac{\dot{Y}\cdot(X^2+Y^2)-Y\cdot\left(X\cdot\dot{X}+Y\cdot\dot{Y}\right)}{(X^2+Y^2)\cdot\sqrt{X^2+Y^2}}\Rightarrow\frac{dg_1}{dt}=\frac{\dot{Y}\cdot X^2-Y\cdot X\cdot\dot{X}}{(X^2+Y^2)\cdot\sqrt{X^2+Y^2}}$$

$$\frac{d\varphi}{dt}=\frac{d\varphi}{dg_1}\cdot\frac{dg_1}{dt}=\frac{1}{\sqrt{1-\frac{Y^2}{X^2+Y^2}}}\cdot\frac{\dot{Y}\cdot X^2-Y\cdot X\cdot\dot{X}}{(X^2+Y^2)\cdot\sqrt{X^2+Y^2}}=\frac{\dot{Y}X-Y\dot{X}}{X^2+Y^2}$$

$$\frac{d\varphi}{dt}=1\Rightarrow1=\frac{\dot{Y}X-Y\dot{X}}{X^2+Y^2}\Rightarrow X^2+Y^2=\dot{Y}X-Y\dot{X}$$

Back to our system cylindrical (ρ, φ, ζ) representation:
$\frac{d\rho}{dt}=\beta_2\cdot\rho+\theta_0\cdot\zeta\cdot\rho;\frac{d\varphi}{dt}=1;\frac{d\zeta}{dt}=\beta_1+\zeta^2+\rho^2$. We can find our fixed point or the radial distance (ρ) and height (ζ) which $d\rho/dt = 0$ & $d\zeta/dt = 0$ (limit cycle). The point (P) go around with $\frac{d\varphi}{dt}=1$. If $\frac{d\rho}{dt}=0\Rightarrow\rho\cdot[\beta_2+\theta_0\cdot\zeta]=0\Rightarrow\rho^{(0)}=0$

$\rho^{(0)} = 0 \,\&\, \frac{d\zeta}{dt} = 0 \Rightarrow \beta_1 + [\zeta^{(0)}]^2 = 0 \Rightarrow \zeta^{(0)} = \pm\sqrt{-\beta_1}\,\forall\,\beta_1 \le 0$. The first fixed point $(\rho^{(0)}, \zeta^{(0)}) = (0, \pm\sqrt{-\beta_1})\,\forall\,\beta_1 \le 0$. If $\frac{d\rho}{dt} = 0 \Rightarrow \rho \cdot [\beta_2 + \theta_0 \cdot \zeta] = 0$ $\Rightarrow \zeta^{(1)} = -\frac{\beta_2}{\theta_0}$

$$\zeta^{(1)} = -\frac{\beta_2}{\theta_0} \,\&\, \frac{d\zeta}{dt} = 0 \Rightarrow \beta_1 + [\frac{\beta_2}{\theta_0}]^2 + [\rho^{(1)}]^2 = 0 \Rightarrow \rho^{(1)} = \pm\sqrt{-\{\beta_1 + [\frac{\beta_2}{\theta_0}]^2\}}\,\forall\,\beta_1 + [\frac{\beta_2}{\theta_0}]^2 \le 0$$

$\beta_1 + [\frac{\beta_2}{\theta_0}]^2 \le 0 \Rightarrow \beta_1 \le -[\frac{\beta_2}{\theta_0}]^2$.

The second fixed point $(\rho^{(1)}, \zeta^{(1)}) = (\pm\sqrt{-\{\beta_1 + [\frac{\beta_2}{\theta_0}]^2\}}, -\frac{\beta_2}{\theta_0})\,\forall\,\beta_1 \le -[\frac{\beta_2}{\theta_0}]^2$. Since the radial distance is non-negative value, $\rho \ge 0$ then the second fixed point $(\rho^{(1)}, \zeta^{(1)}) = (\sqrt{-\{\beta_1 + [\frac{\beta_2}{\theta_0}]^2\}}, -\frac{\beta_2}{\theta_0})\,\forall\,\beta_1 \le -[\frac{\beta_2}{\theta_0}]^2$ [5–9, 71, 72].

A.5 Hopf–Hopf Bifurcation

We consider a two-parameter system $\frac{dV}{dt} = f(V, \alpha), V = (V_1, V_2, \ldots, V_n)^T \in \mathbb{R}^n$

$\alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2$ and f is a sufficiently smooth function of (V, α). If we follow a Hopf bifurcation curve B_H in the system at a typical point on this curve, the system has an equilibrium with a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i \cdot \omega_0$, $\omega_0 > 0$ and no other eigenvalues with Re$\lambda = 0$. The center manifold W^C is two dimensional in this case, and there are polar coordinates (ρ, φ) for which the restriction of $\frac{dV}{dt} = f(V, \alpha), V = (V_1, V_2, \ldots, V_n)^T \in \mathbb{R}^n$ to this manifold is orbit ally equivalent to $\frac{d\rho}{dt} = l_1 \cdot \rho^3 + O(\rho^4)$; $\frac{d\varphi}{dt} = 1 + O(\rho^3)$. $l_1 \neq 0$ at a non-degenerate Hopf point. While moving along the curve, we can encounter two extra complex conjugate eigenvalues $\lambda_{3,4}$ approach the imaginary axis, W^C becomes four-dimensional: $\lambda_{1,2} = \pm i \cdot \omega_0$, $\lambda_{3,4} = \pm i \cdot \omega_1$; $\omega_0 > 0$ and $\omega_1 > 0$. These conditions define the Hopf–Hopf or two pair bifurcation. It is possible only if $n \ge 4$. The Hopf–Hopf bifurcation is a bifurcation of an equilibrium point in a two-parameter family of autonomous ODEs at which the critical equilibrium has two pairs of purely imaginary eigenvalues (double-Hopf bifurcation). The bifurcation point in the parameter plane lies at a transversal intersection of two curves of Andronov–Hopf bifurcations, two branches of torus bifurcations emanate from Hopf–Hopf (HH) point. Back to our two-parameter system $\frac{dV}{dt} = f(V, \alpha), V = (V_1, V_2, \ldots, V_n)^T \in \mathbb{R}^n$; $\alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2$, suppose that at $\alpha = 0$ the system has an equilibrium $V = 0$. Assume that its Jacobian matrix $A = f_V(0, 0)$ has two pairs of purely imaginary eigenvalues $\lambda_{1,2} = \pm i \cdot \omega_1(\alpha = 0)$; $\lambda_{3,4} = \pm i \cdot \omega_2(\alpha = 0)$.

With $\omega_1(\alpha = 0) > \omega_2(\alpha = 0)$. This co-dimension two bifurcation is characterized by the conditions $\mathrm{Re}\lambda_{1,2} = 0\,\&\,\mathrm{Re}\lambda_{3,4} = 0$ and appears in the open sets of two-parameter families of smooth ODEs. Two Andronov–Hopf bifurcation curves intersect transversally at $\alpha = 0$ and two-torus bifurcation curves emanate from the point $\alpha = 0$. In a small fixed neighborhood of $V = 0$ for parameter values sufficiently close to $\alpha = 0$, the system has at most one equilibrium, which can undergo the Andronov–Hopf bifurcations, producing limit cycle. Each torus bifurcation of these limit cycles generates an invariant two-dimensional torus with periodic or quasiperiodic orbits. We consider now system with $n = 4$,

$\frac{dV}{dt} = f(V, \alpha), V = (V_1, .., V_4)^{\mathrm{T}} \in \mathbb{R}^{n=4}$. If the following non-degeneracy conditions hold: $k \cdot \omega_1(\alpha = 0) \neq l \cdot \omega_2(\alpha = 0)\,\forall k, l > 0; k + l \leq 3$. The map $\alpha \rightarrow (\mathrm{Re}\lambda_1(\alpha), \mathrm{Re}\lambda_3(\alpha))$.

Where $\lambda_{1,3}(\alpha)$ are eigenvalues of the continuation of the critical equilibrium for small $\|\alpha\|$ such that $\lambda_1(\alpha = 0) = i \cdot \omega_1(\alpha = 0); \lambda_3(\alpha = 0) = i \cdot \omega_2(\alpha = 0)$, is regular at $\alpha = 0$. This system is locally orbitally smoothly equivalent near the origin to the Poincare normal form:

$$\frac{dv_1}{dt} = (\beta_1 + i \cdot \omega_1(\beta)) \cdot v_1 + G_{2100}(\beta) \cdot v_1 \cdot |v_1|^2 + G_{1011}(\beta) \cdot v_1 \cdot |v_2|^2 + O\Big(||(v_1, \overline{v_1}, v_2, \overline{v_2})||^4\Big)$$

$$\frac{dv_2}{dt} = (\beta_2 + i \cdot \omega_2(\beta)) \cdot v_2 + H_{1110}(\beta) \cdot v_2 \cdot |v_1|^2 + H_{0021}(\beta) \cdot v_2 \cdot |v_2|^2 + O\Big(||(v_1, \overline{v_1}, v_2, \overline{v_2})||^4\Big)$$

Where $v_1, v_2 \in \mathbb{C}; \beta \in \mathbb{R}^2$ and $G_{jklm}(\beta)$, $H_{jklm}(\beta)$ are complex values smooth functions. We will discuss later the formulas for $G_{2100}(\beta = 0)$, $G_{1011}(\beta = 0)$, $H_{1110}(\beta = 0)$, and $H_{0021}(\beta = 0)$. The normal form is particularly simple in polar coordinates (r_k, φ_k), $k = 1, 2$, where it takes the form:

$$\frac{dr_1}{dt} = r_1 \cdot \left[\beta_1 + \alpha_{11}(\beta) \cdot r_1^2 + \alpha_{12}(\beta) \cdot r_2^2\right] + O\Big(\left(r_1^2 + r_2^2\right)^2\Big)$$

$$\frac{dr_2}{dt} = r_2 \cdot \left[\beta_2 + \alpha_{21}(\beta) \cdot r_1^2 + \alpha_{22}(\beta) \cdot r_2^2\right] + O\Big(\left(r_1^2 + r_2^2\right)^3\Big)$$

$$\frac{d\varphi_1}{dt} = \omega_1(\beta) + O\big(r_1^2 + r_2^2\big); \frac{d\varphi_2}{dt} = \omega_2(\beta) + O\big(r_1^2 + r_2^2\big)$$

$$\alpha_{11}(\beta) = \mathrm{Re}G_{2100}(\beta); \alpha_{12}(\beta) = \mathrm{Re}G_{1011}(\beta);\ \alpha_{21}(\beta) = \mathrm{Re}H_{1110}(\beta);\ \alpha_{22}(\beta) = \mathrm{Re}H_{0021}(\beta)$$

Remark The O—terms are 2π periodic in φ_k. The bifurcation diagram of the normal form depends on the O—terms, some of its features are determined by the truncated normal form: $\frac{d\varphi_1}{d_t} = \omega_1(\beta)$; $\frac{d\varphi_2}{d_t} = \omega_2(\beta)$

$$\frac{dr_1}{dt} = r_1 \cdot \left[\beta_1 + \alpha_{11}(\beta) \cdot r_1^2 + \alpha_{12}(\beta) \cdot r_2^2\right]; \frac{dr_2}{dt} = r_2 \cdot \left[\beta_2 + \alpha_{21}(\beta) \cdot r_1^2 + \alpha_{22}(\beta) \cdot r_2^2\right]$$

The first two equations are independent of the last two defining monotone rotations. The local bifurcation diagrams of the planar amplitude system:

$$\frac{dr_1}{dt} = r_1 \cdot \left[\beta_1 + \alpha_{11}(\beta) \cdot r_1^2 + \alpha_{12}(\beta) \cdot r_2^2\right]; \frac{dr_2}{dt} = r_2 \cdot \left[\beta_2 + \alpha_{21}(\beta) \cdot r_1^2 + \alpha_{22}(\beta) \cdot r_2^2\right]$$

Satisfying some extra generality conditions and there are two cases which should be distinguished: first when there is no periodic orbits in the amplitude system $a_{11}(0) \cdot a_{22}(0) > 0$ and second when there is a possible periodic and heteroclinic orbits in the amplitude system $a_{11}(0) \cdot a_{22}(0) < 0$. Each case includes many sub-cases depending on $\theta = \frac{a_{12}(0)}{a_{22}(0)}; \delta = \frac{a_{21}(0)}{a_{11}(0)}$. The equilibrium $r_1 = 0$; $r_2 = 0$ of the amplitude system corresponds to the equilibrium of the 4D system $\frac{dV}{d_t} = f(V, \alpha), V = (V_1, .., V_4)^T \in \mathbb{R}^{n=4}$. Nonzero equilibrium $(r_1, 0)$ and $(0, r_2)$ corresponds to limit cycles, while positive equilibria (r_1, r_2); $r_1 > 0$, $r_2 > 0$ correspond to invariant 2D tori. The limit cycles of the amplitude system correspond to invariant 3D tori. The appearance of an equilibrium $(r_1, 0)$ and $(0, r_2)$ in the amplitude system corresponds to Andronov–Hopf bifurcation, while branching of a positive equilibrium from one of the above implies a torus bifurcation of the corresponding limit cycle. The cubic coefficients in the normal form can be computed for $n \geq 4$. First we write the Taylor expansion of $f(V, \alpha = 0)$ at $V = 0$ as $f(V, \alpha) = A \cdot V + \frac{1}{2} \cdot B(V, V) + \frac{1}{6} \cdot C(V, V, V) + O(||V||^4)$.

Where $B(x, y)$ and $C(x, y, z)$ are the multilinear functions with components:

$$B_j(x, y) = \sum_{k,l=1}^{n} \frac{\partial^2 f_j(\xi, 0)}{\partial \xi_k \partial \xi_l}|_{\xi=0} \cdot x_k \cdot y_l; \; C_j(x, y, z) = \sum_{k,l,m=1}^{n} \frac{\partial^3 f_j(\xi, 0)}{\partial \xi_k \partial \xi_l \partial \xi_m}|_{\xi=0} \cdot x_k \cdot y_l \cdot z_m; \quad j = 1, 2, \ldots, n$$

[5–9, 71, 72].

A.6 Neimark–Sacker (Torus) Bifurcation

Continuous time dynamical systems with phase space dimension $n > 2$ can have invariant tori. An invariant two-dimensional torus T^2 appears through a generic Neimark–Sacker bifurcation. A stable cycle in R^3 can lose stability when a pair of complex conjugate multipliers crosses the unit cycle. Then, provided there are no strong resonances and the cubic normal form coefficient has the proper sign, a smooth, stable, invariant torus bifurcates from the cycle. There are changes of the orbit structure on an invariant two-torus under variation of the parameters of the system. We discuss now the "normal form" of Neimark–Sacker bifurcation. We can describe two-dimensional discrete-time system depending on one parameter by the following transformation:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to (1+\alpha) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Where α is the parameter; $\theta = \theta(\alpha)$; $b = b(\alpha)$; $d = d(\alpha)$ are smooth functions; and $0 < \theta(\alpha = 0) < \pi$; $d(\alpha = 0) \neq 0$. This system has the fixed point $x_1 = x_2 = 0$ for all α with Jacobian matrix $A = (1+\alpha) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The matrix has eigenvalues $\mu_{1,2} = (1+\alpha) \cdot e^{\pm i \cdot \theta}$ which takes the map invertible near the origin for all small $|\alpha|$. The fixed point at the origin is non-hyperbolic at $\alpha = 0$ due to a complex conjugate pair of the eigenvalues on the unit cycle.

$$A = (1+\alpha) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow A - \lambda \cdot I = \begin{pmatrix} (1+\alpha) \cdot \cos\theta - \lambda & -(1+\alpha) \cdot \sin\theta \\ (1+\alpha) \cdot \sin\theta & (1+\alpha) \cdot \cos\theta - \lambda \end{pmatrix}$$

$$\det\{A - \lambda \cdot I\} = 0 \Rightarrow \{(1+\alpha) \cdot \cos\theta - \lambda\}^2 + (1+\alpha)^2 \cdot \sin^2\theta = 0$$

$$\{(1+\alpha) \cdot \cos\theta - \lambda\}^2 + (1+\alpha)^2 \cdot \sin^2\theta = (1+\alpha)^2 \cdot \cos^2\theta - 2 \cdot \lambda \cdot (1+\alpha) \cdot \cos\theta + \lambda^2 + (1+\alpha)^2 \cdot \sin^2\theta$$

$$(1+\alpha)^2 \cdot \cos^2\theta - 2 \cdot \lambda \cdot (1+\alpha) \cdot \cos\theta + \lambda^2 + (1+\alpha)^2 \cdot \sin^2\theta = (1+\alpha)^2 \cdot (\cos^2\theta + \sin^2\theta) - 2 \cdot \lambda \cdot (1+\alpha) \cdot \cos\theta + \lambda^2 = \lambda^2 - 2 \cdot \lambda \cdot (1+\alpha) \cdot \cos\theta + (1+\alpha)^2$$

$$\det\{A - \lambda \cdot I\} = 0 \Rightarrow \lambda^2 - 2 \cdot \lambda \cdot (1+\alpha) \cdot \cos\theta + (1+\alpha)^2 = 0$$

$$\mu_{1,2} = \lambda_{1,2} = \frac{2 \cdot (1+\alpha) \cdot \cos\theta \pm \sqrt{4 \cdot (1+\alpha)^2 \cdot \cos^2\theta - 4 \cdot (1+\alpha)^2}}{2}$$

$$\mu_{1,2} = \lambda_{1,2} = (1+\alpha) \cdot \cos\theta \pm \sqrt{(1+\alpha)^2 \cdot [\cos^2\theta - 1]} = (1+\alpha) \cdot \{\cos\theta \pm \sqrt{[\cos^2\theta - 1]}\}$$

$$\mu_{1,2} = \lambda_{1,2} = (1+\alpha) \cdot \{\cos\theta \pm \sqrt{(-1) \cdot [1 - \cos^2\theta]}\} = (1+\alpha) \cdot \{\cos\theta \pm i \cdot \sin\theta\} = (1+\alpha) \cdot e^{\pm i \cdot \theta}$$

Analyzing the corresponding bifurcation, first we define the complex variable $z = x_1 + i \cdot x_2 \to \bar{z} = x_1 - i \cdot x_2 \to |z|^2 = z \cdot \bar{z} = x_1^2 + x_2^2$ and set $d_1 = d + i \cdot b$. The equation for z: $z \to e^{i \cdot \theta} \cdot z \cdot (1 + \alpha + d_1 \cdot |z|^2) = \mu \cdot z + c_1 \cdot z \cdot |z|^2$ where $\mu = \mu(\alpha) = (1+\alpha) \cdot e^{i \cdot \theta(\alpha)}$.

And $c_1 = c_1(\alpha) = e^{i \cdot \theta(\alpha)} \cdot d_1(\alpha)$ are complex functions of the parameter $\alpha \cdot z \to (1+\alpha) \cdot e^{i \cdot \theta(\alpha)} \cdot z + e^{i \cdot \theta(\alpha)} \cdot d_1(\alpha) \cdot z \cdot |z|^2$. We define $z = \rho \cdot e^{i \cdot \varphi} \Rightarrow \rho = |z|$.

$\rho \to \rho \cdot |1 + \alpha + d_1(\alpha) \cdot \rho^2|$. We can get the mathematical expression:

$$|1 + \alpha + d_1(\alpha) \cdot \rho^2| = (1+\alpha) \cdot \sqrt{1 + \frac{2 \cdot d(\alpha)}{1+\alpha} \cdot \rho^2 + \frac{|d_1(\alpha)|^2}{(1+\alpha)^2} \cdot \rho^4}$$
$$= 1 + \alpha + d(\alpha) \cdot \rho^2 + O(\rho^4)$$

We get the following polar form of the system:

$$\rho \to \rho \cdot \left(1 + \alpha + d(\alpha) \cdot \rho^2\right) + \rho^4 \cdot R_\alpha(\rho);\ \varphi \to \varphi + \theta(\alpha) + \rho^2 \cdot Q_\alpha(\rho)$$

R, Q are smooth functions of (ρ, α). Bifurcations of the systems's phase portrait as α passes through zero can easily be analyzed using the latter form since the mapping for ρ is independent of φ. The first equation $\rho \to \rho \cdot \ldots$

Defines a one-dimensional dynamical system that has the fixed point $\rho = 0$ for all values of α. The point is linearly stable if $\alpha < 0$; for $\alpha > 0$ the point becomes linearly unstable. The stability of the fixed point at $\alpha = 0$ is determined by the sign of the coefficient $d(\alpha = 0)$. Suppose that $d(\alpha = 0) < 0$ then the origin is stable at $\alpha = 0$. The ρ-map has an additional stable fixed point $\rho_0(\alpha) = \sqrt{-\frac{\alpha}{d(\alpha)}} + O(\alpha)$ for $\alpha > 0$. The φ map of $\varphi \to \varphi + \theta(\alpha) + \rho^2 \cdot Q_\alpha(\rho)$ describes a rotation by an angle depending on ρ and α and it is approximately equal to $\theta(\alpha)$. By superposition of the mapping defined by $\rho \to \rho \cdot \ldots$; $\varphi \to \varphi + \ldots$, we obtain the bifurcation diagram for the original two-dimensional system. The system always has a fixed point at the origin. This point is stable for $\alpha < 0$ and unstable for $\alpha > 0$. The invariant curves of the system near the origin look like the orbits near the stable focus of a continuous time system for $\alpha < 0$ and like orbits near the unstable focus for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the point is nonlinear stable. The fixed point is surrounded for $\alpha > 0$ by an isolated closed invariant curve that is unique and stable. The curve is a circle of radius $\rho_0(\alpha)$. All orbits starting outside or inside the closed invariant curve, except at the origin, tend to the curve under iterations

$\rho \to \rho \cdot \ldots; \varphi \to \varphi + \ldots$. This is a Neimark–Sacker bifurcation. This bifurcation can also be presented in (x_1, x_2, α) space. The appearing family of closed invariant curves, parameterized by α, forms a parabolic surface. The case $d(\alpha = 0) > 0$ can be analyzed in the same way. The system undergoes the Neimark–Sacker bifurcation at $\alpha = 0$. There is an unstable closed invariant curve that disappears when α crosses zero from negative to positive values. Any generic two-dimensional system undergoing a Neimark–Sacker bifurcation can be transformed into the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to (1+\alpha) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(x_1^2 + x_2^2\right) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ b & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + O\left(\|x\|^4\right)$$

$O(||x||^4)$ is higher order terms and can depend smoothly on α. $O(||x||^4)$ terms do not affect the bifurcation of the closed invariant curve. That is, a locally unique invariant curve bifurcates from the origin in the same direction and with the same stability as in the system without higher order terms. Consider a system $x \to f(x, \alpha)$; $x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$; $\alpha \in \mathbb{R}^1$ with a smooth function f, which has at $\alpha = 0$ the fixed point $x = 0$ with simple eigenvalues $\mu_{1,2} = \mathrm{e}^{\pm i \cdot \theta_0}$, $0 < \theta_0 < \pi$. By the implicit function theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $|\alpha|$ since $\mu = 1$ is not an eigenvalue of the Jacobian matrix. We can perform a parameter-dependent coordinate shift, placing this fixed point at the origin. We may assume without loss of generality that $x = 0$ is the fixed point of the system for $|\alpha|$ sufficiently small. Thus the system can be written as $x \to A(\alpha) \cdot x + F(x, \alpha)$ where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms, $F(x = 0, \alpha) = 0$ for all sufficiently small $|\alpha|$. The Jacobian matrix $A(\alpha)$ has two multipliers: $\mu_{1,2}(\alpha) = r(\alpha) \cdot \mathrm{e}^{\pm i \cdot \varphi(\alpha)}$ where $r(\alpha = 0) = 1$; $\varphi(\alpha = 0) = \theta_0$. Thus $\mu_{1,2}(\alpha = 0) = \mathrm{e}^{\pm i \cdot \theta_0}$ and $r(\alpha) = 1 + \beta(\alpha)$ for some smooth function $\beta(\alpha)$, $\beta(\alpha = 0) = 0$. If $\beta'(\alpha = 0) \neq 0$ then we can use β as a new parameter and express the multipliers in terms of β, $\mu_1(\beta) = \mu(\beta)$, $\mu_2(\beta) \neq \mu(\beta)$ where $\theta(\beta)$ is a smooth function such that $\theta(\beta = 0) = \theta_0$ and $\mu_1(\beta) = \mu(\beta) = (1+\beta) \cdot \mathrm{e}^{i \cdot \theta(\beta)}$; $\mu_2(\beta) = \bar{\mu}(\beta) = (1+\beta) \cdot \mathrm{e}^{-i \cdot \theta(\beta)}$. We summarize our last discussion by two theorems:

Theorem A Suppose a two-dimensional discrete-time system $x \to f(x, \alpha)$, $x \in \mathbb{R}^2$; $\alpha \in \mathbb{R}^1$ with smooth f function, has, for all sufficient small $|\alpha|$, the fixed point $x = 0$ with multipliers $\mu_{1,2}(\alpha) = r(\alpha) \cdot \mathrm{e}^{\pm i \cdot \varphi(\alpha)}$ where $r(\alpha = 0) = 1$; $\varphi(\alpha = 0) = \theta_0$.

Let the following conditions be satisfied: $r'(\alpha = 0) \neq 0 \,\&\, \mathrm{e}^{i \cdot k \cdot \theta_0} \neq 1 \,\forall\, k = 1, 2, 3, 4$.

Then, there are smooth invertible coordinate and parameter changes transforming the system into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to (1+\beta) \cdot \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$$+ (y_1^2 + y_2^2) \cdot \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \cdot \begin{pmatrix} d(\beta) & -b(\beta) \\ b(\beta) & d(\beta) \end{pmatrix} \cdot \begin{pmatrix} \frac{y_1}{y_2} \end{pmatrix} + O\left(||y||^4\right)$$

$\theta(\beta = 0) = \theta_0; \mathrm{d}(\beta = 0) = \mathrm{Re}\left\{\mathrm{e}_0^{-\mathrm{i}\cdot\theta} \cdot \ \mathrm{c}_1(\beta = 0)\right\}$.

Theorem B For any generic two-dimensional one parameter system $x \to f(x, \alpha)$ having at $\alpha = 0$ the fixed point $x_0 = 0$ with complex multipliers $\mu_{1,2} = \mathrm{e}^{\pm i \cdot \theta_0}$, there is a neighborhood of x_0 in which a unique closed invariant curve bifurcates from x_0 as α passes through zero. (Generic Neimark–Sacker bifurcation [5–9, 71, 72].)

Appendix B
Phototransistor Circuit with Double Coupling LEDs

One of the innovative optoisolation circuit is phototransistor with double coupling LEDs which is implemented in many engineering applications. The circuit's elements are one phototransistor ($Q1$), two LEDs ($D1$, $D2$), two capacitors ($C1$, $C2$), inductor ($L1$), resistor ($R1$) and switch (S1). LEDs light strike in the base window of phototransistor (Q1). We can take Q1-LEDs coupling as two dependent current sources which inject current to Q1's base. $I_{\text{BQ1}} = k_1 \cdot I_{D1}; I_{BQ2} = k_2 \cdot I_{D2}$. The circuit is describe in Fig. B.1.

Capacitor $C1$ is initially charge to high voltage, bigger than phototransistor Collector–Emitter breakover voltage. Capacitor $C2$ voltage is initially close to zero. $V_{C1}(t=0) > 0; V_{C1}(t=0) \gg V_{CEQ1(\text{break})}; I_{L1}(t=0) = 0; V_{C2}(t=0) \to \varepsilon$. At t = 0 we move switch S1 from OFF to ON state and analyze our circuit dynamics.

$$V_{D1} = V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right]; V_{D2} = V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right]; V(t) = V_{L1} + V_{D1} + V_{CEQ1} + V_{D2}.$$

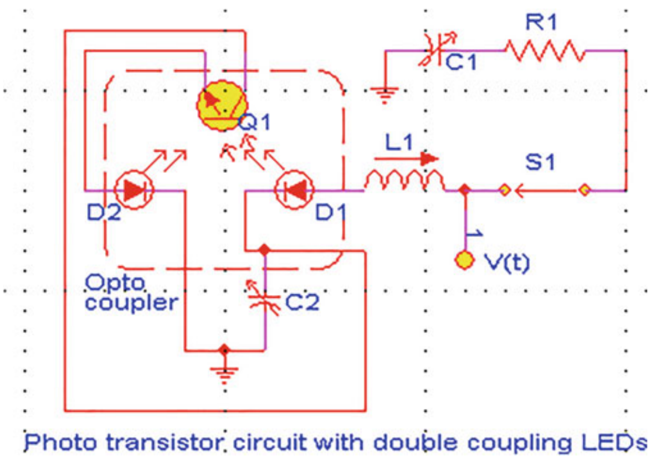

Fig. B.1 Photo transistor circuit with double coupling LEDs

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Now we need to implement the Regular Ebers–Moll Model to the Opto coupler circuit.

$Ide = \frac{\alpha r \cdot Ic - Ie}{\alpha r \cdot \alpha f - 1}$ and $Idc = \frac{Ic - Ie \cdot \alpha f}{\alpha r \cdot \alpha f - 1}$, We now can get the expression for Vbe and Vbc.

$V_{BEQ1} = Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$ and $V_{BCQ1} = Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{Isc \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$

$Vce = Vcb + Vbe$, but $Vcb = -Vbc$, then $Vce = Vbe - Vbc$

$$Vce = Vt \cdot \ln Vt \cdot \ln\left[\left(\frac{\alpha r \cdot I_{CQ1} - I_{EQ1}}{Ise \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right] - Vt \cdot \ln\left[\left(\frac{I_{CQ1} - I_{EQ1} \cdot \alpha f}{Isc \cdot (\alpha r \cdot \alpha f - 1)}\right) + 1\right]$$

$$V_{CEQ1} = Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right] + Vt \cdot \ln\left(\frac{Isc}{Ise}\right), \; I_{R1} = I_{D1} = I_{C1}$$

$$\frac{Isc}{Ise} \to 1 \Rightarrow Vt \cdot \ln\left(\frac{Isc}{Ise}\right) \to \varepsilon \Rightarrow V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{(\alpha r \cdot I_{CQ1} - I_{EQ1}) + Ise \cdot (\alpha r \cdot \alpha f - 1)}{(I_{CQ1} - I_{EQ1} \cdot \alpha f) + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$I_{BQ1} = k_1 \cdot I_{D1} + k_2 \cdot I_{D2}; \; I_{CQ1} = I_{D1} - I_{C2} \Rightarrow I_{D1} = I_{CQ1} + I_{C2}; I_{D2} = I_{EQ1} = I_{BQ1} + I_{CQ1}$$

The equivalent circuit of our photo transistor with double coupling LEDs is in Fig. B.2:

$$I_{D2} = I_{EQ1} = I_{BQ1} + I_{CQ1}; I_{D1} = I_{L1} = I_{C1} = I_{R1} = I_{C2} + I_{CQ1};$$
$$I_{BQ1} = k_1 \cdot (I_{C2} + I_{CQ1}) + k_2 \cdot I_{EQ1}$$

$$I_{EQ1} = I_{BQ1} + I_{CQ1} = k_1 \cdot (I_{CQ1} + I_{C2}) + k_2 \cdot I_{EQ1} + I_{CQ1}$$
$$\Rightarrow I_{EQ1} \cdot (1 - k_2) = k_1 \cdot (I_{CQ1} + I_{C2}) + I_{CQ1}$$

$$I_{EQ1} = \frac{1}{(1 - k_2)} \cdot \left\{k_1 \cdot (I_{CQ1} + I_{C2}) + I_{CQ1}\right\} = I_{CQ1} \cdot \frac{(k_1 + 1)}{(1 - k_2)} + I_{C2} \cdot \frac{k_1}{(1 - k_2)}$$

$$V(t) = V_{L1} + V_{D1} + V_{CEQ1} + V_{D2}; V_{C2} = V_{CEQ1} + V_{D2} = V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right]$$

$$V_{C2} = V_{CEQ1} + V_{D2} = V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{EQ1}}{I_0} + 1\right]; \quad I_{C2} = C_2 \cdot \frac{dV_{C2}}{dt}$$

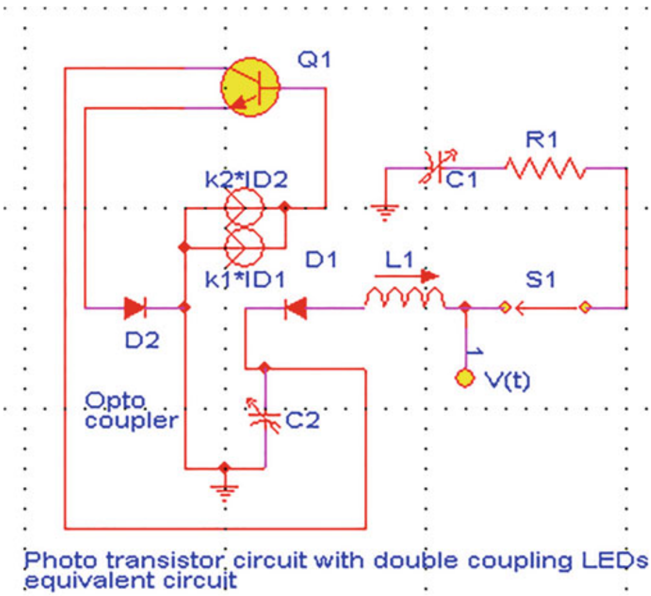

Fig. B.2 Photo transistor circuit with double coupling LEDs equivalent circuit

$$\alpha r \cdot I_{CQ1} - I_{EQ1} = \alpha r \cdot I_{CQ1} - \left(I_{CQ1} \cdot \frac{(k_1+1)}{(1-k_2)} + I_{C2} \cdot \frac{k_1}{(1-k_2)} \right)$$
$$= I_{CQ1} \cdot \left(\alpha r - \frac{(1+k_1)}{(1-k_2)} \right) - I_{C2} \cdot \frac{k_1}{(1-k_2)}$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f = I_{CQ1} - \left\{ I_{CQ1} \cdot \frac{(k_1+1)}{(1-k_2)} + I_{C2} \cdot \frac{k_1}{(1-k_2)} \right\} \cdot \alpha f$$
$$= I_{CQ1} \cdot \left[1 - \frac{(1+k_1) \cdot \alpha f}{(1-k_2)} \right] - I_{C2} \cdot \frac{k_1 \cdot \alpha f}{(1-k_2)}$$

We define the following parameters for simplicity: $A_1 = \frac{(1+k_1)}{(1-k_2)}; A_2 = \frac{k_1}{(1-k_2)}$

$$A_1(k_1, k_2) = \frac{(1+k_1)}{(1-k_2)}; A_2(k_1, k_2) = \frac{k_1}{(1-k_2)};$$
$$\alpha r \cdot I_{CQ1} - I_{EQ1} = I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2$$

$$I_{CQ1} - I_{EQ1} \cdot \alpha f = I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f; \alpha r \cdot I_{CQ1} - I_{EQ1}$$
$$= I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2$$

$$V_{C2} = V_{CEQ1} + V_{D2} = V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{D2}}{I_0} + 1\right] = V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{EQ1}}{I_0} + 1\right]$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)}{I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

$$I_{C2} = C_2 \cdot \frac{dV_{C2}}{dt} = C_2 \cdot \frac{d}{dt}\left\{V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{EQ1}}{I_0} + 1\right]\right\};$$

$$I_{EQ1} = I_{CQ1} \cdot \frac{(k_1 + 1)}{(1 - k_2)} + I_{C2} \cdot \frac{k_1}{(1 - k_2)} = I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2$$

$$I_{C2} = C_2 \cdot \frac{dV_{C2}}{dt} = C_2 \cdot \frac{d}{dt}\left\{V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{EQ1}}{I_0} + 1\right]\right\}$$
$$= C_2 \cdot \frac{d}{dt}\left\{V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2}{I_0} + 1\right]\right\}$$

$$I_{C1} = I_{R1} = I_{L1} = I_{D1}; V_{C1} = R_1 \cdot I_{C1} + V_{D1} + V_{L1} + V_{C2}$$
$$= R_1 \cdot I_{C1} + V_{D1} + V_{L1} + V_{CEQ1} + V_{D2}$$

$$V_{C1} = R_1 \cdot I_{C1} + V_{D1} + V_{L1} + V_{C2}$$
$$= R_1 \cdot C_1 \cdot \frac{dV_{C1}}{dt} + V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] + V_{L1} + V_{C2}$$

$$I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt} \Rightarrow V_{C1} = \frac{1}{C_1} \cdot \int I_{C1} \cdot dt; V_{L1} = L_1 \cdot \frac{dI_{L1}}{dt}$$

$$V_{C1} = R_1 \cdot C_1 \cdot \frac{dV_{C1}}{dt} + V_t \cdot \ln\left[\frac{I_{D1}}{I_0} + 1\right] + V_{L1}$$
$$+ V_{C2} = R_1 \cdot I_{C1} + V_t \cdot \ln\left[\frac{I_{C2} + I_{CQ2}}{I_0} + 1\right] + L_1 \cdot \frac{dI_{C1}}{dt} + V_{C2}$$

$$V_{C1} = R_1 \cdot I_{C1} + V_t \cdot \ln\left[\frac{I_{C2} + I_{CQ2}}{I_0} + 1\right] + L_1 \cdot \frac{dI_{C1}}{dt}$$
$$+ V_{C2} \Rightarrow \frac{1}{C_1} \cdot \int I_{C1} \cdot dt = R_1 \cdot I_{C1} + V_t \cdot \ln\left[\frac{I_{C2} + I_{CQ2}}{I_0} + 1\right] + L_1 \cdot \frac{dI_{C1}}{dt} + V_{C2}$$

$$\frac{d}{dt}\left\{\frac{1}{C_1} \cdot \int I_{C1} \cdot dt\right\} = \frac{d}{dt}\left\{R_1 \cdot I_{C1} + V_t \cdot \ln\left[\frac{I_{C2} + I_{CQ2}}{I_0} + 1\right] + L_1 \cdot \frac{dI_{C1}}{dt} + V_{C2}\right\}$$

$$\frac{1}{C_1} \cdot I_{C1} = R_1 \cdot \frac{dI_{C1}}{dt} + V_t \cdot \frac{1}{\frac{I_{C2} + I_{CQ2}}{I_0} + 1} \cdot \frac{1}{I_0} \cdot \left[\frac{dI_{C2}}{dt} + \frac{dI_{CQ2}}{dt}\right] + L_1 \cdot \frac{d^2 I_{C1}}{dt^2} + \frac{dV_{C2}}{dt}$$

$$(*)\ \frac{1}{C_1} \cdot I_{C1} = R_1 \cdot \frac{dI_{C1}}{dt} + V_t \cdot \frac{\left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ2}}{dt}\right)}{(I_{C2} + I_{CQ2} + I_0)} + L_1 \cdot \frac{d^2 I_{C1}}{dt^2} + \frac{dV_{C2}}{dt}$$

$$V_{C2} = V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{EQ1}}{I_0} + 1\right] = V_{CEQ1} + V_t \cdot \ln\left[\frac{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2}{I_0} + 1\right]$$

$$\frac{\mathrm{d}V_{C2}}{\mathrm{d}t} = \frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t} + V_t \cdot \frac{1}{\frac{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2}{I_0} + 1} \cdot \frac{1}{I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]$$

$$(**)\ \frac{\mathrm{d}V_{C2}}{\mathrm{d}t} = \frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t} + V_t \cdot \frac{1}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]$$

$$(**) \Rightarrow (*) \Rightarrow \frac{1}{C_1} \cdot I_{C1} = R_1 \cdot \frac{\mathrm{d}I_{C1}}{\mathrm{d}t} + V_t \cdot \frac{\left(\frac{\mathrm{d}I_{C2}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ2}}{\mathrm{d}t}\right)}{(I_{C2} + I_{CQ2} + I_0)} + L_1 \cdot \frac{\mathrm{d}^2 I_{C1}}{\mathrm{d}t^2} + \frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t}$$
$$+ V_t \cdot \frac{1}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]$$

$$I_{C1} = I_{C2} + I_{CQ1} \Rightarrow \frac{1}{C_1} \cdot (I_{C2} + I_{CQ1}) = R_1 \cdot \frac{\mathrm{d}(I_{C2} + I_{CQ1})}{\mathrm{d}t} + V_t \cdot \frac{\left(\frac{\mathrm{d}I_{C2}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ2}}{\mathrm{d}t}\right)}{(I_{C2} + I_{CQ2} + I_0)}$$
$$+ L_1 \cdot \frac{\mathrm{d}^2(I_{C2} + I_{CQ1})}{\mathrm{d}t^2} + \frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t} + V_t \cdot \frac{1}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]$$

$$I_{C1} = I_{C2} + I_{CQ1} \Rightarrow \frac{1}{C_1} \cdot (I_{C2} + I_{CQ1}) = R_1 \cdot \left(\frac{\mathrm{d}I_{C2}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\right) + V_t \cdot \frac{\left(\frac{\mathrm{d}I_{C2}}{\mathrm{d}t} + \frac{\mathrm{d}I_{CQ2}}{\mathrm{d}t}\right)}{(I_{C2} + I_{CQ2} + I_0)}$$
$$+ L_1 \cdot \left(\frac{\mathrm{d}^2 I_{C2}}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 I_{CQ1}}{\mathrm{d}t^2}\right) + \frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t} + V_t \cdot \frac{1}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]$$

$$I_{C2} = C_2 \cdot \frac{\mathrm{d}V_{C2}}{\mathrm{d}t} = C_2 \cdot \left\{\frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t}\right.$$
$$\left. + V_t \cdot \frac{1}{\frac{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2}{I_0} + 1} \cdot \frac{1}{I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]\right\}$$

$$I_{C2} = C_2 \cdot \frac{\mathrm{d}V_{C2}}{\mathrm{d}t} = C_2 \cdot \left\{\frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t}\right.$$
$$\left. + V_t \cdot \frac{1}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]\right\}$$

$$I_{C2} = C_2 \cdot \frac{\mathrm{d}V_{C2}}{\mathrm{d}t} = C_2 \cdot \frac{\mathrm{d}V_{CEQ1}}{\mathrm{d}t} + \frac{C_2 \cdot V_t \cdot \left[\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t} \cdot A_1 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t} \cdot A_2\right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0}$$

Back to our V_{CEQ1} expression:

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)}{I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)}\right]$$

We define two functions g_1, g_2: $g_1 = g_1(I_{CQ1}, I_{C2}); g_2 = g_2(I_{CQ1}, I_{C2})$

$$g_1 = g_1(I_{CQ1}, I_{C2}) = I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)$$

$$g_2(I_{CQ1}, I_{C2}) = I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)$$

$$V_{CEQ1} \simeq Vt \cdot \ln\left[\frac{g_1}{g_2}\right] \Rightarrow \frac{dV_{CEQ1}}{dt} = Vt \cdot \frac{g_2}{g_1} \cdot \frac{\left\{\frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1\right\}}{g_2^2}$$
$$= \frac{Vt}{g_1 \cdot g_2} \cdot \left\{\frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1\right\}$$

We define the following (***) $V_{CEQ1} \simeq \frac{Vt}{g_1 \cdot g_2} \cdot \left\{\frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1\right\}$

$$I_{C2} = C_2 \cdot \frac{dV_{C2}}{dt} \Rightarrow (****)\, I_{C2} \cdot \frac{1}{C_2} = \frac{dV_{CEQ1}}{dt} + \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2\right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0}$$

$$\frac{dV_{CEQ1}}{dt} = I_{C2} \cdot \frac{1}{C_2} - \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2\right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0}$$

$$(****) \Rightarrow (*)\, \frac{1}{C_1} \cdot (I_{C2} + I_{CQ1}) = R_1 \cdot \left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt}\right) + V_t \cdot \frac{\left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ2}}{dt}\right)}{(I_{C2} + I_{CQ2} + I_0)}$$
$$+ L_1 \cdot \left(\frac{d^2 I_{C2}}{dt^2} + \frac{d^2 I_{CQ1}}{dt^2}\right) + I_{C2} \cdot \frac{1}{C_2}$$

$$I_{C2} \cdot \left(\frac{1}{C_1} - \frac{1}{C_2}\right) + \frac{1}{C_1} \cdot I_{CQ1} = R_1 \cdot \left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt}\right) + V_t \cdot \frac{\left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ2}}{dt}\right)}{(I_{C2} + I_{CQ2} + I_0)}$$
$$+ L_1 \cdot \left(\frac{d^2 I_{C2}}{dt^2} + \frac{d^2 I_{CQ1}}{dt^2}\right)$$

We use the approximation: $\frac{1}{C_1} \gg \frac{1}{C_2} \Rightarrow C_1 \ll C_2 \Rightarrow \frac{1}{C_1} - \frac{1}{C_2} \approx \frac{1}{C_1}$

$$\frac{1}{C_1} \cdot \{I_{C2} + I_{CQ1}\} = R_1 \cdot \left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt}\right) + V_t \cdot \frac{\left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ2}}{dt}\right)}{(I_{C2} + I_{CQ2} + I_0)}$$
$$+ L_1 \cdot \left(\frac{d^2 I_{C2}}{dt^2} + \frac{d^2 I_{CQ1}}{dt^2}\right)$$

We define a new variable $X = I_{C2} + I_{CQ1}$; $\frac{dX}{dt} = \frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt}$ $\frac{d^2X}{dt^2} = \frac{d^2I_{C2}}{dt^2} + \frac{d^2I_{CQ1}}{dt^2}$ and get the last equation which expressed as a function of X variable.

$$\frac{1}{C_1} \cdot X = R_1 \cdot \frac{dX}{dt} + V_t \cdot \frac{\frac{dX}{dt}}{(X + I_0)} + L_1 \cdot \frac{d^2X}{dt^2} \Rightarrow \frac{1}{C_1} \cdot X \cdot (X + I_0)$$
$$= R_1 \cdot (X + I_0) \cdot \frac{dX}{dt} + V_t \cdot \frac{dX}{dt} + L_1 \cdot (X + I_0) \cdot \frac{d^2X}{dt^2}$$

$$\frac{1}{C_1} \cdot X^2 + \frac{I_0}{C_1} \cdot X = R_1 \cdot X \cdot \frac{dX}{dt} + R_1 \cdot I_0 \cdot \frac{dX}{dt} + V_t \cdot \frac{dX}{dt} + L_1 \cdot X \cdot \frac{d^2X}{dt^2}$$
$$+ L_1 \cdot I_0 \cdot \frac{d^2X}{dt^2}$$

$$\frac{1}{C_1} \cdot X^2 + \frac{I_0}{C_1} \cdot X = R_1 \cdot X \cdot \frac{dX}{dt} + (R_1 \cdot I_0 + V_t) \cdot \frac{dX}{dt} + L_1 \cdot X \cdot \frac{d^2X}{dt^2} + L_1 \cdot I_0 \cdot \frac{d^2X}{dt^2}$$

$$(***) \Rightarrow (****) \Rightarrow I_{C2} \cdot \frac{1}{C_2}$$
$$= \frac{Vt}{g_1 \cdot g_2} \cdot \left\{ \frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1 \right\} + \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2 \right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0}$$

We add $I_{CQ1} \cdot \frac{1}{C_2}$ term to above equation two sides.

$$I_{C2} \cdot \frac{1}{C_2} + I_{CQ1} \cdot \frac{1}{C_2} = \frac{Vt}{g_1 \cdot g_2} \cdot \left\{ \frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1 \right\}$$
$$+ \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2 \right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} + I_{CQ1} \cdot \frac{1}{C_2}$$

We define $\xi = \frac{Vt}{g_1 \cdot g_2} \cdot \left\{ \frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1 \right\} + \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2 \right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0} + I_{CQ1} \cdot \frac{1}{C_2}$

$$I_{C2} \cdot \frac{1}{C_2} + I_{CQ1} \cdot \frac{1}{C_2} = \xi \Rightarrow \frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt} = C_2 \cdot \frac{d\xi}{dt}$$

We previously get the expression for g_1 and g_2 functions.

$$g_1 = g_1(I_{CQ1}, I_{C2}) = I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)$$

$$g_2 = g_2(I_{CQ1}, I_{C2}) = I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)$$

$$g_1 \cdot g_2 = \{I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)\} \cdot \{I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)\}$$

$$\begin{aligned} g_1 \cdot g_2 = {} & I_{CQ1}^2 \cdot (\alpha r - A_1) \cdot [1 - A_1 \cdot \alpha f] - I_{CQ1} \cdot (\alpha r - A_1) \cdot I_{C2} \cdot A_2 \cdot \alpha f \\ & + I_{CQ1} \cdot (\alpha r - A_1) \cdot Isc \cdot (\alpha r \cdot \alpha f - 1) - I_{C2} \cdot A_2 \cdot I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] \\ & + I_{C2}^2 \cdot A_2^2 \cdot \alpha f - I_{C2} \cdot A_2 \cdot Isc \cdot (\alpha r \cdot \alpha f - 1) Ise \cdot (\alpha r \cdot \alpha f - 1) \\ & \cdot I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot I_{C2} \cdot A_2 \cdot \alpha f + Ise \cdot Isc \cdot (\alpha r \cdot \alpha f - 1)^2 \end{aligned}$$

$$\begin{aligned} g_1 \cdot g_2 = {} & I_{CQ1}^2 \cdot (\alpha r - A_1) \cdot [1 - A_1 \cdot \alpha f] + I_{C2}^2 \cdot A_2^2 \cdot \alpha f \\ & - I_{CQ1} \cdot I_{C2} \cdot \{(\alpha r - A_1) \cdot A_2 \cdot \alpha f + A_2 \cdot [1 - A_1 \cdot \alpha f]\} \\ & - I_{C2} \cdot \{A_2 \cdot Isc \cdot (\alpha r \cdot \alpha f - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot A_2 \cdot \alpha f\} \\ & + I_{CQ1} \cdot \{(\alpha r - A_1) \cdot Isc \cdot (\alpha r \cdot \alpha f - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot [1 - A_1 \cdot \alpha f]\} \\ & + Ise \cdot Isc \cdot (\alpha r \cdot \alpha f - 1)^2 \end{aligned}$$

We define six parameters: n_i (i = 1, 2, 3, …, 6).

$$n_1 = (\alpha r - A_1) \cdot [1 - A_1 \cdot \alpha f] = \alpha r - A_1 \cdot \alpha f \cdot \alpha r - A_1 + A_1^2 \cdot \alpha f; n_2 = A_2^2 \cdot \alpha f$$

$$\begin{aligned} n_3 & = (\alpha r - A_1) \cdot A_2 \cdot \alpha f + A_2 \cdot [1 - A_1 \cdot \alpha f] = A_2 \cdot \{(\alpha r - A_1) \cdot \alpha f + 1 - A_1 \cdot \alpha f\} \\ & = A_2 \cdot \{\alpha r \cdot \alpha f - A_1 \cdot \alpha f + 1 - A_1 \cdot \alpha f\} = A_2 \cdot \{\alpha r \cdot \alpha f + 1 - 2 \cdot A_1 \cdot \alpha f\} \end{aligned}$$

$$\begin{aligned} n_4 & = A_2 \cdot Isc \cdot (\alpha r \cdot \alpha f - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot A_2 \cdot \alpha f \\ & = (\alpha r \cdot \alpha f - 1) \cdot A_2 \cdot [Isc + Ise \cdot \alpha f] \end{aligned}$$

$$\begin{aligned} n_5 & = (\alpha r - A_1) \cdot Isc \cdot (\alpha r \cdot \alpha f - 1) + Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot [1 - A_1 \cdot \alpha f] \\ & = (\alpha r \cdot \alpha f - 1) \cdot \{(\alpha r - A_1) \cdot Isc + Ise \cdot [1 - A_1 \cdot \alpha f]\} \end{aligned}$$

$$n_6 = Ise \cdot Isc \cdot (\alpha r \cdot \alpha f - 1)^2$$

$$g_1 \cdot g_2 = I_{CQ1}^2 \cdot n_1 + I_{C2}^2 \cdot n_2 - I_{CQ1} \cdot I_{C2} \cdot n_3 - I_{C2} \cdot n_4 + I_{CQ1} \cdot n_5 + n_6$$

We already defined a variable $X = I_{C2} + I_{CQ1}$ and define $Y = I_{C2} \cdot A_2 + I_{CQ1} \cdot A_1$

$$\begin{aligned} X & = I_{C2} + I_{CQ1} \Rightarrow I_{C2} = X - I_{CQ1}; Y = I_{C2} \cdot A_2 + I_{CQ1} \cdot A_1 \\ & = (X - I_{CQ1}) \cdot A_2 + I_{CQ1} \cdot A_1 \end{aligned}$$

$$Y = X \cdot A_2 - I_{CQ1} \cdot A_2 + I_{CQ1} \cdot A_1 \Rightarrow I_{CQ1} = \frac{Y - X \cdot A_2}{A_1 - A_2}; I_{C2} = X - I_{CQ1} = X - \frac{Y - X \cdot A_2}{A_1 - A_2}$$

$$I_{C2} = X - I_{CQ1} = X - \frac{Y - X \cdot A_2}{A_1 - A_2} = \frac{X \cdot A_1 - Y}{A_1 - A_2}$$

We can summarize our last results: $I_{CQ1} = \frac{Y - X \cdot A_2}{A_1 - A_2}; I_{C2} = \frac{X \cdot A_1 - Y}{A_1 - A_2}$

$$\frac{dI_{CQ1}}{dt} = \frac{\frac{dY}{dt} - \frac{dX}{dt} \cdot A_2}{A_1 - A_2} = \frac{1}{(A_1 - A_2)} \cdot \frac{dY}{dt} - \frac{A_2}{(A_1 - A_2)} \cdot \frac{dX}{dt}$$

$$\frac{dI_{C2}}{dt} = \frac{\frac{dX}{dt} \cdot A_1 - \frac{dY}{dt}}{A_1 - A_2} = \frac{A_1}{(A_1 - A_2)} \cdot \frac{dX}{dt} - \frac{1}{(A_1 - A_2)} \cdot \frac{dY}{dt}$$

Finally we get two equations:

$$\frac{1}{C_1} \cdot (I_{C2} + I_{CQ1}) = R_1 \cdot \left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt} \right) + V_t \cdot \frac{\left(\frac{dI_{C2}}{dt} + \frac{dI_{CQ2}}{dt} \right)}{(I_{C2} + I_{CQ2} + I_0)} + L_1 \cdot \left(\frac{d^2 I_{C2}}{dt^2} + \frac{d^2 I_{CQ1}}{dt^2} \right) + I_{C2} \cdot \frac{1}{C_2}$$

$$\frac{1}{C_1} \cdot X = R_1 \cdot \frac{dX}{dt} + V_t \cdot \frac{\frac{dX}{dt}}{(X + I_0)} + L_1 \cdot \frac{d^2 X}{dt^2} + \left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot \frac{1}{C_2}$$

$$I_{C2} \cdot \frac{1}{C_2} = \frac{Vt}{g_1 \cdot g_2} \cdot \left\{ \frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1 \right\} + \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2 \right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0}$$

Back to our g_1, g_2 equations:

$$g_1 = g_1(I_{CQ1}, I_{C2}) = I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)$$

$$g_2 = g_2(I_{CQ1}, I_{C2}) = I_{CQ1} \cdot [1 - A_1 \cdot \alpha f] - I_{C2} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)$$

$$\frac{dg_1}{dt} = \frac{dI_{CQ1}}{dt} \cdot (\alpha r - A_1) - \frac{dI_{C2}}{dt} \cdot A_2; \frac{dg_2}{dt} = \frac{dI_{CQ1}}{dt} \cdot [1 - A_1 \cdot \alpha f] - \frac{dI_{C2}}{dt} \cdot A_2 \cdot \alpha f$$

$$g_1 = g_1(I_{CQ1}, I_{C2}) = \left\{ \frac{Y - X \cdot A_2}{A_1 - A_2} \right\} \cdot (\alpha r - A_1) - \left\{ \frac{X \cdot A_1 - Y}{A_1 - A_2} \right\} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)$$

$$g_2 = g_2(I_{CQ1}, I_{C2}) = \left\{ \frac{Y - X \cdot A_2}{A_1 - A_2} \right\} \cdot [1 - A_1 \cdot \alpha f] - \left\{ \frac{X \cdot A_1 - Y}{A_1 - A_2} \right\} \cdot A_2 \cdot \alpha f + Isc \cdot (\alpha r \cdot \alpha f - 1)$$

$$\frac{dg_1}{dt} = \left\{ \frac{\frac{dY}{dt} - \frac{dX}{dt} \cdot A_2}{A_1 - A_2} \right\} \cdot (\alpha r - A_1) - \left\{ \frac{\frac{dX}{dt} \cdot A_1 - \frac{dY}{dt}}{A_1 - A_2} \right\} \cdot A_2$$

$$\frac{dg_2}{dt} = \left\{ \frac{\frac{dY}{dt} - \frac{dX}{dt} \cdot A_2}{A_1 - A_2} \right\} \cdot [1 - A_1 \cdot \alpha f] - \left\{ \frac{\frac{dX}{dt} \cdot A_1 - \frac{dY}{dt}}{A_1 - A_2} \right\} \cdot A_2 \cdot \alpha f$$

$$g_1 \cdot g_2 = I_{CQ1}^2 \cdot n_1 + I_{C2}^2 \cdot n_2 - I_{CQ1} \cdot I_{C2} \cdot n_3 - I_{C2} \cdot n_4 + I_{CQ1} \cdot n_5 + n_6$$

$$g_1 \cdot g_2 = \left(\frac{Y - X \cdot A_2}{A_1 - A_2} \right)^2 \cdot n_1 + \left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right)^2 \cdot n_2 - \left(\frac{(Y - X \cdot A_2) \cdot (X \cdot A_1 - Y)}{(A_1 - A_2)^2} \right) \cdot n_3 - \left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot n_4 + \left(\frac{Y - X \cdot A_2}{A_1 - A_2} \right) \cdot n_5 + n_6$$

$$(Y - X \cdot A_2)^2 = Y^2 - 2 \cdot Y \cdot X \cdot A_2 + X^2 \cdot A_2^2; (X \cdot A_1 - Y)^2 = X^2 \cdot A_1^2 - 2 \cdot X \cdot A_1 \cdot Y + Y^2$$

$$(Y - X \cdot A_2) \cdot (X \cdot A_1 - Y) = Y \cdot X \cdot (A_1 + A_2) - Y^2 - X^2 \cdot A_1 \cdot A_2$$

$$\left(\frac{Y - X \cdot A_2}{A_1 - A_2} \right)^2 \cdot n_1 = \frac{Y^2 - 2 \cdot Y \cdot X \cdot A_2 + X^2 \cdot A_2^2}{(A_1 - A_2)^2} \cdot n_1 = \frac{Y^2}{(A_1 - A_2)^2} \cdot n_1 - \frac{2 \cdot Y \cdot X \cdot A_2}{(A_1 - A_2)^2} \cdot n_1 + \frac{X^2 \cdot A_2^2}{(A_1 - A_2)^2} \cdot n_1$$

$$\left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right)^2 \cdot n_2 = \frac{X^2 \cdot A_1^2 - 2 \cdot X \cdot A_1 \cdot Y + Y^2}{(A_1 - A_2)^2} \cdot n_2 = \frac{X^2 \cdot A_1^2}{(A_1 - A_2)^2} \cdot n_2 - \frac{2 \cdot X \cdot A_1 \cdot Y}{(A_1 - A_2)^2} \cdot n_2 + \frac{Y^2}{(A_1 - A_2)^2} \cdot n_2$$

$$- \left(\frac{(Y - X \cdot A_2) \cdot (X \cdot A_1 - Y)}{(A_1 - A_2)^2} \right) \cdot n_3 = - \frac{Y \cdot X \cdot (A_1 + A_2) - Y^2 - X^2 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2} \cdot n_3$$
$$= - \frac{Y \cdot X \cdot (A_1 + A_2)}{(A_1 - A_2)^2} \cdot n_3 + \frac{Y^2}{(A_1 - A_2)^2} \cdot n_3 + \frac{X^2 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2} \cdot n_3$$
$$= - \frac{Y \cdot X}{(A_1 - A_2)} \cdot n_3 + \frac{Y^2}{(A_1 - A_2)^2} \cdot n_3 + \frac{X^2 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2} \cdot n_3$$

$$- \left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot n_4 = - \frac{X \cdot A_1}{(A_1 - A_2)} \cdot n_4 + \frac{Y}{(A_1 - A_2)} \cdot n_4$$

$$\left(\frac{Y - X \cdot A_2}{A_1 - A_2}\right) \cdot n_5 = \frac{Y}{(A_1 - A_2)} \cdot n_5 - \frac{X \cdot A_2}{(A_1 - A_2)} \cdot n_5$$

Back to $g_1 \cdot g_2$ expression:

$$g_1 \cdot g_2 = Y^2 \cdot \left[\frac{n_1}{(A_1 - A_2)^2} + \frac{n_2}{(A_1 - A_2)^2} + \frac{n_3}{(A_1 - A_2)^2}\right] + X^2 \cdot \left[\frac{n_1 \cdot A_2^2}{(A_1 - A_2)^2} + \frac{n_2 \cdot A_1^2}{(A_1 - A_2)^2} + \frac{n_3 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2}\right]$$
$$+ Y \cdot X \cdot \left[-\frac{2 \cdot A_2 \cdot n_1}{(A_1 - A_2)^2} - \frac{2 \cdot A_1 \cdot n_2}{(A_1 - A_2)^2} - \frac{n_3}{(A_1 - A_2)}\right] + Y \cdot \left[\frac{n_4}{(A_1 - A_2)} + \frac{n_5}{(A_1 - A_2)}\right]$$
$$- X \cdot \left[\frac{n_4 \cdot A_1}{(A_1 - A_2)} + \frac{n_5 \cdot A_2}{(A_1 - A_2)}\right] + n_6$$

$$g_1 \cdot g_2 = Y^2 \cdot \frac{\sum_{i=1}^{3} n_i}{(A_1 - A_2)^2} + X^2 \cdot \left[\frac{n_1 \cdot A_2^2 + n_2 \cdot A_1^2 + n_3 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2}\right]$$
$$- Y \cdot X \cdot \left[\frac{2 \cdot (A_2 \cdot n_1 + A_1 \cdot n_2)}{(A_1 - A_2)^2} + \frac{n_3}{(A_1 - A_2)}\right]$$
$$+ Y \cdot \left[\frac{n_4 + n_5}{(A_1 - A_2)}\right] - X \cdot \left[\frac{n_4 \cdot A_1 + n_5 \cdot A_2}{(A_1 - A_2)}\right] + n_6$$

For simplicity we define the following parameters:

$$\eta_1 = \frac{\sum_{i=1}^{3} n_i}{(A_1 - A_2)^2}\,;\ \eta_2 = \frac{n_1 \cdot A_2^2 + n_2 \cdot A_1^2 + n_3 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2};$$
$$\eta_3 = -\left[\frac{2 \cdot (A_2 \cdot n_1 + A_1 \cdot n_2)}{(A_1 - A_2)^2} + \frac{n_3}{(A_1 - A_2)}\right]$$
$$\eta_4 = \frac{n_4 + n_5}{(A_1 - A_2)}; \eta_5 = -\left[\frac{n_4 \cdot A_1 + n_5 \cdot A_2}{(A_1 - A_2)}\right]; \eta_6 = n_6$$

$$g_1 \cdot g_2 = Y^2 \cdot \eta_1 + X^2 \cdot \eta_2 + Y \cdot X \cdot \eta_3 + Y \cdot \eta_4 + X \cdot \eta_5 + \eta_6; A_1 - A_2 = \frac{1}{(1 - k_2)}; (A_1 - A_2)^2 = \frac{1}{(1 - k_2)^2}$$

Next, we need to find the expressions for $\frac{dg_1}{dt} \cdot g_2, \frac{dg_2}{dt} \cdot g_1$

$$\frac{\mathrm{d}g_1}{\mathrm{d}t}\cdot g_2 = \left(\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot(\alpha r - A_1) - \frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot A_2\right)\cdot \left\{I_{CQ1}\cdot[1 - A_1\cdot \alpha f]\right.$$
$$\left. - I_{C2}\cdot A_2\cdot \alpha f + Isc\cdot(\alpha r\cdot \alpha f - 1)\right\} = \frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot I_{CQ1}\cdot(\alpha r - A_1)\cdot[1 - A_1\cdot \alpha f]$$
$$-\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot I_{C2}\cdot(\alpha r - A_1)\cdot A_2\cdot \alpha f + \frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot(\alpha r - A_1)\cdot Isc\cdot(\alpha r\cdot \alpha f - 1)$$
$$-\frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot I_{CQ1}\cdot A_2\cdot[1 - A_1\cdot \alpha f] + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot I_{C2}\cdot A_2^2\cdot \alpha f - \frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot A_2\cdot Isc\cdot(\alpha r\cdot \alpha f - 1)$$

$$\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot I_{CQ1} = \frac{1}{(A_1 - A_2)}\cdot\left[\frac{\mathrm{d}Y}{\mathrm{d}t} - A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\right]\cdot\left[\frac{Y - X\cdot A_2}{A_1 - A_2}\right]$$
$$= \frac{Y\cdot\frac{\mathrm{d}Y}{\mathrm{d}t} - A_2\cdot\left(\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X + \frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y\right) + \frac{\mathrm{d}X}{\mathrm{d}t}\cdot X\cdot A_2^2}{(A_1 - A_2)^2}$$

$$\frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot I_{C2} = \frac{1}{(A_1 - A_2)}\cdot\left[\frac{\mathrm{d}Y}{\mathrm{d}t} - A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\right]\cdot\left[\frac{X\cdot A_1 - Y}{A_1 - A_2}\right]$$
$$= \frac{\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X\cdot A_1 - \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y - A_2\cdot A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X + A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y}{(A_1 - A_2)^2}$$

$$\frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot I_{CQ1} = \frac{1}{(A_1 - A_2)^2}\cdot\left[A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t} - \frac{\mathrm{d}Y}{\mathrm{d}t}\right]\cdot[Y - X\cdot A_2]$$
$$= \frac{A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y - A_1\cdot A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X - \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y + \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X\cdot A_2}{(A_1 - A_2)^2}$$

$$\frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot I_{C2} = \frac{1}{(A_1 - A_2)}\cdot\left[A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t} - \frac{\mathrm{d}Y}{\mathrm{d}t}\right]\cdot\left[\frac{X\cdot A_1 - Y}{A_1 - A_2}\right]$$
$$= \frac{A_1^2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X - A_1(\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y + \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X) + \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y}{(A_1 - A_2)^2}$$

We define for simplicity the following parameters:

$$P_1 = (\alpha r - A_1)\cdot[1 - A_1\cdot \alpha f];\ P_2 = -(\alpha r - A_1)\cdot A_2\cdot \alpha f;$$
$$P_3 = (\alpha r - A_1)\cdot Isc\cdot(\alpha r\cdot \alpha f - 1) P_4 = -A_2\cdot[1 - A_1\cdot \alpha f];$$
$$P_5 = A_2^2\cdot \alpha f;\ P_6 = -A_2\cdot Isc\cdot(\alpha r\cdot \alpha f - 1)$$

$$\frac{\mathrm{d}g_1}{\mathrm{d}t}\cdot g_2 = \frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot I_{CQ1}\cdot P_1 + \frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot I_{C2}\cdot P_2 + \frac{\mathrm{d}I_{CQ1}}{\mathrm{d}t}\cdot P_3 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot I_{CQ1}\cdot P_4 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t}$$
$$\cdot I_{C2}\cdot P_5 + \frac{\mathrm{d}I_{C2}}{\mathrm{d}t}\cdot P_6$$

$$\frac{\mathrm{d}g_1}{\mathrm{d}t} \cdot g_2 = \left(\frac{Y \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} - A_2 \cdot \left(\frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y \right) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot A_2^2}{(A_1 - A_2)^2} \right) \cdot P_1$$

$$+ \left(\frac{\frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \cdot A_1 - \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y - A_2 \cdot A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X + A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y}{(A_1 - A_2)^2} \right) \cdot P_2$$

$$+ \frac{1}{(A_1 - A_2)} \cdot \left(\frac{\mathrm{d}Y}{\mathrm{d}t} - A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \right) \cdot P_3$$

$$+ \left(\frac{A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y - A_1 \cdot A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X - \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \cdot A_2}{(A_1 - A_2)^2} \right) \cdot P_4$$

$$+ \left(\frac{A_1^2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X - A_1 \left(\cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \right) + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y}{(A_1 - A_2)^2} \right) \cdot P_5$$

$$+ \frac{1}{(A_1 - A_2)} \cdot \left(A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} - \frac{\mathrm{d}Y}{\mathrm{d}t} \right) \cdot P_6$$

$$\frac{\mathrm{d}g_1}{\mathrm{d}t} \cdot g_2 = \frac{1}{(A_1 - A_2)^2} \cdot Y \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot P_1 - \frac{1}{(A_1 - A_2)^2} \cdot A_2 \cdot \left(\frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y \right) \cdot P_1$$

$$+ \frac{1}{(A_1 - A_2)^2} \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot A_2^2 \cdot P_1 + \frac{1}{(A_1 - A_2)^2} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \cdot A_1 \cdot P_2$$

$$- \frac{1}{(A_1 - A_2)^2} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y \cdot P_2 - \frac{1}{(A_1 - A_2)^2} \cdot A_2 \cdot A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot P_2$$

$$+ \frac{1}{(A_1 - A_2)^2} \cdot A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y \cdot P_2 + \frac{1}{(A_1 - A_2)} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot P_3$$

$$- \frac{1}{(A_1 - A_2)} \cdot A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot P_3 + \frac{1}{(A_1 - A_2)^2} \cdot A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y \cdot P_4$$

$$- \frac{1}{(A_1 - A_2)^2} \cdot A_1 \cdot A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot P_4 - \frac{1}{(A_1 - A_2)^2} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y \cdot P_4$$

$$+ \frac{1}{(A_1 - A_2)^2} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \cdot A_2 \cdot P_4 + \frac{1}{(A_1 - A_2)^2} \cdot A_1^2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot P_5$$

$$- \frac{1}{(A_1 - A_2)^2} \cdot A_1 \left(\frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \right) \cdot P_5 + \frac{1}{(A_1 - A_2)^2} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y \cdot P_5$$

$$+ \frac{1}{(A_1 - A_2)} \cdot A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} \cdot P_6 - \frac{1}{(A_1 - A_2)} \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot P_6$$

$$\begin{aligned}
\frac{\mathrm{d}g_1}{\mathrm{d}t}\cdot g_2 = {} & \frac{1}{(A_1-A_2)^2}\cdot Y\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot P_1 - \frac{1}{(A_1-A_2)^2}\cdot A_2\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X\cdot P_1 \\
& - \frac{1}{(A_1-A_2)^2}\cdot A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y\cdot P_1 + \frac{1}{(A_1-A_2)^2}\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X\cdot A_2^2\cdot P_1 \\
& + \frac{1}{(A_1-A_2)^2}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X\cdot A_1\cdot P_2 - \frac{1}{(A_1-A_2)^2}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y\cdot P_2 \\
& - \frac{1}{(A_1-A_2)^2}\cdot A_2\cdot A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X\cdot P_2 + \frac{1}{(A_1-A_2)^2}\cdot A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y\cdot P_2 \\
& + \frac{1}{(A_1-A_2)}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot P_3 - \frac{1}{(A_1-A_2)}\cdot A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot P_3 \\
& + \frac{1}{(A_1-A_2)^2}\cdot A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y\cdot P_4 - \frac{1}{(A_1-A_2)^2}\cdot A_1\cdot A_2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X\cdot P_4 \\
& - \frac{1}{(A_1-A_2)^2}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y\cdot P_4 + \frac{1}{(A_1-A_2)^2}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X\cdot A_2\cdot P_4 \\
& + \frac{1}{(A_1-A_2)^2}\cdot A_1^2\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot X\cdot P_5 - \frac{1}{(A_1-A_2)^2}\cdot A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y\cdot P_5 \\
& - \frac{1}{(A_1-A_2)^2}\cdot A_1\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X\cdot P_5 + \frac{1}{(A_1-A_2)^2}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y\cdot P_5 \\
& + \frac{1}{(A_1-A_2)}\cdot A_1\cdot\frac{\mathrm{d}X}{\mathrm{d}t}\cdot P_6 - \frac{1}{(A_1-A_2)}\cdot\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot P_6
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}g_1}{\mathrm{d}t}\cdot g_2 = {} & \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot Y\cdot\frac{(P_1-P_2-P_4+P_5)}{(A_1-A_2)^2} + \frac{\mathrm{d}X}{\mathrm{d}t}\cdot X \\
& \cdot\frac{(A_2^2\cdot P_1 - A_2\cdot A_1\cdot P_2 - A_1\cdot A_2\cdot P_4 + A_1^2\cdot P_5)}{(A_1-A_2)^2} \quad \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot X \\
& \cdot\frac{1}{(A_1-A_2)^2}\cdot(-A_2\cdot P_1 + A_1\cdot P_2 + A_2\cdot P_4 - A_1\cdot P_5) + \frac{\mathrm{d}X}{\mathrm{d}t}\cdot Y \\
& \cdot\frac{1}{(A_1-A_2)^2}\cdot(-A_2\cdot P_1 + A_2\cdot P_2 + A_1\cdot P_4 - A_1\cdot P_5)\frac{\mathrm{d}X}{\mathrm{d}t}\cdot\frac{1}{(A_1-A_2)} \\
& \cdot(-A_2\cdot P_3 + A_1\cdot P_6) + \frac{\mathrm{d}Y}{\mathrm{d}t}\cdot\frac{1}{(A_1-A_2)}\cdot(P_3-P_6)
\end{aligned}$$

We define the following functions:

$$\phi_1 = \frac{(P_1 - P_2 - P_4 + P_5)}{(A_1 - A_2)^2}; \phi_2 = \frac{(A_2^2 \cdot P_1 - A_2 \cdot A_1 \cdot P_2 - A_1 \cdot A_2 \cdot P_4 + A_1^2 \cdot P_5)}{(A_1 - A_2)^2}$$

$$\phi_3 = \frac{1}{(A_1 - A_2)^2} \cdot (-A_2 \cdot P_1 + A_1 \cdot P_2 + A_2 \cdot P_4 - A_1 \cdot P_5); \phi_4 = \frac{1}{(A_1 - A_2)^2} \cdot (-A_2 \cdot P_1 + A_2 \cdot P_2 + A_1 \cdot P_4 - A_1 \cdot P_5)$$

$$\phi_5 = \frac{1}{(A_1 - A_2)} \cdot (P_3 - P_6); \phi_6 = \frac{1}{(A_1 - A_2)} \cdot (-A_2 \cdot P_3 + A_1 \cdot P_6)$$

$$\frac{dg_1}{dt} \cdot g_2 = \frac{dY}{dt} \cdot Y \cdot \phi_1 + \frac{dX}{dt} \cdot X \cdot \phi_2 + \frac{dY}{dt} \cdot X \cdot \phi_3 + \frac{dX}{dt} \cdot Y \cdot \phi_4 + \frac{dX}{dt} \cdot \phi_6 + \frac{dY}{dt} \cdot \phi_5$$

$$\begin{aligned} g_1 \cdot \frac{dg_2}{dt} &= [I_{CQ1} \cdot (\alpha r - A_1) - I_{C2} \cdot A_2 + Ise \cdot (\alpha r \cdot \alpha f - 1)] \cdot \left[\frac{dI_{CQ1}}{dt} \cdot [1 - A_1 \cdot \alpha f] - \frac{dI_{C2}}{dt} \cdot A_2 \cdot \alpha f\right] = I_{CQ1} \\ &\cdot \frac{dI_{CQ1}}{dt} \cdot (\alpha r - A_1) \cdot [1 - A_1 \cdot \alpha f] - I_{CQ1} \cdot \frac{dI_{C2}}{dt} \cdot (\alpha r - A_1) \cdot A_2 \cdot \alpha f \\ &- I_{C2} \cdot \frac{dI_{CQ1}}{dt} \cdot A_2 \cdot [1 - A_1 \cdot \alpha f] + I_{C2} \cdot \frac{dI_{C2}}{dt} \cdot A_2^2 \cdot \alpha f \\ &+ \frac{dI_{CQ1}}{dt} \cdot Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot [1 - A_1 \cdot \alpha f] \\ &- \frac{dI_{C2}}{dt} \cdot Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot A_2 \cdot \alpha f \end{aligned}$$

We define the following parameters:

$$q_1 = (\alpha r - A_1) \cdot [1 - A_1 \cdot \alpha f]; q_2 = -(\alpha r - A_1) \cdot A_2 \cdot \alpha f; q_3 = -A_2 \cdot [1 - A_1 \cdot \alpha f]$$

$$q_4 = A_2^2 \cdot \alpha f; q_5 = Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot [1 - A_1 \cdot \alpha f]; q_6 = -Ise \cdot (\alpha r \cdot \alpha f - 1) \cdot A_2 \cdot \alpha f$$

$$g_1 \cdot \frac{dg_2}{dt} = I_{CQ1} \cdot \frac{dI_{CQ1}}{dt} \cdot q_1 + I_{CQ1} \cdot \frac{dI_{C2}}{dt} \cdot q_2 + I_{C2} \cdot \frac{dI_{CQ1}}{dt} \cdot q_3 + I_{C2} \cdot \frac{dI_{C2}}{dt} \cdot q_4 + \frac{dI_{CQ1}}{dt} \cdot q_5 + \frac{dI_{C2}}{dt} \cdot q_6$$

We already got the below results:

$$\frac{dI_{CQ1}}{dt} \cdot I_{CQ1} = \frac{1}{(A_1 - A_2)} \cdot \left[\frac{dY}{dt} - A_2 \cdot \frac{dX}{dt}\right] \cdot \left[\frac{Y - X \cdot A_2}{A_1 - A_2}\right]$$
$$= \frac{Y \cdot \frac{dY}{dt} - A_2 \cdot \left(\frac{dY}{dt} \cdot X + \frac{dX}{dt} \cdot Y\right) + \frac{dX}{dt} \cdot X \cdot A_2^2}{(A_1 - A_2)^2}$$

$$\frac{dI_{CQ1}}{dt} \cdot I_{C2} = \frac{1}{(A_1 - A_2)} \cdot \left[\frac{dY}{dt} - A_2 \cdot \frac{dX}{dt}\right] \cdot \left[\frac{X \cdot A_1 - Y}{A_1 - A_2}\right]$$
$$= \frac{\frac{dY}{dt} \cdot X \cdot A_1 - \frac{dY}{dt} \cdot Y - A_2 \cdot A_1 \cdot \frac{dX}{dt} \cdot X + A_2 \cdot \frac{dX}{dt} \cdot Y}{(A_1 - A_2)^2}$$

$$\frac{dI_{C2}}{dt} \cdot I_{CQ1} = \frac{1}{(A_1 - A_2)^2} \cdot \left[A_1 \cdot \frac{dX}{dt} - \frac{dY}{dt}\right] \cdot [Y - X \cdot A_2]$$
$$= \frac{A_1 \cdot \frac{dX}{dt} \cdot Y - A_1 \cdot A_2 \cdot \frac{dX}{dt} \cdot X - \frac{dY}{dt} \cdot Y + \frac{dY}{dt} \cdot X \cdot A_2}{(A_1 - A_2)^2}$$

$$\frac{dI_{C2}}{dt} \cdot I_{C2} = \frac{1}{(A_1 - A_2)} \cdot \left[A_1 \cdot \frac{dX}{dt} - \frac{dY}{dt}\right] \cdot \left[\frac{X \cdot A_1 - Y}{A_1 - A_2}\right]$$
$$= \frac{A_1^2 \cdot \frac{dX}{dt} \cdot X - A_1\left(\cdot \frac{dX}{dt} \cdot Y + \frac{dY}{dt} \cdot X\right) + \frac{dY}{dt} \cdot Y}{(A_1 - A_2)^2}$$

$$g_1 \cdot \frac{dg_2}{dt} = \left[\frac{Y \cdot \frac{dY}{dt} - A_2 \cdot \left(\frac{dY}{dt} \cdot X + \frac{dX}{dt} \cdot Y\right) + \frac{dX}{dt} \cdot X \cdot A_2^2}{(A_1 - A_2)^2}\right] \cdot q_1$$
$$+ \left[\frac{A_1 \cdot \frac{dX}{dt} \cdot Y - A_1 \cdot A_2 \cdot \frac{dX}{dt} \cdot X - \frac{dY}{dt} \cdot Y + \frac{dY}{dt} \cdot X \cdot A_2}{(A_1 - A_2)^2}\right] \cdot q_2$$
$$+ \left[\frac{\frac{dY}{dt} \cdot X \cdot A_1 - \frac{dY}{dt} \cdot Y - A_2 \cdot A_1 \cdot \frac{dX}{dt} \cdot X + A_2 \cdot \frac{dX}{dt} \cdot Y}{(A_1 - A_2)^2}\right] \cdot q_3$$
$$+ \left[\frac{A_1^2 \cdot \frac{dX}{dt} \cdot X - A_1\left(\cdot \frac{dX}{dt} \cdot Y + \frac{dY}{dt} \cdot X\right) + \frac{dY}{dt} \cdot Y}{(A_1 - A_2)^2}\right] \cdot q_4$$
$$+ \left[\frac{1}{(A_1 - A_2)} \cdot \frac{dY}{dt} - \frac{A_2}{(A_1 - A_2)} \cdot \frac{dX}{dt}\right] \cdot q_5$$
$$+ \left[\frac{A_1}{(A_1 - A_2)} \cdot \frac{dX}{dt} - \frac{1}{(A_1 - A_2)} \cdot \frac{dY}{dt}\right] \cdot q_6$$

$$
\begin{aligned}
g_1 \cdot \frac{dg_2}{dt} = {} & \frac{Y \cdot \frac{dY}{dt} \cdot q_1}{(A_1 - A_2)^2} - \frac{A_2 \cdot \frac{dY}{dt} \cdot X \cdot q_1}{(A_1 - A_2)^2} - \frac{A_2 \cdot \frac{dX}{dt} \cdot Y \cdot q_1}{(A_1 - A_2)^2} \\
& + \frac{\frac{dX}{dt} \cdot X \cdot A_2^2 \cdot q_1}{(A_1 - A_2)^2} + \frac{A_1 \cdot \frac{dX}{dt} \cdot Y \cdot q_2}{(A_1 - A_2)^2} - \frac{A_1 \cdot A_2 \cdot \frac{dX}{dt} \cdot X}{(A_1 - A_2)^2} \\
& - \frac{\frac{dY}{dt} \cdot Y \cdot q_2}{(A_1 - A_2)^2} + \frac{\frac{dY}{dt} \cdot X \cdot A_2 \cdot q_2}{(A_1 - A_2)^2} + \frac{\frac{dY}{dt} \cdot X \cdot A_1 \cdot q_3}{(A_1 - A_2)^2} \\
& - \frac{\frac{dY}{dt} \cdot Y \cdot q_3}{(A_1 - A_2)^2} - \frac{A_2 \cdot A_1 \cdot \frac{dX}{dt} \cdot X \cdot q_3}{(A_1 - A_2)^2} + \frac{A_2 \cdot \frac{dX}{dt} \cdot Y \cdot q_3}{(A_1 - A_2)^2} \\
& + \frac{A_1^2 \cdot \frac{dX}{dt} \cdot X \cdot q_4}{(A_1 - A_2)^2} - \frac{A_1 \cdot \frac{dX}{dt} \cdot Y \cdot q_4}{(A_1 - A_2)^2} - \frac{A_1 \cdot \frac{dY}{dt} \cdot X \cdot q_4}{(A_1 - A_2)^2} \\
& + \frac{\frac{dY}{dt} \cdot Y \cdot q_4}{(A_1 - A_2)^2} + \frac{\frac{dY}{dt} \cdot q_5}{(A_1 - A_2)} - \frac{A_2 \cdot \frac{dX}{dt} \cdot q_5}{(A_1 - A_2)} + \frac{A_1 \cdot \frac{dX}{dt} \cdot q_6}{(A_1 - A_2)} - \frac{\frac{dY}{dt} \cdot q_6}{(A_1 - A_2)}
\end{aligned}
$$

$$
\begin{aligned}
g_1 \cdot \frac{dg_2}{dt} = {} & Y \cdot \frac{dY}{dt} \cdot \left\{ \frac{q_1 - q_2 - q_3 + q_4}{(A_1 - A_2)^2} \right\} + \frac{dX}{dt} \cdot X \\
& \cdot \left\{ \frac{A_2^2 \cdot q_1 - A_1 \cdot A_2 \cdot q_2 - A_2 \cdot A_1 \cdot q_3 + A_1^2 \cdot q_4}{(A_1 - A_2)^2} \right\} \\
& + \frac{dX}{dt} \cdot Y \cdot \left\{ \frac{-A_2 \cdot q_1 + A_1 \cdot q_2 + A_2 \cdot q_3 - A_1 \cdot q_4}{(A_1 - A_2)^2} \right\} + \frac{dY}{dt} \cdot X \\
& \cdot \left\{ \frac{-A_2 \cdot q_1 + A_2 \cdot q_2 - A_1 \cdot q_4 + A_1 \cdot q_3}{(A_1 - A_2)^2} \right\} \\
& + \frac{dY}{dt} \cdot \left\{ \frac{q_5 - q_6}{(A_1 - A_2)} \right\} + \frac{dX}{dt} \cdot \left\{ \frac{-A_2 \cdot q_5 + A_1 \cdot q_6}{(A_1 - A_2)} \right\}
\end{aligned}
$$

We define the following parameters:

$$
\psi_1 = \frac{q_1 - q_2 - q_3 + q_4}{(A_1 - A_2)^2};\ \psi_2 = \frac{A_2^2 \cdot q_1 - A_1 \cdot A_2 \cdot q_2 - A_2 \cdot A_1 \cdot q_3 + A_1^2 \cdot q_4}{(A_1 - A_2)^2}
$$

$$
\psi_3 = \frac{-A_2 \cdot q_1 + A_1 \cdot q_2 + A_2 \cdot q_3 - A_1 \cdot q_4}{(A_1 - A_2)^2};\ \psi_4 = \frac{-A_2 \cdot q_1 + A_2 \cdot q_2 - A_1 \cdot q_4 + A_1 \cdot q_3}{(A_1 - A_2)^2}
$$

$$
\psi_5 = \frac{q_5 - q_6}{(A_1 - A_2)};\ \psi_6 = \frac{-A_2 \cdot q_5 + A_1 \cdot q_6}{(A_1 - A_2)}
$$

$$g_1 \cdot \frac{dg_2}{dt} = Y \cdot \frac{dY}{dt} \cdot \psi_1 + \frac{dX}{dt} \cdot X \cdot \psi_2 + \frac{dX}{dt} \cdot Y \cdot \psi_3 + \frac{dY}{dt} \cdot X \cdot \psi_4 + \frac{dY}{dt} \cdot \psi_5 + \frac{dX}{dt} \cdot \psi_6$$

We can summarize our last results:

$$\frac{dg_1}{dt} \cdot g_2 = \frac{dY}{dt} \cdot Y \cdot \phi_1 + \frac{dX}{dt} \cdot X \cdot \phi_2 + \frac{dY}{dt} \cdot X \cdot \phi_3 + \frac{dX}{dt} \cdot Y \cdot \phi_4 + \frac{dX}{dt} \cdot \phi_6 + \frac{dY}{dt} \cdot \phi_5$$

$$g_1 \cdot \frac{dg_2}{dt} = Y \cdot \frac{dY}{dt} \cdot \psi_1 + \frac{dX}{dt} \cdot X \cdot \psi_2 + \frac{dX}{dt} \cdot Y \cdot \psi_3 + \frac{dY}{dt} \cdot X \cdot \psi_4 + \frac{dY}{dt} \cdot \psi_5 + \frac{dX}{dt} \cdot \psi_6$$

We already present the equation:

$$I_{C2} \cdot \frac{1}{C_2} = \frac{Vt}{g_1 \cdot g_2} \cdot \left\{ \frac{dg_1}{dt} \cdot g_2 - \frac{dg_2}{dt} \cdot g_1 \right\} + \frac{V_t \cdot \left[\frac{dI_{CQ1}}{dt} \cdot A_1 + \frac{dI_{C2}}{dt} \cdot A_2 \right]}{I_{CQ1} \cdot A_1 + I_{C2} \cdot A_2 + I_0}$$

$$\left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot \frac{1}{C_2} \frac{Vt \cdot \left\{ \frac{dY}{dt} \cdot Y \cdot \phi_1 + \frac{dX}{dt} \cdot X \cdot \phi_2 + \frac{dY}{dt} \cdot X \cdot \phi_3 + \frac{dX}{dt} \cdot Y \cdot \phi_4 + \frac{dX}{dt} \cdot \phi_6 + \frac{dY}{dt} \cdot \phi_5 - Y \cdot \frac{dY}{dt} \cdot \psi_1 - \frac{dX}{dt} \cdot X \cdot \psi_2 - \frac{dX}{dt} \cdot Y \cdot \psi_3 - \frac{dY}{dt} \cdot X \cdot \psi_4 - \frac{dY}{dt} \cdot \psi_5 - \frac{dX}{dt} \cdot \psi_6 \right\}}{\left\{ Y^2 \cdot \frac{\sum_{i=1}^{3} n_i}{(A_1 - A_2)^2} + X^2 \cdot \left[\frac{n_1 \cdot A_2^2 + n_2 \cdot A_1^2 + n_3 \cdot A_1 \cdot A_2}{(A_1 - A_2)^2} \right] - Y \cdot X \cdot \left[\frac{2 \cdot (A_2 \cdot n_1 + A_1 \cdot n_2)}{(A_1 - A_2)^2} + \frac{n_3}{(A_1 - A_2)} \right] + Y \cdot \left[\frac{n_4 + n_5}{(A_1 - A_2)} \right] - X \cdot \left[\frac{n_4 \cdot A_1 + n_5 \cdot A_2}{(A_1 - A_2)} \right] + n_6 \right\}}$$
$$+ \frac{V_t \cdot \left[\left(\frac{1}{(A_1 - A_2)} \cdot \frac{dY}{dt} - \frac{A_2}{(A_1 - A_2)} \cdot \frac{dX}{dt} \right) \cdot A_1 + \left(\frac{A_1}{(A_1 - A_2)} \cdot \frac{dX}{dt} - \frac{1}{(A_1 - A_2)} \cdot \frac{dY}{dt} \right) \cdot A_2 \right]}{\left(\frac{Y - X \cdot A_2}{A_1 - A_2} \right) \cdot A_1 + \left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot A_2 + I_0}$$

$$\left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot \frac{1}{C_2} = \frac{Vt \cdot \left\{ \frac{dY}{dt} \cdot Y \cdot (\phi_1 - \psi_1) + \frac{dX}{dt} \cdot X \cdot (\phi_2 - \psi_2) + \frac{dY}{dt} \cdot X \cdot (\phi_3 - \psi_4) + \frac{dX}{dt} \cdot Y \cdot (\phi_4 - \psi_3) + \frac{dX}{dt} \cdot (\phi_6 - \psi_6) + \frac{dY}{dt} \cdot (\phi_5 - \psi_5) \right\}}{\{ Y^2 \cdot \eta_1 + X^2 \cdot \eta_2 + Y \cdot X \cdot \eta_3 + Y \cdot \eta_4 + X \cdot \eta_5 + \eta_6 \}}$$
$$+ \frac{V_t \cdot \left[\left(\frac{dY}{dt} - A_2 \cdot \frac{dX}{dt} \right) \cdot A_1 + \left(A_1 \cdot \frac{dX}{dt} - \frac{dY}{dt} \right) \cdot A_2 \right]}{(Y - X \cdot A_2) \cdot A_1 + (X \cdot A_1 - Y) \cdot A_2 + I_0 \cdot (A_1 - A_2)}$$

We already present $\frac{1}{C_1} \cdot X = R_1 \cdot \frac{dX}{dt} + V_t \cdot \frac{\frac{dX}{dt}}{(X + I_0)} + L_1 \cdot \frac{d^2X}{dt^2} + \left(\frac{X \cdot A_1 - Y}{A_1 - A_2} \right) \cdot \frac{1}{C_2}$

Multiple two sides by $(A_1 - A_2) \cdot C_2$ term gives the below equation.

$$\frac{(A_1 - A_2) \cdot C_2}{C_1} \cdot X = (A_1 - A_2) \cdot C_2 \cdot R_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{V_t \cdot (A_1 - A_2) \cdot C_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t}}{(X + I_0)} + (A_1 - A_2) \cdot C_2 \cdot L_1 \cdot \frac{\mathrm{d}^2X}{\mathrm{d}t^2} + X \cdot A_1 - Y$$

$$Y = X \cdot \left\{A_1 - (A_1 - A_2) \cdot \frac{C_2}{C_1}\right\} + (A_1 - A_2) \cdot C_2 \cdot R_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{V_t \cdot (A_1 - A_2) \cdot C_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t}}{(X + I_0)} + (A_1 - A_2) \cdot C_2 \cdot L_1 \cdot \frac{\mathrm{d}^2X}{\mathrm{d}t^2}$$

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}t} - A_2 \cdot \frac{\mathrm{d}X}{\mathrm{d}t}\right) \cdot A_1 + \left(A_1 \cdot \frac{\mathrm{d}X}{\mathrm{d}t} - \frac{\mathrm{d}Y}{\mathrm{d}t}\right) \cdot A_2 = \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot (A_1 - A_2); (Y - X \cdot A_2) \cdot A_1 + (X \cdot A_1 - Y) \cdot A_2 = Y \cdot (A_1 - A_2)$$

$$\left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{1}{C_2} = \frac{Vt\left\{\frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y \cdot (\phi_1 - \psi_1) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot (\phi_2 - \psi_2) + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \cdot (\phi_3 - \psi_4) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y \cdot (\phi_4 - \psi_3) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot (\phi_6 - \psi_6) + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot (\phi_5 - \psi_5)\right\}}{\{Y^2 \cdot \eta_1 + X^2 \cdot \eta_2 + Y \cdot X \cdot \eta_3 + Y \cdot \eta_4 + X \cdot \eta_5 + \eta_6\}} + \frac{V_t \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot (A_1 - A_2)}{Y \cdot (A_1 - A_2) + I_0 \cdot (A_1 - A_2)}$$

We define the following functions:

$$\xi_1 = \xi_1\left(X, \frac{\mathrm{d}X}{\mathrm{d}t}, Y, \frac{\mathrm{d}Y}{\mathrm{d}t}\right) = \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot Y \cdot (\phi_1 - \psi_1) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot X \cdot (\phi_2 - \psi_2) + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot X \cdot (\phi_3 - \psi_4) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot Y \cdot (\phi_4 - \psi_3) + \frac{\mathrm{d}X}{\mathrm{d}t} \cdot (\phi_6 - \psi_6) + \frac{\mathrm{d}Y}{\mathrm{d}t} \cdot (\phi_5 - \psi_5)$$

$$\xi_2 = \xi_2(X, Y) = Y^2 \cdot \eta_1 + X^2 \cdot \eta_2 + Y \cdot X \cdot \eta_3 + Y \cdot \eta_4 + X \cdot \eta_5 + \eta_6; \left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{1}{C_2 \cdot Vt} = \frac{\xi_1}{\xi_2} + \frac{\frac{\mathrm{d}Y}{\mathrm{d}t}}{Y + I_0}$$

We define the following setting: $Z = \frac{\mathrm{d}X}{\mathrm{d}t}; \frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{\mathrm{d}^2X}{\mathrm{d}t^2}$

$$Y = X \cdot \left\{A_1 - (A_1 - A_2) \cdot \frac{C_2}{C_1}\right\} + (A_1 - A_2) \cdot C_2 \cdot R_1 \cdot Z + \frac{V_t \cdot (A_1 - A_2) \cdot C_2 \cdot Z}{(X + I_0)} + (A_1 - A_2) \cdot C_2 \cdot L_1 \cdot \frac{\mathrm{d}Z}{\mathrm{d}t}$$

We define the following new parameters: $W_1 = (A_1 - A_2) \cdot C_2 \cdot R_1$; $W_2 = V_t \cdot (A_1 - A_2) \cdot C_2$

$$W_3 = (A_1 - A_2) \cdot C_2 \cdot L_1; W_4 = A_1 - (A_1 - A_2) \cdot \frac{C_2}{C_1}; Y = X \cdot W_4 + W_1$$

$$\cdot Z + \frac{W_2 \cdot Z}{(X + I_0)} + W_3 \cdot \frac{dZ}{dt} Y = X \cdot W_4 + W_1$$

$$\cdot Z + \frac{W_2 \cdot Z}{(X + I_0)} + W_3 \cdot \frac{dZ}{dt} \Rightarrow \frac{dZ}{dt} = \frac{1}{W_3} \cdot \left[Y - X \cdot W_4 - W_1 \cdot Z - \frac{W_2 \cdot Z}{(X + I_0)} \right]$$

$$\xi_1 = \xi_1\left(X, Z, Y, \frac{dY}{dt}\right) = \frac{dY}{dt} \cdot Y \cdot (\phi_1 - \psi_1) + Z \cdot X \cdot (\phi_2 - \psi_2) + \frac{dY}{dt} \cdot X \cdot (\phi_3 - \psi_4) + Z \cdot Y \cdot (\phi_4 - \psi_3) + Z \cdot (\phi_6 - \psi_6) + \frac{dY}{dt} \cdot (\phi_5 - \psi_5)$$

$$\left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{1}{C_2 \cdot Vt} = \frac{\xi_1}{\xi_2} + \frac{\frac{dY}{dt}}{Y + I_0}$$

$$\Rightarrow \left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{1}{C_2 \cdot Vt} = \frac{\left[\frac{dY}{dt} \cdot Y \cdot (\phi_1 - \psi_1) + Z \cdot X \cdot (\phi_2 - \psi_2) + \frac{dY}{dt} \cdot X \cdot (\phi_3 - \psi_4) + Z \cdot Y \cdot (\phi_4 - \psi_3) + Z \cdot (\phi_6 - \psi_6) + \frac{dY}{dt} \cdot (\phi_5 - \psi_5)\right]}{Y^2 \cdot \eta_1 + X^2 \cdot \eta_2 + Y \cdot X \cdot \eta_3 + Y \cdot \eta_4 + X \cdot \eta_5 + \eta_6} + \frac{\frac{dY}{dt}}{Y + I_0}$$

$$\left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{\xi_2(X, Y)}{C_2 \cdot Vt} = \frac{dY}{dt} \cdot Y \cdot (\phi_1 - \psi_1) + Z \cdot X \cdot (\phi_2 - \psi_2) + \frac{dY}{dt} \cdot X \cdot (\phi_3 - \psi_4) + Z \cdot Y \cdot (\phi_4 - \psi_3) + Z \cdot (\phi_6 - \psi_6) + \frac{dY}{dt} \cdot (\phi_5 - \psi_5) + \frac{\frac{dY}{dt} \cdot \xi_2(X, Y)}{Y + I_0}$$

$$\frac{dY}{dt} \cdot \left\{ Y \cdot (\phi_1 - \psi_1) + \phi_5 - \psi_5 + X \cdot (\phi_3 - \psi_4) + \frac{\xi_2(X, Y)}{Y + I_0} \right\} = \left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{\xi_2(X, Y)}{C_2 \cdot Vt} - Z \cdot Y \cdot (\phi_4 - \psi_3) - Z \cdot (\phi_6 - \psi_6) - Z \cdot X \cdot (\phi_2 - \psi_2)$$

$$\frac{dY}{dt} \cdot \left\{ \frac{Y \cdot (\phi_1 - \psi_1) + \phi_5 - \psi_5 + X \cdot (\phi_3 - \psi_4)}{\xi_2(X, Y)} + \frac{1}{Y + I_0} \right\} = \left(\frac{X \cdot A_1 - Y}{A_1 - A_2}\right) \cdot \frac{1}{C_2 \cdot Vt} + \frac{-Z \cdot Y \cdot (\phi_4 - \psi_3) - Z \cdot (\phi_6 - \psi_6) - Z \cdot X \cdot (\phi_2 - \psi_2)}{\xi_2(X, Y)}$$

We define the following function:

$$\chi(X, Y) = \frac{Y \cdot (\phi_1 - \psi_1) + \phi_5 - \psi_5 + X \cdot (\phi_3 - \psi_4)}{\xi_2(X, Y)} + \frac{1}{Y + I_0}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t}\cdot \chi(X,Y) = \left(\frac{X\cdot A_1 - Y}{A_1 - A_2}\right)\cdot\frac{1}{C_2\cdot Vt} + \frac{-Z\cdot Y\cdot(\phi_4-\psi_3) - Z\cdot(\phi_6-\psi_6) - Z\cdot X\cdot(\phi_2-\psi_2)}{\xi_2(X,Y)}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{1}{\chi(X,Y)}\cdot\left\{\left(\frac{X\cdot A_1 - Y}{A_1 - A_2}\right)\cdot\frac{1}{C_2\cdot Vt} + \frac{-Z\cdot Y\cdot(\phi_4-\psi_3) - Z\cdot(\phi_6-\psi_6) - Z\cdot X\cdot(\phi_2-\psi_2)}{\xi_2(X,Y)}\right\}$$

We can summarize our system three differential equations:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = Z$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{1}{\chi(X,Y)}\cdot\left\{\left(\frac{X\cdot A_1 - Y}{A_1 - A_2}\right)\cdot\frac{1}{C_2\cdot Vt} + \frac{-Z\cdot Y\cdot(\phi_4-\psi_3) - Z\cdot(\phi_6-\psi_6) - Z\cdot X\cdot(\phi_2-\psi_2)}{\xi_2(X,Y)}\right\}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \frac{1}{W_3}\cdot\left[Y - X\cdot W_4 - W_1\cdot Z - \frac{W_2\cdot Z}{(X+I_0)}\right]$$

To find system fixed points we set $\mathrm{d}X/\mathrm{d}t = 0$; $\mathrm{d}Y/\mathrm{d}t = 0$; $\mathrm{d}Z/\mathrm{d}t = 0$; $i = 0, 1, 2$

$$Z^{(i)} = 0; \frac{1}{\chi(X^{(i)},Y^{(i)})}\cdot\left(\frac{X^{(i)}\cdot A_1 - Y^{(i)}}{A_1 - A_2}\right)\cdot\frac{1}{C_2\cdot Vt} = 0; \frac{1}{W_3}\cdot\left[Y^{(i)} - X^{(i)}\cdot W_4\right] = 0$$

$$\frac{1}{W_3}\cdot\left[Y^{(i)} - X^{(i)}\cdot W_4\right] = 0 \Rightarrow Y^{(i)} = X^{(i)}\cdot W_4 \Rightarrow \frac{1}{\chi(X^{(i)},X^{(i)}\cdot W_4)}\cdot\left(\frac{X^{(i)}\cdot[A_1 - W_4]}{A_1 - A_2}\right)\cdot\frac{1}{C_2\cdot Vt} = 0$$

$$\frac{1}{\chi(X^{(i)},X^{(i)}\cdot W_4)}\cdot\left(\frac{X^{(i)}\cdot[A_1 - W_4]}{A_1 - A_2}\right)\cdot\frac{1}{C_2\cdot Vt} = 0 \Rightarrow X^{(i)} = 0 \text{ or } \chi(X^{(i)},X^{(i)}\cdot W_4)\to\infty \text{ or } A_1 = W_4$$

$\frac{1}{C_2\cdot Vt} \neq 0$. We get one possible fixed point: $E^{(i=0)}(X^{(i=0)}, Y^{(i=0)}, Z^{(i=0)}) = (0,0,0)$

$$\chi(X,Y)\to\infty \Rightarrow \left[\frac{Y\cdot(\phi_1-\psi_1)+\phi_5-\psi_5+X\cdot(\phi_3-\psi_4)}{\xi_2(X,Y)} + \frac{1}{Y+I_0}\right]\to\infty$$

$$\left\{A_1 = W_4\ \&\ W_4 = A_1 - (A_1 - A_2)\cdot\frac{C_2}{C_1}\right\}$$

$$\Rightarrow W_4 = W_4 - (W_4 - A_2)\cdot\frac{C_2}{C_1} \Rightarrow -(W_4 - A_2)\cdot\frac{C_2}{C_1} = 0$$

$$-(W_4 - A_2)\cdot\frac{C_2}{C_1} = 0 \Rightarrow \frac{C_2}{C_1} \neq 0 \Rightarrow W_4 - A_2 = 0 \Rightarrow W_4 = A_2 \,\&\, W_4 = A_1 - (A_1 - A_2)\cdot\frac{C_2}{C_1}$$
$$\Rightarrow A_2 = A_1 - (A_1 - A_2)\cdot\frac{C_2}{C_1} \Rightarrow A_2 - A_2\cdot\frac{C_2}{C_1} = A_1 - A_1\cdot\frac{C_2}{C_1} \Rightarrow A_2 = A_1 \Rightarrow \frac{k_1}{(1-k_2)} = \frac{(1+k_1)}{(1-k_2)}$$
$$\Rightarrow k_1 = 1 + k_1 \Rightarrow 0 = 1 \text{ (Impossible)}$$

We choose the below circuit parameters (Table B.1):
Then we calculate our system global parameters.

$$A_1 = 1.051; A_2 = 0.0206; n_1 = 0.0165; n_2 = 0.000415; n_3 = -0.0117; n_4 = -0.0313\times 10^{-6}$$
$$n_5 = 0.577\times 10^{-6}; n_6 = 0.5202\times 10^{-12}; \eta_1 = 0.00491; \eta_2 = 0.0002; \eta_3 = 0.00988; \eta_4 = 0.5298\times 10^{-6}$$
$$\eta_5 = 0.0203\times 10^{-6}; \eta_6 = 0.5202\times 10^{-12}; P_1 = 0.0165; P_2 = -0.0111; P_3 = 0.562\times 10^{-6}$$

$$P_4 = 0.000617; P_5 = 0.000415; P_6 = 0.021\times 10^{-6};$$
$$\phi_1 = 0.0257; \phi_2 = 6.529\times 10^{-4}; \phi_3 = -0.01166;$$
$$\phi_4 = -3.349\times 10^{-4}; \phi_5 = 0.525\times 10^{-6}; \phi_6 = 0.0101\times 10^{-6};$$
$$q_1 = 0.0159; q_2 = 0.0111; q_3 = 0.000617;$$
$$q_4 = 0.000415; q_5 = 0.0152\times 10^{-6}; q_6 = 0.0102\times 10^{-6};$$
$$\psi_1 = 0.00433; \psi_2 = -6.646\times 10^{-4}\psi_3 = -0.0158;$$
$$\psi_4 = 1.07\times 10^{-4}; \psi_5 = 0.00485\times 10^{-6}; \psi_6 = 0.01\times 10^{-6};$$
$$W_1 = 0.10304; W_2 = 2.679\times 10^{-6}; W_3 = 1.0304\times 10^{-7}; W_4 = -1029.349;$$
$$A_1 - A_2 = 1.051 - 0.0206 = 1.0304$$

We run MATLAB script to get phase portrait and time series of X, Y, Z variables in time for our system. We define the following parameters in our MATLAB script: $\chi \to h$; η_i (i = 1, 2, …, 6). $\xi_2 \to r$. $1/[C_2 \cdot V_t]$ = 384,615.38 (Fig. B.3).

$$\phi_1 - \psi_1 = 0.0257 - 0.00433 = 0.02137; \phi_5 - \psi_5 = 0.525\times 10^{-6} - 0.00485\times 10^{-6} = 0.52\times 10^{-6}$$
$$\phi_3 - \psi_4 = -0.01166 - 1.07\times 10^{-4} = -0.01176; \phi_4 - \psi_3 = -3.349\times 10^{-4} + 0.0158 = 0.0154$$

B.1 Circuit parameters

V_t	0.026	α_f	0.98
I_0	1E−6	β_f	49
R_1	1 k Ω	α_r	0.5
C_1	0.1 μF	β_r	1
C_2	100 μF	k_2	0.03
L_1	1 mH	k_1	0.02
Ise	1 μA	I_{sc}	2 μA

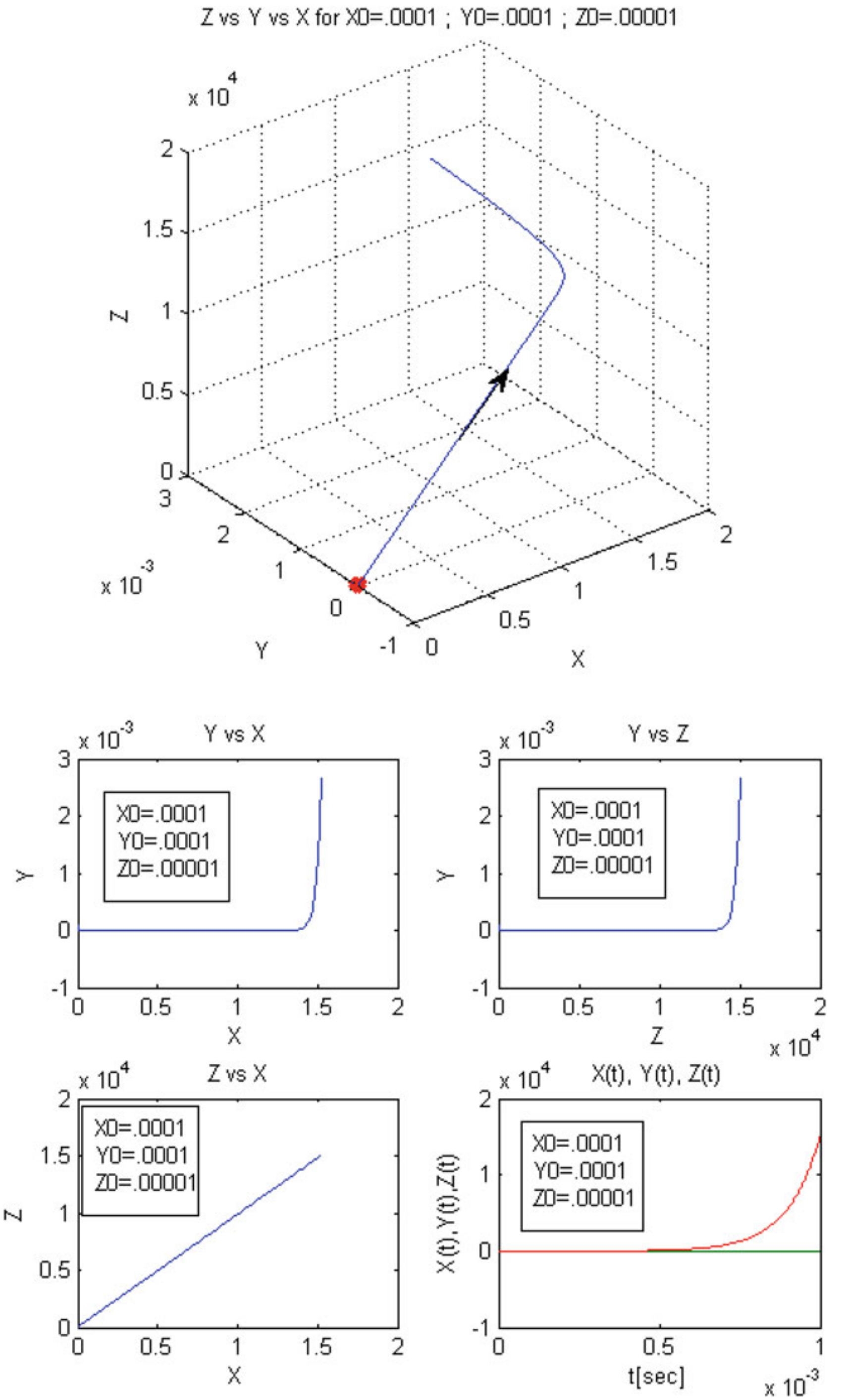

Fig. B.3 *X* vs *Y* vs *X* graph and *Y* vs *X* graph and *Y* vs *Z* graph and *Z* vs *X* graph and *X*(*t*), *Y*(*t*), *Z*(*t*) graph for our system

$$\phi_6 - \psi_6 = 0.0101 \times 10^{-6} - 0.01 \times 10^{-6} = 0.0001 \times 10^{-6}; \phi_2 - \psi_2 = 6.529 \times 10^{-4} + 6.646 \times 10^{-4} = 13.175 \times 10^{-4}$$

MATLAB Script

```
function g=doublcoupling(t,x)
g=zeros(3,1);
g(1)=x(3);
r=x(2)*x(2)*0.00491+x(1)*x(1)*0.0002+x(1)*x(2)
*0.00988+x(2)*0.5298*0.000001+x(1)*0.0203*0.000001
+0.5202e-12;
h=(x(2)*0.02137+0.525e-6-x(1)*0.01176)./r+1/(x(2)
+1e-6);
g(2)=(1./h).*((x(1)*1.051-x(2))/1.0304*384615.38-(x
(3)*x(2)*0.0154+x(3)*0.0001*1e-6+x(3)*x(1)
*13.175e-4)./r);
g(3)=1/1.0304e-7*(x(2)+x(1)*1029.349-0.10304*x(3)-
2.679e-6*x(3)./(x(1)+1e-6));
```

```
function h=doublcoupling1(x0,y0,z0)
[t,x]=ODE45 (@doublcoupling,[0,.001],[x0,y0,z0],[]);
plot3 (x(:,1),x(:,2),x(:,3));
%xlabel ('X')
%ylabel ('Y')
%zlabel ('Z')
grid on
axis square
%subplot(2,2,1);plot(x(:,1),x(:,2));subplot(2,2,2);
plot(x(:,3),x(:,2));subplot(2,2,3);plot(x(:,1),x
(:,3));subplot(2,2,4);plot(t,x);
```

$$X = I_{C2} + I_{CQ1}; Y = I_{C2} \cdot A_2 + I_{CQ1} \cdot A_1; Z = \frac{dI_{C2}}{dt} + \frac{dI_{CQ1}}{dt}$$

Appendix C
Optoisolation Elements Bridges Investigation and Stability Analysis

Optoisolation devices are used in many industrial applications and can be widely implemented in many circuit's structures and topologies. One set of engineering topologies is optoisolation Elements Bridge. The structure is a bridge of four LEDs and each LED is coupled with photo transistor. We activate the circuit in an alternate fashion by using square wave. We can divide our square wave source to frequently positive value (duration T_A) and negative value (duration T_B). We assume that $T_A \neq T_B$. The middle capacitor C_m is charged in one direction and charged in opposite direction according to the positive and negative values period of the square wave through resistor R_m (Fig. C.1).

$$V_B < 0; V_A > 0; V_A = |V_B|; V_1 \gg V_A \gg V_2; V_2 \ll |V_B|; V_1 > 0; V_2 > 0; V_1 \neq V_2$$

$V_s(t)$ is infinite series of square waves, positive amplitude V_A ($V_a = V_A$) and negative amplitude V_B ($V_b = V_B$). The duration of the square wave positive portion is T_A ($T_a = T_A$) and the negative portion is T_B ($T_b = T_B$). Complete cycle is define as T ($T = T_A + T_B$), Positive duty cycle is define as D_+ and negative duty cycle is define as D_- ($D_+ = \frac{1}{1+T_B/T_A}$; $D_- = \frac{1}{1+T_A/T_B}$). Series of square waves is define as $X_n(t)$. $n = 1, 2, \ldots, k$.

Remark $U(t - \Delta)$ is a step function. It is equal to 1 for $t \geq \Delta$ and equal to zero for $t < \Delta$.

$$X_n(t) = [U(t-(n-1)\cdot(T_A+T_B)) - U(t-n\cdot T_A-(n-1)\cdot T_B)]\cdot V_A + [U(t-n\cdot T_A-(n-1)\cdot T_B) - U(t-n\cdot(T_A+T_B))]\cdot V_B$$

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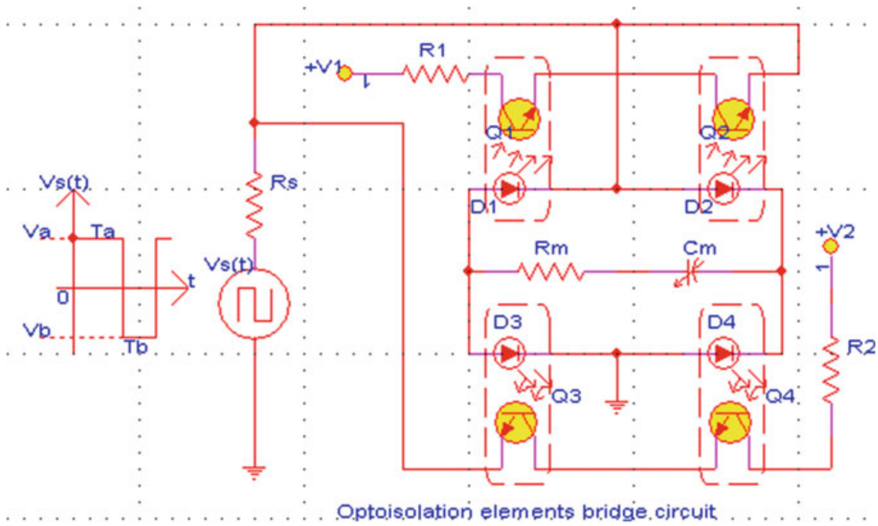


Fig. C.1 Optoisolation elements bridge circuit

$$n = 1 \Rightarrow X_1(t) = [U(t) - U(t - T_A)] \cdot V_A + [U(t - T_A) - U(t - (T_A + T_B))] \cdot V_B$$
$$n = 2 \Rightarrow X_2(t) = [U(t - (T_A + T_B)) - U(t - 2T_A - T_B)] \cdot V_A + [U(t - 2 \cdot T_A - T_B) - U(t - 2 \cdot (T_A + T_B))] \cdot V_B$$
$$n = 3 \Rightarrow X_3(t) = [U(t - 2 \cdot (T_A + T_B)) - U(t - 3 \cdot T_A - 2 \cdot T_B)] \cdot V_A + [U(t - 3 \cdot T_A - 2 \cdot T_B) - U(t - 3 \cdot (T_A + T_B))] \cdot V_B$$

$$V_s(t) = \sum_{n=1}^{\infty} X_n(t) = V_A \cdot \sum_{n=1}^{\infty} [U(t - (n-1) \cdot (T_A + T_B)) - U(t - n \cdot T_A - (n-1) \cdot T_B)] + V_B \cdot \sum_{n=1}^{\infty} [U(t - n \cdot T_A - (n-1) \cdot T_B) - U(t - n \cdot (T_A + T_B))]; V_A > 0; V_B < 0; V_A = |V_B|$$

The mutual relationship between V_A, V_B, V_1, and V_2 can be describe in the below levels figure (Fig. C.2).

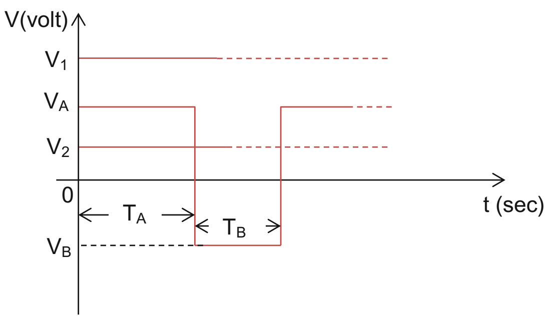


Fig. C.2 Optoisolation elements bridge mutual relationship between V_A, V_B, V_1, and V_2

Circuit Operation We divide our analysis into two sections. The first section is which $V_s(t) = V_A$ for $(n-1)\cdot(T_A+T_B) \le t \le n\cdot T_A + T_B\cdot(n-1); n = 1,2,3,\ldots$ and the second section is which $V_s(t) = V_B$ for $n\cdot T_A + (n-1)\cdot T_B \le t \le (T_A+T_B)\cdot n; n = 1,2,3,\ldots$.

If $V_s(t) = V_A$ then D_2 and D_3 are ON state and D_1 and D_4 are OFF state. if $V_s(t) = V_B$ then D_2 and D_3 are OFF state and D_1 and D_4 are ON state. If $V_s(t) = V_A$ then capacitor C_m is charged through D_2, D_3 and resistor R_m. If $V_s(t) = V_B$ then capacitor C_m is charged through D_1, D_4 and resistor R_m. If $V_s(t) = 0$ then all bridge's LEDs (D_1, …, D_4) are in OFF state and all phototransistors (Q_1, …, Q_4) are in OFF state. Once the current starts to flow through specific LED the coupled phototransistor enters to saturation accordingly (It depends on the amount of current that flows through the LED). The final capacitor C_m voltage after k cycles of input square wave signal is important in our analysis.

We consider that all optocouplers in the circuit are not identical. The circuit model for analysis is based on the assumption that the LED light which strikes the photo transistor base window can be represented as a dependent current source. The dependent current source depends on the LED forward current, with proportional k_i constant (Index i = 1 for the first optocoupler, i = 2 for the second … etc.,). $I_{B-i} = I_{\mathrm{LED}-i}\cdot k_i$ and is the photo transistor (I_{B-i}) base current. The mathematical analysis is based on the basic transistor Ebers–Moll equations. V_{CEQ_i}—Transistor (i) collector emitter voltage. I_{CQ_i}—Transistor (i) collector current. I_{EQ_i}—Transistor (i) emitter current. The main benefit of using optocoupler as an element in our circuit is the ability to choose circuit characteristic by switching from one analog optocoupler to another one. Each optocoupler has unique specific parameters $(I_0, I_{se}, I_{sc}, \alpha_r, \alpha_f, k)$.

$$V_{CEQ_i} = V_t\cdot\ln\left[\frac{(\alpha_{rQ_i}\cdot I_{CQ_i} - I_{EQ_i}) + I_{se}\cdot(\alpha_{rQ_i}\cdot\alpha_{fQ_i} - 1)}{(I_{CQ_i} - I_{EQ_i}\cdot\alpha_{fQ_i}) + I_{sc}\cdot(\alpha_{rQ_i}\cdot\alpha_{fQ_i} - 1)}\right] + V_t\cdot\ln\left[\frac{I_{sc}}{I_{se}}\right]; \quad i = 1,2,3,4$$

$$\frac{I_{sc}}{I_{se}} \approx 1 \Rightarrow \ln\left[\frac{I_{sc}}{I_{se}}\right] \to \varepsilon; I_{EQ_i} = I_{CQ_i} + I_{BQ_i}; I_{EQ_i} = I_{CQ_i} + I_{\mathrm{LED}-i}\cdot k_i; \frac{\alpha_{fQ_i}}{\alpha_{rQ_i}} = \frac{I_{sc}}{I_{se}}$$

The equivalent circuit for our analysis is describing (Fig. C.3).

A bipolar junction phototransistor (BJT or bipolar phototransistor) can be represents as Ebers–Moll model and we define some parameters. V_t is the thermal voltage $\frac{k\cdot T}{q}$ (approximately 26 mV at 300 K at room temperature). α_{fQ_i} is the common base forward short circuit gain of phototransistor (i), 0.98–0.998. α_{rQ_i} = 0.1 − 0.5 typically. I_{se}, I_{sc} are photo transistor saturation currents. I_{se}: reverse saturation current of the base emitter diode. I_{sc}: reverse saturation current of the base–collector diode.

$$I_{EQ_1} = I_{CQ_2}; I_{EQ_4} = I_{CQ_3}$$

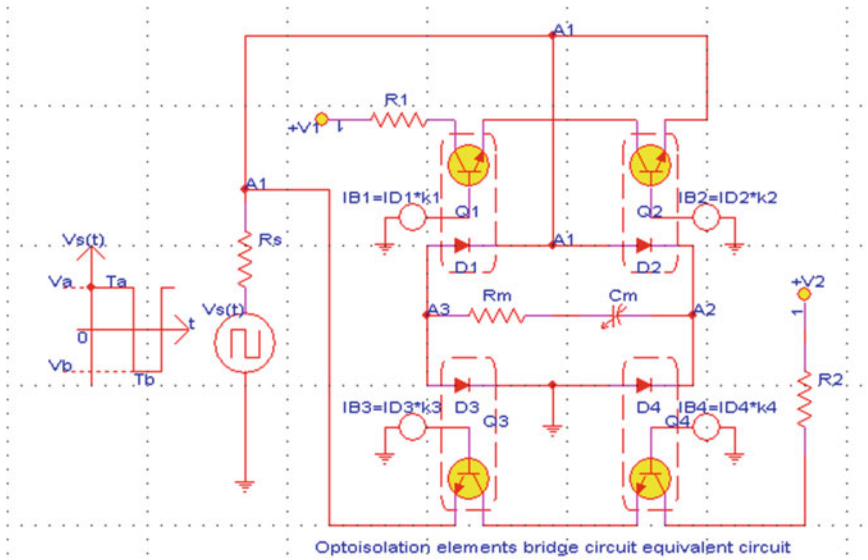


Fig. C.3 Optoisolation elements bridge circuit equivalent circuit

$$V_{CEQ_i} = V_t \cdot \ln\left[\frac{(\alpha_{rQ_i} \cdot I_{CQ_i} - (I_{CQ_i} + I_{\text{LED}-i} \cdot k_i)) + I_{se} \cdot (\alpha_{rQ_i} \cdot \alpha_{fQ_i} - 1)}{(I_{CQ_i} - (I_{CQ_i} + I_{\text{LED}-i} \cdot k_i) \cdot \alpha_{fQ_i}) + I_{sc} \cdot (\alpha_{rQ_i} \cdot \alpha_{fQ_i} - 1)}\right] + V_t \cdot \ln\left[\frac{I_{sc}}{I_{se}}\right] \to \varepsilon; \quad i = 1,2,3,4$$

$$V_{CEQ_i} = V_t \cdot \ln\left[\frac{([\alpha_{rQ_i} - 1] \cdot I_{CQ_i} - I_{\text{LED}-i} \cdot k_i) + I_{se} \cdot (\alpha_{rQ_i} \cdot \alpha_{fQ_i} - 1)}{(I_{CQ_i} \cdot [1 - \alpha_{fQ_i}] - I_{\text{LED}-i} \cdot k_i \cdot \alpha_{fQ_i}) + I_{sc} \cdot (\alpha_{rQ_i} \cdot \alpha_{fQ_i} - 1)}\right]$$

Case I $V_s(t) = V_A$ for $(n-1) \cdot (T_A + T_B) \le t \le n \cdot T_A + T_B \cdot (n-1);\ n = 1,2,3,\ldots$

KCL @ A_1: $I_{R_s} + I_{EQ_3} + I_{EQ_2} = I_{\text{LED}-2}$. The assumption is D_1 and D_4 are in OFF state, and we consider as disconnected elements. $I_{\text{LED}-1} \to \varepsilon; I_{\text{LED}-4} \to \varepsilon$.

$$I_{\text{LED}-2} = I_{C_m} = I_{R_m} = I_{\text{LED}-3}; I_{R_s} = \frac{V_s(t) - V_{A_1}}{R_s}\Big|_{V_s(t)=V_A > 0} = \frac{V_A - V_{A_1}}{R_s};$$
$$V_{A_1} = V_{\text{LED}-2} + V_{C_m} + V_{R_m} + V_{\text{LED}-3}$$

$$I_{C_m} = C_m \cdot \frac{\mathrm{d}V_{C_m}}{\mathrm{d}t}; V_{R_m} = I_{R_m} \cdot R_m; V_{\text{LED}-2} = V_t \cdot \ln\left[\frac{I_{\text{LED}-2}}{I_0} + 1\right];$$
$$V_{\text{LED}-3} = V_t \cdot \ln\left[\frac{I_{\text{LED}-3}}{I_0} + 1\right]$$

$$V_1 - V_{A_1} = I_{CQ_1} \cdot R_1 + V_{CEQ_1} + V_{CEQ_2}; V_1 \gg V_A; V_2 - V_{A_1} = I_{CQ_4} \cdot R_2 + V_{CEQ_4} + V_{CEQ_3}; V_2 \ll V_A$$

$$V_{CEQ_1} = V_t \cdot \ln\left[\frac{([\alpha_{rQ_1} - 1] \cdot I_{CQ_1} - I_{\text{LED}-1} \cdot k_1) + I_{se} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)}{(I_{CQ_1} \cdot [1 - \alpha_{fQ_1}] - I_{\text{LED}-1} \cdot k_1 \cdot \alpha_{fQ_1}) + I_{sc} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)}\right]$$

$$V_{CEQ_2} = V_t \cdot \ln\left[\frac{([\alpha_{rQ_2} - 1] \cdot I_{CQ_2} - I_{\text{LED}-2} \cdot k_2) + I_{se} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)}{(I_{CQ_2} \cdot [1 - \alpha_{fQ_2}] - I_{\text{LED}-2} \cdot k_2 \cdot \alpha_{fQ_2}) + I_{sc} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)}\right]$$

$$V_{CEQ_3} = V_t \cdot \ln\left[\frac{([\alpha_{rQ_3} - 1] \cdot I_{CQ_3} - I_{\text{LED}-3} \cdot k_3) + I_{se} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}{(I_{CQ_3} \cdot [1 - \alpha_{fQ_3}] - I_{\text{LED}-3} \cdot k_3 \cdot \alpha_{fQ_3}) + I_{sc} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}\right]$$

$$V_{CEQ_4} = V_t \cdot \ln\left[\frac{([\alpha_{rQ_4} - 1] \cdot I_{CQ_4} - I_{\text{LED}-4} \cdot k_4) + I_{se} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}{(I_{CQ_4} \cdot [1 - \alpha_{fQ_4}] - I_{\text{LED}-4} \cdot k_4 \cdot \alpha_{fQ_4}) + I_{sc} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}\right]$$

$$V_{A_1} = V_t \cdot \ln\left[\frac{I_{\text{LED}-2}}{I_0} + 1\right] + V_{C_m} + I_{R_m} \cdot R_m + V_t \cdot \ln\left[\frac{I_{\text{LED}-3}}{I_0} + 1\right];$$

$$I_{C_m} = C_m \cdot \frac{\mathrm{d}V_{C_m}}{\mathrm{d}t} \Rightarrow V_{C_m} = \frac{1}{C_m} \cdot \int I_{C_m} \cdot \mathrm{d}t$$

$$V_{A_1} = V_t \cdot \ln\left[\frac{I_{\text{LED}-2}}{I_0} + 1\right] + \frac{1}{C_m} \cdot \int I_{C_m} \cdot dt + I_{R_m} \cdot R_m + V_t \cdot \ln\left[\frac{I_{\text{LED}-3}}{I_0} + 1\right]; I_{\text{LED}-2} = I_{C_m} = I_{R_m} = I_{\text{LED}-3}$$

$$I_{R_s} = \frac{V_s(t) - V_{A_1}}{R_s}\Big|_{V_s(t) = V_A > 0} = \frac{V_A - V_{A_1}}{R_s} \Rightarrow V_{A_1} = V_A - I_{R_s} \cdot R_s; I_{R_s} = I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}$$

$$V_{A_1} = V_A - (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s$$

$$V_A - (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s = 2 \cdot V_t \cdot \ln\left[\frac{I_{\text{LED}-2}}{I_0} + 1\right] + \frac{1}{C_m} \cdot \int I_{\text{LED}-2} \cdot \mathrm{d}t + I_{\text{LED}-2} \cdot R_m$$

$$V_1 - V_A + (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s = I_{CQ_1} \cdot R_1 + V_t \cdot \ln\left[\frac{([\alpha_{rQ_1} - 1] \cdot I_{CQ_1} - I_{\text{LED}-1} \cdot k_1) + I_{se} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)}{(I_{CQ_1} \cdot [1 - \alpha_{fQ_1}] - I_{\text{LED}-1} \cdot k_1 \cdot \alpha_{fQ_1}) + I_{sc} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)}\right] + V_t \cdot \ln\left[\frac{([\alpha_{rQ_2} - 1] \cdot I_{CQ_2} - I_{\text{LED}-2} \cdot k_2) + I_{se} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)}{(I_{CQ_2} \cdot [1 - \alpha_{fQ_2}] - I_{\text{LED}-2} \cdot k_2 \cdot \alpha_{fQ_2}) + I_{sc} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)}\right]$$

$$V_2 - V_A + (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s = I_{CQ_4} \cdot R_2 + V_t \cdot \ln\left[\frac{([\alpha_{rQ_4} - 1] \cdot I_{CQ_4} - I_{\text{LED}-4} \cdot k_4) + I_{se} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}{(I_{CQ_4} \cdot [1 - \alpha_{fQ_4}] - I_{\text{LED}-4} \cdot k_4 \cdot \alpha_{fQ_4}) + I_{sc} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}\right]$$
$$+ V_t \cdot \ln\left[\frac{([\alpha_{rQ_3} - 1] \cdot I_{CQ_3} - I_{\text{LED}-3} \cdot k_3) + I_{se} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}{(I_{CQ_3} \cdot [1 - \alpha_{fQ_3}] - I_{\text{LED}-3} \cdot k_3 \cdot \alpha_{fQ_3}) + I_{sc} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}\right]$$

We derivate the two side of previous expression ($\frac{\mathrm{d}V_A}{\mathrm{d}t} = 0$) by

$$\frac{\mathrm{d}}{\mathrm{d}t}\{V_A - (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s\} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{2 \cdot V_t \cdot \ln\left[\frac{I_{\text{LED}-2}}{I_0} + 1\right] + \frac{1}{C_m} \cdot \int I_{\text{LED}-2} \cdot \mathrm{d}t + I_{\text{LED}-2} \cdot R_m\right\}$$
$$-\left(\frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} - \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t}\right) \cdot R_s = 2 \cdot V_t \cdot \frac{\mathrm{d}}{\mathrm{d}t}\ln\left[\frac{I_{\text{LED}-2}}{I_0} + 1\right] + \frac{1}{C_m} \cdot I_{\text{LED}-2} + \frac{\mathrm{d}I_{LED-2}}{\mathrm{d}t} \cdot R_m$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left[\frac{I_{LED-2}}{I_0} + 1\right] = \frac{1}{\left(\frac{I_{\text{LED}-2}}{I_0} + 1\right)} \cdot \frac{1}{I_0} \cdot \frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} = \frac{1}{I_{\text{LED}-2} + I_0} \cdot \frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t}$$

$$-\left(\frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} - \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t}\right) \cdot R_s = \frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} \cdot \frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} + \frac{1}{C_m} \cdot I_{\text{LED}-2} + \frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} \cdot R_m$$

$$\left(-\frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} + \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} + \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t}\right) \cdot R_s = \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m\right] \cdot \frac{\mathrm{d}I_{\text{LED}-2}}{\mathrm{d}t} + \frac{1}{C_m} \cdot I_{\text{LED}-2}$$

Based on the assumption: $I_{LED-1} \to \varepsilon; I_{\text{LED}-4} \to \varepsilon; I_{\text{LED}-2} = I_{\text{LED}-3} = I_{C_m} = I_{R_m}$

$$V_1 - V_A + (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s = I_{CQ_1} \cdot R_1 + V_t \cdot \ln\left[\frac{([\alpha_{rQ_1} - 1] \cdot I_{CQ_1}) + I_{se} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)}{(I_{CQ_1} \cdot [1 - \alpha_{fQ_1}]) + I_{sc} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)}\right]$$
$$+ V_t \cdot \ln\left[\frac{([\alpha_{rQ_2} - 1] \cdot I_{CQ_2} - I_{\text{LED}-2} \cdot k_2) + I_{se} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)}{(I_{CQ_2} \cdot [1 - \alpha_{fQ_2}] - I_{\text{LED}-2} \cdot k_2 \cdot \alpha_{fQ_2}) + I_{sc} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)}\right]$$

$$V_2 - V_A + (I_{\text{LED}-2} - I_{EQ_3} - I_{EQ_2}) \cdot R_s = I_{CQ_4} \cdot R_2 + V_t \cdot \ln\left[\frac{([\alpha_{rQ_4} - 1] \cdot I_{CQ_4}) + I_{se} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}{(I_{CQ_4} \cdot [1 - \alpha_{fQ_4}]) + I_{sc} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}\right]$$
$$+ V_t \cdot \ln\left[\frac{([\alpha_{rQ_3} - 1] \cdot I_{CQ_3} - I_{\text{LED}-2} \cdot k_3) + I_{se} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}{(I_{CQ_3} \cdot [1 - \alpha_{fQ_3}] - I_{\text{LED}-2} \cdot k_3 \cdot \alpha_{fQ_3}) + I_{sc} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}\right]$$

$$\frac{dI_{EQ_3}}{dt}\cdot R_s+\frac{dI_{EQ_2}}{dt}\cdot R_s=\left[\frac{2\cdot V_t}{I_{\text{LED}-2}+I_0}+R_m+R_s\right]\cdot\frac{dI_{\text{LED}-2}}{dt}+\frac{1}{C_m}\cdot I_{\text{LED}-2};$$
$$I_{EQ_1}=I_{CQ_2};I_{EQ_4}=I_{CQ_3}$$

$$I_{EQ_1}=I_{CQ_1}+I_{\text{LED}-1}\cdot k_1;I_{EQ_2}=I_{CQ_2}+I_{\text{LED}-2}\cdot k_2;I_{EQ_3}=I_{CQ_3}+I_{\text{LED}-3}\cdot k_3;I_{EQ_4}=I_{CQ_4}+I_{\text{LED}-4}\cdot k_4$$
$$I_{\text{LED}-1}\to\varepsilon;I_{\text{LED}-4}\to\varepsilon\Rightarrow I_{EQ_1}\approx I_{CQ_1};I_{EQ_4}\approx I_{CQ_4};I_{\text{LED}-2}=I_{\text{LED}-3}\Rightarrow I_{EQ_3}=I_{CQ_3}+I_{\text{LED}-2}\cdot k_3$$

We do the following transformation: $I_{CQ_1}\approx I_{EQ_1}=I_{CQ_2}=I_{EQ2}-I_{\text{LED}-2}\cdot k_2$

$$I_{CQ_2}=I_{EQ2}-I_{\text{LED}-2}\cdot k_2;I_{CQ_3}=I_{EQ_3}-I_{\text{LED}-2}\cdot k_3;I_{CQ_4}\approx I_{EQ_4}=I_{CQ_3}=I_{EQ_3}-I_{\text{LED}-2}\cdot k_3$$

We get the following two equations, which include variables $I_{EQ2},I_{EQ3},I_{\text{LED}-2}$:

$$\begin{aligned}V_1-V_A+(I_{\text{LED}-2}-I_{EQ_3}-I_{EQ_2})\cdot R_s&=[I_{EQ_2}-I_{\text{LED}-2}\cdot k_2]\cdot R_1\\&+V_t\cdot\ln\left[\frac{([\alpha_{rQ_1}-1]\cdot[I_{EQ_2}-I_{\text{LED}-2}\cdot k_2])+I_{se}\cdot(\alpha_{rQ_1}\cdot\alpha_{fQ_1}-1)}{([I_{EQ_2}-I_{\text{LED}-2}\cdot k_2]\cdot[1-\alpha_{fQ_1}])+I_{sc}\cdot(\alpha_{rQ_1}\cdot\alpha_{fQ_1}-1)}\right]\\&+V_t\cdot\ln\left[\frac{([\alpha_{rQ_2}-1]\cdot[I_{EQ_2}-I_{\text{LED}-2}\cdot k_2]-I_{\text{LED}-2}\cdot k_2)+I_{se}\cdot(\alpha_{rQ_2}\cdot\alpha_{fQ_2}-1)}{([I_{EQ_2}-I_{\text{LED}-2}\cdot k_2]\cdot[1-\alpha_{fQ_2}]-I_{\text{LED}-2}\cdot k_2\cdot\alpha_{fQ_2})+I_{sc}\cdot(\alpha_{rQ_2}\cdot\alpha_{fQ_2}-1)}\right]\end{aligned}$$

$$\begin{aligned}V_2-V_A+(I_{\text{LED}-2}-I_{EQ_3}-I_{EQ_2})\cdot R_s&=[I_{EQ_3}-I_{\text{LED}-2}\cdot k_3]\cdot R_2\\&+V_t\cdot\ln\left[\frac{([\alpha_{rQ_4}-1]\cdot[I_{EQ_3}-I_{\text{LED}-2}\cdot k_3])+I_{se}\cdot(\alpha_{rQ_4}\cdot\alpha_{fQ_4}-1)}{([I_{EQ_3}-I_{\text{LED}-2}\cdot k_3]\cdot[1-\alpha_{fQ_4}])+I_{sc}\cdot(\alpha_{rQ_4}\cdot\alpha_{fQ_4}-1)}\right]\\&+V_t\cdot\ln\left[\frac{([\alpha_{rQ_3}-1]\cdot[I_{EQ_3}-I_{\text{LED}-2}\cdot k_3]-I_{\text{LED}-2}\cdot k_3)+I_{se}\cdot(\alpha_{rQ_3}\cdot\alpha_{fQ_3}-1)}{([I_{EQ_3}-I_{\text{LED}-2}\cdot k_3]\cdot[1-\alpha_{fQ_3}]-I_{\text{LED}-2}\cdot k_3\cdot\alpha_{fQ_3})+I_{sc}\cdot(\alpha_{rQ_3}\cdot\alpha_{fQ_3}-1)}\right]\end{aligned}$$

We can represent the above equations as:

$$\begin{aligned}V_1-V_A+I_{\text{LED}-2}\cdot[R_s+k_2\cdot R_1]-I_{EQ_3}\cdot R_s-I_{EQ_2}\cdot[R_s+R_1]&=V_t\\&\cdot\ln\left[\frac{([\alpha_{rQ_1}-1]\cdot[I_{EQ_2}-I_{LED-2}\cdot k_2])+I_{se}\cdot\left(\alpha_{rQ_1}\cdot\alpha_{fQ_1}-1\right)}{\left([I_{EQ_2}-I_{LED-2}\cdot k_2]\cdot\left[1-\alpha_{fQ_1}\right]\right)+I_{sc}\cdot\left(\alpha_{rQ_1}\cdot\alpha_{fQ_1}-1\right)}\right]\\&+V_t\cdot\ln\left[\frac{([\alpha_{rQ_2}-1]\cdot[I_{EQ_2}-I_{\text{LED}-2}\cdot k_2]-I_{\text{LED}-2}\cdot k_2)+I_{se}\cdot\left(\alpha_{rQ_2}\cdot\alpha_{fQ_2}-1\right)}{\left([I_{EQ_2}-I_{\text{LED}-2}\cdot k_2]\cdot\left[1-\alpha_{fQ_2}\right]-I_{\text{LED}-2}\cdot k_2\cdot\alpha_{fQ_2}\right)+I_{sc}\cdot\left(\alpha_{rQ_2}\cdot\alpha_{fQ_2}-1\right)}\right]\end{aligned}$$

$$V_2 - V_A + I_{\text{LED}-2} \cdot [R_s + k_3 \cdot R_2] - I_{EQ_3} \cdot [R_s + R_2] - I_{EQ_2} \cdot R_s = V_t \cdot \ln\left[\frac{([\alpha_{rQ_4} - 1] \cdot [I_{EQ_3} - I_{\text{LED}-2} \cdot k_3]) + I_{se} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}{([I_{EQ_3} - I_{\text{LED}-2} \cdot k_3] \cdot [1 - \alpha_{fQ_4}]) + I_{sc} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)}\right] + V_t \cdot \ln\left[\frac{([\alpha_{rQ_3} - 1] \cdot [I_{EQ_3} - I_{\text{LED}-2} \cdot k_3] - I_{\text{LED}-2} \cdot k_3) + I_{se} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}{([I_{EQ_3} - I_{\text{LED}-2} \cdot k_3] \cdot [1 - \alpha_{fQ_3}] - I_{\text{LED}-2} \cdot k_3 \cdot \alpha_{fQ_3}) + I_{sc} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)}\right]$$

We define four new functions for simplicity of our analysis.

$$\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) = ([\alpha_{rQ_1} - 1] \cdot [I_{EQ_2} - I_{\text{LED}-2} \cdot k_2]) + I_{se} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)$$
$$\xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) = ([I_{EQ_2} - I_{\text{LED}-2} \cdot k_2] \cdot [1 - \alpha_{fQ_1}]) + I_{sc} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)$$
$$\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) = ([\alpha_{rQ_2} - 1] \cdot [I_{EQ_2} - I_{\text{LED}-2} \cdot k_2] - I_{\text{LED}-2} \cdot k_2) + I_{se} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)$$
$$\xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) = ([I_{EQ_2} - I_{\text{LED}-2} \cdot k_2] \cdot [1 - \alpha_{fQ_2}] - I_{\text{LED}-2} \cdot k_2 \cdot \alpha_{fQ_2}) + I_{sc} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)$$

$$\xi_{21}(I_{EQ_3}, I_{\text{LED}-2}) = ([\alpha_{rQ_4} - 1] \cdot [I_{EQ_3} - I_{\text{LED}-2} \cdot k_3]) + I_{se} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)$$
$$\xi_{22}(I_{EQ_3}, I_{\text{LED}-2}) = ([I_{EQ_3} - I_{\text{LED}-2} \cdot k_3] \cdot [1 - \alpha_{fQ_4}]) + I_{sc} \cdot (\alpha_{rQ_4} \cdot \alpha_{fQ_4} - 1)$$
$$\xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) = ([\alpha_{rQ_3} - 1] \cdot [I_{EQ_3} - I_{\text{LED}-2} \cdot k_3] - I_{\text{LED}-2} \cdot k_3) + I_{se} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)$$
$$\xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) = ([I_{EQ_3} - I_{\text{LED}-2} \cdot k_3] \cdot [1 - \alpha_{fQ_3}] - I_{\text{LED}-2} \cdot k_3 \cdot \alpha_{fQ_3}) + I_{sc} \cdot (\alpha_{rQ_3} \cdot \alpha_{fQ_3} - 1)$$

$$\frac{d\xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{dt} = [\alpha_{rQ_1} - 1] \cdot \left[\frac{dI_{EQ_2}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_2\right]$$
$$\frac{d\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{dt} = \left[\frac{dI_{EQ_2}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_2\right] \cdot [1 - \alpha_{fQ_1}]$$
$$\frac{d\xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{dt} = [\alpha_{rQ_2} - 1] \cdot \left[\frac{dI_{EQ_2}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_2\right] - \frac{dI_{\text{LED}-2}}{dt} \cdot k_2$$
$$\frac{d\xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{dt} = \left[\frac{dI_{EQ_2}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_2\right] \cdot [1 - \alpha_{fQ_2}] - \frac{dI_{\text{LED}-2}}{dt} \cdot k_2 \cdot \alpha_{fQ_2}$$

$$\frac{d\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})}{dt} = [\alpha_{rQ_4} - 1] \cdot \left[\frac{dI_{EQ_3}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_3\right]$$
$$\frac{d\xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}{dt} = \left[\frac{dI_{EQ_3}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_3\right] \cdot [1 - \alpha_{fQ_4}]$$
$$\frac{d\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{dt} = [\alpha_{rQ_3} - 1] \cdot \left[\frac{dI_{EQ_3}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_3\right] - \frac{dI_{\text{LED}-2}}{dt} \cdot k_3$$
$$\frac{d\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{dt} = \left[\frac{dI_{EQ_3}}{dt} - \frac{dI_{\text{LED}-2}}{dt} \cdot k_3\right] \cdot [1 - \alpha_{fQ_3}] - \frac{dI_{\text{LED}-2}}{dt} \cdot k_3 \cdot \alpha_{fQ_3}$$

We can rewrite our equations by using new functions.

$$V_1 - V_A + I_{\text{LED}-2} \cdot [R_s + k_2 \cdot R_1] - I_{EQ_3} \cdot R_s - I_{EQ_2} \cdot [R_s + R_1] = V_t \cdot \ln\left[\frac{\xi_{11}(I_{EQ2}, I_{\text{LED}-2})}{\xi_{12}(I_{EQ2}, I_{\text{LED}-2})}\right] + V_t \cdot \ln\left[\frac{\xi_{13}(I_{EQ2}, I_{\text{LED}-2})}{\xi_{14}(I_{EQ2}, I_{\text{LED}-2})}\right]$$

$$V_2 - V_A + I_{\text{LED}-2} \cdot [R_s + k_3 \cdot R_2] - I_{EQ_3} \cdot [R_s + R_2] - I_{EQ_2} \cdot R_s = V_t \cdot \ln\left[\frac{\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}\right] + V_t \cdot \ln\left[\frac{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}\right]$$

Hint: If $X = X(t)$ and $Y = Y(t)$ are differentiable at t and $Z = \Psi(X(t), Y(t))$ is differentiable at $(X(t), Y(t))$ then $Z = \Psi(X(t), Y(t))$ is differentiable at t and

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial X} \cdot \frac{dX}{dt} + \frac{\partial Z}{\partial Y} \cdot \frac{dY}{dt}.$$

First we derive the two sides of the first equation: $I_{\text{LED}-2} = I_{\text{LED}-2}(t)$

$$I_{EQ_2} = I_{EQ_2}(t); I_{EQ_3} = I_{EQ_3}(t); \frac{d}{dt}(V_1 - V_A) = 0$$

$$\frac{d}{dt}\{V_1 - V_A + I_{\text{LED}-2} \cdot [R_s + k_2 \cdot R_1] - I_{EQ_3} \cdot R_s - I_{EQ_2} \cdot [R_s + R_1]\} = \frac{d}{dt}\left\{V_t \cdot \ln\left[\frac{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}\right] + V_t \cdot \ln\left[\frac{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]\right\}$$

$$\frac{dI_{\text{LED}-2}}{dt} \cdot [R_s + k_2 \cdot R_1] - \frac{dI_{EQ_3}}{dt} \cdot R_s - \frac{dI_{EQ_2}}{dt} \cdot [R_s + R_1] = V_t \cdot \frac{d}{dt}\left\{\ln\left[\frac{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}\right] + \ln\left[\frac{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]\right\}$$

$$Z_1 = \ln\left[\frac{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}\right]; \frac{dZ_1}{dt} = \frac{\partial Z_1}{\partial I_{EQ_2}} \cdot \frac{dI_{EQ_2}}{dt} + \frac{\partial Z_1}{\partial I_{\text{LED}-2}} \cdot \frac{dI_{\text{LED}-2}}{dt}$$

$$\frac{\partial Z_1}{\partial I_{EQ_2}} = \left[\frac{\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}\right] \cdot \left[\frac{\frac{\partial \xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}}}{[\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})]^2}\right]$$

$$\frac{\partial Z_1}{\partial I_{EQ_2}} = \frac{\frac{\partial \xi_{11}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{EQ_2}} \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \frac{\partial \xi_{12}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{EQ_2}}}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}$$

$$\frac{\partial Z_1}{\partial I_{EQ_2}} = \frac{\frac{\partial \xi_{11}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{EQ_2}} \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \frac{\partial \xi_{12}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{EQ_2}}}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}$$

$$\frac{\partial \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = [\alpha_{rQ_1} - 1]; \frac{\partial \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = \left[1 - \alpha_{fQ_1}\right]$$

$$\frac{\partial Z_1}{\partial I_{EQ_2}} = \frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot [1 - \alpha_{fQ_1}]}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}$$

$$\frac{\partial Z_1}{\partial I_{\mathrm{LED}-2}} = \left[\frac{\xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right] \cdot \left[\frac{\frac{\partial \xi_{11}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{\mathrm{LED}-2}} \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \frac{\partial \xi_{12}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{\mathrm{LED}-2}}}{[\xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})]^2}\right]$$

$$\frac{\partial Z_1}{\partial I_{\mathrm{LED}-2}} = \left[\frac{\frac{\partial \xi_{11}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{\mathrm{LED}-2}} \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \frac{\partial \xi_{12}\left(I_{EQ_2}, I_{\mathrm{LED}-2}\right)}{\partial I_{\mathrm{LED}-2}}}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right]$$

$$\frac{\partial \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{\mathrm{LED}-2}} = -k_2 \cdot [\alpha_{rQ_1} - 1]; \frac{\partial \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{\mathrm{LED}-2}} = -k_2 \cdot \left[1 - \alpha_{fQ_1}\right]$$

$$\frac{\partial Z_1}{\partial I_{\mathrm{LED}-2}} = -k_2 \cdot \left\{\frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \left[1 - \alpha_{fQ_1}\right]}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right\}$$

$$\frac{\mathrm{d}Z_1}{\mathrm{d}t} = \left\{\frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \left[1 - \alpha_{fQ_1}\right]}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right\} \cdot \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} - k_2 \cdot \left\{\frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \left[1 - \alpha_{fQ_1}\right]}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right\} \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}Z_1}{\mathrm{d}t} = \left\{\frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \left[1 - \alpha_{fQ_1}\right]}{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right\} \cdot \left(\frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} - k_2 \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}\right)$$

$$Z_2 = \ln\left[\frac{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right]; \frac{\mathrm{d}Z_2}{\mathrm{d}t} = \frac{\partial Z_2}{\partial I_{EQ_2}} \cdot \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} + \frac{\partial Z_2}{\partial I_{\mathrm{LED}-2}} \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}$$

$$\frac{\partial Z_2}{\partial I_{EQ_2}} = \left[\frac{\xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}\right] \cdot \left[\frac{\frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}}}{\xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]$$

$$\frac{\partial Z_2}{\partial I_{EQ_2}} = \left[\frac{\frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}}}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]$$

$$\frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = [\alpha_{rQ_2} - 1]; \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = [1 - \alpha_{fQ_2}]$$

$$\frac{\partial Z_2}{\partial I_{EQ_2}} = \left[\frac{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_2}]}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]$$

$$\frac{\partial Z_2}{\partial I_{\text{LED}-2}} = \left[\frac{\frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]$$

$$\frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = -k_2 \cdot ([\alpha_{rQ_2} - 1] + 1) = -k_2 \cdot \alpha_{rQ_2}; \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = [-k_2] \cdot [1 - \alpha_{fQ_2}] - k_2 \cdot \alpha_{fQ_2} = -k_2$$

$$\frac{\partial Z_2}{\partial I_{\text{LED}-2}} = -k_2 \cdot \left[\frac{\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]$$

$$\frac{dZ_2}{dt} = \left[\frac{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_2}]}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right] \cdot \frac{dI_{EQ_2}}{dt} - k_2 \cdot \left[\frac{\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right] \cdot \frac{dI_{\text{LED}-2}}{dt}$$

We can summarize our results:

$$\frac{dZ_1}{dt} = \left\{\frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_1}]}{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}\right\} \cdot \left(\frac{dI_{EQ_2}}{dt} - k_2 \cdot \frac{dI_{\text{LED}-2}}{dt}\right)$$

$$\frac{dZ_2}{dt} = \left[\frac{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_2}]}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right] \cdot \frac{dI_{EQ_2}}{dt}$$
$$- k_2 \cdot \left[\frac{\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right] \cdot \frac{dI_{\text{LED}-2}}{dt}$$

We define for simplicity three functions:

$$\Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_1}]}{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}$$

$$\Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_2}]}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}$$

$$\Upsilon_3(I_{EQ_2}, I_{\text{LED}-2}) = k_2 \cdot \left[\frac{\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right]$$

$$\frac{dZ_1}{dt} = \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) \cdot \left(\frac{dI_{EQ_2}}{dt} - k_2 \cdot \frac{dI_{\text{LED}-2}}{dt}\right);$$
$$\frac{dZ_2}{dt} = \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{dI_{EQ_2}}{dt} - \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{dI_{\text{LED}-2}}{dt}$$

We already found that

$$Z_1 = \ln\left[\frac{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}\right] \Rightarrow \frac{dZ_1}{dt} = \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) \cdot \left(\frac{dI_{EQ_2}}{dt} - k_2 \cdot \frac{dI_{\text{LED}-2}}{dt}\right)$$

$$Z_2 = \ln\left[\frac{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}\right] \Rightarrow \frac{dZ_2}{dt}$$
$$= \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{dI_{EQ_2}}{dt} - \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{dI_{\text{LED}-2}}{dt}$$

$$\frac{dI_{\text{LED}-2}}{dt} \cdot [R_s + k_2 \cdot R_1] - \frac{dI_{EQ_3}}{dt} \cdot R_s - \frac{dI_{EQ_2}}{dt} \cdot [R_s + R_1]$$
$$= V_t \cdot \left\{\Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) \cdot \left(\frac{dI_{EQ_2}}{dt} - k_2 \cdot \frac{dI_{\text{LED}-2}}{dt}\right)\right.$$
$$\left. + \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{dI_{EQ_2}}{dt} - \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{dI_{\text{LED}-2}}{dt}\right\}$$

Second we derive the two sides of the second equation: $I_{\text{LED}-2} = I_{\text{LED}-2}(t)$

$$I_{EQ_2} = I_{EQ_2}(t); I_{EQ_3} = I_{EQ_3}(t); \frac{\mathrm{d}}{\mathrm{d}t}(V_2 - V_A) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{V_2 - V_A + I_{\mathrm{LED-2}} \cdot [R_s + k_3 \cdot R_2] - I_{EQ_3} \cdot [R_s + R_2] - I_{EQ_2} \cdot R_s\} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{V_t \cdot \ln\left[\frac{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right] + V_t \cdot \ln\left[\frac{\xi_{23}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\xi_{24}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right]\right\}$$

$$\frac{\mathrm{d}I_{\mathrm{LED-2}}}{\mathrm{d}t} \cdot [R_s + k_3 \cdot R_2] - \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} \cdot [R_s + R_2] - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot R_s = V_t \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left[\frac{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right]\right\} + V_t \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left\{\ln\left[\frac{\xi_{23}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\xi_{24}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right]\right\}$$

$$Z_3 = \ln\left[\frac{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right]; \frac{\mathrm{d}Z_3}{\mathrm{d}t} = \frac{\partial Z_3}{\partial I_{EQ_3}} \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} + \frac{\partial Z_3}{\partial I_{\mathrm{LED-2}}} \cdot \frac{\mathrm{d}I_{\mathrm{LED-2}}}{\mathrm{d}t}$$

$$\frac{\partial Z_3}{\partial I_{EQ_3}} = \frac{\frac{\partial \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{EQ_3}} \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}}) - \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \frac{\partial \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{EQ_3}}}{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}$$

$$\frac{\partial \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{EQ_3}} = [\alpha_{rQ_4} - 1]; \frac{\partial \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{EQ_3}} = \left[1 - \alpha_{fQ_4}\right]$$

$$\frac{\partial Z_3}{\partial I_{EQ_3}} = \frac{[\alpha_{rQ_4} - 1] \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}}) - \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \left[1 - \alpha_{fQ_4}\right]}{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}$$

$$\frac{\partial Z_3}{\partial I_{\mathrm{LED-2}}} = \frac{\frac{\partial \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{\mathrm{LED-2}}} \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}}) - \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \frac{\partial \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{\mathrm{LED-2}}}}{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}$$

$$\frac{\partial \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{\mathrm{LED-2}}} = -k_3 \cdot [\alpha_{rQ_4} - 1]; \frac{\partial \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}{\partial I_{\mathrm{LED-2}}} = -k_3 \cdot \left[1 - \alpha_{fQ_4}\right]$$

$$\frac{\partial Z_3}{\partial I_{\mathrm{LED-2}}} = -k_3 \cdot \left(\frac{[\alpha_{rQ_4} - 1] \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}}) - \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \left[1 - \alpha_{fQ_4}\right]}{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right)$$

$$\frac{\mathrm{d}Z_3}{\mathrm{d}t} = \left(\frac{[\alpha_{rQ_4} - 1] \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}}) - \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \left[1 - \alpha_{fQ_4}\right]}{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - k_3 \cdot \left(\frac{[\alpha_{rQ_4} - 1] \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}}) - \xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \left[1 - \alpha_{fQ_4}\right]}{\xi_{21}(I_{EQ_3}, I_{\mathrm{LED-2}}) \cdot \xi_{22}(I_{EQ_3}, I_{\mathrm{LED-2}})}\right) \cdot \frac{\mathrm{d}I_{\mathrm{LED-2}}}{\mathrm{d}t}$$

$$\frac{dZ_3}{dt} = \left(\frac{[\alpha_{rQ_4} - 1]\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\left[1 - \alpha_{fQ_4}\right]}{\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}\right) \cdot\left(\frac{dI_{EQ_3}}{dt} - k_3\cdot\frac{dI_{\text{LED}-2}}{dt}\right)$$

$$Z_4 = \ln\left[\frac{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}\right]; \frac{dZ_4}{dt} = \frac{\partial Z_4}{\partial I_{EQ_3}}\cdot\frac{dI_{EQ_3}}{dt} + \frac{\partial Z_4}{\partial I_{\text{LED}-2}}\cdot\frac{dI_{\text{LED}-2}}{dt}$$

$$\frac{\partial Z_4}{\partial I_{EQ_3}} = \frac{\frac{\partial\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}}\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\frac{\partial\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}}}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = [\alpha_{rQ_3} - 1]; \frac{\partial\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = \left[1 - \alpha_{fQ_3}\right]$$

$$\frac{\partial Z_4}{\partial I_{EQ_3}} = \frac{[\alpha_{rQ_3} - 1]\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\left[1 - \alpha_{fQ_3}\right]}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial Z_4}{\partial I_{\text{LED}-2}} = \frac{\frac{\partial\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\frac{\partial\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = [\alpha_{rQ_3} - 1]\cdot[-k_3] - k_3 = -\alpha_{rQ_3}\cdot k_3; \frac{\partial\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = -k_3$$

$$\frac{\partial Z_4}{\partial I_{\text{LED}-2}} = -k_3\cdot\left[\frac{\alpha_{rQ_3}\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}\right]$$

$$\frac{dZ_4}{dt} = \frac{[\alpha_{rQ_3} - 1]\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\left[1 - \alpha_{fQ_3}\right]}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}\cdot\frac{dI_{EQ_3}}{dt} - k_3\cdot\left[\frac{\alpha_{rQ_3}\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}\right]\cdot\frac{dI_{\text{LED}-2}}{dt}$$

We can summarize our results:

$$\frac{dZ_3}{dt} = \left(\frac{[\alpha_{rQ_4} - 1]\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot[1 - \alpha_{fQ_4}]}{\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}\right) \cdot\left(\frac{dI_{EQ_3}}{dt} - k_3\cdot\frac{dI_{\text{LED}-2}}{dt}\right)$$

$$\frac{dZ_4}{dt} = \frac{[\alpha_{rQ_3} - 1] \cdot \xi_{24}(I_{EQ_3}, I_{LED-2}) - \xi_{23}(I_{EQ_3}, I_{LED-2}) \cdot [1 - \alpha_{fQ_3}]}{\xi_{23}(I_{EQ_3}, I_{LED-2}) \cdot \xi_{24}(I_{EQ_3}, I_{LED-2})} \cdot \frac{dI_{EQ_3}}{dt} - k_3 \cdot \left[\frac{\alpha_{rQ_3} \cdot \xi_{24}(I_{EQ_3}, I_{LED-2}) - \xi_{23}(I_{EQ_3}, I_{LED-2})}{\xi_{23}(I_{EQ_3}, I_{LED-2}) \cdot \xi_{24}(I_{EQ_3}, I_{LED-2})}\right] \cdot \frac{dI_{LED-2}}{dt}$$

We define for simplicity three functions:

$$\Upsilon_4(I_{EQ_3}, I_{LED-2}) = \frac{[\alpha_{rQ_4} - 1] \cdot \xi_{22}(I_{EQ_3}, I_{LED-2}) - \xi_{21}(I_{EQ_3}, I_{LED-2}) \cdot [1 - \alpha_{fQ_4}]}{\xi_{21}(I_{EQ_3}, I_{LED-2}) \cdot \xi_{22}(I_{EQ_3}, I_{LED-2})}$$

$$\Upsilon_5(I_{EQ_3}, I_{LED-2}) = \frac{[\alpha_{rQ_3} - 1] \cdot \xi_{24}(I_{EQ_3}, I_{LED-2}) - \xi_{23}(I_{EQ_3}, I_{LED-2}) \cdot [1 - \alpha_{fQ_3}]}{\xi_{23}(I_{EQ_3}, I_{LED-2}) \cdot \xi_{24}(I_{EQ_3}, I_{LED-2})}$$

$$\Upsilon_6(I_{EQ_3}, I_{LED-2}) = k_3 \cdot \left[\frac{\alpha_{rQ_3} \cdot \xi_{24}(I_{EQ_3}, I_{LED-2}) - \xi_{23}(I_{EQ_3}, I_{LED-2})}{\xi_{23}(I_{EQ_3}, I_{LED-2}) \cdot \xi_{24}(I_{EQ_3}, I_{LED-2})}\right]$$

$$\frac{dZ_3}{dt} = \Upsilon_4(I_{EQ_3}, I_{LED-2}) \cdot \left(\frac{dI_{EQ_3}}{dt} - k_3 \cdot \frac{dI_{LED-2}}{dt}\right);$$

$$\frac{dZ_4}{dt} = \Upsilon_5(I_{EQ_3}, I_{LED-2}) \cdot \frac{dI_{EQ_3}}{dt} - \Upsilon_6(I_{EQ_3}, I_{LED-2}) \cdot \frac{dI_{LED-2}}{dt}$$

We already found that

$$Z_3 = \ln\left[\frac{\xi_{21}(I_{EQ_3}, I_{LED-2})}{\xi_{22}(I_{EQ_3}, I_{LED-2})}\right] \Rightarrow \frac{dZ_3}{dt} = \Upsilon_4(I_{EQ_3}, I_{LED-2}) \cdot \left(\frac{dI_{EQ_3}}{dt} - k_3 \cdot \frac{dI_{LED-2}}{dt}\right)$$

$$Z_4 = \ln\left[\frac{\xi_{23}(I_{EQ_3}, I_{LED-2})}{\xi_{24}(I_{EQ_3}, I_{LED-2})}\right] \Rightarrow \frac{dZ_4}{dt} = \Upsilon_5(I_{EQ_3}, I_{LED-2}) \cdot \frac{dI_{EQ_3}}{dt} - \Upsilon_6(I_{EQ_3}, I_{LED-2}) \cdot \frac{dI_{LED-2}}{dt}$$

$$\frac{dI_{LED-2}}{dt} \cdot [R_s + k_3 \cdot R_2] - \frac{dI_{EQ_3}}{dt} \cdot [R_s + R_2] - \frac{dI_{EQ_2}}{dt} \cdot R_s = V_t \cdot \Upsilon_4(I_{EQ_3}, I_{LED-2}) \cdot \left(\frac{dI_{EQ_3}}{dt} - k_3 \cdot \frac{dI_{LED-2}}{dt}\right) + V_t \cdot \left(\Upsilon_5(I_{EQ_3}, I_{LED-2}) \cdot \frac{dI_{EQ_3}}{dt} - \Upsilon_6(I_{EQ_3}, I_{LED-2}) \cdot \frac{dI_{LED-2}}{dt}\right)$$

We can summarize our system three differential equations:

$$(*)\ \frac{dI_{EQ_3}}{dt} \cdot R_s + \frac{dI_{EQ_2}}{dt} \cdot R_s = \left[\frac{2 \cdot V_t}{I_{LED-2} + I_0} + R_m + R_s\right] \cdot \frac{dI_{LED-2}}{dt} + \frac{1}{C_m} \cdot I_{LED-2}$$

$$(**)\ \begin{aligned}&\frac{dI_{\text{LED}-2}}{dt}\cdot[R_s+k_2\cdot R_1]-\frac{dI_{EQ_3}}{dt}\cdot R_s-\frac{dI_{EQ_2}}{dt}\cdot[R_s+R_1]=V_t\cdot\Big\{\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\\&\cdot\left(\frac{dI_{EQ_2}}{dt}-k_2\cdot\frac{dI_{\text{LED}-2}}{dt}\right)+\Upsilon_2(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{EQ_2}}{dt}-\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{\text{LED}-2}}{dt}\Big\}\end{aligned}$$

$$(***)\ \begin{aligned}&\frac{dI_{\text{LED}-2}}{dt}\cdot[R_s+k_3\cdot R_2]-\frac{dI_{EQ_3}}{dt}\cdot[R_s+R_2]-\frac{dI_{EQ_2}}{dt}\cdot R_s=V_t\cdot\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})\\&\cdot\left(\frac{dI_{EQ_3}}{dt}-k_3\cdot\frac{dI_{\text{LED}-2}}{dt}\right)+V_t\cdot\left(\Upsilon_5(I_{EQ_3},I_{LED-2})\cdot\frac{dI_{EQ_3}}{dt}-\Upsilon_6(I_{EQ_3},I_{\text{LED}-2})\cdot\frac{dI_{LED-2}}{dt}\right)\end{aligned}$$

We need to get thee functions for our stability analysis:

$$\frac{dI_{\text{LED}-2}}{dt}=g_1(I_{\text{LED}-2},I_{EQ_2},I_{EQ_3});\frac{dI_{EQ_2}}{dt}=g_2(I_{\text{LED}-2},I_{EQ_2},I_{EQ_3});$$
$$\frac{dI_{EQ_3}}{dt}=g_3(I_{\text{LED}-2},I_{EQ_2},I_{EQ_3})$$

First we work on equation (**).

$$\begin{aligned}&\frac{dI_{LED-2}}{dt}\cdot[R_s+k_2\cdot R_1]-\frac{dI_{EQ_3}}{dt}\cdot R_s-\frac{dI_{EQ_2}}{dt}\cdot[R_s+R_1]=V_t\cdot\Big\{\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\\&\cdot\left(\frac{dI_{EQ_2}}{dt}-k_2\cdot\frac{dI_{\text{LED}-2}}{dt}\right)+\Upsilon_2(I_{EQ_2},I_{LED-2})\cdot\frac{dI_{EQ_2}}{dt}-\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{\text{LED}-2}}{dt}\end{aligned}$$

$$\begin{aligned}&\frac{dI_{\text{LED}-2}}{dt}\cdot[R_s+k_2\cdot R_1]-\frac{dI_{EQ_3}}{dt}\cdot R_s-\frac{dI_{EQ_2}}{dt}\cdot[R_s+R_1]=V_t\cdot\Upsilon_1(I_{EQ_2},I_{LED-2})\cdot\frac{dI_{EQ_2}}{dt}\\&\quad-V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\cdot k_2\cdot\frac{dI_{\text{LED}-2}}{dt}+V_t\cdot\Upsilon_2(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{EQ_2}}{dt}-V_t\cdot\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{\text{LED}-2}}{dt}\\&\frac{dI_{\text{LED}-2}}{dt}\cdot[R_s+k_2\cdot R_1]+V_t\cdot\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{\text{LED}-2}}{dt}+V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\cdot k_2\cdot\frac{dI_{\text{LED}-2}}{dt}\\&\quad=V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{EQ_2}}{dt}+V_t\cdot\Upsilon_2(I_{EQ_2},I_{\text{LED}-2})\cdot\frac{dI_{EQ_2}}{dt}+\frac{dI_{EQ_2}}{dt}\cdot[R_s+R_1]+\frac{dI_{EQ_3}}{dt}\cdot R_s\end{aligned}$$

$$\begin{aligned}&\frac{dI_{\text{LED}-2}}{dt}\cdot\{R_s+k_2\cdot R_1+V_t\cdot\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})+V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\cdot k_2\}\\&\quad=\frac{dI_{EQ_2}}{dt}\cdot\{V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})+V_t\cdot\Upsilon_2(I_{EQ_2},I_{\text{LED}-2})+R_s+R_1\}+\frac{dI_{EQ_3}}{dt}\cdot R_s\end{aligned}$$

$$\begin{aligned}&\frac{dI_{EQ_3}}{dt}\cdot R_s=\frac{dI_{\text{LED}-2}}{dt}\cdot\{R_s+k_2\cdot R_1+V_t\cdot\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})+V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\cdot k_2\}\\&\quad-\frac{dI_{EQ_2}}{dt}\cdot\{V_t\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})+V_t\cdot\Upsilon_2(I_{EQ_2},I_{\text{LED}-2})+R_s+R_1\}\end{aligned}$$

$$\begin{aligned}&\frac{dI_{EQ_3}}{dt}=\frac{dI_{\text{LED}-2}}{dt}\cdot\{1+k_2\cdot\frac{R_1}{R_s}+\frac{V_t}{R_s}\cdot\Upsilon_3(I_{EQ_2},I_{\text{LED}-2})+\frac{V_t}{R_s}\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})\cdot k_2\}\\&\quad-\frac{dI_{EQ_2}}{dt}\cdot\{\frac{V_t}{R_s}\cdot\Upsilon_1(I_{EQ_2},I_{\text{LED}-2})+\frac{V_t}{R_s}\cdot\Upsilon_2(I_{EQ_2},I_{\text{LED}-2})+1+\frac{R_1}{R_s}\}\end{aligned}$$

We define for simplicity the following functions:

$$\frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} = \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot \Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot \Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})$$

$$\Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) = 1 + k_2 \cdot \frac{R_1}{R_s} + \frac{V_t}{R_s} \cdot \Upsilon_3(I_{EQ_2}, I_{\mathrm{LED}-2}) + \frac{V_t}{R_s} \cdot \Upsilon_1(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot k_2$$

$$\Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) = \frac{V_t}{R_s} \cdot \Upsilon_1(I_{EQ_2}, I_{\mathrm{LED}-2}) + \frac{V_t}{R_s} \cdot \Upsilon_2(I_{EQ_2}, I_{\mathrm{LED}-2}) + 1 + \frac{R_1}{R_s}$$

Second we work on equation (***).

$$\frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot [R_s + k_3 \cdot R_2] - \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} \cdot [R_s + R_2] - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot R_s = V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \left(\frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - k_3 \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}\right) + V_t \cdot \left(\Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}\right)$$

$$\frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot [R_s + k_3 \cdot R_2] - \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} \cdot [R_s + R_2] - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot R_s = V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \left(\frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - k_3 \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}\right) + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot [R_s + k_3 \cdot R_2] - \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} \cdot [R_s + R_2] - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot R_s = V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot k_3 \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} - V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot [R_s + k_3 \cdot R_2] + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot k_3 \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} + V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} = \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot R_s + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} + \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} \cdot [R_s + R_2]$$

$$\frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot \{R_s + k_3 \cdot R_2 + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot k_3 + V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2})\} = \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot R_s + \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} \cdot \{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) + R_s + R_2\}$$

$$\frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} = \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} \cdot \left\{\frac{R_s + k_3 \cdot R_2 + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot k_3 + V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2})}{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) + R_s + R_2}\right\} - \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} \cdot \frac{R_s}{\{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) + R_s + R_2\}}$$

We define for simplicity the following functions:

$$\frac{dI_{EQ_3}}{dt} = \frac{dI_{LED-2}}{dt} \cdot \Omega_{21}(I_{EQ_3}, I_{LED-2}) - \frac{dI_{EQ_2}}{dt} \cdot \Omega_{22}(I_{EQ_3}, I_{LED-2})$$

$$\Omega_{21}(I_{EQ_3}, I_{LED-2}) = \frac{R_s + k_3 \cdot R_2 + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{LED-2}) \cdot k_3 + V_t \cdot \Upsilon_6(I_{EQ_3}, I_{LED-2})}{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{LED-2}) + \Upsilon_5(I_{EQ_3}, I_{LED-2}) + R_s + R_2}$$

$$\Omega_{22}(I_{EQ_3}, I_{LED-2}) = \frac{R_s}{\{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{LED-2}) + \Upsilon_5(I_{EQ_3}, I_{LED-2}) + R_s + R_2\}}$$

Outcome of the above discussion:

$$\frac{dI_{LED-2}}{dt} \cdot \Omega_{11}(I_{EQ_2}, I_{LED-2}) - \frac{dI_{EQ_2}}{dt} \cdot \Omega_{12}(I_{EQ_2}, I_{LED-2}) = \frac{dI_{LED-2}}{dt} \cdot \Omega_{21}(I_{EQ_3}, I_{LED-2}) - \frac{dI_{EQ_2}}{dt} \cdot \Omega_{22}(I_{EQ_3}, I_{LED-2})$$

$$\frac{dI_{EQ_2}}{dt} \cdot [\Omega_{12}(I_{EQ_2}, I_{LED-2}) - \Omega_{22}(I_{EQ_3}, I_{LED-2})] = \frac{dI_{LED-2}}{dt} \cdot [\Omega_{11}(I_{EQ_2}, I_{LED-2}) - \Omega_{21}(I_{EQ_3}, I_{LED-2})]$$

$$\frac{dI_{EQ_2}}{dt} = \frac{dI_{LED-2}}{dt} \cdot \left[\frac{\Omega_{11}(I_{EQ_2}, I_{LED-2}) - \Omega_{21}(I_{EQ_3}, I_{LED-2})}{\Omega_{12}(I_{EQ_2}, I_{LED-2}) - \Omega_{22}(I_{EQ_3}, I_{LED-2})}\right];$$

$$\frac{dI_{EQ_2}}{dt} = \frac{dI_{LED-2}}{dt} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{LED-2})$$

$$\Xi_1(I_{EQ_2}, I_{EQ_3}, I_{LED-2}) = \frac{\Omega_{11}(I_{EQ_2}, I_{LED-2}) - \Omega_{21}(I_{EQ_3}, I_{LED-2})}{\Omega_{12}(I_{EQ_2}, I_{LED-2}) - \Omega_{22}(I_{EQ_3}, I_{LED-2})}$$

$$\frac{dI_{EQ_3}}{dt} = \frac{dI_{LED-2}}{dt} \cdot \Omega_{11}(I_{EQ_2}, I_{LED-2}) - \frac{dI_{LED-2}}{dt} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{LED-2}) \cdot \Omega_{12}(I_{EQ_2}, I_{LED-2})$$

$$\frac{dI_{EQ_3}}{dt} = \frac{dI_{LED-2}}{dt} \cdot \left\{\Omega_{11}(I_{EQ_2}, I_{LED-2}) - \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{LED-2}) \cdot \Omega_{12}(I_{EQ_2}, I_{LED-2})\right\}$$

$$\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{LED-2}) = \Omega_{11}(I_{EQ_2}, I_{LED-2}) - \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{LED-2}) \cdot \Omega_{12}(I_{EQ_2}, I_{LED-2})$$

Back to equation (*) $\frac{dI_{EQ_3}}{dt} = \frac{dI_{LED-2}}{dt} \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{LED-2})$

$$\frac{dI_{EQ_3}}{dt} \cdot R_s + \frac{dI_{EQ_2}}{dt} \cdot R_s = \left[\frac{2 \cdot V_t}{I_{LED-2} + I_0} + R_m + R_s\right] \cdot \frac{dI_{LED-2}}{dt} + \frac{1}{C_m} \cdot I_{LED-2}$$

$$\frac{dI_{LED-2}}{dt}\cdot\left\{\Omega_{11}(I_{EQ_2},I_{LED-2})-\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot\Omega_{12}(I_{EQ_2},I_{LED-2})\right\}\cdot R_s + \frac{dI_{LED-2}}{dt}\cdot\left[\frac{\Omega_{11}(I_{EQ_2},I_{LED-2})-\Omega_{21}(I_{EQ_3},I_{LED-2})}{\Omega_{12}(I_{EQ_2},I_{LED-2})-\Omega_{22}(I_{EQ_3},I_{LED-2})}\right]\cdot R_s = \left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\cdot\frac{dI_{LED-2}}{dt}+\frac{1}{C_m}\cdot I_{LED-2}$$

$$\frac{dI_{LED-2}}{dt}\cdot\left\{\Omega_{11}(I_{EQ_2},I_{LED-2})-\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot\Omega_{12}(I_{EQ_2},I_{LED-2})\right\}\cdot R_s+\frac{dI_{LED-2}}{dt}\cdot\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s=\left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\cdot\frac{dI_{LED-2}}{dt}+\frac{1}{C_m}\cdot I_{LED-2}$$

$$\frac{dI_{LED-2}}{dt}\cdot\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s+\frac{dI_{LED-2}}{dt}\cdot\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s=\left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\cdot\frac{dI_{LED-2}}{dt}+\frac{1}{C_m}\cdot I_{LED-2}$$

$$\frac{dI_{LED-2}}{dt}\cdot\left\{\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s+\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s-\left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\right\}=\frac{1}{C_m}\cdot I_{LED-2}$$

$$\frac{dI_{LED-2}}{dt}=\frac{1}{C_m\cdot\left\{\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s+\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s-\left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\right\}}\cdot I_{LED-2}$$

$$\frac{dI_{EQ_2}}{dt}=\frac{I_{LED-2}\cdot\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})}{C_m\cdot\left\{\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s+\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s-\left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\right\}}$$

$$\frac{dI_{EQ_3}}{dt}=\frac{I_{LED-2}\cdot\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})}{C_m\cdot\{\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s+\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s-[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s]\}}$$

$$\frac{dI_{LED-2}}{dt}=g_1(I_{LED-2},I_{EQ_2},I_{EQ_3});\frac{dI_{EQ_2}}{dt}=g_2(I_{LED-2},I_{EQ_2},I_{EQ_3});\frac{dI_{EQ_3}}{dt}=g_3(I_{LED-2},I_{EQ_2},I_{EQ_3})$$

$$g_1(I_{LED-2},I_{EQ_2},I_{EQ_3})=\frac{1}{C_m\cdot\left\{\Xi_2(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s+\Xi_1(I_{EQ_2},I_{EQ_3},I_{LED-2})\cdot R_s-\left[\frac{2\cdot V_t}{I_{LED-2}+I_0}+R_m+R_s\right]\right\}}$$

$$g_2(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{I_{\text{LED}-2} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{C_m \cdot \left\{\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{LED-2} + I_0} + R_m + R_s\right]\right\}}$$

$$g_3(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{I_{\text{LED}-2} \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{C_m \cdot \left\{\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s\right]\right\}}$$

To analyze our system fixed points for the case $V_s(t) = V_A$ for $(n-1) \cdot (T_A + T_B) \leq t \leq n \cdot T_A + T_B \cdot (n-1); n = 1, 2, 3, \ldots$ we need to consider $T_A \to \infty$ otherwise there is no fixed points and the analysis is done only until T_A seconds (dynamical behavior which is restricted in time). $T_A \to \infty$ then at fixed points (equilibrium points)

$$\frac{dI_{\text{LED}-2}}{dt} = 0; \frac{dI_{EQ_2}}{dt} = 0; \frac{dI_{EQ_3}}{dt} = 0; \frac{dI_{\text{LED}-2}}{dt} = 0 \Rightarrow I^*_{\text{LED}-2} = 0.$$

$$\frac{dI_{EQ_2}}{dt} = 0 \Rightarrow I^*_{\text{LED}-2} = 0 \text{ or } \Xi^*_1(I^*_{EQ_2}, I^*_{EQ_3}, I^*_{\text{LED}-2}) = 0;$$

$$\frac{dI_{EQ_3}}{dt} = 0 \Rightarrow I^*_{\text{LED}-2} = 0 \text{ or } \Xi^*_2(I^*_{EQ_2}, I^*_{EQ_3}, I^*_{\text{LED}-2}) = 0$$

$$\{I^*_{\text{LED}-2} = 0\} \cap \{I^*_{\text{LED}-2} = 0 \text{ or } \Xi^*_1(I^*_{EQ_2}, I^*_{EQ_3}, I^*_{\text{LED}-2}) = 0\}$$
$$\cap \{I^*_{\text{LED}-2} = 0 \text{ or } \Xi^*_2(I^*_{EQ_2}, I^*_{EQ_3}, I^*_{\text{LED}-2}) = 0\} = \{I^*_{\text{LED}-2} = 0\}$$

Web approximate the phase portrait near our system fixed points by that of a corresponding linear system. Our system differential equations:

$$\frac{dI_{\text{LED}-2}}{dt} = g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}); \frac{dI_{EQ_2}}{dt} = g_2(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3});$$
$$\frac{dI_{EQ_3}}{dt} = g_3(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})$$

And suppose that $(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})$ is a fixed point$g_i(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}) = 0$; $i = 1, 2, 3$.

Let $u = I_{\text{LED}-2} - I^*_{\text{LED}-2}; v = I_{EQ_2} - I^*_{EQ_2}; w = I_{EQ_3} - I^*_{EQ_3}$ denote the components of a small disturbance from the fixed point (equilibrium point). To see whether the disturbance grows or decays, we need to derive differential equations for u, v, and w. Let's do u equation first: $\frac{du}{dt} = \frac{dI_{\text{LED}-2}}{dt}$ since $I^*_{\text{LED}-2}$ is a constant and by substitution $\frac{du}{dt} = \frac{dI_{\text{LED}-2}}{dt} = g_1(u + I^*_{\text{LED}-2}, v + I^*_{EQ_2}, w + I^*_{EQ_3})$ and by using Taylor expansion $\frac{du}{dt} = \frac{dI_{\text{LED}-2}}{dt} = g_1(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}) + u \cdot \frac{\partial g_1}{\partial I_{\text{LED}-2}} + v \cdot \frac{\partial g_1}{\partial I_{EQ_2}} + w \cdot \frac{\partial g_1}{\partial I_{EQ_3}} + O(u^2, v^2, w^2, \ldots)$.

$$g_1(I^*_{\mathrm{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}) = 0 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}I_{\mathrm{LED}-2}}{\mathrm{d}t} = u \cdot \frac{\partial g_1}{\partial I_{\mathrm{LED}-2}} + v \cdot \frac{\partial g_1}{\partial I_{EQ_2}} + w \cdot \frac{\partial g_1}{\partial I_{EQ_3}} + O(u^2, v^2, w^2, \ldots)$$

To simplify the notation, we have written $\frac{\partial g_1}{\partial I_{\mathrm{LED}-2}}, \frac{\partial g_1}{\partial I_{EQ_2}}, \frac{\partial g_1}{\partial I_{EQ_3}}$. These partial derivatives are to be evaluated at the fixed point $(I^*_{\mathrm{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})$; thus they are numbers, not functions. The shorthand notation $O(u^2, v^2, w^2, \ldots)$ denotes quadratic terms in u, v, and w. Since u, v, and w variables are small, these quadratic terms are extremely small. In the same way we find the following:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}I_{EQ_2}}{\mathrm{d}t} = g_2\left(I^*_{\mathrm{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right) + u \cdot \frac{\partial g_2}{\partial I_{\mathrm{LED}-2}} + v \cdot \frac{\partial g_2}{\partial I_{EQ_2}} + w \cdot \frac{\partial g_2}{\partial I_{EQ_3}} + O(u^2, v^2, w^2, \ldots)$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}I_{EQ_3}}{\mathrm{d}t} = g_3\left(I^*_{LED-2}, I^*_{EQ_2}, I^*_{EQ_3}\right) + u \cdot \frac{\partial g_3}{\partial I_{\mathrm{LED}-2}} + v \cdot \frac{\partial g_3}{\partial I_{EQ_2}} + w \cdot \frac{\partial g_3}{\partial I_{EQ_3}} + O(u^2, v^2, w^2, \ldots)$$

Hence the disturbance (u, v, w) evolves according to

$$\begin{pmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} \\ \frac{\mathrm{d}v}{\mathrm{d}t} \\ \frac{\mathrm{d}w}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_1}{\partial I_{EQ_2}} & \frac{\partial g_1}{\partial I_{EQ_3}} \\ \frac{\partial g_2}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_2}{\partial I_{EQ_2}} & \frac{\partial g_2}{\partial I_{EQ_3}} \\ \frac{\partial g_3}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_3}{\partial I_{EQ_2}} & \frac{\partial g_3}{\partial I_{EQ_3}} \end{pmatrix} + \text{quadratic term}$$

The matrix $A = \begin{pmatrix} \frac{\partial g_1}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_1}{\partial I_{EQ_2}} & \frac{\partial g_1}{\partial I_{EQ_3}} \\ \frac{\partial g_2}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_2}{\partial I_{EQ_2}} & \frac{\partial g_2}{\partial I_{EQ_3}} \\ \frac{\partial g_3}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_3}{\partial I_{EQ_2}} & \frac{\partial g_3}{\partial I_{EQ_3}} \end{pmatrix}_{(I^*_{\mathrm{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})}$ is called the system Jacobian matrix at the fixed point $(I^*_{\mathrm{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})$. Since the quadratic terms are tiny, it's tempting to neglect them altogether. We get our optoisolation elements bridge circuit linearized system.

$$\begin{pmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} \\ \frac{\mathrm{d}v}{\mathrm{d}t} \\ \frac{\mathrm{d}w}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_1}{\partial I_{EQ_2}} & \frac{\partial g_1}{\partial I_{EQ_3}} \\ \frac{\partial g_2}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_2}{\partial I_{EQ_2}} & \frac{\partial g_2}{\partial I_{EQ_3}} \\ \frac{\partial g_3}{\partial I_{\mathrm{LED}-2}} & \frac{\partial g_3}{\partial I_{EQ_2}} & \frac{\partial g_3}{\partial I_{EQ_3}} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$g_1(I_{\mathrm{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{1}{C_m \cdot \left\{\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{LED-2} + I_0} + R_m + R_s\right]\right\}} \cdot I_{\mathrm{LED}-2}$$

$$\frac{\partial g_1}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c} C_m \cdot \left\{ \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s \right. \\ \left. - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s \right] \right\} - I_{\text{LED}-2} \cdot C_m \cdot \left\{ \frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot R_s \right. \\ \left. + \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot R_s - \frac{\partial}{\partial I_{\text{LED}-2}} \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} \right] \right\} \end{array}}{\begin{array}{c} C_m^2 \cdot \left\{ \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s \right. \\ \left. - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s \right] \right\}^2 \end{array}}$$

$$\frac{\partial}{\partial I_{\text{LED}-2}} \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} \right] = \frac{-2 \cdot V_t}{[I_{\text{LED}-2} + I_0]^2}$$

$$\frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})$$
$$- \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot \frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{V_t}{R_s} \cdot \left[\frac{\partial \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} + \frac{\partial \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot k_2 \right]$$

$$\Upsilon_3(I_{EQ_2}, I_{\text{LED}-2}) = k_2 \cdot \left[\frac{\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})} \right]$$

$$\frac{\partial \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = k_2 \cdot \left[\frac{\begin{array}{c} \left[\alpha_{rQ_2} \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \right] \cdot \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) \\ - \frac{\partial}{\partial I_{\text{LED}-2}} \left\{ \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) \right\} \cdot \left\{ \alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) \right. \\ \left. - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \right\} \end{array}}{[\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})]^2} \right]$$

$$\frac{\partial}{\partial I_{\text{LED}-2}} \left\{ \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) \right\} = \frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})$$
$$+ \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}; \frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = -k_2 \cdot \alpha_{rQ_2}; \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = -k_2$$

$$\Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_1}]}{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}$$

$$\frac{\partial \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c} \left\{ [\alpha_{rQ_1} - 1] \cdot \frac{\partial \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial \xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot [1 - \alpha_{fQ_1}] \right\} \\ \cdot \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \left\{ [\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ2}, I_{\text{LED}-2}) \right. \\ \left. - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_1}] \right\} \cdot \frac{\partial}{\partial I_{\text{LED}-2}} \left\{ \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) \right\} \end{array}}{[\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{\text{LED}-2}}\{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})\} = \frac{\partial \xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) + \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\frac{\partial \xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = -k_2 \cdot [\alpha_{rQ_1} - 1]; \frac{\partial \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = -k_2 \cdot [1 - \alpha_{fQ_1}]$$

$$\Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) = \frac{\Omega_{11}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c}\left[\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\right] \cdot [\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) \\ -\Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})] - [\Omega_{11}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})] \\ \cdot \left[\frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\right]\end{array}}{[\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) = \frac{V_t}{R_s} \cdot \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) + \frac{V_t}{R_s} \cdot \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) + 1 + \frac{R_1}{R_s}$$

$$\frac{\partial \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{V_t}{R_s} \cdot \left[\frac{\partial \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} + \frac{\partial \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\right]$$

$$\Omega_{21}(I_{EQ_3}, I_{\text{LED}-2}) = \frac{R_s + k_3 \cdot R_2 + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) \cdot k_3 + V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\text{LED}-2})}{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2}) + R_s + R_2}$$

$$\frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c}V_t \cdot \left[\frac{\partial \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot k_3 + \frac{\partial \Upsilon_6(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\right] \cdot [V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) \\ + \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2}) + R_s + R_2] \\ -[R_s + k_3 \cdot R_2 + V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) \cdot k_3 + V_t \cdot \Upsilon_6(I_{EQ_3}, I_{\text{LED}-2})] \\ \cdot \left[V_t \cdot \frac{\partial \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} + \frac{\partial \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\right]\end{array}}{[V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2}) + R_s + R_2]^2}$$

$$\Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_2}]}{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}$$

$$\frac{\partial \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c}\left\{[\alpha_{rQ_2} - 1] \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} \cdot [1 - \alpha_{fQ_2}]\right\} \cdot \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \\ \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_2}]\} \\ \cdot \frac{\partial}{\partial I_{\text{LED}-2}}\{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})\}\end{array}}{[\xi_{13}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{\text{LED}-2}}\left\{\xi_{13}(I_{EQ_2}, I_{\text{LED}-2})\cdot\ \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})\right\} = \frac{\partial \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\cdot\ \xi_{14}(I_{EQ_2}, I_{\text{LED}-2}) + \xi_{13}(I_{EQ_2}, I_{\text{LED}-2})\cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_4} - 1]\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot[1-\alpha_{fQ_4}]}{\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c}\left\{[\alpha_{rQ_4}-1]\cdot\frac{\partial\xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\cdot\left[1-\alpha_{fQ_4}\right]\right\} \\ \cdot\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{22}(I_{EQ_3}, I_{\text{LED}-2}) - \left\{[\alpha_{rQ_4}-1]\cdot\xi_{22}(I_{EQ_3}, I_{\text{LED}-2})\right. \\ \left.-\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot[1-\alpha_{fQ_4}]\right\}\cdot\frac{\partial}{\partial I_{\text{LED}-2}}\left\{\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{22}(I_{EQ_3}, I_{\text{LED}-2})\right\}\end{array}}{[\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{22}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{\text{LED}-2}}\{\xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2})\} = \frac{\partial \xi_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\cdot\ \xi_{22}(I_{EQ_3}, I_{\text{LED}-2}) + \xi_{21}(I_{EQ_3}, I_{\text{LED}-2})\cdot \frac{\partial \xi_{22}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\Upsilon_5(I_{EQ_3}, I_{LED-2}) = \frac{[\alpha_{rQ_3} - 1]\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot[1-\alpha_{fQ_3}]}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = \frac{\begin{array}{c}\left\{[\alpha_{rQ_3}-1]\cdot\frac{\partial\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\cdot[1-\alpha_{fQ_3}]\right\} \\ \cdot\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \left\{[\alpha_{rQ_3}-1]\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})\right. \\ \left.-\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot[1-\alpha_{fQ_3}]\right\}\cdot\frac{\partial}{\partial I_{\text{LED}-2}}\left\{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})\right\}\end{array}}{[\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{\text{LED}-2}}\left\{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LEDLED}-2})\right\} = \frac{\partial \xi_{23}(I_{EQ_3}, I_{LED-2})}{\partial I_{\text{LED}-2}}\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) + \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot \frac{\partial \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\Upsilon_6(I_{EQ_3}, I_{\text{LED}-2}) = k_3\cdot\left[\frac{\alpha_{rQ_3}\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\ \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}\right]$$

$$\frac{\partial \Upsilon_6(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} = k_3\cdot\frac{\begin{array}{c}\left[\alpha_{rQ_3}\cdot\frac{\partial\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}} - \frac{\partial\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}\right] \\ \cdot\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - [\alpha_{rQ_3}\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) \\ -\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})]\cdot\frac{\partial}{\partial I_{\text{LED}-2}}\left\{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})\right\}\end{array}}{[\xi_{23}(I_{EQ_3}, I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{1}{C_m \cdot \left\{ \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s\right] \right\}} \cdot I_{\text{LED}-2}$$

$$\frac{\partial g_1}{\partial I_{EQ_2}} = \frac{-I_{\text{LED}-2} \cdot C_m \cdot \left\{ \frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot R_s + \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot R_s \right\}}{C_m^2 \cdot \left\{ \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s\right] \right\}^2}$$

$$\frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = \frac{\left[\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} - \frac{\partial \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}}\right] \cdot [\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})] - [\Omega_{11}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})] \cdot \left[\frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} - \frac{\partial \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}}\right]}{[\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = 0; \frac{\partial \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = 0$$

$$\frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = \frac{\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot [\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})] - [\Omega_{11}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\text{LED}-2})] \cdot \frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}}}{[\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$\Omega_{11}(I_{EQ_2}, I_{\text{LED}-2}) = 1 + k_2 \cdot \frac{R_1}{R_s} + \frac{V_t}{R_s} \cdot \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2}) + \frac{V_t}{R_s} \cdot \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) \cdot k_2$$

$$\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = \frac{V_t}{R_s} \cdot \left[\frac{\partial \Upsilon_3(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} + \frac{\partial \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot k_2\right]$$

$$\Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) = \frac{V_t}{R_s} \cdot \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) + \frac{V_t}{R_s} \cdot \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2}) + 1 + \frac{R_1}{R_s}$$

$$\frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = \frac{V_t}{R_s} \cdot \left[\frac{\partial \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} + \frac{\partial \Upsilon_2(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}}\right]$$

$$\Upsilon_1(I_{EQ_2}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_1}]}{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}$$

$$\frac{\partial \Upsilon_1(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} = \frac{\left([\alpha_{rQ_1} - 1] \cdot \frac{\partial \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} - \frac{\partial \xi_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_2}} \cdot [1 - \alpha_{fQ_1}]\right) \cdot \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - ([\alpha_{rQ_1} - 1] \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_1}]) \cdot \frac{\partial}{\partial I_{EQ_2}} \{\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})\}}{[\xi_{11}(I_{EQ_2}, I_{\text{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{EQ_2}}\{\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})\} = \frac{\partial \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} \cdot \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) + \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \frac{\partial \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}}$$

$$\xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) = ([\alpha_{rQ_1} - 1] \cdot [I_{EQ_2} - I_{\mathrm{LED}-2} \cdot k_2]) + I_{se} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)$$
$$\xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) = ([I_{EQ_2} - I_{\mathrm{LED}-2} \cdot k_2] \cdot [1 - \alpha_{fQ_1}]) + I_{sc} \cdot (\alpha_{rQ_1} \cdot \alpha_{fQ_1} - 1)$$

$$\frac{\partial \xi_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = [1 - \alpha_{fQ_1}]; \frac{\partial \xi_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = [\alpha_{rQ_1} - 1]$$

$$\Upsilon_2(I_{EQ_2}, I_{\mathrm{LED}-2}) = \frac{[\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot [1 - \alpha_{fQ_2}]}{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}$$

$$\frac{\partial \Upsilon_2(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = \frac{\begin{array}{c}\left([\alpha_{rQ_2} - 1] \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} - \frac{\partial \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} \cdot [1 - \alpha_{fQ_2}]\right) \\ \cdot \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) - ([\alpha_{rQ_2} - 1] \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) \\ - \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot [1 - \alpha_{fQ_2}]) \cdot \frac{\partial}{\partial I_{EQ_2}}\{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})\}\end{array}}{[\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{EQ_2}}\{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})\} = \frac{\partial \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) + \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}}$$

$$\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) = ([\alpha_{rQ_2} - 1] \cdot [I_{EQ_2} - I_{\mathrm{LED}-2} \cdot k_2] - I_{\mathrm{LED}-2} \cdot k_2) + I_{se} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)$$
$$\xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) = ([I_{EQ_2} - I_{\mathrm{LED}-2} \cdot k_2] \cdot [1 - \alpha_{fQ_2}] - I_{\mathrm{LED}-2} \cdot k_2 \cdot \alpha_{fQ_2}) + I_{sc} \cdot (\alpha_{rQ_2} \cdot \alpha_{fQ_2} - 1)$$

$$\frac{\partial \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = [\alpha_{rQ_2} - 1]; \frac{\partial \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = [1 - \alpha_{fQ_2}]$$

$$\Upsilon_3(I_{EQ_2}, I_{\mathrm{LED}-2}) = k_2 \cdot \left[\frac{\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}\right]$$

$$\frac{\partial \Upsilon_3(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = k_2 \cdot \left[\frac{\begin{array}{c}\left[\alpha_{rQ_2} \cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} - \frac{\partial \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}}\right] \\ \cdot \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) - [\alpha_{rQ_2} \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) \\ - \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})] \cdot \frac{\partial}{\partial I_{EQ_2}}\{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})\}\end{array}}{[\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2}) \cdot \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})]^2}\right]$$

$$\frac{\partial}{\partial I_{EQ_2}}\left\{\xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})\cdot\ \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})\right\} = \frac{\partial \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}}\cdot\ \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2}) + \xi_{13}(I_{EQ_2}, I_{\mathrm{LED}-2})\cdot \frac{\partial \xi_{14}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}}$$

$$\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2}) = \Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2}) \cdot\ \Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})$$

$$\frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} = \frac{\partial \Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} - \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}} \cdot\ \Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})\cdot \frac{\partial \Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_2}}$$

$$g_1(I_{\mathrm{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{I_{\mathrm{LED}-2}}{C_m\cdot\left\{\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})\cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})\cdot R_s - \left[\frac{2\cdot V_t}{I_{\mathrm{LED}-2}+I_0} + R_m + R_s\right]\right\}}$$

$$\frac{\partial g_1}{\partial I_{EQ_3}} = \frac{-I_{\mathrm{LED}-2}\cdot C_m\cdot\left\{\frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}}\cdot R_s + \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}}\cdot R_s\right\}}{C_m^2\cdot\left\{\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})\cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})\cdot R_s - \left[\frac{2\cdot V_t}{I_{\mathrm{LED}-2}+I_0} + R_m + R_s\right]\right\}^2}$$

$$\Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2}) = \frac{\Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\mathrm{LED}-2})}{\Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})}$$

$$\frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}} = \frac{\left[\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}} - \frac{\partial \Omega_{21}(I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}}\right]\cdot[\Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})] - [\Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\mathrm{LED}-2})]\cdot\left[\frac{\partial \Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}} - \frac{\partial \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}}\right]}{[\Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})]^2}$$

$$\frac{\partial \Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}} = 0; \frac{\partial \Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}} = 0$$

$$\frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}} = \frac{-\frac{\partial \Omega_{21}(I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}}\cdot[\Omega_{12}(I_{EQ_2}, I_{LED-2}) - \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})] + [\Omega_{11}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{21}(I_{EQ_3}, I_{\mathrm{LED}-2})]\cdot\frac{\partial \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})}{\partial I_{EQ_3}}}{[\Omega_{12}(I_{EQ_2}, I_{\mathrm{LED}-2}) - \Omega_{22}(I_{EQ_3}, I_{\mathrm{LED}-2})]^2}$$

$$\Omega_{21}(I_{EQ_3}, I_{\mathrm{LED}-2}) = \frac{R_s + k_3\cdot R_2 + V_t\cdot\ \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2})\cdot k_3 + V_t\cdot\ \Upsilon_6(I_{EQ_3}, I_{\mathrm{LED}-2})}{V_t\cdot\ \Upsilon_4(I_{EQ_3}, I_{\mathrm{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\mathrm{LED}-2}) + R_s + R_2}$$

$$\frac{\partial\Omega_{21}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=\frac{\begin{array}{c}\left[V_t\cdot\frac{\partial\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\cdot k_3+V_t\cdot\frac{\partial\Upsilon_6(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\right]\cdot[V_t\cdot\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})\\+\Upsilon_5(I_{EQ_3},I_{\text{LED}-2})+R_s+R_2]-[R_s+k_3\cdot R_2+V_t\cdot\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})\cdot k_3\\+V_t\cdot\Upsilon_6(I_{EQ_3},I_{\text{LED}-2})]\cdot\left[V_t\cdot\frac{\partial\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}+\frac{\partial\Upsilon_5(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\right]\end{array}}{[V_t\cdot\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})+\Upsilon_5(I_{EQ_3},I_{\text{LED}-2})+R_s+R_2]^2}$$

$$\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})=\frac{[\alpha_{rQ_4}-1]\cdot\,\Xi_{22}(I_{EQ_3},I_{\text{LED}-2})-\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot[1-\alpha_{fQ_4}]}{\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\,\xi_{22}(I_{EQ_3},I_{\text{LED}-2})}$$

$$\frac{\partial\Upsilon_4(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=\frac{\begin{array}{c}\left([\alpha_{rQ_4}-1]\cdot\frac{\partial\xi_{22}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}-\frac{\partial\xi_{21}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\cdot[1-\alpha_{fQ_4}]\right)\\\cdot\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\xi_{22}(I_{EQ_3},I_{\text{LED}-2})\\-\left([\alpha_{rQ_4}-1]\cdot\xi_{22}(I_{EQ_3},I_{\text{LED}-2})-\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\left[1-\alpha_{fQ_4}\right]\right)\\\cdot\frac{\partial}{\partial I_{EQ_3}}\left\{\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\xi_{22}(I_{EQ_3},I_{\text{LED}-2})\right\}\end{array}}{[\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\xi_{22}(I_{EQ_3},I_E)]^2}$$

$$\frac{\partial}{\partial I_{EQ_3}}\left\{\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\,\xi_{22}(I_{EQ_3},I_{\text{LED}-2})\right\}=\frac{\partial\xi_{21}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\cdot\,\xi_{22}(I_{EQ_3},I_{\text{LED}-2})+\xi_{21}(I_{EQ_3},I_{\text{LED}-2})\cdot\frac{\partial\xi_{22}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}$$

$$\frac{\partial\xi_{22}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=[1-\alpha_{fQ_4}];\frac{\partial\xi_{21}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=[\alpha_{rQ_4}-1]$$

$$\Upsilon_6(I_{EQ_3},I_{\text{LED}-2})=k_3\cdot\left[\frac{\alpha_{rQ_3}\cdot\,\xi_{24}(I_{EQ_3},I_{\text{LED}-2})-\xi_{23}(I_{EQ_3},I_{\text{LED}-2})}{\xi_{23}(I_{EQ_3},I_{\text{LED}-2})\cdot\,\xi_{24}(I_{EQ_3},I_{\text{LED}-2})}\right]$$

$$\frac{\partial\Upsilon_6(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=k_3\cdot\left[\frac{\begin{array}{c}\left[\alpha_{rQ_3}\cdot\frac{\partial\xi_{24}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}-\frac{\partial\xi_{23}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\right]\cdot\xi_{23}(I_{EQ_3},I_{LED-2})\\\cdot\xi_{24}(I_{EQ_3},I_{\text{LED}-2})-[\alpha_{rQ_3}\cdot\xi_{24}(I_{EQ_3},I_{\text{LED}-2})\\-\xi_{23}(I_{EQ_3},I_{\text{LED}-2})]\cdot\frac{\partial}{\partial I_{EQ_3}}\left\{\xi_{23}(I_{EQ_3},I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3},I_{\text{LED}-2})\right\}\end{array}}{[\xi_{23}(I_{EQ_3},I_{\text{LED}-2})\cdot\xi_{24}(I_{EQ_3},I_{\text{LED}-2})]^2}\right]$$

$$\frac{\partial}{\partial I_{EQ_3}}\left\{\xi_{23}(I_{EQ_3},I_{\text{LED}-2})\cdot\,\xi_{24}(I_{EQ_3},I_{\text{LED}-2})\right\}=\frac{\partial\xi_{23}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}\cdot\,\xi_{24}(I_{EQ_3},I_{\text{LED}-2})+\xi_{23}(I_{EQ_3},I_{\text{LED}-2})\cdot\frac{\partial\xi_{24}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}$$

$$\frac{\partial\xi_{23}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=[\alpha_{rQ_3}-1];\frac{\partial\xi_{24}(I_{EQ_3},I_{\text{LED}-2})}{\partial I_{EQ_3}}=[1-\alpha_{fQ_3}]$$

$$\Upsilon_5(I_{EQ_3}, I_{\text{LED}-2}) = \frac{[\alpha_{rQ_3} - 1] \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_3}]}{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}$$

$$\frac{\partial \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = \frac{\begin{array}{c}\left([\alpha_{rQ_3} - 1] \cdot \frac{\partial \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} - \frac{\partial \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} \cdot [1 - \alpha_{fQ_3}]\right) \\ \cdot \xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) - ([\alpha_{rQ_3} - 1] \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) \\ - \xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot [1 - \alpha_{fQ_3}]) \cdot \frac{\partial}{\partial I_{EQ_3}} \{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})\}\end{array}}{[\xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})]^2}$$

$$\frac{\partial}{\partial I_{EQ_3}} \{\xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})\} = \frac{\partial \xi_{23}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} \cdot \xi_{24}(I_{EQ_3}, I_{\text{LED}-2}) + \xi_{23}(I_{EQ_3}, I_{\text{LED}-2}) \cdot \frac{\partial \xi_{24}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}}$$

$$\Omega_{22}(I_{EQ_3}, I_{\text{LED}-2}) = \frac{R_s}{\{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2}) + R_s + R_2\}}$$

$$\frac{\partial \Omega_{22}(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = \frac{-R_s \cdot \left(V_t \cdot \frac{\partial \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} + \frac{\partial \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}}\right)}{\{V_t \cdot \Upsilon_4(I_{EQ_3}, I_{\text{LED}-2}) + \Upsilon_5(I_{EQ_3}, I_{\text{LED}-2}) + R_s + R_2\}^2}$$

$$\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) = \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2}) - \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})$$

$$\frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = \frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_3}} - \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} \cdot \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2}) - \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot \frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_3}}; \frac{\partial \Omega_{11}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = 0; \frac{\partial \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = 0$$

$$\frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} = -\frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}} \cdot \Omega_{12}(I_{EQ_2}, I_{\text{LED}-2})$$

$$g_2(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{I_{\text{LED}-2} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{C_m \cdot \left\{\Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s\right]\right\}}$$

$$g_2(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})$$

$$\frac{\partial g_2}{\partial I_{\text{LED}-2}} = \frac{\partial g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})}{\partial I_{\text{LED}-2}} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) + g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\frac{\partial g_2}{\partial I_{EQ_2}} = \frac{\partial g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})}{\partial I_{EQ_2}} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) + g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}}$$

$$\frac{\partial g_2}{\partial I_{EQ_3}} = \frac{\partial g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})}{\partial I_{EQ_3}} \cdot \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) + g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \frac{\partial \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}}$$

$$g_3(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = \frac{I_{\text{LED}-2} \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{C_m \cdot \left\{ \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s + \Xi_1(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) \cdot R_s - \left[\frac{2 \cdot V_t}{I_{\text{LED}-2} + I_0} + R_m + R_s\right] \right\}}$$

$$g_3(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) = g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})$$

$$\frac{\partial g_3}{\partial I_{\text{LED}-2}} = \frac{\partial g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})}{\partial I_{\text{LED}-2}} \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) + g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{\text{LED}-2}}$$

$$\frac{\partial g_3}{\partial I_{EQ_2}} = \frac{\partial g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})}{\partial I_{EQ_2}} \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) + g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_2}}$$

$$\frac{\partial g_3}{\partial I_{EQ_3}} = \frac{\partial g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3})}{\partial I_{EQ_3}} \cdot \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2}) + g_1(I_{\text{LED}-2}, I_{EQ_2}, I_{EQ_3}) \cdot \frac{\partial \Xi_2(I_{EQ_2}, I_{EQ_3}, I_{\text{LED}-2})}{\partial I_{EQ_3}}$$

Remark Our stability analysis is done based on the particular assumption that $T_A \to \infty; D_+ = \frac{1}{1 + T_B/(T_A \to \infty)} \to 1; D_- = \frac{1}{1 + (T_A \to \infty)/T_B} \to \varepsilon$ otherwise our system dynamical analysis is restricted in time.

For stability analysis we need to find our system characteristic equation. It is done by finding the determinant of system Jacobian matrix at the fixed point $(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})$ minus the unity matrix (3 × 3) multiples by eigenvalue λ.

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial I_{\text{LED}-2}} & \frac{\partial g_1}{\partial I_{EQ_2}} & \frac{\partial g_1}{\partial I_{EQ_3}} \\ \frac{\partial g_2}{\partial I_{\text{LED}-2}} & \frac{\partial g_2}{\partial I_{EQ_2}} & \frac{\partial g_2}{\partial I_{EQ_3}} \\ \frac{\partial g_3}{\partial I_{\text{LED}-2}} & \frac{\partial g_3}{\partial I_{EQ_2}} & \frac{\partial g_3}{\partial I_{EQ_3}} \end{pmatrix}_{(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})};$$

$$A - \lambda \cdot I = \begin{pmatrix} \frac{\partial g_1}{\partial I_{\text{LED}-2}} - \lambda & \frac{\partial g_1}{\partial I_{EQ_2}} & \frac{\partial g_1}{\partial I_{EQ_3}} \\ \frac{\partial g_2}{\partial I_{\text{LED}-2}} & \frac{\partial g_2}{\partial I_{EQ_2}} - \lambda & \frac{\partial g_2}{\partial I_{EQ_3}} \\ \frac{\partial g_3}{\partial I_{\text{LED}-2}} & \frac{\partial g_3}{\partial I_{EQ_2}} & \frac{\partial g_3}{\partial I_{EQ_3}} - \lambda \end{pmatrix}_{(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3})}$$

$$\begin{aligned} \det|A - \lambda \cdot I| = {} & \left(\frac{\partial g_1}{\partial I_{\text{LED}-2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \right) \\ & \cdot \det \begin{pmatrix} \frac{\partial g_2}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda & \frac{\partial g_2}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \\ \frac{\partial g_3}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} & \frac{\partial g_3}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \end{pmatrix} \\ & - \frac{\partial g_1}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \cdot \det \begin{pmatrix} \frac{\partial g_2}{\partial I_{\text{LED}-2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} & \frac{\partial g_2}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \\ \frac{\partial g_3}{\partial I_{\text{LED}-2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} & \frac{\partial g_3}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \end{pmatrix} \\ & + \frac{\partial g_1}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \cdot \det \begin{pmatrix} \frac{\partial g_2}{\partial I_{\text{LED}-2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} & \frac{\partial g_2}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \\ \frac{\partial g_3}{\partial I_{\text{LED}-2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} & \frac{\partial g_3}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \det \begin{pmatrix} \frac{\partial g_2}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda & \frac{\partial g_2}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \\ \frac{\partial g_3}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} & \frac{\partial g_3}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \end{pmatrix} = \left(\frac{\partial g_2}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \right) \\ \cdot \left(\frac{\partial g_3}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} - \lambda \right) - \frac{\partial g_3}{\partial I_{EQ_2}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \cdot \frac{\partial g_2}{\partial I_{EQ_3}} \Big|_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} \end{aligned}$$

$$\det\begin{pmatrix} \frac{\partial g_2}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} & \frac{\partial g_2}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \\ \frac{\partial g_3}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} & \frac{\partial g_3}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda \end{pmatrix} = \frac{\partial g_2}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)}$$

$$\cdot\left(\frac{\partial g_3}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right) - \frac{\partial g_3}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \frac{\partial g_2}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)}$$

$$\det\begin{pmatrix} \frac{\partial g_2}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} & \frac{\partial g_2}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda \\ \frac{\partial g_3}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} & \frac{\partial g_3}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \end{pmatrix} = \frac{\partial g_2}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)}$$

$$\cdot\frac{\partial g_3}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \frac{\partial g_3}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \left(\frac{\partial g_2}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right)$$

Finally we get the eigenvalues characteristic equation:

$$\begin{aligned}\det|A - \lambda\cdot I| = {} & \left(\frac{\partial g_1}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right)\cdot\left\{\left(\frac{\partial g_2}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right)\right. \\ & \left.\cdot\left(\frac{\partial g_3}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right) - \frac{\partial g_3}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \frac{\partial g_2}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)}\right\} \\ & - \frac{\partial g_1}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \left\{\frac{\partial g_2}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \left(\frac{\partial g_3}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right)\right. \\ & \left. - \frac{\partial g_3}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \frac{\partial g_2}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)}\right\} + \frac{\partial g_1}{\partial I_{EQ_3}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \\ & \cdot\left\{\frac{\partial g_2}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \frac{\partial g_3}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)}\right. \\ & \left. - \frac{\partial g_3}{\partial I_{\text{LED}-2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} \cdot \left(\frac{\partial g_2}{\partial I_{EQ_2}}|_{\left(I^*_{\text{LED}-2},I^*_{EQ_2},I^*_{EQ_3}\right)} - \lambda\right)\right\}\end{aligned}$$

We define for simplicity the following characteristic equation parameters:

$$B_1 = \frac{\partial g_1}{\partial I_{\text{LED}-2}} |_{\left(I^*_{LED-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ; B_2 = \frac{\partial g_2}{\partial I_{EQ_2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ;$$

$$B_3 = \frac{\partial g_3}{\partial I_{EQ_3}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)}$$

$$B_4 = \frac{\partial g_3}{\partial I_{EQ_2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ; B_5 = \frac{\partial g_2}{\partial I_{EQ_3}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ;$$

$$B_6 = \frac{\partial g_1}{\partial I_{EQ_2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)}$$

$$B_7 = \frac{\partial g_2}{\partial I_{\text{LED}-2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ; B_8 = \frac{\partial g_3}{\partial I_{EQ_3}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ;$$

$$B_9 = \frac{\partial g_3}{\partial I_{\text{LED}-2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)}$$

$$B_{10} = \frac{\partial g_2}{\partial I_{EQ_3}} |_{\left(I^*_{LED-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ; B_{11} = \frac{\partial g_1}{\partial I_{EQ_3}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ;$$

$$B_{12} = \frac{\partial g_2}{\partial I_{\text{LED}-2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)}$$

$$B_{13} = \frac{\partial g_3}{\partial I_{EQ_2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ; B_{14} = \frac{\partial g_3}{\partial I_{\text{LED}-2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)} ;$$

$$B_{15} = \frac{\partial g_2}{\partial I_{EQ_2}} |_{\left(I^*_{\text{LED}-2}, I^*_{EQ_2}, I^*_{EQ_3}\right)}$$

$$\det|A - \lambda \cdot I| = (B_1 - \lambda) \cdot \{(B_2 - \lambda) \cdot (B_3 - \lambda) - B_4 \cdot B_5\} - B_6 \cdot \{B_7 \cdot (B_8 - \lambda) - B_9 \cdot B_{10}\} + B_{11} \cdot \{B_{12} \cdot B_{13} - B_{14} \cdot (B_{15} - \lambda)\}$$

$$\begin{aligned} \det|A - \lambda \cdot I| = {} & B_1 \cdot B_2 \cdot B_3 - B_1 \cdot (B_2 + B_3) \cdot \lambda + B_1 \cdot \lambda^2 \\ & - B_1 \cdot B_4 \cdot B_5 - \lambda \cdot B_2 \cdot B_3 + (B_2 + B_3) \cdot \lambda^2 - \lambda^3 \\ & + \lambda \cdot B_4 \cdot B_5 - B_6 \cdot B_7 \cdot B_8 + B_6 \cdot B_7 \cdot \lambda \\ & + B_6 \cdot B_9 \cdot B_{10} + B_{11} \cdot B_{12} \cdot B_{13} - B_{11} \cdot B_{14} \cdot B_{15} + B_{11} \cdot B_{14} \cdot \lambda \end{aligned}$$

$$\det|A-\lambda\cdot I| = B_1\cdot B_2\cdot B_3 - B_1\cdot B_4\cdot B_5 - B_6\cdot B_7\cdot B_8 + B_6\cdot B_9\cdot B_{10} + B_{11}\cdot B_{12}\cdot B_{13} - B_{11}\cdot B_{14}\cdot B_{15} + \lambda \cdot [B_4\cdot B_5 + B_6\cdot B_7 + B_{11}\cdot B_{14} - B_1\cdot(B_2+B_3) - B_2\cdot B_3] + [B_1+B_2+B_3]\cdot\lambda^2 - \lambda^3$$

$$\Phi_0 = B_1\cdot B_2\cdot B_3 - B_1\cdot B_4\cdot B_5 - B_6\cdot B_7\cdot B_8 + B_6\cdot B_9\cdot B_{10} + B_{11}\cdot B_{12}\cdot B_{13} - B_{11}\cdot B_{14}\cdot B_{15}$$
$$\Phi_1 = B_4\cdot B_5 + B_6\cdot B_7 + B_{11}\cdot B_{14} - B_1\cdot(B_2+B_3) - B_2\cdot B_3; \Phi_2 = B_1+B_2+B_3; \Phi_3 = -1$$

$$\det|A-\lambda\cdot I| = \sum_{k=0}^{3}\Phi_k\cdot\lambda^k; \det|A-\lambda\cdot I| = 0 \Rightarrow \sum_{k=0}^{3}\Phi_k\cdot\lambda^k = 0$$

The system characteristic cubic function is $\sum_{k=0}^{3}\Phi_k\cdot\lambda^k = \Phi_3\cdot\lambda^3+\Phi_2\cdot\lambda^2+\Phi_1\cdot\lambda+\Phi_0 = 0$
with $\Phi_3 \neq 0$. The coefficients $\Phi_0, \Phi_1, \Phi_2, \Phi_3$ are assumed to be real numbers. Our system characteristic cubic function with real coefficients has at least one eigenvalue solution (λ) among the real numbers and this is a consequence of the intermediate value theorem. We can distinguish several possible cases using the discriminant ($\Delta(\phi_0, \phi_1, \phi_2, \phi_3)$) by the following expression:

$$\Delta(\Phi_0,\Phi_1,\Phi_2,\Phi_3) = 18\cdot\prod_{k=0}^{3}\Phi_k - 4\cdot\Phi_2^3\cdot\Phi_0 + \Phi_2^2\cdot\Phi_1^2 - 4\cdot\Phi_3\cdot\Phi_1^3 - 27\cdot\Phi_3^2\cdot\Phi_0^2$$

The following cases need to be considered: If $\Delta(\phi_0,\phi_1,\phi_2,\phi_3) > 0$, then the system characteristic equation has three distinct eigenvalue real roots. If $\Delta(\phi_0, \phi_1, \phi_2, \phi_3) = 0$, then the characteristic equation has a multiple eigenvalue root and all its root real. If $\Delta(\phi_0, \phi_1, \phi_2, \phi_3) < 0$, then the system characteristic equation has one eigenvalue real root and two nonreal eigenvalue complex conjugate roots. For our system characteristic cubic function equation $\sum_{k=0}^{3}\Phi_k\cdot\lambda^k = 0$ the general formula for the eigenvalue roots in terms of the coefficients ($\phi_0, \phi_1, \phi_2, \phi_3$) is as follow: $\lambda_k = \frac{-1}{3\cdot\Phi_3}\cdot(\Phi_2+n_k\cdot T+\frac{\Delta_0}{n_k\cdot T})\,\forall\, k\in\{1,2,3\}$

Where $n_1 = 1; n_2 = \frac{-1+i\cdot\sqrt{3}}{2}; n_3 = \frac{-1-i\cdot\sqrt{3}}{2}$ are the three characteristic cube roots of unity, and where $T = \sqrt[3]{\frac{\Delta_1+\sqrt{\Delta_1^2-4\cdot\Delta_0^3}}{2}}; \Delta_0(\Phi_1,\Phi_2,\Phi_3) = \Phi_2^2 - 3\cdot\Phi_3\cdot\Phi_1$

$$\Delta_1(\Phi_1,\Phi_2,\Phi_3,\Phi_4) = 2\cdot\Phi_2^3 - 9\cdot\prod_{k=1}^{3}\Phi_k + 27\cdot\Phi_3^2\cdot\Phi_0; \Delta_0 = \Delta_0(\Phi_1,\Phi_2,\Phi_3)$$

$$\Delta_1 = \Delta_1(\Phi_1,\Phi_2,\Phi_3,\Phi_4); T = T(\Phi_1,\Phi_2,\Phi_3,\Phi_4); \Delta_1^2 - 4\cdot\Delta_0^3 = -27\cdot\Phi_3^2\cdot\Delta.$$

Table. C.1 Optoisolation element's bridge system eigenvalues and fixed points stability classification

System eigenvalues (λ_1, λ_2, λ_3)	System fixed point stability classification ($T_A \to \infty$)
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0$ $\lambda_i > 0\,\forall\, i = 1,2,3$	Unstable node
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0$	Saddle point
$\lambda_1 > 0,\ \lambda_2 < 0,\ \lambda_3 < 0$	Saddle point
$\lambda_1 < 0,\ \lambda_2 < 0,\ \lambda_3 < 0$ $\lambda_i < 0\,\forall\, i = 1,2,3$	Stable node
$\mathrm{Re}(\lambda_1) > 0,\ \lambda_2 = \lambda_1^*,\ \lambda_3 > 0$ $\lambda_1 = \Gamma_1 + i\cdot\Gamma_2; \lambda_2 = \Gamma_1 - i\cdot\Gamma_2$ $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \Gamma_1$	Unstable spiral node
$\mathrm{Re}(\lambda_1) > 0,\ \lambda_2 = \lambda_1^*,\ \lambda_3 < 0$ $\lambda_1 = \Gamma_1 + i\cdot\Gamma_2; \lambda_2 = \Gamma_1 - i\cdot\Gamma_2$ $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \Gamma_1$	Unstable spiral saddle point
$\mathrm{Re}(\lambda_1) < 0,\ \lambda_2 = \lambda_1^*,\ \lambda_3 > 0$ $\lambda_1 = \Gamma_1 + i\cdot\Gamma_2; \lambda_2 = \Gamma_1 - i\cdot\Gamma_2$ $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \Gamma_1$	Unstable spiral saddle point
$\mathrm{Re}(\lambda_1) < 0,\ \lambda_2 = \lambda_1^*,\ \lambda_3 < 0$ $\lambda_1 = \Gamma_1 + i\cdot\Gamma_2; \lambda_2 = \Gamma_1 - i\cdot\Gamma_2$ $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \Gamma_1$	Stable spiral node

$\Delta = \Delta(\Phi_0, \Phi_1, \Phi_2, \Phi_3)$. In these formulas, $\sqrt{\ }, \sqrt[3]{\ }$ denote any choice for the square or eigenvalue cube roots. The changing of choice for the square root amounts to exchanging λ_2 and λ_3. The changing of choice for the cube root amounts to circularly permuting the roots. Thus the freeness of choosing a determination of the square or eigenvalue cube roots corresponds exactly to the freeness for numbering the eigenvalue roots of the characteristic equation.

There are special cases in our analysis: If $\Delta \neq 0$ and $\Delta_0 = 0$, the sign of $\sqrt{\Delta_1^2 - 4\cdot\Delta_0^3} = \sqrt{\Delta_1^2}$ has to be chosen to have $T \neq 0$, that is one should define $\sqrt{\Delta_1^2} = \Delta_1$, which even is the sign of Δ_1. If $\Delta = 0$ and $\Delta_0 = 0$, the three eigenvalue roots are equal: $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{\Phi_2}{3\cdot\Phi_3}$. If $\Delta = 0$ and $\Delta_0 \neq 0$, the above expression for the eigenvalue roots is correct but misleading. In this case there is a double eigenvalue root $\lambda_1 = \lambda_2 = \frac{9\cdot\Phi_3\cdot\Phi_0 - \Phi_2\cdot\Phi_1}{2\cdot\Delta_0}$ and a simple eigenvalue root $\lambda_3 = \frac{4\cdot\prod_{k=1}^{3}\Phi_k - 9\cdot\Phi_3^2\cdot\Phi_0 - \Phi_2^3}{\Phi_3\cdot\Delta_0}$. We can classify our optoisolation elements bridge system's fixed points for stability analysis (Table C.1).

Case II $V_s(t) = V_B < 0$ for $n\cdot T_A + (n-1)\cdot T_B \leq t \leq (T_A + T_B)\cdot n; n = 1,2,3,\ldots$.
KCL @ A_1: $I_{EQ_3} + I_{EQ_2} + I_{\mathrm{LED}-1} = I_{R_s}$. The assumption is D_1 and D_4 are in ON state, and D_2 and D_3 are in OFF state. We consider D_2 and D_3 as disconnected elements. $I_{\mathrm{LED}-2} \to \varepsilon; I_{\mathrm{LED}-3} \to \varepsilon$. $V_s(t) = V_B < 0 \Rightarrow V_{A_1} < 0$
$V_B < 0; V_A > 0; V_A = |V_B|; V_1 \gg V_A \gg V_2; V_2 \ll |V_B|; V_1 > 0; V_2 > 0; V_1 \neq V_2$

$$I_{\text{LED}-1} = I_{C_m} = I_{R_m} = I_{\text{LED}-4} \ ; \ I_{R_s} = \frac{V_{A_1} - V_s(t)}{R_s}\Big|_{V_s(t)=V_B<0} = \frac{V_{A_1} - V_B}{R_s} \ ;$$

$$|V_{A_1}| = V_{\text{LED}-1} + V_{C_m} + V_{R_m} + V_{\text{LED}-4}$$

$$I_{C_m} = C_m \cdot \frac{\mathrm{d}V_{C_m}}{\mathrm{d}t} \ ;$$

$$V_{R_m} = I_{R_m} \cdot R_m \ ; \ V_{\text{LED}-1} = V_t \cdot \ln\left[\frac{I_{\text{LED}-1}}{I_0} + 1\right] \ ;$$

$$V_{\text{LED}-4} = V_t \cdot \ln\left[\frac{I_{\text{LED}-4}}{I_0} + 1\right]$$

$$V_1 - V_{A_1} = I_{CQ_1} \cdot R_1 + V_{CEQ_1} + V_{CEQ_2} \ ;$$

$$V_1 \gg V_B \ ; \ V_2 \gg V_B \ ; \ V_2 - V_{A_1} = I_{CQ_4} \cdot R_2 + V_{CEQ_4} + V_{CEQ_3}$$

Remark It is reader task to analyze stability of our system for $V_s(t) = V_B < 0$. In our analysis we consider $T_B \to \infty$.

Appendix D
BJT Transistor Ebers–Moll Model and MOSFET Model

A bipolar junction transistor (BJT or bipolar transistor) is a type of transistor that relies on the contact of two types of semiconductor for its operation. BJTs can be used as amplifiers, optoisolation circuits, switches, or in oscillators in many industrial and commercial applications. BJTs can be found either as individual discrete components, or in large numbers as parts of integrated circuits. The operation of bipolar transistor involves both electron and holes. There are two kinds of charge carriers which characteristic of the two kinds of doped semiconductor material. Electrons are majority charge carriers in n-type semiconductors, whereas holes are majority charge carriers in p-type semiconductors. Unipolar transistors such as the field-effect transistors have only one kind of charge carrier. Charge flow in a BJT is due to diffusion of charge carriers across a junction between two regions of different charge concentrations. The regions of a BJT are called emitter, collector, and base. A discrete transistor has three leads for connection to these regions. Typically, the emitter region is heavily doped compared to the other two layers, whereas the majority charge carrier concentrations in base and collector layers are about the same. By design, most of the BJT collector current is due to the flow of charges injected from a high-concentration emitter into the base where there are minority carriers that diffuse toward the collector, and so BJTs are classified as minority-carrier devices. There are two types of BJT transistors, PNP and NPN based on the doping types of the three main terminal regions. An NPN transistor comprises two semiconductor junctions that share a thin p-doped anode region, and a PNP transistor comprises two semiconductor junctions that share a thin n-doped cathode region. In an NPN transistor, when positive bias is applied to the base–emitter junction, the equilibrium is disturbed between the thermally generated carriers and the repelling electric field of the n-doped emitter depletion region. This allows thermally excited electrons to inject from the emitter into the base region. These electrons diffuse through the base from the region of high concentration near the emitter towards the region of low concentration near the collector. The electrons in the base are called minority carriers because the base is doped p-type, which makes holes the majority carrier in the base.

The collector–emitter current can be viewed as being controlled by the base–emitter current (current control), or by the base–emitter voltage (voltage control). These views are related by the current–voltage relation of the base–emitter junction,

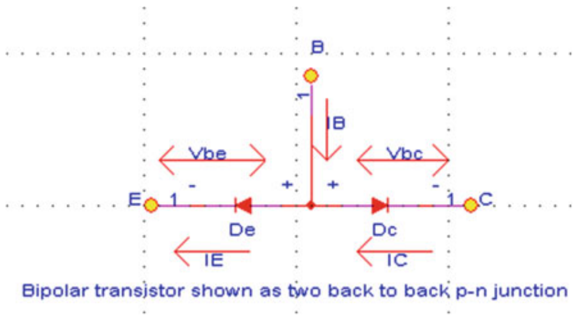
O. Aluf, *Advance Elements of Optoisolation Circuits*,
DOI 10.1007/978-3-319-55316-0

which is just the usual exponential current–voltage curve of a p–n junction (diode). The Bipolar transistor exhibits a few delay characteristics when turning on and off. Most transistors and especially power transistors, exhibit long base-storage times that limit maximum frequency of operation in switching applications. One method for reducing this storage time is by using a Baker clamp. The proportion of electrons able to cross the base and reach the collector is a measure of the BJT efficiency. The heavy doping of the emitter region and light doping of the base region causes many more electrons to be injected from the emitter into the base than holes to be injected from the base into the emitter. The common–emitter current gain is represented by $\beta_F(\beta_f)$ or the h-parameter h_{FE}. It is approximately the ratio of the DC collector current to the DC base current in forward-active region. It is typically greater than 50 for small signal transistors but can be smaller in transistors designed for high-power applications. Another important parameter is the common-base current gain $\alpha_F(\alpha_f)$. The common-base current gain is approximately the gain of current from emitter to collector in the forward-active region. This ratio usually has a value close to unity; between 0.98 and 0.998. It is less than unity due to recombination of charge carriers as they cross the base region.

$$\alpha_F = \frac{I_C}{I_B}; \beta_F = \frac{I_C}{I_B}; \beta = \frac{I_C}{I_E - I_C} = \frac{I_C/I_E}{1 - I_C/I_E} = \frac{\alpha_F}{1 - \alpha_F}; \alpha_F = \frac{\beta_F}{\alpha_F + 1}$$

Transistors can be thought of as two diodes (P–N junctions) sharing a common region that minority carriers can move through. A PNP BJT will function like two diodes that share an N-type cathode region, and the NPN like two diodes sharing a P-type anode region. Connecting two diodes with wires will not make a transistor, since minority carriers will not be able to get from one P–N junction to the other through the wire. Both types of BJT function by letting a small current input to the base control an amplified output from the collector. The result is that the transistor makes a good switch that is controlled by its base input. The BJT also makes a good amplifier, since it can multiply a weak input signal to about 100 times its original strength. Networks of transistors are used to make powerful amplifiers with many different applications. In the discussion below, focus is on the NPN bipolar transistor. In the NPN transistor in what is called active mode, the base–emitter voltage V_{BE} and collector–base voltage V_{CB} are positive, forward biasing the emitter–base junction and reverse-biasing the collector–base junction. In the active mode of operation, electrons are injected from the forward biased n-type emitter region into the p-type base where they diffuse as minority carriers to the reverse-biased n-type collector and are swept away by the electric field in the reverse-biased collector–base junction. For a figure describing forward and reverse bias, see semiconductor diodes. The bipolar junction transistor can be considered essentially as two p–n junctions placed back to back, with the base p-type region being common to both diodes. This can be viewed as two diodes having a common third terminal as shown in Fig. D.1. The two diodes are not in isolation but are interdependent. This means that the total current flowing in each diode is influenced by the conditions

Fig. D.1 Bipolar transistor shown as two back to back p–n junction



prevailing in the other. In isolation, the two junctions would be characterized by the normal diode equation with a suitable notation used to differentiate between the two junctions as can be seen. When the two junctions are combined, to form a transistor, the base region is shared internally by both diodes even though there is an external connection to it.

In the forward active mode, α_F of the emitter current reaches the collector. This means that α_F of the diode current passing through the base–emitter junction contributes to the current flowing through the base–collector junction. Typically, α_F has a value of between 0.98 and 0.99. This is shown as the forward component of current as it applies to the normal forward active mode of operation of the device. This current is shown as a conventional current. It is equally possible to reverse the biases on the junctions to operate the transistor in the "reverse active mode". In this case, $\alpha_R(\alpha_r)$ times the collector current will contribute to the emitter current. For the doping ratios normally used the transistor will be much less efficient in the reverse mode and α_R would typically be in the range 0.1–0.5. The Ebers–Moll transistor model is an attempt to create an electrical model of the device as two diodes whose currents are determined by the normal diode law but with additional transfer ratios to quantify the interdependency of the junctions. Two dependent current sources are used to indicate the interaction of the junctions. Figure D.2 describes NPN Bipolar transistor Ebers–Moll model.

Applying Kirchoff's laws to the model gives the terminal current as:

$$I_{DE} = I_E + \alpha_r \cdot I_{DC}; I_C + I_{DC} = \alpha_f \cdot I_{DE}; \alpha_f \cdot I_{se} = \alpha_r \cdot I_{sc} = I_s; I_E = I_C + I_B$$

$\alpha_f = 0.98 - 0.99$ typically. $\alpha_r = 0.1 - 0.5$ typically. I_{se}: reverse saturation current of the base emitter diode. I_{sc}: reverse saturation current of the base collector diode. $I_{DC} = I_{sc} \cdot (e^{\frac{V_{BC}}{V_T}} - 1); I_{DE} = I_{se} \cdot (e^{\frac{V_{BE}}{V_T}} - 1)$. V_T—the thermal voltage $V_T \simeq \frac{k \cdot T}{q}$ (approximately 26 mV at 300 K ($\sim$room temperature). I_E is the transistor's emitter current. I_C is the transistor's collector current. I_B is the transistor's base current. The base internal current is mainly by diffusion (see Fick's law) and $J_{n(base)} = \frac{q \cdot D_n \cdot n_{bo}}{W} \cdot e^{\frac{V_{BE}}{V_T}}$. W is the base width. D_n is the diffusion constant for electron in the p type base.

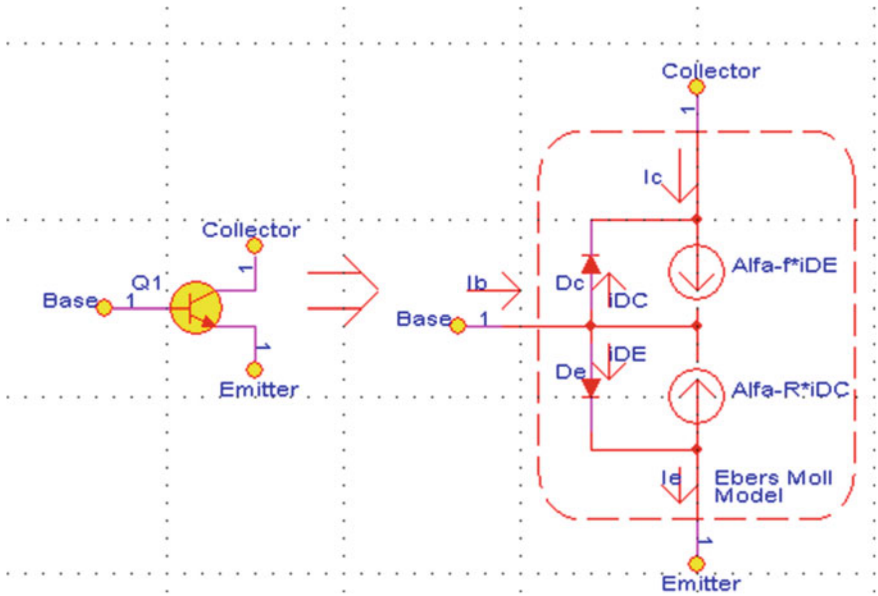


Fig. D.2 NPN Bipolar transistor Ebers–Moll model

$$I_{DE} = I_E + \alpha_r \cdot I_{DC} \Rightarrow I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) = I_E + \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right) \Rightarrow I_E = I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right)$$

$I_C + I_{DC} = \alpha_f \cdot I_{DE} \Rightarrow I_C + I_{sc} \cdot (e^{\frac{V_{BC}}{V_T}} - 1) = \alpha_f \cdot I_{se} \cdot (e^{\frac{V_{BE}}{V_T}} - 1) \Rightarrow I_C = \alpha_f \cdot I_{se} \cdot (e^{\frac{V_{BE}}{V_T}} - 1) - I_{sc} \cdot (e^{\frac{V_{BC}}{V_T}} - 1)$ $I_B = I_E - I_C = (1 - \alpha_f)\cdot$ $I_{se} \cdot (e^{\frac{V_{BE}}{V_T}} - 1) + (1 - \alpha_r) \cdot I_{sc} \cdot (e^{\frac{V_{BC}}{V_T}} - 1)$ If we use the notation $\alpha_f \cdot I_{se} = \alpha_r \cdot I_{sc} = I_s; I_{sc} = \frac{I_s}{\alpha_r}; I_{se} = \frac{I_s}{\alpha_f}$ the following Ebers–Moll equations:

$$I_E = \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_E = \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_s \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_C = \alpha_f \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_C = I_s \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_B = (1 - \alpha_f) \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right); \frac{1}{\beta_f} = \frac{1 - \alpha_f}{\alpha_f}; \frac{1}{\beta_r} = \frac{1 - \alpha_r}{\alpha_r}$$

$$I_B = (1 - \alpha_f) \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_B = \frac{I_s}{\beta_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + \frac{I_s}{\beta_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

The expressions for V_{BE}, V_{BC}, and V_{CE} are as follow:

$$I_E = I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow \left(e^{\frac{V_{EB}}{V_T}} - 1\right) = \frac{I_E + \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)}{I_{se}}$$

$$I_C = \alpha_f \cdot I_{se} \cdot \left(\frac{I_E + \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)}{I_{se}}\right) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$\Rightarrow I_C = \alpha_f \cdot I_E + \alpha_f \cdot \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_C = \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow e^{\frac{V_{CB}}{V_T}} = \frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1$$

$$e^{\frac{V_{CB}}{V_T}} = \frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1 \Rightarrow V_{CB} = V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1\right]$$

$$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow \left(e^{\frac{V_{CB}}{V_T}} - 1\right) = \frac{\alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_C}{I_{sc}}$$

$$e^{\frac{V_{BC}}{V_T}} = \frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1 \Rightarrow V_{BC} = V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1\right]$$

$$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right) \Rightarrow \left(e^{\frac{V_{BC}}{V_T}} - 1\right) = \frac{\alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - I_C}{I_{sc}}$$

$$I_E = I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(\frac{\alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - I_C}{I_{sc}}\right)$$

$$\Rightarrow I_E = I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot \alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) + \alpha_r \cdot I_C$$

$$I_E = (1 - \alpha_r \cdot \alpha_f) \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) + \alpha_r \cdot I_C \Rightarrow e^{\frac{V_{BE}}{V_T}} = \frac{I_E - \alpha_r \cdot I_C}{(1 - \alpha_r \cdot \alpha_f) \cdot I_{se}} + 1$$

$$e^{\frac{V_{BE}}{V_T}} = \frac{I_E - \alpha_r \cdot I_C}{(1 - \alpha_r \cdot \alpha_f) \cdot I_{se}} + 1 \Rightarrow V_{BE} = V_T \cdot \ln\left[\frac{I_E - \alpha_r \cdot I_C}{(1 - \alpha_r \cdot \alpha_f) \cdot I_{se}} + 1\right]; V_{BE} = V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}} + 1\right]$$

We can summarize our intermediate results:

$$V_{BC} = V_T \cdot \ln\left[\left(\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}\right) + 1\right]; V_{BE} = V_T \cdot \ln\left[\left(\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right) + 1\right]$$

$$V_{CE} = V_{CB} + V_{BE}, \text{ but } V_{CB} = -V_{BC}. \text{ Then } V_{CE} = V_{BE} - V_{BC}.$$

Remark there is a use with capital and small letters in the Appendix as compared to Chapter 1, consider the terminology is the same.

$$I_e = I_E; I_c = I_C; I_b = I_B; V_t = V_T;\ V_{be} = V_{BE}; V_{cb} = V_{CB}; V_{ce} = V_{CE}$$

$$V_{CB} = -V_{BC} = -V_T \cdot \ln\left[\left(\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}\right) + 1\right]; V_{CE} = V_{BE} - V_{BC}$$

$$V_{CE} = V_{BE} - V_{BC} = V_T \cdot \ln\left[\left(\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right) + 1\right] - V_T \cdot \ln\left[\left(\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}\right) + 1\right]$$

$$V_{CE} = V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right] - V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}\right]$$

$$V_{CE} = V_T \cdot \ln\left\{\frac{[\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}]}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}} \cdot \frac{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{[I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}]}\right\}$$

$$V_{CE} = V_T \cdot \ln\left\{\frac{[\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}]}{[I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}]} \cdot \frac{I_{sc}}{I_{se}}\right\}$$

$$V_{CE} = V_T \cdot \ln\left\{\frac{[\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}]}{[I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}]}\right\} + V_T \cdot \ln\left(\frac{I_{sc}}{I_{se}}\right); \frac{I_{sc}}{I_{se}} \approx 1 \Rightarrow \ln\left(\frac{I_{sc}}{I_{se}}\right) \to \varepsilon$$

$$V_{CE} \approx V_T \cdot \ln\left\{\frac{[\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}]}{[I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}]}\right\}$$

Figure D.3 describes PNP Bipolar transistor Ebers–Moll model.

$$I_{DE} = I_E + \alpha_r \cdot I_{DC} \Rightarrow I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) = I_E + \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_E = I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$I_C + I_{DC} = \alpha_f \cdot I_{DE} \Rightarrow I_C + I_{sc} \cdot (e^{\frac{V_{CB}}{V_T}} - 1) = \alpha_f \cdot I_{se} \cdot (e^{\frac{V_{EB}}{V_T}} - 1) \Rightarrow I_C = \alpha_f \cdot I_{se} \cdot (e^{\frac{V_{EB}}{V_T}} - 1) - I_{sc} \cdot (e^{\frac{V_{CB}}{V_T}} - 1)$ $I_B = I_E - I_C = (1 - \alpha_f) \cdot I_{se} \cdot (e^{\frac{V_{EB}}{V_T}} - 1) + (1 - \alpha_r) \cdot I_{sc} \cdot (e^{\frac{V_{CB}}{V_T}} - 1)$. If we use the notation $\alpha_f \cdot I_{se} = \alpha_r \cdot I_{sc} = I_s; I_{sc} = \frac{I_s}{\alpha_r}; I_{se} = \frac{I_s}{\alpha_f}$ the following Ebers–Moll equations:

$$I_E = \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_E = \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_s \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_C = \alpha_f \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_C = I_s \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_B = (1 - \alpha_f) \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right); \frac{1}{\beta_f} = \frac{1 - \alpha_f}{\alpha_f}; \frac{1}{\beta_r} = \frac{1 - \alpha_r}{\alpha_r}$$

$$I_B = (1 - \alpha_f) \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow I_B = \frac{I_s}{\beta_f} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + \frac{I_s}{\beta_r} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

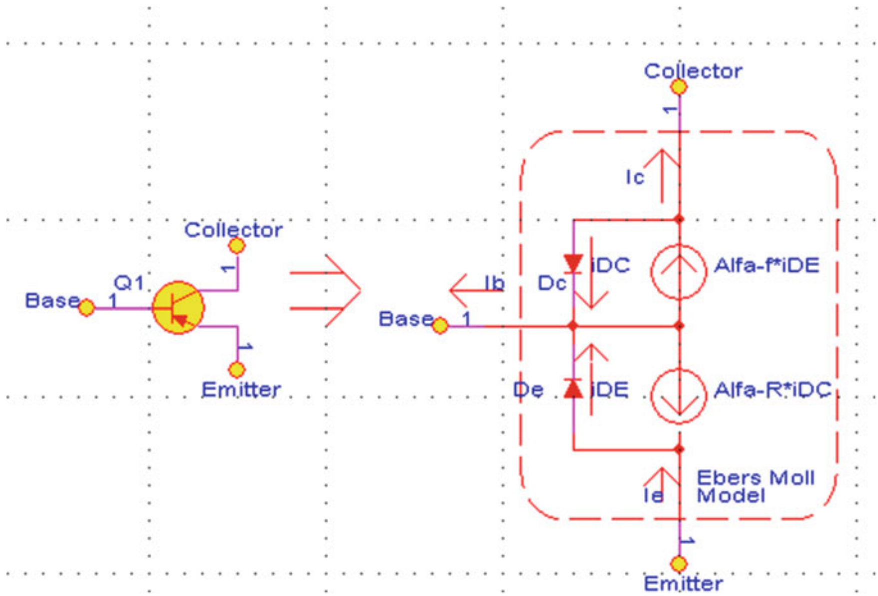


Fig. D.3 PNP Bipolar transistor Ebers–Moll model

The expressions for V_{EB}, V_{CB}, and V_{EC} are as follow:

$$I_E = I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow \left(e^{\frac{V_{EB}}{V_T}} - 1\right) = \frac{I_E + \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)}{I_{se}}$$

$$I_C = \alpha_f \cdot I_{se} \cdot (\frac{I_E + \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)}{I_{se}}) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$\Rightarrow I_C = \alpha_f \cdot I_E + \alpha_f \cdot \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_C = \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) \Rightarrow e^{\frac{V_{CB}}{V_T}} = \frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1$$

$$e^{\frac{V_{CB}}{V_T}} = \frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1 \Rightarrow V_{CB} = V_T \cdot \ln[\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1]$$

$$I_E = I_{se} \cdot (e^{\frac{V_{EB}}{V_T}} - 1) - \alpha_r \cdot I_{sc} \cdot \left(\frac{\alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_C}{I_{sc}}\right)$$

$$\Rightarrow I_E = I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot \alpha_f \cdot I_{se} \cdot (e^{\frac{V_{EB}}{V_T}} - 1) + \alpha_r \cdot I_C$$

$$I_E = (1 - \alpha_r \cdot \alpha_f) \cdot I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + \alpha_r \cdot I_C \Rightarrow e^{\frac{V_{EB}}{V_T}} = \frac{I_E - \alpha_r \cdot I_C}{(1 - \alpha_r \cdot \alpha_f) \cdot I_{se}} + 1$$

$$e^{\frac{V_{EB}}{V_T}} = \frac{I_E - \alpha_r \cdot I_C}{(1 - \alpha_r \cdot \alpha_f) \cdot I_{se}} + 1 \Rightarrow V_{EB} = V_T \cdot \ln\left[\frac{I_E - \alpha_r \cdot I_C}{(1 - \alpha_r \cdot \alpha_f) \cdot I_{se}} + 1\right]; V_{EB} = V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}} + 1\right]$$

We can summarize our result regarding I_C and I_E:

$$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right); I_E = I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)$$

$$I_B = I_E - I_C = I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right) - \left[\alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right)\right]$$

$$I_B = I_E - I_C = (1 - \alpha_f) I_{se} \cdot \left(e^{\frac{V_{EB}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot I_{sc} \cdot \left(e^{\frac{V_{CB}}{V_T}} - 1\right); V_{BE} = -V_{EB}; V_{BC} = -V_{CB}$$

$$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{-V_{BE}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{-V_{BC}}{V_T}} - 1\right); I_E = I_{se} \cdot \left(e^{\frac{-V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{-V_{BC}}{V_T}} - 1\right)$$

$$I_B = I_E - I_C = (1 - \alpha_f) I_{se} \cdot \left(e^{\frac{-V_{BE}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot I_{sc} \cdot \left(e^{\frac{-V_{BC}}{V_T}} - 1\right)$$

$$V_{CE} = V_{CB} + V_{BE}, \text{ but } V_{CB} = -V_{BC}. \text{ Then } V_{CE} = V_{BE} - V_{BC}.$$

$$V_{CB} = V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1\right]; V_{EB} = V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}} + 1\right]$$

$$V_{BC} = -V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1\right]; V_{BE} = -V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}} + 1\right]$$

$$V_{CE} = V_{CB} + V_{BE} = V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}} + 1\right] - V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}} + 1\right]$$

$$V_{CE} = V_{CB} + V_{BE} = V_T \cdot \ln\left[\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{(\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}\right] - V_T \cdot \ln\left[\frac{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}{(\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right]$$

$$V_{CE} = V_{CB} + V_{BE} = V_T \cdot \ln\left[\left\{\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right\} \cdot \left\{\frac{I_{se}}{I_{sc}}\right\}\right]$$

$$V_{CE} = V_{CB} + V_{BE} = V_T \cdot \ln\left\{\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right\} + V_T \cdot \ln\left\{\frac{I_{se}}{I_{sc}}\right\}; I_{se} \approx I_{sc}; \ln\left\{\frac{I_{se}}{I_{sc}}\right\} \to \varepsilon$$

Table. D.1 Summary of our BJT NPN and PNP transistors Ebers–Moll equations

	BJT NPN transistor	BJT PNP transistor
I_C	$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right)$	$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{-V_{BE}}{V_T}} - 1\right) - I_{sc} \cdot (e^{\frac{-V_{BC}}{V_T}} - 1)$
I_E	$I_E = I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right)$	$I_E = I_{se} \cdot \left(e^{\frac{-V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{-V_{BC}}{V_T}} - 1\right)$
I_B	$I_B = (1 - \alpha_f) \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right)$	$I_B = (1 - \alpha_f) I_{se} \cdot \left(e^{\frac{-V_{BE}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot I_{sc} \cdot \left(e^{\frac{-V_{BC}}{V_T}} - 1\right)$
V_{CE}	$V_{CE} \approx V_T \cdot \ln\left\{\frac{[\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}]}{[I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}]}\right\}$	$V_{CE} \approx V_T \cdot \ln\left\{\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right\}$

$$V_{CE} = V_{CB} + V_{BE} \approx V_T \cdot \ln\left\{\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right\}$$

$$V_{CE-NPN} \approx V_T \cdot \ln\left\{\frac{[\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}]}{[I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}]}\right\}; V_{CE-PNP} \approx V_T \cdot \ln\left\{\frac{I_C - \alpha_f \cdot I_E + (\alpha_f \cdot \alpha_r - 1) \cdot I_{sc}}{\alpha_r \cdot I_C - I_E + (\alpha_r \cdot \alpha_f - 1) \cdot I_{se}}\right\}$$

Summary of our BJT NPN and PNP transistors Ebers–Moll equations (Table D.1):

There are three basic circuit configurations to connect bipolar junction transistor. *First* Common Base (CB) , both the input and output share the base "in common". *Second* Common Emitter (CE) , both the input and output share the emitter "in common". *Third* Common Collector (CC) , both the input and output share the collector "in common". There are four bipolar junction transistor biasing modes. Active biasing is useful for amplifiers (most common mode). Saturation biasing mode is equivalent to an on state when transistor is used as a switch. Cutoff biasing mode is equivalent to an off state when transistor is used as a switch. Inverted biasing mode is rarely if ever used (Table D.2).

The BJT transistor base current is much smaller that the emitter and collector currents in forward active mode. If the collector of an NPN transistor was open circuit, it would look like a diode. When forward biased, the circuit in the base–emitter junction would consist of holes injected into the emitter from the base and electrons injected into the base from the emitter. But since there are many more electrons in the emitter than holes in the base, the vast majority of the current will be due to electrons. When the reverse-biased collector is added, It "sucks" the

Table D.2 Summary of BJT transistor biasing modes

Biasing mode (NPN)	E–B junction bias (NPN)	C–B junction bias (NPN)	Applied voltages (NPN)
Saturation	Forward	Forward	$V_E < V_B > V_C$
Active (forward active)	Forward	Reverse	$V_E < V_B < V_C$
Inverted (reverse active)	Reverse	Forward	$V_E > V_B > V_C$
Cutoff	Reverse	Reverse	$V_E > V_B < V_C$
Biasing mode (PNP)	E–B junction bias (PNP)	C–B junction bias (PNP)	Applied voltages (PNP)
Saturation	Forward	Forward	$V_E > V_B < V_C$
Active (forward active)	Forward	Reverse	$V_E > V_B > V_C$
Inverted (reverse active)	Reverse	Forward	$V_E < V_B < V_C$
Cutoff	Reverse	Reverse	$V_E < V_B > V_C$

electrons out of the base. Thus, the base-emitter current is due predominantly to hole current (the smaller current component) while the collector-emitter current is due to electrons (larger current component due to more electrons from the n+ emitter doping). We define two BJT transistor performance parameters: emitter efficiency (γ) and base transport factor (α_T). Emitter efficiency parameter characterizes how effective the large hole current is controlled by the small electron current. Unity is best, zero is worst. Base transport factor characterizes how much of the injected hole current is lost to recombination in the base. Unity is best, zero is worst. $\gamma = \frac{I_{Ep}}{I_E} = \frac{I_{Ep}}{I_{Ep} + I_{En}}\ \alpha_T = \frac{I_{Cp}}{I_{Ep}}$. We define some equations in active mode, common base characteristics. I_{CBo} is defined as the collector current when the emitter is open circuit. It is the collector–base junction saturation current. I_C is the fraction of emitter current making it across the base + leakage current.

$I_C = \alpha_{dc} \cdot I_E + I_{CBo}$, where α_{dc} is the common base DC current gain.

$$I_{Cp} = \alpha_T \cdot I_{Ep} = \gamma \cdot\ \alpha_T \cdot I_E; I_C = I_{Cp} + I_{Cn} = \alpha_T \cdot I_{Ep} + I_{Cn} = \gamma \cdot\ \alpha_T \cdot I_E + I_{Cn}; \alpha_{dc} = \gamma \cdot\ \alpha_T$$

and $I_{CBo} = I_{Cn}$. We define some equations in active mode, common emitter characteristics. I_{CEo} is defined as the collector current when the base is open circuit. I_C is multiple of the base current making it across the base + leakage current. $I_C = \beta_{dc} \cdot I_B + I_{CEo}$; Where β_{dc} is the common emitter DC current gain. I_{CEo} is defines as the collector current when the base is open circuit.

$\alpha_F = \alpha_{dc}$ is common base current gain. $I_E = \alpha_R \cdot I_C; \alpha_R \neq \alpha_{DC}$. In inverse mode, the emitter current is the fraction of the collector current "collected".

$$I_E = I_C + I_B; I_C = \alpha_{dc} \cdot (I_C + I_B) + I_{CBo}; I_C = \frac{\alpha_{dc}}{1 - \alpha_{dc}} \cdot I_B + \frac{I_{CBo}}{1 - \alpha_{dc}}$$

$$\beta_{dc} = \frac{\alpha_{dc}}{1 - \alpha_{dc}}; I_{CEo} = \frac{I_{CBo}}{1 - \alpha_{dc}}; \beta_{dc} = \frac{I_C}{I_B}$$

We can break the BJT transistor up into a large signal analysis and a small signal analysis and "linearize" the nonlinear behavior of the Ebers–Moll model. Small signal models are only useful for forward active mode and thus, are derived under this condition. Saturation and cutoff are used for switches which involve very large voltage/current swings from on to off states.

Small signal models are used to determine amplifier characteristics ("Gain" = increase in the magnitude of a signal at the output of a circuit relative to its magnitude at the input of the circuit). Just like when a diode voltage exceeds a certain value, the nonlinear behavior of the diode leads to distortion of the current/voltage curves, if the inputs/outputs exceed certain limits, the full Ebers–Moll model must be used. There are physical meanings of β_f (β_F) and β_r (β_R). β_F is the current gain (I_C/I_B) of the device when it is operating with the emitter as the emitter and the collector as the collector in the active mode. β_R is the current gain of the device when it is operating with the emitter as a collector and the collector as an emitter in the reverse mode. The BJT device is made to have higher forward current gain than reverse current gain. The terminals for emitter and collector are not completely interchangeable due to different doping of the collector and emitter.

BJT Transistor Modes of Operation

The Ebers–Moll BJT model is a good large signal, steady-state model of the transistor and allows the state of conduction of the device to be easily determined for different modes of operation of the device. The different modes of operation are determined by the manner in which the junctions are biased. BJT NPN transistor Ebers–Moll BJT model:

$$I_C = \alpha_f \cdot I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right); I_E = I_{se} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) - \alpha_r \cdot I_{sc} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right)$$

$$I_B = (1 - \alpha_f) \cdot \frac{I_s}{\alpha_f} \cdot \left(e^{\frac{V_{BE}}{V_T}} - 1\right) + (1 - \alpha_r) \cdot \frac{I_s}{\alpha_r} \cdot \left(e^{\frac{V_{BC}}{V_T}} - 1\right); \alpha_f \cdot I_{se} = \alpha_r \cdot I_{sc} = I_s;$$

$$I_{sc} = \frac{I_s}{\alpha_r}; I_{se} = \frac{I_s}{\alpha_f}$$

$$I_B = (1 - \alpha_f) \cdot I_{se} \cdot (e^{\frac{V_{BE}}{V_T}} - 1) + (1 - \alpha_r) \cdot I_{sc} \cdot (e^{\frac{V_{BC}}{V_T}} - 1)$$

(A) Forward-Active mode:

B–E forward biased, V_{BE} positive $e^{\frac{V_{BE}}{V_T}} \gg 1; (e^{\frac{V_{BE}}{V_T}} - 1) \approx e^{\frac{V_{BE}}{V_T}}$. B–C reverse biased, V_{BC} negative $e^{\frac{V_{BC}}{V_T}} \ll 1; (e^{\frac{V_{BC}}{V_T}} - 1) \approx -1$. Then from the Ebers–Moll model equations we get the following results:

$I_E \simeq I_{se} \cdot e^{\frac{V_{BE}}{V_T}} + \alpha_r \cdot I_{sc} \approx I_{se} \cdot e^{\frac{V_{BE}}{V_T}}; I_{se} \cdot e^{\frac{V_{BE}}{V_T}} \gg \alpha_r \cdot I_{sc}$; Relatively large.

$I_C \simeq \alpha_f \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} + I_{sc} \approx \alpha_f \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} = \alpha_f \cdot I_E; \alpha_f \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} \gg I_{sc}$; Relatively large.

$$I_B \simeq (1 - \alpha_f) \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} - (1 - \alpha_r) \cdot I_{sc} \approx (1 - \alpha_f) \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}}$$
$$= (1 - \alpha_f) \cdot I_E; (1 - \alpha_f) \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} \gg (1 - \alpha_r) \cdot I_{sc}$$

(B) Reverse active mode:

B–E reverse biased, V_{BE} negative $e^{\frac{V_{BE}}{V_T}} \ll 1; (e^{\frac{V_{BE}}{V_T}} - 1) \approx -1$. B–C forward biased, V_{BC} positive $e^{\frac{V_{BC}}{V_T}} \gg 1; (e^{\frac{V_{BC}}{V_T}} - 1) \approx e^{\frac{V_{BC}}{V_T}}$. The transistor conducts in the opposite direction. Then from the Eber–Moll model equations we get the following results:

$I_E \simeq -I_{se} - \alpha_r \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}} \approx -\alpha_r \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}}; \alpha_r \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}} \gg I_{se}$; Moderately high.

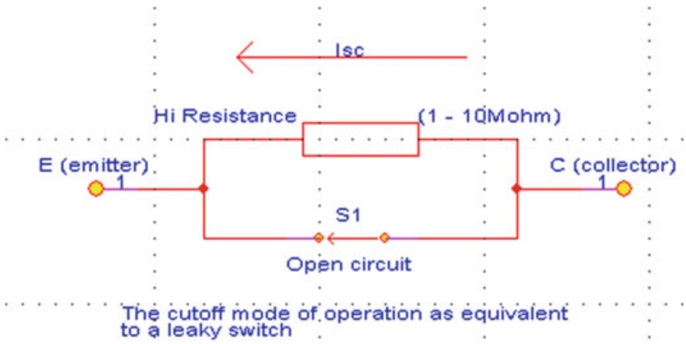


Fig. D.4 The cutoff mode of operation as equivalent to a leaky switch

$$I_C \simeq -\alpha_f \cdot I_{se} - I_{sc} \cdot e^{\frac{V_{BC}}{V_{T^n}}} \approx -I_{sc} \cdot e^{\frac{V_{BC}}{V_T}}; I_{sc} \cdot e^{\frac{V_{BC}}{V_T}} \gg \alpha_f \cdot I_{se} \text{ Moderate.}$$

$$I_B \simeq -(1-\alpha_f) \cdot I_{se} + (1-\alpha_r) \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}} \approx (1-\alpha_r) \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}}; (1-\alpha_r) \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}} \gg (1-\alpha_f) \cdot I_{se}$$

It is as high as $0.5 \cdot |I_C|$. This mode does not provide useful amplification but is used, mainly, for current steering in switching circuits, e.g., TTL.

(C) Cut-off mode:

B–E is unbiased, $V_{BE} = 0$ V. *B–C* is reverse biased, V_{BC} negative. $e^{\frac{V_{BE}}{V_T}} = 1; (e^{\frac{V_{BE}}{V_T}} - 1) \to \varepsilon = 0; e^{\frac{V_{BC}}{V_T}} \ll 1; (e^{\frac{V_{BC}}{V_T}} - 1) \approx -1$

$I_E \simeq \alpha_r \cdot I_{sc}$; Leakage current nA. $I_C \simeq I_{sc}$; Leakage current nA. $I_B \simeq -(1-\alpha_r) \cdot I_{sc}$.

This is equivalent to a very low conductance between collector and emitter, i.e. open switch (Fig. D.4).

(D) Saturation mode:

B–E is forward biased, V_{BE} is positive $e^{\frac{V_{BE}}{V_T}} \gg 1; (e^{\frac{V_{BE}}{V_T}} - 1) \approx e^{\frac{V_{BE}}{V_T}}$ and both junctions are forward biased. B–C is forward biased, V_{BC} $e^{\frac{V_{BC}}{V_T}} \gg 1; (e^{\frac{V_{BC}}{V_T}} - 1) \approx e^{\frac{V_{BC}}{V_T}}$. We get the following currents expressions:

$$I_C \approx \alpha_f \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} - I_{sc} \cdot e^{\frac{V_{BC}}{V_T}}; I_E \approx I_{se} \cdot e^{\frac{V_{BE}}{V_T}} - \alpha_r \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}}$$

$$I_B \approx (1-\alpha_f) \cdot I_{se} \cdot e^{\frac{V_{BE}}{V_T}} + (1-\alpha_r) \cdot I_{sc} \cdot e^{\frac{V_{BC}}{V_T}}$$

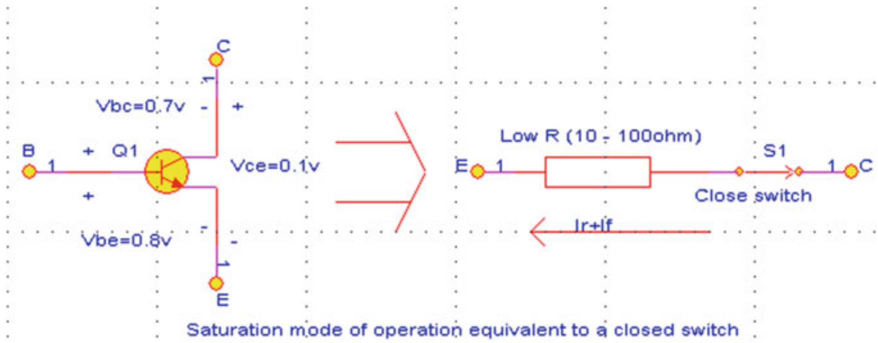


Fig. D.5 Saturation mode of operation equivalent to a closed switch

In this case, with both junctions forward biased.

$$V_{BE} \approx 0.8\,\text{V};\ V_{BC} \approx 0.7\,\text{V};\ V_{CE} = V_{CB} + V_{BE}; V_{CB} = -V_{BC}; V_{CE} = V_{BE} - V_{BC} = 0.1\,\text{V}$$

There is a 0.1 V drop across the transistor from collector to emitter which is quite low while a substantial current flows through the device. In this mode it can be considered as having a very high conductivity and acts as a closed switch with a finite resistance and conductivity (Fig. D.5).

BJT Transistor Avalanche Breakdown Region of Operation

An avalanche transistor is a bipolar junction transistor designed for operation in the region of its collector-current/collector-to-emitter voltage characteristics beyond the collector-to-emitter breakdown voltage, called avalanche breakdown region. This region characterized by avalanche breakdown, a phenomenon similar to Negative Differential Resistance (NDR) . Operation in the avalanche breakdown region is called avalanche-mode operation. It gives avalanche transistors the ability to switch very high currents with less than nanosecond rise and fall times (transition times). Transistors not specifically designed for the purpose can have reasonably consistent avalanche properties.

Static Avalanche Region Characteristics

The static characteristic of an navalanche transistor is $I_C - V_{CE}$. The static characteristic of an avalanche NPN transistor is the same as PNP devices only changing sign to voltages and currents accordingly. The avalanche breakdown multiplication is present only across the collector–base junction. The first step of the calculation is to determine collector current as a sum of various component currents through the collector since only those fluxes of charge are subject to this phenomenon. Applying Kirchhoff's current law (KCL) to a bipolar junction transistor, implies the following relation which satisfied by the collector current I_C ($I_C = I_E - I_B$) while for the same device working in the active region. $\alpha = \alpha_f; \beta = \beta_f I_C = \beta \cdot I_B + (\beta + 1) \cdot I_{CBo}$, I_B is the base current. I_{CBo} is the collector–base reverse leakage current. I_E is the emitter current. B is the common emitter current gain of the transistor. Equating the two formulas for I_C gives the following result

$I_E = (\beta + 1) \cdot I_B + (\beta + 1) \cdot I_{CBo}$ and since $\alpha = \frac{\beta}{\beta+1}$; α is the common base current gain of the transistor, then $\alpha \cdot I_E = \beta \cdot I_B + \beta \cdot I_{CBo} = I_C - I_{CBo} \Rightarrow I_C = \alpha \cdot I_E + I_{CBo}$.

When the avalanche effects in a transistor collector are considered, the collector current I_C is given by $I_C = M \cdot (\alpha \cdot I_E + I_{CBo})$. M is miller's avalanche multiplication coefficient. It is the most important parameter in avalanche mode operation $M = \frac{1}{1-(\frac{V_{CB}}{BV_{CBo}})^n}$. BV_{CBo} is the collector–base breakdown voltage. n is a constant depending on the semiconductor used for the construction of the transistor and doping profile of the collector–base junction. V_{CB} is the collector–base voltage. Using Kirchhoff's current law (KCL) for the bipolar junction transistor and the expression for M, the resulting expression for I_C is the following: $I_C = \frac{M}{(1-\alpha \cdot M)} \cdot (\alpha \cdot I_B + I_{CBo}) \Rightarrow I_C = \frac{\alpha \cdot I_B + I_{CBo}}{1-\alpha-(\frac{V_{CB}}{BV_{CBo}})^n}$

$V_{CB} = V_{CE} - V_{BE}; V_{BE} = V_{BE}(I_B)$ where V_{BE} is the base-emitter voltage.

$I_C = \frac{\alpha \cdot I_B + I_{CBo}}{1-\alpha-(\frac{V_{CE}-V_{BE}(I_B)}{BV_{CBo}})^n} \simeq \frac{\alpha \cdot I_B + I_{CBo}}{1-\alpha-(\frac{V_{CE}}{BV_{CBo}})^n}$ Since $V_{CE} \gg V_{BE}$. This is the expression of the parametric family of the collector characteristics $I_C - V_{CE}$ with parameter I_B (I_C increases without limit if $(1-\alpha) = (\frac{V_{CE}}{BV_{CBo}})^n \Rightarrow V_{CE} = BV_{CEo} = BV_{CBo} \cdot \sqrt[n]{(1-\alpha)} = \frac{BV_{CBo}}{\sqrt[n]{\beta+1}}$; $1 - \alpha = 1 - (\frac{\beta}{\beta+1}) = \frac{1}{\beta+1} \beta \gg 1 \Rightarrow V_{CE} = BV_{CEo} = BV_{CBo} \cdot \sqrt[n]{(1-\alpha)}|_{\beta \gg 1} = \frac{BV_{CBo}}{\sqrt[n]{\beta}}$. Where, BV_{CEo} is the collector–emitter breakdown voltage.

Avalanche Multiplication

The maximum reverse-biasing voltage which may be applied before breakdown between the collector and base terminals of the transistor, under the condition that the emitter lead be open circuited, is represented by the symbol BV_{CBo}. This breakdown voltage is a characteristic of the transistor alone. The breakdown may occur because of avalanche multiplication of the current I_{CO} that crosses the collector junction. As a result of this multiplication, the current becomes $M \cdot I_{CO}$, in which M is the factor by which the original I_{CO} is multiplies by the avalanche effect. It is possible to neglect leakage current, which does not flow through the junction and is therefore not subject to avalanche multiplication. At a high enough BV_{CBo}, the multiplication factor M becomes nominally infinite and the region of breakdown is then attained. The current rises abruptly, and large changes in current accompany small changes in applied voltage. The avalanche multiplication factor depends on the voltage V_{CB} between transistor's collector and base. If a current I_E is caused to flow across the emitter junction, then, neglecting the avalanche effect, a fraction$\alpha \cdot I_E$, where α is the common–base current gain, reaches the collector junction. If we take multiplication into account, I_C has the magnitude$M \cdot \alpha \cdot I_E$. In presence of avalanche multiplication, the transistor behaves as though its common base current gain where $M \cdot \alpha$. The maximum allowable collector to emitter voltage depends not only upon the transistor, but also upon the circuit in which it is used.

BJT Transistor Second Breakdown Avalanche Mode

When the collector current rises above the data sheet limit I_{Cmax}, a new breakdown mechanism happened, the second breakdown. This phenomenon is caused by excessive heating of some points (hot spots) in the base-emitter region of the bipolar junction transistor, which give rise to an exponentially increasing current through these points. This exponential rise of current in turn gives rise to even more overheating, originating a positive thermal feedback mechanism. While analyzing the $I_C - V_{CE}$ static characteristic, the presence of this phenomenon is seen as a sharp collector voltage drop and a corresponding almost vertical rise of the collector current. While this phenomenon is destructive for bipolar junction transistors working in the usual way, it can be used to push up further the current and voltage limits of a device working in avalanche mode by limiting its time duration . The switching speed of the device is not negatively affected.

Small signal model of the BJT, base charging capacitance (diffusion capacitance). In active mode when the emitter–base is forward biased, the capacitance of the emitter–base junction is dominated by the diffusion capacitance (not depletion capacitance). Recall for a diode we define the following: $C_{\text{Diffusion}} = \frac{dQ_D}{dv'_D} = \frac{dQ_D}{dt} \cdot \frac{dt}{dv'_D}$. The sum up all minority carrier charges on either side of the junction.

$$Q_D = q \cdot A \cdot \int_0^{\infty} p_{no} \cdot \left(e^{\frac{v'_D}{V_T}} - 1 \right) \cdot e^{-\frac{X}{L_P}} \cdot dX + q \cdot A \cdot \int_0^{\infty} n_{po} \cdot \left(e^{\frac{v'_D}{V_T}} - 1 \right) \cdot e^{-\frac{X}{L_n}} \cdot dX$$

If we neglect charge injected from the base into the emitter due to *p*+ emitter in PNP then $Q_D = q \cdot A \cdot \int_0^{\infty} p_{no} \cdot (e^{\frac{v'_D}{V_T}} - 1) \cdot e^{-\frac{X}{L_P}} \cdot dX$. Excess charge stored is due almost entirely to the charge injected from the emitter. The BJT acts like a very efficient “siphon,” As majority carriers from the emitter are injected into the base and become “excess minority carriers”, the collector “siphons them” out of the base. We can view the collector current as the amount of excess charge in the base collected by the collector per unit time and we can express the charge due to the excess hole concentration in the base as: $Q_B = i_c \cdot \tau_F$ or the excess charge in the base depends on the magnitude of current flowing and the “forward” base transport time, τ_F, the average time the carriers spend in the base. $\tau_F = \frac{W^2}{2 \cdot D_B}$, W is the base quasi-neutral region width. D_B is the minority carrier diffusion coefficient. Thus, the diffusion capacitance is $C_B = \frac{\partial Q_B}{\partial v_{BE}}|_{Q-\text{point}} = (\frac{W^2}{2 \cdot D_B}) \cdot \frac{\partial i_c}{\partial v_{BE}}|_{Q-\text{point}}$; $C_B = \tau_F \cdot \frac{I_C}{V_T} = \tau_F \cdot g_m$. The upper operational frequency of the transistor is limited by the forward base transport time $f \leq \frac{1}{2 \cdot \pi \cdot \tau_F}$. It is the similarity to the diode diffusion capacitance.

$$C_{\text{Diffusion}} = g_d \cdot \tau_t; \tau_t = \frac{|p_{no} \cdot L_p + n_{po} \cdot L_n| \cdot q \cdot A}{I_S}; C_{\text{Diffusion}} = g_d \cdot \frac{|p_{no} \cdot L_p + n_{po} \cdot L_n| \cdot q \cdot A}{I_S}$$

τ_t is the transit time. In active mode for small forward biases the depletion capacitance of the base–emitter junction can contribute to the total capacitance.
$C_{jE} = \frac{C_{jEo}}{\sqrt{1 + \frac{V_{EB}}{V_{\text{bi for emitter-base}}}}}$. $C_{jE} \equiv$ zero bias depletion capacitance.

$V_{\text{bi for emitter–base}} \equiv$ built in voltage for E-B junction. Thus, the emitter–base capacitance is $C_\pi = C_B + C_{jE}$. In active mode when the collector–base is reverse biased, the capacitance of the collector–base junction is dominated by the depletion capacitance (not diffusion capacitance).
$C_\mu = \frac{C_{\mu o}}{\sqrt{1 + \frac{V_{CB}}{V_{\text{bi for collector-base}}}}}$. $C_{\mu o} \equiv$ zero bias depletion capacitance.

$V_{\text{bi for collector–base}} \equiv$ built in voltage for the B–C junction. In some integrated BJTs (lateral BJTs in particular) the device has a capacitance to the substrate wafer it is fabricated in. This results from a "buried" reverse-biased junction. Thus, the collector–substrate junction is reverse biased and the capacitance of the collector–substrate junction is dominated by the depletion capacitance (not diffusion capacitance).
$C_{cs} = \frac{C_{cso}}{\sqrt{1 + \frac{V_{cs}}{V_{\text{bi for collector-substrate}}}}}$. $C_{cs} \equiv$ zero bias depletion capacitance.
$V_{\text{bi for collector–substrate}} \equiv$ built in voltage for the C substrate junction.
Small signal model of the BJT, parasitic resistances:

r_b = base resistance between metal inter connect and B–E junction.
r_c = parasitic collector resistance.
r_{ex} = emitter resistance due to polysilicon contact.

Complete BJT Small Signal Model

What set the maximum limits of operation of the BJT circuit? Forward-active mode lies between saturation and cutoff. Thus, the maximum voltage extremes that one can operate an amplifier over can easily be found by examining the boundaries between forward active and cutoff and the boundaries between forward active and saturation. Output signals that exceed the voltage range that would keep the transistor within its forward-active mode will result in "clipping" of the signal leading to distortion. The maximum voltage swing allowed without clipping depends on the DC bias points (Fig. D.6).

MOSFET Transistor Model

The basic static model of MOSFET transistor (Shichman and Hodges) is as follow:
$I_{DS} = \mu_0 \cdot C_{\text{ox}} \cdot \frac{W}{L_{\text{eff}}} \cdot \left[(V_{GS} - V_{TH}) \cdot V_{DS} - \frac{V_{DS}^2}{2} \right]$

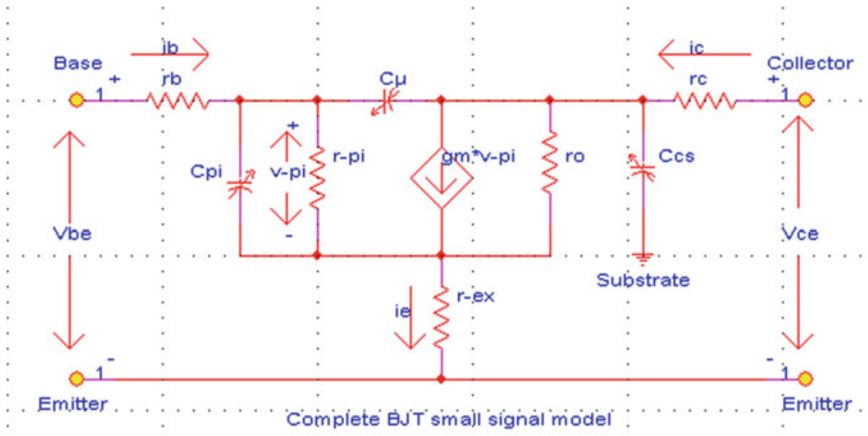


Fig. D.6 Complete BJT small signal model

$$I_{DSsat} = \frac{1}{2} \cdot \mu_0 \cdot C_{\mathrm{ox}} \cdot \frac{W}{L_{\mathrm{eff}}} \cdot (V_{GS} - V_{TH})^2$$

There is an empirical correction to these equations to account for the channel length modulation (Fig. D.7):

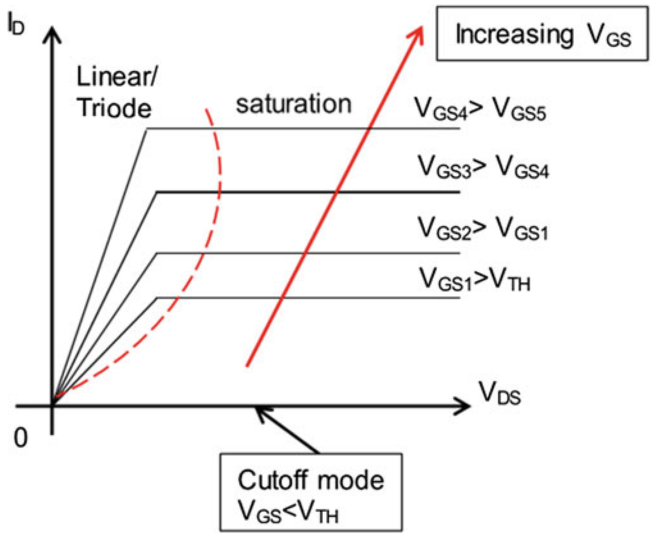


Fig. D.7 MOSFET transistor model graph

$$I_{DS} = \mu_0 \cdot C_{ox} \cdot \frac{W}{L_{eff}} \cdot \left[(V_{GS} - V_{TH}) \cdot V_{DS} - \frac{V_{DS}^2}{2} \right] \cdot [1 + \lambda \cdot V_{DS}]$$

$$I_{\text{DSsat}} = \frac{1}{2} \cdot \mu_0 \cdot C_{ox} \cdot \frac{W}{L_{\text{eff}}} \cdot (V_{GS} - V_{TH})^2 \cdot [1 + \lambda \cdot V_{DS}]$$

In the linear region:

$$I_{DS} = KP \cdot \frac{W}{(L - 2 \cdot X_{jl})} \cdot \left[(V_{GS} - V_{TH}) \cdot V_{DS} - \frac{V_{DS}^2}{2} \right] \cdot [1 + \lambda \cdot V_{DS}]$$

In the saturation region:

$$I_{\text{Dsat}} = \frac{KP}{2} \cdot \frac{W}{(L - 2 \cdot X_{jl})} \cdot (V_{GS} - V_{TH})^2 \cdot [1 + \lambda \cdot V_{DS}]$$

X_{jl} is the lateral diffusion parameter (Fig. D.8).

Threshold Voltage (V_{TH})

The threshold voltage changes with changes in body-source voltage, V_{BS}. The expression for threshold voltage $V_{TH} = V_{TO} + \gamma \cdot (\sqrt{2 \cdot \phi_p - V_{BS}} - \sqrt{2 \cdot \phi_p})$ Where V_{TO} is the threshold voltage when the body-source voltage is zero, γ is the body effect parameter and Φ_p is the surface inversion potential. If the bulk is connected to the source (i.e. the MOSFET is acting as a 3 terminal device, the threshold voltage is always equal to the value V_{TO}). There is a depletion layer which grows into the accumulation region and thus for a given V_{GS}, cuts off the channel. Need to add more V_{GS} to re-establish the channel when we stacked transistors in

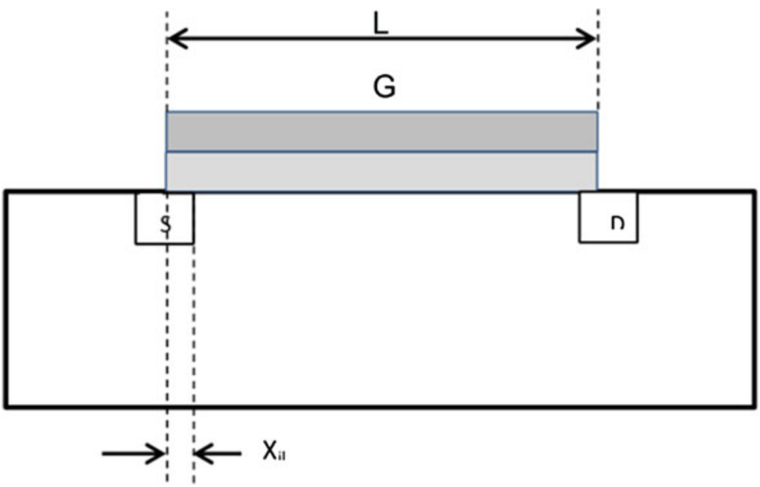


Fig. D.8 MOSFET transistor structure and important parameters

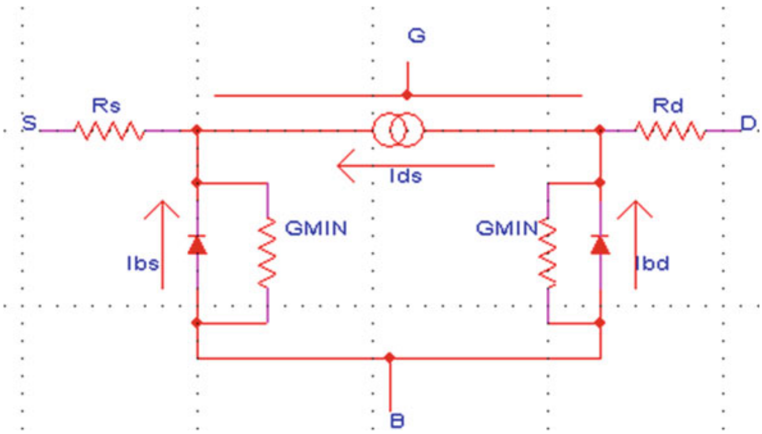


Fig. D.9 MOSFET complete DC model

integrated circuits. If you connected bulk to source on each transistor in an integrated circuit you would end up shorting many points in the circuit to ground.

Complete DC Model

The model includes body-source and body-drain diodes. Equations used for the diode model, for forward bias on the body-source/body-drain diodes:

$$I_{BS} = I_{SS} \cdot \left[e^{\frac{V_{BS}}{V_t}} - 1\right] + \text{GMIN} \cdot V_{BS}; I_{BD} = I_{SD} \cdot \left[e^{\frac{V_{BD}}{V_t}} - 1\right] + \text{GMIN} \cdot V_{BD}$$

For the negative reverse bias on those diodes (Fig. D.9):

$$I_{BS} = I_{SS} \cdot \frac{V_{BS}}{V_t} + \text{GMIN} \cdot V_{BS}; I_{BD} = I_{SD} \cdot \frac{V_{BD}}{V_t} + \text{GMIN} \cdot V_{BD}$$

MOSFET body diodes: The reverse bias terms are simply the first terms in a power series expansion of the exponential term. The GMIN convergence resistance. I_{SS} and are taken to be one constant in simulation.

DC MOSFET parameters: L = channel length, W = channel width, $KP(kp)$ = The trans-conductance parameter, V_{TO} = Threshold voltage under zero bias conditions, GAMMA (γ) = Body effect parameter, PHI (Φ_p) = surface inversion potential, RS(R_S) = source contact resistance, RD(R_D) = Drain contact resistance, LAMBDA(λ) = channel length modulation parameter, $XJ(X_{jl})$ = lateral diffusion parameter. IS (I_{SS}, I_{SD}) = reverse saturation current of body-drain/source diodes.

Large signal transient model: We add some capacitances to the DC model to create the transient model to form the final transient model, as shown in Fig. D.10:

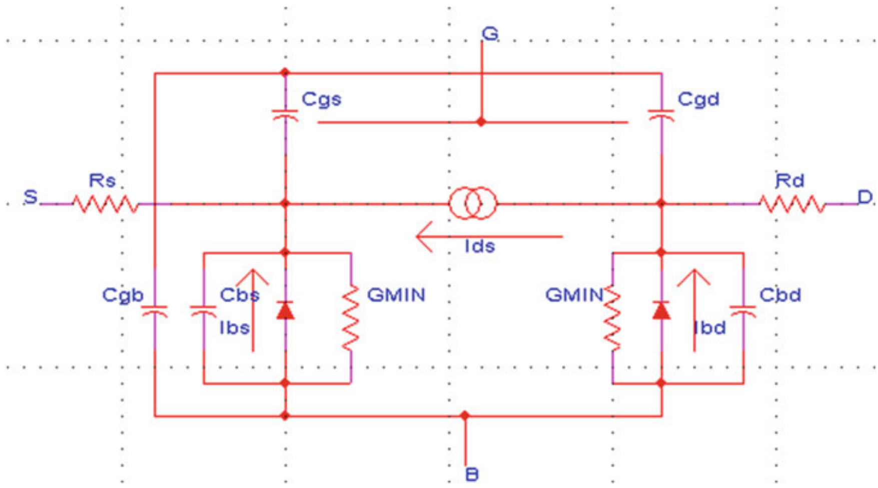


Fig. D.10 MOSFET large signal transient model

Capacitances: Static overlap capacitances between gate and drain (C_{GB0}), gate and source (C_{GS0}), and gate and bulk (C_{GB0}). These are fixed values, and are specified per unit width. In saturation, $C_{GS} = \frac{2}{3} \cdot C_{0x} + C_{GS0} \cdot W; C_{GD} = C_{GD0} \cdot W$ In saturation after pinch-off, it is assumed that altering the drain voltage does not have any effect on stored charge in the channel and thus the only capacitance between gate and drain is the overlap capacitance.

In the linear/triode region, in this region the following equations are used:

$$C_{GS} = C_{0x} \cdot \left\{1 - \left[\frac{V_{GS} - V_{DS} - V_{TH}}{2 \cdot (V_{GS} - V_{TH}) - V_{DS}}\right]^2\right\} + C_{GS0} \cdot W$$

$$C_{GD} = C_{0x} \cdot \left\{1 - \left[\frac{V_{GS} - V_{TH}}{2 \cdot (V_{GS} - V_{TH}) - V_{DS}}\right]^2\right\} + C_{GD0} \cdot W$$

As the device is moved further into the linear region, V_{GS} becomes large compared to ($V_{DS} - V_{TH}$) then the values of C_{GS} and C_{GD} become close to $C_{0x}/2$ (plus the relevant overlap capacitance).

The body diode capacitances: The capacitances of the body diodes are given by slightly modified expressions for junction capacitances of the diode model: The expression for a PN diode capacitance: $C_j = \frac{C_j(0)}{\sqrt{1-\frac{V}{V_0}}}$. The MOSFET equation is based on the following slightly modified equation:

$C_j = \frac{C_j(0)}{\sqrt{1-\frac{V}{V_0}}} + \frac{C_{jSW}(0)}{\sqrt{1-\frac{V}{V_0}}}$. The junction capacitance is made up of two components. The main component, due to $C_j(0)$ is the normal junction capacitance. The second parameter is the perimeter junction capacitance of the diffused source. The diffusion capacitance is zero in reverse bias and the MOSFET must be operated with the bulk-drain and bulk-source diodes in reverse bias to stop large bulk currents flowing. The additional parameters required for specifying the transient model in addition to those required by the DC model are thus:

CGD0(C_{GD0}) = Gate drain overlap capacitance per unit width of device.
CGS0(C_{GS0}) = Gate source overlap capacitance per unit width of device.
CJ(C_j) = Zero bias depletion capacitance for body diodes.
CJSW(C_{jsw}) = Zero bias depletion perimeter capacitance for body diodes.
TOX(t_{ox}) = Oxide thickness (used for calculating C_{ox}).

Bipolar Transistor Metrology and Theory
The interest topics regarding bipolar junction transistor (BJT) are operation, I–V characteristics, current gain and output conductance. High-level injection and heavy doping induced band narrowing. SiGe transistor, transit time, and cutoff frequency are important parameters. There are several bipolar transistor models which are used (Ebers–Moll model, Small signal model, and charge control model). Each model has its own areas of applications. The metal–oxide–semiconductor (MOS) ICs have high density and low power advantages. The BJTs are preferred in some high frequency and analog applications because of their high speed, low noise, and high output power advantages such as in some cell phone amplifier circuits. A small number of BJTs are integrated into a high density complementary MOS (CMOS) chip integration of BJT and CMOS is known as the BiCMOS technology. The term bipolar refers to the fact that both electrons and holes are involved in the operation of a BJT. Minority carrier diffusion plays the leading role as in the PN diode junction diode. A BJT is made of a heavily doped emitter, a P-type base, and an N-type collector. This device is an NPN BJT, a PNP BJT would have a P^+ emitter, N-type base, and P-type collector. NPN transistor exhibit higher trans-conductance and speed than PNP transistors because the electron mobility is larger than the hole mobility, BJTs are almost exclusively of the NPN type since high performance is BJT's competitive edge over MOSFETs (Fig. D.11).

When the base-emitter junction is forward biased, electrons are injected into the more lightly doped base. They diffuse across the base to the reverse-biased base–

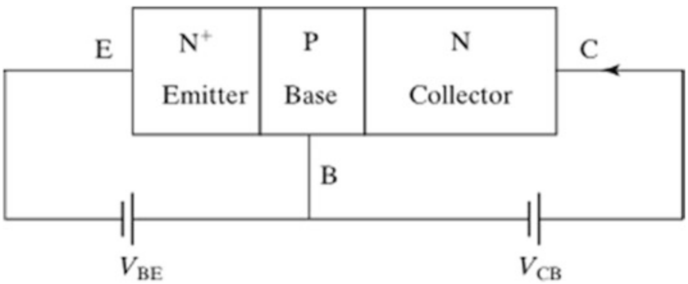


Fig. D.11 NPN BJT transistor voltages connection collector junction which is the edge of the depletion layer and get swept into the collector. This produces a collector current, I_C. I_C is independent of V_{CB} as long as V_{CB} is a reverse bias or a small forward bias. I_C is determined by the rate of electron injection from the emitter into the base, determined by V_{BE}. The rate of electron injection is proportional to $e^{\frac{q \cdot V_{BE}}{k \cdot T}}$. The emitter is often connected to ground. The emitter and collector are the equivalents of source and drain of a MOSFET when the base is the equivalent of the gate. The I_C curve is usually plotted against V_{CE}. $V_{CE} = V_{CB} + V_{BE}$, below $V_{CE} = 0.3$ V the base–collector junction is strongly forward biased and I_C decreases. Because of the parasitic IR drops, it is difficult to accurately ascertain the true base–emitter junction voltage. The easily measurable base current I_B is commonly used as the variable parameter in lieu of V_{BE}, I_C is proportional to I_B (Fig. D.12).

Collector current: The collector current is the output current of a BJT transistor. Applying the electron diffusion equation to the base region gives the following $\frac{d^2 n'}{dx^2} = \frac{n'}{L_B^2}$; $L_B = \sqrt{\tau_B \cdot D_B}$; $\frac{d^2 n'}{dx^2} = \frac{n'}{\tau_B \cdot D_B}$ (Fig. D.13).

τ_B and D_B are the recombination lifetime and the minority carrier (electron) diffusion constant in the base, respectively. The boundary conditions are as follow: $n'(0) = n_{B0} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1)$; $n'(W_B) = n_{B0} \cdot (e^{\frac{q \cdot V_{BC}}{k \cdot T}} - 1) \approx -n_{B0} \approx 0$. Where $n_{B0} = \frac{n_i^2}{N_B}$ and N_B is the base doping concentration. V_{BE} is normally a forward bias (positive value) and V_{BC} is a reverse bias (negative value).

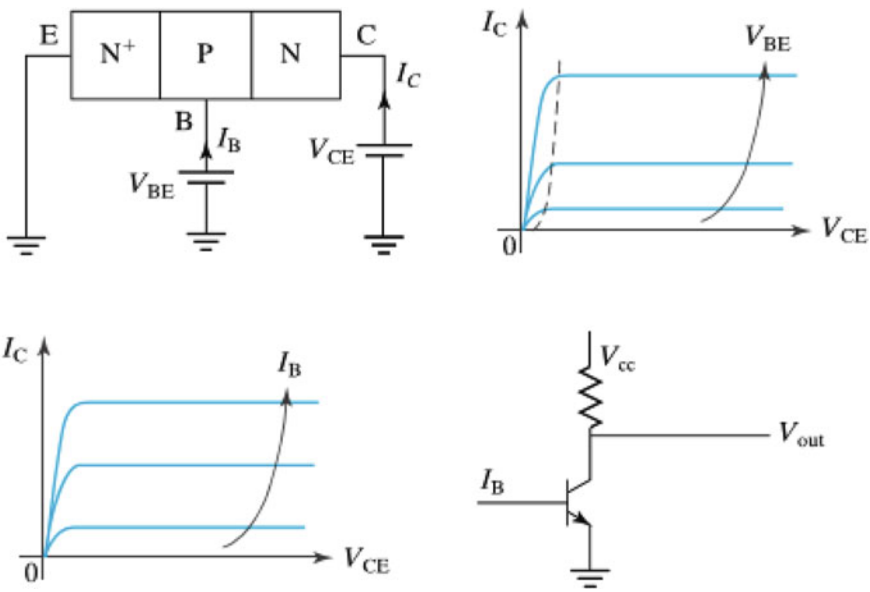


Fig. D.12 NPN transistor structure, connections and graphs

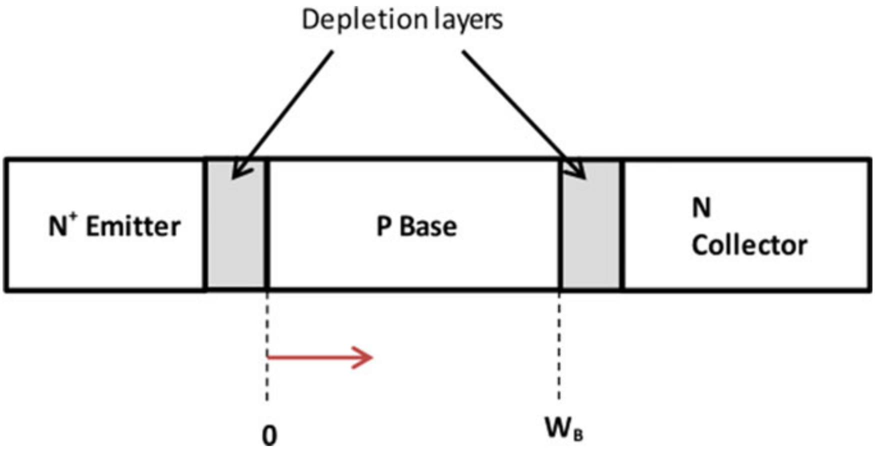


Fig. D.13 NPN transistor structure with depletion layers

We get the following expression for $n'(x) = n_{B0} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1) \cdot \frac{\sinh(\frac{W_B - x}{L_B})}{\sinh(\frac{W_B}{L_B})}$. Modern BJTs have base widths of about 0.1 μm. This is much smaller than the typical diffusion length of tens of microns. In the case of $W_B \ll L_B$ we get the $n'(x)$ expression: $n'(x) = n'(0) \cdot (1 - \frac{x}{W_B}) = \frac{n_{iB}^2}{N_B} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1) \cdot (1 - \frac{x}{W_B})$. n_{iB} is the intrinsic carrier concentration of the base material. The subscript *B*, is added to n_i because the base may be made of a different semiconductor such as SiGe alloy, which has a smaller band gap and therefore a larger n_i than the emitter and collector material. The minority carrier current is dominated by the diffusion current. The sign of I_C is positive and defined in the expression:

$I_C = |A_E \cdot q \cdot D_B \cdot \frac{dn}{dx}| = A_E \cdot q \cdot D_B \cdot \frac{n'(0)}{W_B} = A_E \cdot q \cdot \frac{D_B}{W_B} \cdot \frac{n_{iB}^2}{N_B} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1)$. A_E is the area of the BJT specifically the emitter area. There is a similarity between BJT transistor I_C current and the PN diode IV relation. Both are proportional to $(e^{\frac{q \cdot V}{k \cdot T}} - 1)$ and to $\frac{D \cdot n_i^2}{N}$. The only difference is that $\frac{dn'}{dx}$ has produced the $\frac{1}{W_B}$ term due to the linear n' profile. We can condense the expression of I_C to $I_C = I_s \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1)$, where I_s is the saturation current. $I_C = A_E \cdot \frac{q \cdot n_i^2}{G_B} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1)$ and $G_B = \frac{n_i^2}{n_{iB}^2} \cdot \frac{N_B}{D_B} \cdot W_B = \frac{n_i^2}{n_{iB}^2} \cdot \frac{p}{D_B} \cdot W_B$, where *p* is the majority carrier concentration in the base. It is valid even for no uniform base and high-level injection condition if G_b is generalized to 1. $G_B = \int_0^{W_B} \frac{n_i^2}{n_{iB}^2} \cdot \frac{p}{D_B} \cdot dx$, G_B has the unusual dimension of s/cm^4 and is known as the base Gummel number. In the special case of $n_{iB} = n_i$, D_B is a constant, and $p(x) = N_B(x)$ which is low level injection. $G_B = \frac{1}{D_B} \cdot \int_0^{W_B} N_B(x) \cdot dx = \frac{1}{D_B} \times$ base dopant atoms per unit area. The base Gummel number is basically proportional to the base dopant density per area. The

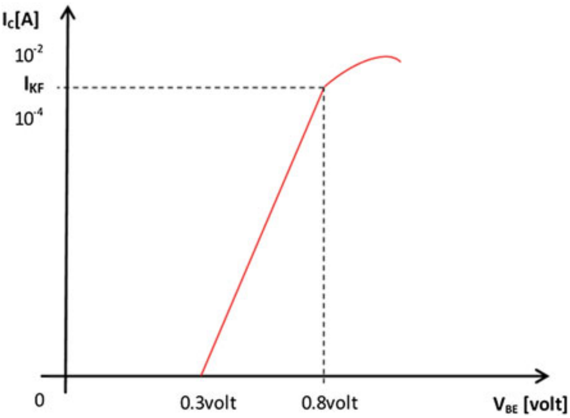


Fig. D.14 NPN transistor I_c [A] versus V_{BE} [V]

higher the base dopant density is, the lower the I_C will be for a given V_{BE}. The concept of a Gummel number simplifies the I_C model because it contains all the subtleties of transistor design that affect I_C; changing base material through $n_{iB}(x)$, nonconstant D_B, non-uniform base dopant concentration through $p(x) = N_B(x)$ and even the high-level injection condition, where $p > N_B$. Although many factors affect G_B, G_B can be easily determined from the Gummel plot. The inverse slope of the straight line can be described as 60 mV per decade. The extrapolated intercept of the straight line and $V_{BE} = 0$ yields I_s. G_B is equal to $A_E \cdot q \cdot n_i^2$ divided by the intercept (Fig. D.14).

The decrease in the slope of the curve at high I_C is called the high-level injection effect. At large V_{BE}, n' can become larger than the base doping concentration N_B, $n' = p' \gg N_B$. The condition of $n' = p' \gg N_B$ is called high-level injection. A consequence is that in the base $n \approx p \approx n_i \cdot e^{\frac{q \cdot V_{BE}}{2 \cdot k \cdot T}}; G_B \propto n_i \cdot e^{\frac{q \cdot V_{BE}}{2 \cdot k \cdot T}}$.

Yield to $I_c \propto n_i \cdot e^{\frac{q \cdot V_{BE}}{2 \cdot k \cdot T}}$. Therefore, at high V_{BE} or high I_C, $I_c \propto e^{\frac{q \cdot V_{BE}}{2 \cdot k \cdot T}}$ and the inverse slope becomes 120 mV/decade. I_{KF}, the knee current, is the current at which the slope changes. It is a useful parameter in the BJT model for circuit simulation. The IR drop in the parasitic resistance significantly increases V_{BE} at very high I_C and further flattens the curve.

Base current: Whenever the base-emitter junction is forward biased, some holes are injected from the P-type into the N^+ emitter. These holes are provided by the base current I_B, I_B is an undesirable but inevitable side effect of producing I_C by forward biasing the BE junction. The analysis of I_B, the base to emitter injection current, is a perfect parallel of the I_C analysis. The base current can be expressed as $I_B = A_E \cdot \frac{q \cdot n_i^2}{G_E} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1); G_E = \int_0^{W_E} \frac{n_i^2}{n_{iE}^2} \cdot \frac{p}{D_E} \cdot dx$. G_E is the emitter Gummel number. In case of uniform emitter, where n_{iE}, N_E(emitter doping concentration) and D_E are not functions of x. $I_B = A_E \cdot q \cdot \frac{D_E}{W_E} \cdot \frac{n_{iE}^2}{N_E} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1)$ (Fig. D.15).

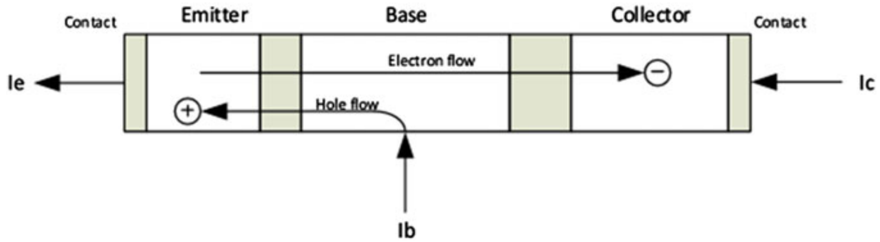


Fig. D.15 NPN transistor structure electron flow and hole flow

Current gain: The most important DC parameter of a BJT is its common emitter current gain β_F. Another current ratio, the common base current gain, is defined by α_F. $\beta_F \equiv \frac{I_C}{I_B}; I_C = \alpha_F \cdot I_E; \alpha_F = \frac{I_C}{I_E} = \frac{I_C}{I_C + I_B} = \frac{I_C/I_{CB}}{I_C/I_{CB}+1} = \frac{\beta_F}{1+\beta_F}$. α_F is typically very close to unity, such as 0.99, because β_F is large. $\alpha_F = \frac{\beta_F}{1+\beta_F}; \beta_F = \frac{\alpha_F}{1-\alpha_F}$. I_B is a load on the input signal source, an undesirable side effect of forward biasing the BE junction. I_B should be minimized (β_F should be maximized). $\beta_F = \frac{G_E}{G_B} = \frac{D_B \cdot W_E \cdot N_E \cdot n_{iB}^2}{D_E \cdot W_B \cdot N_B \cdot n_{iE}^2}$. A typical good β_F is 100. D and W cannot be changed very much. The most obvious way to achieve a high β_F, is to use a large N_E and a small N_B. A small N_B, would introduce too large a base resistance, which degrades the BJT's ability to operate at high current and high frequencies. Typically N_B is around 10^{18} cm^{-3}. An emitter is said to be efficient if the emitter current is mostly the useful electron current injected into the base with little useless hole current (the base current). The emitter efficiency is defined as $\gamma_E = \frac{I_E - I_B}{I_E} = \frac{I_C}{I_C + I_B} = \frac{1}{1 + G_B/G_E}$. To raise β_F, N_E is typically made larger than 10^{20} cm^{-3}. When N_E is very large, n_{iE}^2 becomes larger than n_i^2. This is called the heavy doping effect. $n_i^2 = N_C \cdot N_V \cdot e^{-\frac{E_g}{k \cdot T}}$, heavy doping can modify the Si crystal sufficient to reduce E_g and cause n_i^2 to increase significantly. Therefore, the heavy doping effect is also known as band gap narrowing. $n_{iE}^2 = n_i^2 \cdot e^{\frac{\Delta E_{gE}}{k \cdot T}}$, ΔE_{gE} is the narrowing of the emitter band gap relative to lightly doped Si and is negligible for $N_E < 10^{18}$ cm^{-3}, 50 meV at 10^{19} cm^{-3}, 95 meV cm^{-3} at 10^{20} cm^{-3}, and 140 meV at 10^{21} cm^{-3}. To further elevate β_F, we can raise n_{iB} by using a base material that has a smaller band gap than the emitter material. $Si_{1-\eta}Ge_\eta$ is an excellent base material candidate for an Si emitter. With η = 0.2, E_{gB} is reduced by 0.1 eV. In a SiGe BJT, the base is made of high quality P-type epitaxial SiGe. In practice, η is graded such that $\eta = 0$ at the emitter end of the base and 0.2 at the drain end to create a built in field that improves the speed of the BJT. Because the emitter and base junction is made of two different semiconductors, the device is known as a heterojunction bipolar transistor or HBT. HBTs made of InP emitter (E_g = 1.35 eV) and InGaAs base (E_g = 0.68 eV) and GaAlAs emitter with GaAs base are other examples of well-studied HBTs. The ternary semiconductors are used to achieve lattice constant matching at the heterojunction. Whether the base material is SiGe or plain Si, a high-performance BJT would have a relatively thick (>100 nm) layer of As-doped N^+ poly-Si film film in the emitter. Arsenic is thermally driven into the "base" by $\sim$20 nm and converts that single crystalline layer into a part of the N^+ emitter. This way, β_F is larger due to the large W_E, mostly made of the N^+ poly-Si. This is the poly-Silicon emitter technology. The simpler alternative, a deeper implanted or diffused N^+ emitter without the poly-Si film film, is known to produce a higher density of crystal defects in the thin base causing excessive emitters to collector leakage current or even shorts in a small number of the BJTs. High speed circuits operate at high I_C, and low power circuits may operate at low I_C. Current gain β, drops at both high I_C and at low I_C. In Gummel plot the I_C flattens at high V_{BE} due to the high-level injection effect in the base. That I_C curve arising from hole injection into the emitter, does not flatten due to this effect because the emitter is very heavily doped, and it is practically impossible to inject a higher density of holes than N_E. Over a wide mid-range of I_C, I_C and I_B are parallel, indicating that the ratio I_C/I_B, i.e., β_F is a constant. Above 1 mA, the slope of I_C drops due to high-level injection. Consequently, the I_C/I_B ratio or β_F decreases rapidly. This fall-off of current gain unfortunately degrades the performance of BJTs at high current where the BJTs speed is the highest. I_B is the base emitter junction forward bias current. The forward bias current slope decreases at low V_{BE} or very low current due to the Space Charge Region (SCR) current. As a result, the I_C/I_B ratio or β_F decreases at very low I_C.

As in MOSFETs, a large output conductance, $\frac{\partial I_C}{\partial V_{CE}}$, of BJTs is deleterious to the voltage gain of circuits. The cause of the output conductance is base-width modulation. The thick vertical line indicates the location of the base–collector junction. With increasing V_{ce}, the base–collector depletion region widens and the neutral base width decreases. This leads to an increase in I_C. If the curves $I_C - V_{CE}$ are extrapolated, they intercept the $I_C = 0$ axis at approximately the same point. V_A is defined as early voltage. V_A is a parameter that describes the flatness of the I_C curves. Specifically, the output resistance can be expressed as V_A/I_C: $r_0 \equiv (\frac{\partial I_C}{\partial V_{CE}})^{-1} = \frac{V_A}{I_C}$. A large V_A (large r_0) is desirable for high voltage gains. A typical V_A is 50 V. V_A is sensitive to the transistor design. We can except V_A and r_0 to increase, expect the base width modulation to be a smaller fraction of the base width, if we increase the base width, increase the base doping concentration N_B or decrease the collector doping concentration N_C. Increasing the base width would reduce the sensitivity to any given ΔW_B. Increasing the base doping concentration N_B would reduce the depletion region thickness on the base side because the depletion region penetrates less into the more heavily doped side of a PN junction. Decreasing the collector doping concentration N_C would tend to move the depletion region into the collector and thus reduce the depletion region thickness on the base side, too. Both increasing the base width and the base doping concentration N_B would depress β_F. Decreasing the collector doping concentration N_C is the most acceptable course of action. It is also reduces the base–collector junction capacitance, which is a good thing. Therefore, the collector doping is typically ten times lighter than the base doping. The larger slopes at $V_{CE} > 3$ V are caused by impact ionization. The rise of I_C due to base-width modulation is known as the early effect. Model the collector current as a function of the collector voltage: $I_C = \beta_F \cdot I_B$ and differentiating with respect to V_C while I_B was held constant gave, $\frac{\partial I_C}{\partial V_C} = I_B \cdot \frac{\partial \beta_F}{\partial V_C}$. The question is how can β_F change with V_C, the collector depletion layer thickens as collector voltage is raised. The base gets thinner and current gain raises.

Bipolar Transistor Transit Time and Charge Storage

Static IV characteristics are only one part of the BJT theory. Another part is its dynamic behavior or its speed. When the BE junction is forward biased, excess holes are stored in the emitter, the base, and even the depletion layers. The sum of all excess hole charges everywhere Q_F. Q_F is the stored excess carrier charge. If Q_F = 1 pC (Pico coulomb), there is +1 pC of excess hole charge and −1 pC of excess electron charge stored in the BJT. The ratio of Q_F to I_C is called the forward transit time τ_F ($\tau_F \equiv \frac{Q_F}{I_C}$). Ic and Q_F are related by a constant ratio τ_F. Q_F and therefore τ_F are very difficult to predict accurately for a complex device structure. τ_F can be measured experimentally and once τ_F is determined for a given BJT, equation $\tau_F \equiv \frac{Q_F}{I_C}$ becomes a powerful conceptual and mathematical tool giving Q_F as a function of I_C, and vice versa. τ_F sets a high frequency limit of BJT operation. The excess hole charge in the base Q_{FB}: $Q_{FB} = q \cdot A_E \cdot n'(0) \cdot W_B/2; \frac{Q_{FB}}{I_C} \equiv \tau_{FB} = \frac{W_B^2}{2 \cdot D_B}$. The base transit time can be further reduced by building into the base a drift field that aids the flow of electrons from the emitter to the collector. There are two ways of accomplishing this. The classical method is to use graded base doping (a large N_B near the EB junction), which gradually decreases toward the CB junction. Such a doping gradient is automatically achieved if the base is produced by dopant diffusion. The changing N_B creates a $dE_v/\mathrm{d}x$ and a$dE_c/\mathrm{d}x$. This means that there is a drift field. Any electron injected into the base would drift toward the collector with a base transit time shorter than the diffusion transit time, $\frac{W_B^2}{2 \cdot D_B}$. In a SiGe BJT, P-type epitaxial $Si_{1-\eta}Ge_\eta$ is grown over the Si collector with a constant N_B and η linearly varying from about 0.2 at the collector end to 0 at the emitter end. A large $dE_c/\mathrm{d}x$ can be produced by the grading ofE_{gB}. These high speed BJTs are used in high frequency communication circuits. Drift transistors can have a base transit time several times less than$\frac{W_B^2}{2 \cdot D_B}$, as short as 1 p s. The total forward transit time, τ_F is known as the emitter to collector transit time. τ_{FB} is only one portion of τ_F. The base transit time typically contributes about half of τ_F. To reduce the transit (or storage) time in the emitter and collector, the emitter and the depletion layers must be kept thin. τ_F can be measured. τ_F starts to increase at a current density where the electron density corresponding to the dopant density in the collector (n = N_C) is insufficient to support the collector current even if the dopant induced electrons move at the saturation velocity. This intriguing condition of too few dopant atoms and too much current leads to a reversal of the sign of the charge density in the depletion region.

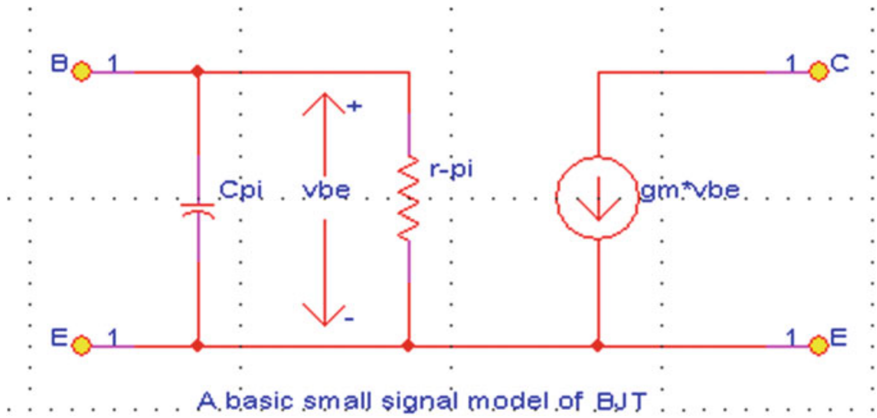


Fig. D.16 Bipolar transistor small signal model

$I_C = A_E \cdot q \cdot n \cdot v_{sat}$; $\rho = q \cdot N_C - q \cdot n = q \cdot N_C - \frac{I_C}{A_E \cdot v_{sat}}$; $\frac{d\wp(x)}{dx} = \frac{\rho}{\varepsilon_s}$. When I_C is small then $\rho(\rho = q \cdot N_C)$ as expected from the PN junction analysis, and the electric field in the depletion layer. The N^+ collector is always present to reduce the series resistance. No depletion layer is shown in the base for simplicity because the base is much more heavily doped than the collector. As I_C increases, ρ decreases and $\frac{d\wp(x)}{dx}$ decreases. The electric field drops to zero in the very heavily doped N^+ collector as expected. Because of the base widening, τ_F increases as a consequence. This is called the Kirk effect. Base widening can be reduced by increasing N_C and V_{CE}. The Kirk effect limits the peak BJT operating speed.

Bipolar Transistor Small Signal Model

The equivalent circuit for the behavior of a BJT in response to a small input signal (10 mV sinusoidal signal, superimposed on the DC bias) is presented in Fig. D.16. BJTs are often operated in this manner in analog circuits.

If V_{BE} is not close to zero, the "1" in $I_C = I_s \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1)$ is negligible; in that case

$I_C = I_s \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1) \approx I_s \cdot e^{\frac{q \cdot V_{BE}}{k \cdot T}}$. When a signal v_{BE} is applied to the BE junction, a collector current $g_m \cdot v_{BE}$ is produced. g_m, the trans-conductance, is

$$g_m \equiv \frac{dI_C}{dV_{BE}} = \frac{d}{dV_{BE}}\left(I_s \cdot e^{\frac{q \cdot V_{BE}}{k \cdot T}}\right) = \frac{q}{k \cdot T} \cdot I_s \cdot e^{\frac{q \cdot V_{BE}}{k \cdot T}} = I_C / \frac{k \cdot T}{q}; g_m = I_C / \frac{k \cdot T}{q}$$

At room temperature, $g_m = I_C/26\,\text{mV}$. The trans-conductance is determined by the collector bias current, I_C. The input node, the base, appears to the input drive circuit as a parallel RC circuit. $\frac{1}{r_\pi} = \frac{dI_B}{dV_{BE}} = \frac{1}{\beta_F} \cdot \frac{dI_C}{dV_{BE}} = \frac{g_m}{\beta_F}$; $r_\pi = \frac{\beta_F}{g_m}$. Q_F is the excess carrier charge stored in the BJT. If $Q_F = 1$ pC, there is +1 pC of excess holes and −1 pC of excess electrons in the BJT. All the excess hole charge, Q_F, is supplied by the base current, I_B. Therefore, the base presents this capacitance to the input drive circuit: $C_\pi = \frac{dQ_F}{dV_{BE}} = \frac{d}{dV_{BE}}[\tau_F \cdot I_C] = \tau_F \cdot g_m$. The capacitance C_Π may be called the charge storage capacitance, known as the diffusion capacitance. There is one charge component that is not proportional to I_C and therefore cannot be included in Q_F. That is the junction depletion layer charge. Therefore, a complete model of C_Π should include the BE junction depletion layer capacitance, C_{dBE}, $C_\pi = \tau_F \cdot g_m + C_{dBE}$. Once the parameters in the basic small signal model of the BJT have been determined, one can use the small signal model to analyze circuits with arbitrary signal source impedance network which composing resistors, capacitors, and inductors, and additionally load impedance network. r_0 is the intrinsic output resistance, V_A/I_C. C_μ also arises from base width modulation; when V_{BC} varies, the base width varies; therefore, the base stored charge varies, thus giving rise to$C_\mu = \frac{dQ_{FB}}{dV_{CB}}$. C_{dBC} is the CB junction depletion layer capacitance. Model parameters are difficult to predict from theory with the accuracy required for commercial circuit design. Therefore, the parameters are routinely determined through comprehensive measurement of the BJT AC and DC characteristics.

The Fig. D.17 describes the small signal model which can be used to analyze a BJT circuit by hand.

Cutoff frequency: We consider small signal model when the load is a short circuit. The signal source is a current source i_b, at a frequency *f*. The question is at what frequency the AC current gain does $\beta \equiv i_c/i_b$ fall to unity?

$$v_{be} = \frac{i_b}{\text{input admittance}} = \frac{i_b}{1/r_\pi + j \cdot \omega \cdot C_\pi}; i_c = g_m \cdot v_{be}$$

$$\beta(\omega) \equiv \left|\frac{i_c}{i_b}\right| = \frac{g_m}{|1/r_\pi + j \cdot \omega \cdot C_\pi|} = \frac{1}{|1/g_m \cdot r_\pi + j \cdot \omega \cdot \tau_F + j \cdot \omega \cdot C_{dBE}/g_m|}$$

$$= \frac{1}{|1/\beta_F + j \cdot \omega \cdot \tau_F + j \cdot \omega \cdot C_{dBE} \cdot k \cdot T/q \cdot I_C|}$$

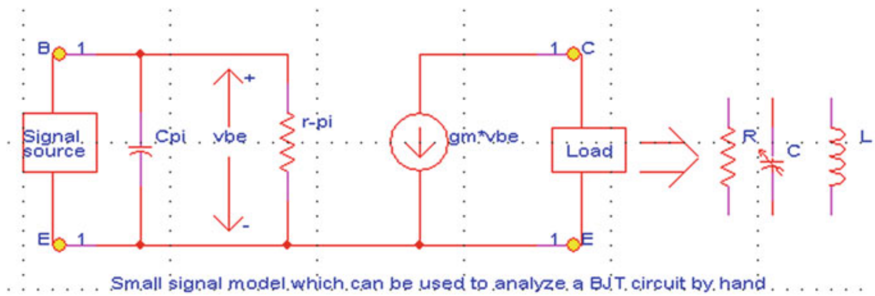


Fig. D.17 Bipolar transistor small signal model which can be used to analyze a BJT circuit by hand

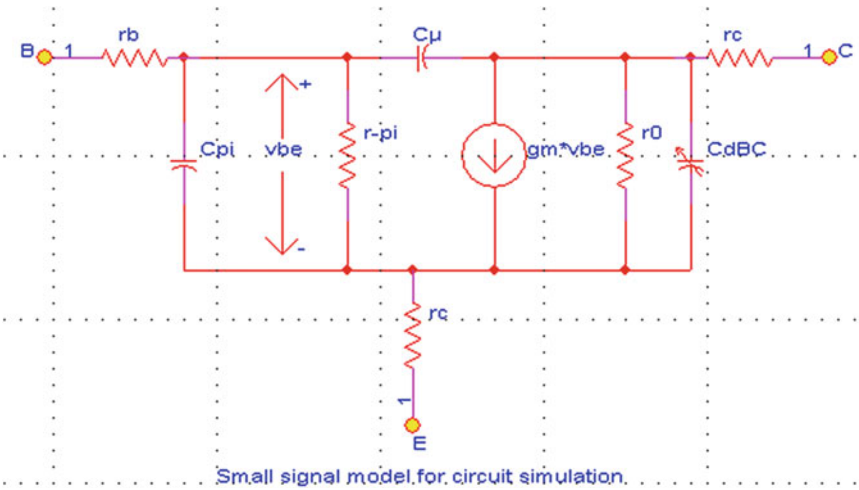


Fig. D.18 Bipolar transistor small signal model for circuit simulation

The Fig. D.18 describes the small signal model for circuit simulation by computer.

At $\omega = 0$, i.e., DC $\beta(\omega) = \ldots$ equation reduces to β_F as expected. As ω increases, β drops. By carefully analyzing the $\beta(\omega)$ data, one can determine τ_F. If $\beta_F \gg 1$ so that $1/\beta_F$ is negligible, $\beta(\omega) \propto \frac{1}{\omega}$ and $\beta = 1$ at f_T ,$f_T = \frac{1}{2\cdot\pi\cdot(\tau_F + C_{dBE}\cdot k\cdot T/q\cdot I_C)}$.

If we use a more complete small signal model, it can be shown that $f_T = \frac{1}{2\cdot\pi\cdot[\tau_F + (C_{dBE} + C_{dBC})\cdot k\cdot T/(q\cdot I_C) + C_{dBC}\cdot(r_e + r_c)]}$. f_T is the cutoff frequency and is commonly used to compare the speed of transistors. The above equations predict that f_T rises with increasing I_C due to increasing g_m, in agreement with the measured f_T. At very high I_C, τ_F increases due to base widening (Kirk effect), and therefore, f_T falls. BJTs are often biased near the I_C where f_T peaks in order to obtain the best high frequency performance. F_T is the frequency of unity power gain. The frequency of unity power gain, called the maximum oscillation frequency. $f_{\max} = \sqrt{\left(\frac{f_T}{8\cdot\pi\cdot r_b\cdot C_{dBC}}\right)}$, it is therefore important to reduce the base resistance, r_b. While MOSFET scaling is motivated by the need for high packing density and large I_{dsat}, BJT scaling is often motivated by the need for high f_T and $f_{\max}$. This involves the reduction of τ_F (thin base, etc,) and the reduction of parasitic $(C_{dBE}, C_{dBC}, r_b, r_e, r_c)$. We interested in BJT with poly-Si emitter, self-aligned base, and deep trench isolation. The base is contacted through two small P^+ regions created by boron diffusion from a P^+ poly-Si film film. The film also provides a low resistance electrical connection to the base without introducing a large P^+ junction area and junction capacitance. To minimizing the base series resistance, the emitter opening is made very narrow. The lightly doped epitaxial N-type collector is contacted through a heavily doped sub-collector in order to minimize the collector series resistance. The substrate is lightly doped to minimize the collector capacitance. Both the shallow trench and the deep trench are filled with dielectrics (SiO_2) and serve the function of electrical isolation. The deep trench forms a rectangular moat that completely surrounds the BJT. It isolates the collector of this transistor from the collectors of neighboring transistors. The structure incorporates many improvements that have been developed over the past decades and have greatly reduced the device size from older BJT design. BJT is a larger transistor than a MOSFET.

Bipolar Transistor Charge Control Model

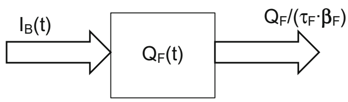
The small signal model is ideal for analyzing circuit response to small sinusoidal signals. If the signal is large, input is step function I_B switching from zero to 20 μA or by any $I_B(t)$ and then $I_C(t)$ is produced. The response is analyzed with the charge control model which is a simple extension of the charge storage concept. $I_C = \frac{Q_F}{\tau_F} \Rightarrow I_C(t) = \frac{Q_F(t)}{\tau_F}$, $I_C(t)$ becomes known if we solve for $Q_F(t)$. τ_F has to be characterized beforehand for the BJT being used. I_C is controlled by Q_F (charge control model). At DC condition$I_B = \frac{I_C}{\beta_F} = \frac{Q_F}{\tau_F \cdot \beta_F}$, the equation has a straightforward physical meaning. In order to sustain a constant excess hole charge in the transistor, holes must be supplied to the transistor through I_B to replenish the holes that are lost to recombination. Therefore, DC I_B is proportional to Q_F. When holes are supplied by I_B at the rate of$Q_F/\tau_F \cdot \beta_F$, the rate of hole supply is exactly equal to the rate of hole loss to recombination and Q_F remains at a constant value. In the case that I_B is larger than $Q_F/\tau_F \cdot \beta_F (I_B > Q_F/\tau_F \cdot \beta_F)$, holes flow into the BJT at a higher rate than the rate of hole loss and the stored hole charge Q_F increases with time ($\frac{dQ_F}{dt} = I_B(t) - \frac{Q_F}{\tau_F \cdot \beta_F}$).

The presented equations together constitute the basic charge control model.

For any given $I_B(t)$, equation $\frac{dQ_F}{dt} = I_B(t) - \frac{Q_F}{\tau_F \cdot \beta_F}$ can be solved for $Q_F(t)$ analytically or by numerical integration. Once $Q_F(t)$ is found, $I_C(t)$ becomes known from equation $I_C(t) = \frac{Q_F(t)}{\tau_F}$. Figure D.19 describes the charge control model. Excess hole charge Q_F rises or falls at the rate of supply current I_B minus loss ($\propto Q_F$).

Q_F is the amount of charges in the vessel, and $\frac{Q_F}{\tau_F \cdot \beta_F}$ is the rate of charge leakage. I_B is the rate of charges flowing into the vessel. The Fig. D.19 is a basic version of the charge control model. We can introduce the junction depletion layer capacitances into equation $\frac{dQ_F}{dt} = I_B(t) - \frac{Q_F}{\tau_F \cdot \beta_F}$. Diverting part of I_B to charge the junction capacitances would produce an additional delay in $I_C(t)$.

Fig. D.19 Bipolar transistor charge control model



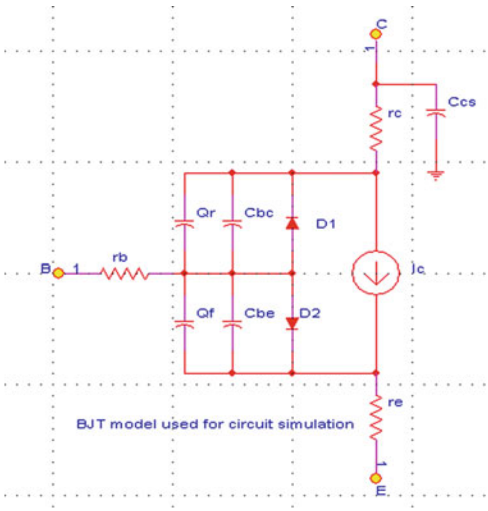


Fig. D.20 Bipolar transistor BJT model used for circuit simulation

Bipolar Transistor Model for Large Signal Circuit Simulation

The BJT model used in circuit simulation can accurately represent the DC and dynamic currents of the transistor in response to $V_{BE}(t)$ and $V_{CE}(t)$. A typical circuit simulation model or compact model is made of the Ebers–Moll model when V_{BE} and V_{BC} are two driving forces for I_C and I_B, plus additional enhancements for high-level injection, voltage dependent capacitances that accurately represent the charge storage in the transistor, and parasitic resistances as shown. This BJT model is known as the Gummel–Poon model. The two diodes represent the two I_B terms due to V_{BE} and V_{BC}. The capacitor labeled Q_F is voltage dependent such that the charge stored in it is equal to the Q_F described in the bipolar transistor transit time and charge storage discussion. Q_R is the counterpart of Q_F produced by a forward bias at the BC junction. Inclusion of Q_R makes the dynamic response of the model accurate even when V_{BC} is sometimes forward biased. C_{BE} and C_{BC} are the junction depletion layer capacitances. C_{CS} is the collector to substrate capacitance (Fig. D.20).

$I_C = I_s' \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - e^{\frac{q \cdot V_{BC}}{k \cdot T}}) \cdot (1 + \frac{V_{CB}}{V_A}) - \frac{I_s}{\beta_R} \cdot (e^{\frac{q \cdot V_{BC}}{k \cdot T}} - 1)$. The $1 + \frac{V_{CB}}{V_A}$ factor is added to represent the early effect – I_C increasing with increasing V_{CB}. I_s' differs from I_s in that I_s' decreases at high V_{BE} due to the high-level injection effect in accordance with equation $G_B \equiv \int_0^{W_B} \frac{n_i^2}{n_{iB}^2} \cdot \frac{p}{D_B} \cdot dx$.

$I_B = \frac{I_s}{\beta_F} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1) + \frac{I_s}{\beta_R} \cdot (e^{\frac{q \cdot V_{BC}}{k \cdot T}} - 1) + I_{SE} \cdot (e^{\frac{q \cdot V_{BE}}{n_E \cdot k \cdot T}} - 1)$. I_{SE} and n_E parameters are determined from the measured BJT data as are all of the several dozens of model parameters. We can summarize the current appendix discussion, that base emitter junction is usually forward biased while the base–collector junction is reverse biased. V_{BE} determines the rate of electron injection from the emitter into the base, and thus uniquely determines the collector current, I_C regardless of the reverse bias V_{CB}. $I_C = A_E \cdot \frac{q \cdot n_i^2}{G_B} \cdot (e^{\frac{q \cdot V_{BE}}{k \cdot T}} - 1); G_B \equiv \int_0^{W_B} \frac{n_i^2}{n_{iB}^2} \cdot \frac{p}{D_B} \cdot dx$ G_B is the base Gummel number, which represents all the subtleties of BJT design that affects I_C; base material, non-uniform base doping, non-uniform material composition, and the high-level injection effect. An undesirable but unavoidable side effect of the application on V_{BE} is a hole current flowing from the base, mostly into the emitter. This base input current, I_B, is related to I_C by the common emitter current gain β_F ($\beta_F = \frac{I_C}{I_B} \approx \frac{G_E}{G_B}$) where G_E is the emitter Gummel number. The common base current gain is $\alpha_F \equiv \frac{I_C}{I_E} = \frac{\beta_F}{1+\beta_F}$. The Gummel plot indicates that β_F falls off in the high I_C region due to high-level injection in the base and also in the low I_C region due to excess base current. Base width modulation by V_{CB} results in a significant slope of the $I_C - V_{CE}$ curve in the active region. This is the early effect. The slope, called the output conductance, limits the voltage gain that can be produced with a BJT. The early effect can be suppressed with a lightly doped collector. A heavily doped sub-collector is routinely used to reduce the collector resistance. Due to the forward bias, V_{BE}, a BJT stores a certain amount of excess hole charge, which is equal but of opposite sign to the excess electron charge. Its magnitude is called the excess carrier charge, Q_F. Q_F is linearly proportional to I_C ($Q_F \equiv I_C \cdot \tau_F$).

τ_F is the forward transit time. If there were no excess carriers stored outside the base $\tau_F = \tau_{FB} = \frac{W_B^2}{2 \cdot D_B}$. τ_{FB} is the base transit time, $\tau_F > \tau_{FB}$ because excess carrier storage in the emitter and in the depletion layer is also significant. All these regions should be made small in order to minimize τ_F. Besides minimizing the base width, W_B, τ_{FB} may be reduced by building a drift field into the base with graded base doping (also with graded Ge content in a SiGe base). τ_{FB} is significantly increased at large I_C due to base widening which known at the Kirk effect. In the Gummel–Poon model, both the DC and the dynamic (charge storage) currents are well modeled. The early effect and high-level injection effect are included. Simpler models consisting of R, C, and current source are used for hand analysis of circuits. The small signal models employ parameters such as trans-conductance $g_m = \frac{dI_C}{dV_{BE}} = I_C / \frac{k \cdot T}{q}$ and input capacitance $C_\pi = \frac{dQ_F}{dV_{BE}} = \frac{d}{dV_{BE}}[\tau_F \cdot I_C] = \tau_F \cdot g_m$ and input resistance$r_\pi = \frac{dV_{BE}}{dI_B} = \frac{\beta_F}{g_m}$. The BJT's unity gain cutoff frequency at which β falls to unity is f_T. In order to raise device speed, device density, or current gain, a modern high-performance BJT usually employs poly-Si emitter, self-aligned poly-Si base contacts, graded Si-Ge base, shallow oxide trench, and deep trench isolation. High-performance BJTs excel over MOSFETs in circuits requiring the highest device g_m and speed.

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